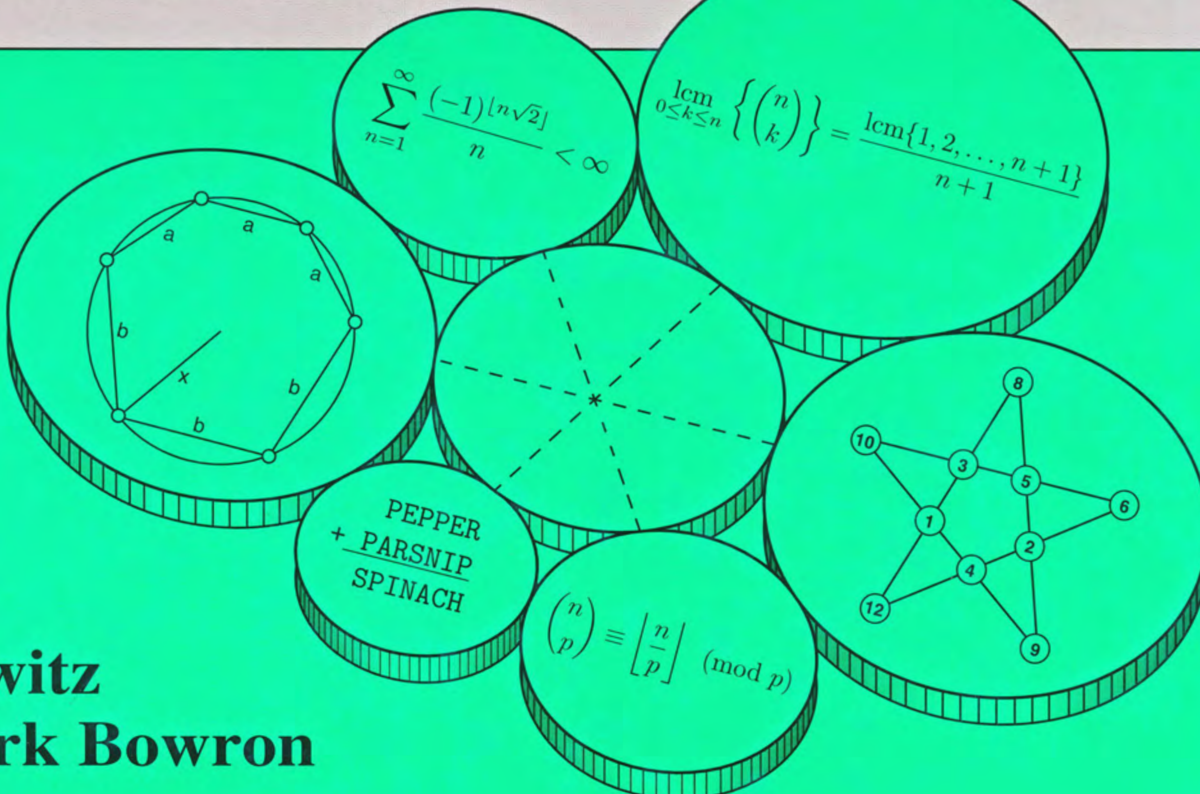
INDEX TO Mathematical Problems 1975–1979



EDITED BY
Stanley Rabinowitz and Mark Bowron

A Compendium of over 5,000 Problems

with Subject, Keyword, Author and Citation Indexes

MathPro Press

Volume 2

Indexes to Mathematical Problems

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INDEXES TO MATHEMATICAL PROBLEMS, VOLUME 2

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edited by

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FOREWORD

A few years ago, MathPro Press published an index to mathematical problems published between 1980 and 1984, and announced an ambitious program to publish other volumes extending this project both forward and backward in time. I was fortunate to have that volume available during my five-year term as Editor of the Problems and Solutions column of the *American Mathematical Monthly*. The system for classifying problems by topic, by itself, adds an important level of organization, as does the section on notation, but the Index goes beyond this to allow related problems to be identified, and locates individuals and journals associated with these related problems. The wealth of information about the problems and their means of publication is an enormous service to anyone facing the task of preparing a list of problems.

In his foreword to the 1980–1984 Index, Murray S. Klamkin was critical of the indexing information on problems provided by journals. Although I was in a position to move one journal in the direction that he indicated, there was little change. It thus falls on me to defend the present state of indexing of problems in journals. It is not really an answer to say that I was never asked to develop a better index, since I am sure that anything along these lines that I produced would have been used. The system has some inertia based on the way that various tasks are assigned to meet publication deadlines, although sweeping changes are often made when there is a change of editor. However, the fans of Problem Sections also tend to have strong opinions. Removing the distinction between Elementary and Advanced problems had already generated strong comments: half opposed to the change, and half in favor. While the subject classification of this volume is useful for organizing thousands of problems, it is not clear that such a classification would be useful for fewer than one hundred problems. Indexing by author has the nice feature that it encourages the reader to use the name of the author as the key to locating a distinctive problem or solution. Those whose skill in formulating problems and writing insightful solutions deserve to be closely identified with their work. Additional indexing may well be better confined to indexes of broader scope. The continuation of this project will raise the general level of awareness of this aspect of doing mathematics, and give a better picture of the high value placed on this activity.

The spectrum of problems runs from routine exercises to the great problems capable of inspiring the development of mathematics for a century or more. Those represented here are chosen from a smaller range from contest problems allowing an hour or so to journal problems for which several months of work are needed for an adequate solution. Although this avoids the extremes of the spectrum, there is still room for significant difference in difficulty. Since the reader is expected to be able to solve these problems, it is reasonable to expect that each problem contains the seeds of its solution. Also, full statements of problems are given, so the Index may be enjoyed by someone interested in the subject, as well as (indeed, probably more than) those who need it in their work. A beginner may need guidance in selecting problems suitable to his present level of training, but an experienced mathematician should develop an irresistible urge to pick up pencil and paper after opening the book to a random page in the Subject Index. I would go so far as to suggest that this Index is the ideal retirement gift for a mathematician to allow the fun of the subject to be rediscovered after a career that has reached the point of research on a highly specialized topic and teaching of the same old subjects.

Richard T. Bumby
professor of mathematics
Rutgers University

ACKNOWLEDGMENT

The effort of many people went into producing this second volume in the *Indexes to Mathematical Problems* series. Their help is greatly appreciated. Data entry and proofreading were done by **Selma Burrows, Mark Buxbaum, Michael Clarke, Anne R. Costa, Joan Duprey, Kathi Duprey, Cheryl Hoffman, Helen Metcalf, Carol Anderson Peters, Stanley Rabinowitz, Dennis Spellman,** and **Mindy Swartz.** The monumental task of producing a book like this could not have been completed without their help.

I want to thank everyone else who performed translations into English, including George Berzsenyi, Roel Lipsch, Susan Oliver, Dennis Spellman, and Jordan Tabov. The editorial board (George Berzsenyi, Clark Kimberling, Murray S. Klamkin, Leroy F. Meyers, and Jordan Tabov) did an exceptional job in reviewing all of the material from my many mailings. They offered good advice and were instrumental in the shaping of this index. Many thanks to Richard T. Bumby for writing the foreword to this volume, and to Murray Klamkin for his continuing encouragement. Problems were classified by Mark Bowron, Joan Duprey, Stanley Rabinowitz, and Dennis Spellman. Special thanks to Mark Bowron for adding the cross-referencing scheme to the keyword index, to Mark Buxbaum for valuable help with that index, and to Herb Jacobs for his help in many phases of the book production. The cover was designed and composed by Kathi Duprey at Ad Infinitum Graphics.

This index gives credit to the many authors of the problems indexed herein and also carefully cites the sources (names of the journals and page numbers) in which these problems were originally posed. Consult the Problem Chronology (page 282) to determine the original source for any problem listed. I wish to thank the following organizations for giving me permission to re-print their problems in this index:

The Mathematical Association of America (for The American Mathematical Monthly, Mathematics Magazine, The Two-Year College Mathematics Journal), The MATYC Journal, Inc. (for The MATYC Journal), the Canadian Mathematical Society (for Crux Mathematicorum and The Canadian Mathematical Bulletin), the Fibonacci Association (for The Fibonacci Quarterly), Baywood Publishers (for The Journal of Recreational Mathematics), the National Council of Teachers of Mathematics (for The Mathematics Student Journal), the University of Waterloo (for The Ontario Secondary School Mathematics Bulletin), Kappa Mu Epsilon (for The Pentagon), the councilors of Pi Mu Epsilon (for The Pi Mu Epsilon Journal), the Society for Industrial and Applied Mathematics (for SIAM Review), School Science and Mathematics Association (for School Science and Mathematics), the editorial committee of Function, the Malaysian Mathematical Society (for Menemui Matematik), the Association of Mathematics Teachers of New York State (for The New York State Mathematics Teachers' Journal), the Ontario Association for Mathematics Education (for the Ontario Mathematics Gazette), the editor of Mathematical Spectrum, and Stickting Mathematisch Centrum (for Nieuw Archief voor Wiskunde).

I am indebted to Donald Knuth for designing the TeX system for typesetting technical text which was used to typeset the mathematical portions of this book, and to Michael Spivak, who designed the MathTime family of fonts.

I wish to thank the many people who supplied details about unsolved problems and papers that reference problems:

Julia Abrahams, Ed Barbeau, Petter Bjørstad, F. S. Cater, Peter Giblin, Jacob E. Goodman, Doug Hensley, James Hirschfeld, Melvin Hochster, David M. Jackson, Kenneth R. Kellum, Dan J. Kleitman, Robert Leslie, John S. Lew, Colin L. Mallows, Armel Mercier, Eric Milner, Thomas E. Moore, Harry Nelson, Louis Nirenberg, Joseph O'Rourke, Carl Pomerance, James Propp, David Singmaster, Dan Sokolowsky, Lloyd N. Trefethen, Peter Ungar, and J. Ernest Wilkins.

The following people helped me in locating journals, contests, pseudonyms, and bibliographic references:

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Finally, I wish to thank all the problemists out there who have enriched my life and the lives of many others by contributing such fine problems to the mathematical literature. I hope they will understand the few cases where I have had to edit their creations in order to remove extraneous matter and fit the space constraints of this index. A great thank you must go to all those problem column editors who endure a thankless task but provide an invaluable service to the mathematics community.

- Stan -

PREFACE TO THIS PDF EDITION

This ebook contains many corrections and revisions to the original book. Hyperlinks and text search eliminate the need for any Problem Locator, so that section has been removed. In the Subject Index, the relatively short Statistics and Symbolic Logic sections have been subsumed under Probability and Set Theory, respectively.

Many solutions are now readily available online. Links colored green below represent free unlimited access. Blue links represent JSTOR, which offers free but limited access (the Putnam page leads to JSTOR). Red links represent paid access.

AMM	American Mathematical Monthly	https://www.jstor.org/journal/amermathmont
CRUX	Crux Mathematicorum	https://cms.math.ca/crux/
FQ	The Fibonacci Quarterly	https://www.fq.math.ca/list-of-issues.html
MM	Mathematics Magazine	https://www.jstor.org/journal/mathmaga
PARAB	Parabola	https://www.parabola.unsw.edu.au/2010-2019/volume-54-2018/issue-1
PENT	The Pentagon	http://www.kappamuepsilon.org/pages/a/pentagon.php
PME	The Pi Mu Epsilon Journal	http://www.pme-math.org/journal/issues.html
SIAM	SIAM Review	https://epubs.siam.org/toc/siread/current
SPECT	Mathematical Spectrum	http://www.appliedprobability.org/content.aspx?Group=ms&Page=allmsissues
SSM	School Science and Mathematics	https://onlinelibrary.wiley.com/loi/19498594
TYCMJ	The Two-Year College Mathematics J.	https://www.jstor.org/journal/twoyearcollmathj
PUTNAM	The Putnam Mathematical Competition	https://kskedlaya.org/putnam-archive/
USA	U.S.A. Mathematical Olympiad	https://mks.mff.cuni.cz/kalva/usa.html

Separate permissions were obtained from all of the rights holders for this electronic version to be made available at no charge, for which we are grateful. We may do the same for Volume 1, which covers the years 1980-1984.

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PREFACE TO THE ORIGINAL BOOK

In the preface to the first edition of their book *Concrete Mathematics* published in 1989, Graham, Knuth, and Patashnik state that *they have not been able to pin down the sources of many problems that have become part of the folklore.* The same year that book was published, a new company called MathPro Press was founded by Stanley Rabinowitz to reference the world's problem literature in a comprehensive way. In 1992, Rabinowitz introduced a series of books called *Indexes to Mathematical Problems.* The first volume in that series, *Index to Mathematical Problems 1980-1984,* represented a milestone in problem indexing: never before had so many problems from so many sources been gathered together into a single volume. To problemists worldwide, the introduction of this remarkable book brought the hope that problems and their references might eventually become very easy to locate.

As one might guess, the production of such an index requires an enormous amount of work. But it was not expected that nine years would pass from the time the first volume was published until the time this second one was published. The main culprit behind the delay was a seemingly endless variety of loose ends and details that needed tending to. In addition, the editors were not able to devote as much attention to the project as they would have liked during the past few years, because their full-time jobs demanded most of their time. A great deal of work was performed by generous volunteers. Given the unique and valuable nature of this project, it is hoped that some sort of alliance may be forged with one or more of the various mathematical associations that publish problem columns and contests, in order to accelerate book production in the future.

We were saddened by the passing of our friend and advisor Leroy F. Meyers in 1995. He played an important role in the success of these indexes, offering many detailed comments (especially with respect to the proper spelling of people's names) and providing English translations of problems originally published in Dutch. He is missed by all of us at MathPro Press.

To the many people who offered encouragement and miscellaneous help, we salute

> Gary Barna, Cathy Bence, George and Roberta Berry, Mark Buxbaum, Anton and Peggy Chernoff, Anne and Peter Costa, Bill and Tricia Fisher, Peter Gilbert, Tim and Cheryl Hoffman, Herb Jacobs, Clark Kimberling, Murray Klamkin, Joe Konhauser, Hank Lieberman, Erwin Lutwak, Walter Mientka, Peter O'Halloran, Bill Perkins, Susan Perkins, Eric and Carol Peters, Jack and Fay Rabinowitz, Jim and Sharon Ravan, Leanne Robertson, Josh Rosen, Léo Sauvé, Dave Scheifler, Leo Schneider, Larry Somer, Rob and Marty Spence, Judy Swank, Rick Swift, Craig Thomas, and Marijke van Gans.

Except for the specific problems indexed, there are few differences between the first two volumes. A new cross-referencing feature was added to the keyword index that allows users to readily browse all classifications containing a given keyword. This feature adds a new dimension to the classification scheme: while still grouping similar problems together as before, now it can also be used to conduct keyword searches. Partly as a result of this, and partly because the overall database is growing, more detail was added to the lower-level classification categories in this volume.

Another difference between the two volumes is in the typefaces used. The first volume was typeset with the Computer Modern family of fonts created by Donald Knuth, whereas this volume was typeset with the MathTime family of fonts created by Michael Spivak. Finally, Volume 1 contained a very comprehensive list of journals with problem columns; this section has been omitted from the present volume to avoid redundancy. (The list can also be viewed online at our website address: www.mathpropress.com.)

It should be noted that the collection of contests referenced in this volume is by no means comprehensive. There are dozens of fine contests held around the globe each year. Ones missing from this volume were either too difficult to obtain, or unknown to us. Readers are encouraged to submit information regarding any contests missing from this series (or journals missing from the list mentioned above), so they may be included in future volumes.

It is hoped that users will find this volume at least as useful as Volume 1, if not more useful. The editors must apologize for any errors or omissions in the presentation. It seemed better to publish with errors than to err by not publishing at all. Please report any typos or mistakes to MathPro Press so they may be corrected in future editions. We are constantly striving to improve our methods of problem indexing, so all comments and suggestions for future enhancements are heartily welcomed.

Stanley Rabinowitz — Chelmsford, MA

Mark Bowron — Laughlin, NV

For Roma the librarian,
who so dutifully reshelved all those hundreds of
books and journals wildly torn from the shelf
during frantic problem-hunting sprees...

HOW TO USE THIS INDEX

WHAT IS INDEXED?

To determine which journals and contests have been indexed, see the list on page 17. A more detailed list can be found in the Journal Issue Checklist (page 401). That section also tells you which columns in these journals were indexed.

SEARCH BY TOPIC

Given a topic that you are interested in, consult the Subject Classification Scheme (page 7) to find the classification closest to your topic of interest. Click on that topic and scan the problems looking for those of interest to you. A quick overview of the classification scheme can be found in the Table of Contents.

SEARCH BY KEYWORD

You can look in the Keyword Index (page 440) under various keywords that pertain to your topic of interest to locate specific problems associated with this keyword. This is particularly useful when you remember a memorable word or phrase from the problem you are searching for. You can also check the Title Index (page 355) to see if the keyword appears in the title of a problem.

SEARCH BY AUTHOR

If you know the author of a problem (or are interested in problems by a specific author), use the Author Index (page 316). For references to biographical information about problemists, see page 430.

LOCATING A PROBLEM

Given a problem number (for example, one listed in the Keyword Index), click on it to jump to the page in this index where the text of the problem is printed. Scanning around on that page may also show you related problems that may be of interest to you.

Note that the text for the problem as printed in this index may only be a summary of the full text as originally printed. We have omitted extraneous information and may have reworded the problem to make the notation consistent. To find the original complete wording for this problem, consult the Problem Chronology (page 282) to find the journal, volume, and page number where the problem was proposed.

When a problem number is not a hyperlink, this means that the text of the problem does not appear in this index (because the problem was not published during the years 1975–1979). Consult the Problem Chronology (page 282) to find a reference to a solution or comment concerning this problem.

To find problems of a certain type or difficulty level, determine which journals or contests normally publish problems of the kind that you are interested in. Then scan the appropriate portion of the subject index for problems from these journals.

LOCATING A SOLUTION

Once you find a specific problem that interests you, click the problem number to jump to the Problem Chronology (page 282). You will then find references to where the problem was published (journal, volume, issue, and page number) as well as references to all published solutions, partial solutions, and comments related to this problem. Additional references may be found in the Citation Index (page 423), which lists journal articles that reference problems covered by this index.

CONTESTS

Consult the Citation Index (page 423) to find references to specific contest problems. Articles about complete contests (frequently reprinting the problems from the contest and often containing solutions) can be located in the Contest References section of the Citation Index (page 430).

LOCATING A JOURNAL

The list of abbreviations for the journal names can be found on page 17. A more complete list of journal abbreviations is given in Volume 1, on page 437 of that volume. If you want to examine a problem or solution from some journal and that journal is not in your library, consult the Journal Information section (page 435 of Volume 1) for data about the journal, such as the ISSN number. Your librarian should be able to help you locate a library that carries this journal from the bibliographic in-formation given. The name of the publisher is also given, along with the address to write to for subscription information if the journal is still active. (The list of journals can also be viewed online at our website address: www.mathpropress.com.)

NOTATION

For unfamiliar terms or notation, consult the Notation (page 3) or the Glossary (page 438).

UNSOLVED PROBLEMS

A convenient compilation of those problems proposed during the years 1975–1979 that remain unsolved as of 1991 can be found on page 413. An author index to the proposers of these unsolved problems can be found on page 422. Consult the Problem Chronology (page 282) to locate references to partial solutions to these problems. Additional references to these problems in the literature can be found in the Citation Index (page 423).

PROBLEM BOOKS

A list of problem books that have been reviewed during the years 1975–1979 can be found on page 430.

BIBLIOGRAPHY

References to journal articles appear in square brackets. See page 431 for the full bibliographic citation.

ADDITIONAL INFORMATION

Each section of this index also includes additional details on how to use that section.

Notation

We have attempted to use a common notation and may therefore have modified the statement of a problem for the purposes of consistency. The most frequently appearing symbols are listed below. Consult the glossary on page 438 for additional information about some of the terms used herein. Since the problems covered by this index encompass a large portion of the field of mathematics, it is not possible to list every symbol used. For symbols not appearing in this list, consult the individual problem in question and/or go back to the original source of the problem where more detail about the notation may be given. For problems dealing with very specialized topics, it is assumed that the reader is familiar with the specialized notation. Consult any standard textbook on the subject if you need further information about such specialized notation.

α	In problems about Fibonacci numbers, $\alpha = (1+\sqrt{5})/2$.
β	In problems about Fibonacci numbers, $\beta = (1-\sqrt{5})/2$.
$B(m,n)$	Beta function: $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$.
γ	Euler's constant: $\gamma = \lim_{n\to\infty}(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n)$.
$\Gamma(x)$	gamma function: $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$. If n is a nonnegative integer, $\Gamma(n+1) = n!$.
$\Delta f(x)$	First difference: $\Delta f(x) = f(x+1) - f(x)$.
$\Delta^n f(x)$	nth difference: $\Delta^n f(x) = \Delta\left(\Delta^{n-1} f(x)\right)$.
$\zeta(s)$	Riemann Zeta Function: $\zeta(s) = \sum_{n=1}^\infty 1/n^s$.
$\mu(n)$	Möbius mu function: $\mu(1) = 1$, $\mu(n) = 0$ if n has a squared factor, $\mu(p_1p_2\dots p_k) = (-1)^k$ if all the primes $p_1, p_2, \dots, p_k$ are different.
π	pi: ratio of circumference of a circle to its diameter.
$\displaystyle\prod_{k=m}^{n} f(k)$	product of terms of the form $f(k)$ as the integer k ranges from m to n. Also written as $\prod_{k=m}^n f(k)$.
$\sigma(n)$	sum of the divisors of n (including 1 and n).
$\displaystyle\sum_{k=m}^{n} f(k)$	sum of terms of the form $f(k)$ as the integer k ranges from m to n. Also written as $\sum_{k=m}^n f(k)$.
$\displaystyle\sum_{k\in S} f(k)$	sum of terms of the form $f(k)$ as k ranges through all elements in set S.
$\displaystyle\sum_{\text{sym}} f(x_1,\dots,x_n)$	Symmetric sum: sum of terms of the form $f(x_{\sigma(1)},\dots,x_{\sigma(n)})$ as σ ranges through all permutations of $(1,\dots,n)$. Can also denote the sum, over all $\binom{n}{r}$ subsets of r variables among n given variables, of a symmetric function f of r variables.
$\tau(n)$	number of divisors of the positive integer n.
$\phi(n)$	Euler's totient function: number of positive integers less than or equal to n that are relatively prime to n.
$\varnothing$	the null set.
$\psi(z)$	digamma function: $\psi(z) = \Gamma'(z)/\Gamma(z)$.
ω	Brocard angle of a triangle ABC: $\cot\omega = \cot A + \cot B + \cot C$.
∞	infinity.
$\pm$	plus or minus.
$x \times y$	x times y, also written $x \cdot y$ or just xy.
$m \times n$	m by n (as in an $m \times n$ array).
$\angle ABC$	angle ABC.
$\triangle ABC$	triangle ABC.
$[ABC]$	area of triangle ABC.
AB	line segment AB or line AB. May also refer to the length of line segment AB. Sometimes written as $\overline{AB}$.
$\overrightarrow{AB}$	vector AB or ray from A through B.
$\vec{x}$	vector x.
AB	arc of circle from A to B.
$\triangle ABC \cong \triangle XYZ$	triangles ABC and XYZ are congruent.
$\triangle ABC \sim \triangle XYZ$	triangles ABC and XYZ are similar.
$f(x) \sim g(x)$	f is asymptotic to g: $f(x)/g(x) \to 1$.
$AB \| CD$	AB is parallel to CD.
$AB \perp CD$	AB is perpendicular to CD.
$\|\|A\|\|$	norm of matrix A.
A^{T}	transpose of matrix A.
A^*	conjugate transpose of matrix A.
$d \mid n$	d divides n.
$d \nmid n$	d does not divide n.
$m : n$	ratio of m to n.
m/n	m divided by n. Usage note: a/bc means $a/(bc)$.
$m \div n$	same as m/n.
$\left(\frac{n}{p}\right)$	Also written as (n/p). Legendre symbol: If p is an odd prime, $(n/p) = 1$ if there is an x such that $x^2 \equiv n \pmod{p}$ and $(n/p) = -1$ otherwise.
$\|x\|$	absolute value of real number x: $\|x\| = x$ if $x \ge 0$ and $\|x\| = -x$ if $x < 0$.
$\|z\|$	norm of complex number z: If $z = a + bi$ where a and b are real, then $\|z\| = \sqrt{a^2+b^2}$.
$x < y$	x is less than y.
$x \le y$	x is less than or equal to y.
$x > y$	x is greater than y.
$x \ge y$	x is greater than or equal to y.
$x \prec y$	x precedes y in some ordering.
$x \succ y$	x follows y in some ordering.
$x = y$	x equals y.
$x \ne y$	x is not equal to y.
$\{x \mid \text{condition}\}$	the set of all x such that the specified condition is true. Also written as $\{x : \text{condition}\}$.

Notation

$\{x_n\}$	the sequence $x_1, x_2, x_3 \ldots$.
(a, b)	point with coordinates a and b. Also ordered pair.
$(a_1, a_2, \ldots, a_n)$	point with specified coordinates. Also ordered n-tuple.
(a, b)	open interval from a to b.
$[a, b]$	closed interval from a to b.
$(a, b]$, $[a, b)$	half-open intervals.
$n!$	Factorial: $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. Usage note: $2n!$ means $2(n!)$. By definition, $0! = 1$.
$x \in A$	x is an element of the set A.
$x \notin A$	x is not an element of the set A.
¢	monetary amount in cents.
\$	monetary amount in dollars.
£	monetary amount in pounds sterling.
$x \approx y$	x is approximately equal to y.
$y \propto x$	y is proportional to x: there is a constant k such that $y = kx$.
$A \cup B$	union of sets A and B. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
$A \cap B$	intersection of sets A and B. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
$AB \cap CD$	point that is the intersection of lines AB and CD.
$A \setminus B$	set difference: $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.
$\overline{A}$	topological closure of set A.
$x \equiv y \pmod{m}$	$m \mid (x - y)$.
$x \not\equiv y \pmod{m}$	$m \nmid (x - y)$.
$x \bmod y$	remainder: $x - y\lfloor x/y \rfloor$.
$\{x\}$	fractional part: $x \bmod 1$.
$\lfloor x \rfloor$	floor: greatest integer not larger than x: $\lfloor x \rfloor = \max\{n \mid n \in \mathbb{Z} \text{ and } n \le x\}$.
$\lceil x \rceil$	ceiling: smallest integer not less than x: $\lceil x \rceil = \min\{n \mid n \in \mathbb{Z} \text{ and } x \le n\}$.
$f^n(x)$	nth iterate of the function f, i.e. $f(f(f(\ldots f(x)\ldots)))$. Exception: For trigonometric functions, the superscript represents an exponent. For example, $\sin^n \theta$ means $(\sin\theta)^n$.
$f\colon A \to B$	a function f that maps A into B.
$[G : H]$	index of subgroup H in group G.
G/H	quotient group by normal subgroup H.
$G \cong H$	groups G and H are isomorphic.
$\lvert G \rvert$	order of group G: the number of elements in the group.
$f'(x)$	first derivative of $f(x)$ with respect to x.
$f''(x)$	second derivative of $f(x)$ with respect to x.
$f'''(x)$	third derivative of $f(x)$ with respect to x.
$f^{(n)}(x)$	nth derivative of the function f at x.
$\dot{x}$	derivative of x with respect to t.
$\ddot{x}$	second derivative of x with respect to t.
$D^n f$	nth derivative of f.
$\frac{df(x)}{dx}$	derivative of $f(x)$ with respect to x.
$\frac{d^n f(x)}{dx^n}$	nth derivative of $f(x)$ with respect to x.
$\frac{\partial f}{\partial x}$	partial derivative of f with respect to x.
∂S	boundary of set S.
$[f(x)]_{x=a}$	function $f(x)$ evaluated when $x = a$.
$\int f(x)\,dx$	indefinite integral of $f(x)$.
$\int_a^b f(x)\,dx$	definite integral of $f(x)$ from a to b.
$\int_{\mathbb{R}}$	integral over the real line.
$\int_{\mathbb{C}}$	integral over the complex plane.
$\overline{z}$	complex conjugate of the complex number z: If $z = a + bi$ where a and b are real, then $\overline{z} = a - bi$.
$A \Rightarrow B$	A implies B.
n°	n degrees.
`ABCDE`	This font indicates that the letters represent successive digits of a number written in the usual radix form. If no radix is specified, base 10 is assumed.
$\texttt{ABCDE}_b$	a number written to base b.
$\binom{n}{k}$	binomial coefficient ("n choose k"): $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
$\left[{n \atop k}\right]$	Stirling number of the first kind ("Stirling cycle number"): $x^{\overline{n}} = \sum_k \left[{n \atop k}\right] x^k$.
$\left\{{n \atop k}\right\}$	Stirling number of the second kind ("Stirling subset number"): $x^n = \sum_k \left\{{n \atop k}\right\} x^{\underline{k}}$.
$\sqrt{x}$	square root of x.
$\sqrt[n]{x}$	nth root of x.
x^n	x to the nth power.
$x^{\underline{n}}$	falling factorial: $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$.
$x^{\overline{n}}$	rising factorial: $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$. Sometimes written as $(x)_k$ ("Pochhammer's symbol").
$K[x]$	ring of polynomials in the variable x with coefficients from the field K.
$K[x, y]$	ring of polynomials in the variables x and y with coefficients from the field K.
A_n	alternating group on n elements.
a, b, c	In problems about $\triangle ABC$, a, b, and c denote the lengths of the sides of the triangle.
A, B, C	In problems about $\triangle ABC$, A, B, and C denote the angles of the triangle.
$\forall x$	for all x.
$\operatorname{adj} A$	(classical) adjoint of matrix A.
$\arccos x$	principal value of angle whose cosine is x. Also written as $\cos^{-1} x$.
$\operatorname{arccot} x$	principal value of angle whose cotangent is x. Also written as $\cot^{-1} x$.

Notation

$\arcsin x$	principal value of angle whose sine is x. Also written as $\sin^{-1} x$.
$\arctan x$	principal value of angle whose tangent is x. Also written as $\tan^{-1} x$.
$\arg z$	argument of complex number z: If $z = r(\cos\theta + i \sin\theta)$, then $\arg z = \theta$.
B_n	Bernoulli number: $\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$.
$\mathbb{C}$	the set of complex numbers.
C^k	the set of k times differentiable functions.
C^∞	the set of infinitely differentiable functions.
$\mathrm{card}(A)$	cardinality of a set A: the number of elements in A. Sometimes written as $\lvert A\rvert$.
$\cos\theta$	cosine of the angle θ.
$\cosh\theta$	hyperbolic cosine of the angle θ.
$\cot\theta$	cotangent of the angle θ.
$\csc\theta$	cosecant of the angle θ.
$\mathrm{csch}\,\theta$	hyperbolic cosecant of the angle θ.
$\det(A)$	determinant of square matrix A.
$\mathrm{diag}(a_1,\ldots,a_n)$	$n \times n$ diagonal matrix with elements $a_1, a_2, \ldots, a_n$ along the diagonal.
e	base of natural logarithms: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.
E^n	Euclidean n-space.
$E[x]$	expected value.
$\exists x$	there exists an x such that.
$\mathrm{erf}(x)$	error function: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$.
$\exp(x)$	exponential function: e^x.
F_n	Fibonacci number: nth term in the sequence 1, 1, 2, 3, 5, 8, 13, ... defined by the recurrence: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$.
$f_n(x)$	Fibonacci polynomial defined by the recurrence $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$ with initial conditions $f_1(x) = 1$ and $f_2(x) = x$.
${}_mF_n\left(\begin{smallmatrix} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{smallmatrix} \middle\vert z\right)$	also written as ${}_mF_n(a_1,\ldots,a_m; b_1,\ldots,b_n; z)$. hypergeometric function: $F\left(\begin{smallmatrix} a_1,\ldots,a_m \\ b_1,\ldots,b_n \end{smallmatrix} \middle\vert z\right) = \sum_{k\ge 0} \frac{a_1^{\overline{k}}\cdots a_m^{\overline{k}}}{b_1^{\overline{k}}\cdots b_n^{\overline{k}}}\frac{z^k}{k!}$. The subscripts m and n may be omitted if their values are clear.
$\gcd(m,n)$	greatest common divisor of integers m and n.
$\mathrm{GF}(q^n)$	Galois field with q^n elements.
H_n	Harmonic number: $H_n = \sum_{k=1}^n \frac{1}{k}$.
h_a, h_b, h_c	In problems about $\triangle ABC$, h_a, h_b, and h_c denote the lengths of the altitudes of the triangle.
i	imaginary unit: $i = \sqrt{-1}$.
iff	if and only if.
$\mathrm{Im}(z)$	imaginary part of the complex number z: If $z = a + bi$ where a and b are real, then $\mathrm{Im}(z) = b$.
$\inf$	infimum.
$J_\nu(z)$	Bessel function (of the first kind): $J_\nu(z) = \frac{(x/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty \cos(x\cos(t))(\sin^{2\nu} t)\,dt$.
K	In problems about $\triangle ABC$, K denotes the area of the triangle. (F is sometimes used in the literature.)
L_n	Lucas number: nth term in the sequence 1, 3, 4, 7, 11, 18, 29, ... defined by the recurrence: $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$.
$\mathrm{L}^p[a,b]$	space of p-times differentiable functions on the interval $[a,b]$.
$\mathrm{lcm}[m,n]$	least common multiple of integers m and n.
$\lg x$	binary logarithm: $\log_2 x$.
$\lim\limits_{x\to a} f(x)$	limit of $f(x)$ as x approaches a.
$\lim\limits_{x\to a+} f(x)$	limit of $f(x)$ as x approaches a from above.
$\lim\limits_{x\to a-} f(x)$	limit of $f(x)$ as x approaches a from below.
$\liminf$	greatest lower limit. Also written as $\underline{\lim}$.
$\limsup$	least upper limit. Also written as $\overline{\lim}$.
$\ln x$	natural logarithm: $\log_e x$.
$\log x$	common logarithm: $\log_{10} x$. Usage note: $\log x/\log y$ means $(\log x)/(\log y)$.
$\log_b x$	logarithm of x to the base b.
$m(A)$	Lebesgue measure of the set A.
m_a, m_b, m_c	In problems about $\triangle ABC$, m_a, m_b, and m_c denote the lengths of the medians of the triangle.
$\max(a,b,\ldots)$	maximum of a set of numbers.
$\min(a,b,\ldots)$	minimum of a set of numbers.
$\mathbb{N}$	the set of natural numbers (integers larger than 0).
$o(n)$	$k = o(n)$ means that $k/n \to 0$.
$O(f(n))$	$g(n) = O(f(n))$ means that there is a constant C such that $\lvert g(n)\rvert \le C\lvert f(n)\rvert$.
P_n	Pell number (of the first kind): nth term in the sequence defined by the recurrence: $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$.
$P_n(x)$	Legendre polynomial: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.
$P(n,k)$	Permutation: $P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$.
$\mathrm{per}(A)$	permanent of square matrix A.
$P(x)$	probability that event x is true.
$\mathbb{Q}$	the set of rational numbers.
Q_n	Pell number of the second kind: nth term in the sequence defined by the recurrence: $Q_0 = 1$, $Q_1 = 1$, and $Q_n = 2Q_{n-1} + Q_{n-2}$.
$\mathbb{R}$	the set of real numbers.
$\mathbb{R}^n$	for our purposes, same as E^n, Euclidean n-space.

Notation

R	In problems about $\triangle ABC$, R denotes the length of the circumradius of the triangle.
r	In problems about $\triangle ABC$, r denotes the length of the inradius of the triangle.
r_a, r_b, r_c	In problems about $\triangle ABC$, r_a, r_b, and r_c denote the lengths of the exradii of the triangle.
$\mathrm{Re}(z)$	real part of the complex number z: If $z = a + bi$ where a and b are real, then $\mathrm{Re}(z) = a$.
S_n	symmetric group on n elements.
s	In problems about $\triangle ABC$, s denotes the semiperimeter of the triangle: $s = (a + b + c)/2$.
$\sec\theta$	secant of the angle θ.
$\operatorname{sech}\theta$	hyperbolic secant of the angle θ.
$\sin\theta$	sine of the angle θ.
$\sinh\theta$	hyperbolic sine of the angle θ.
$\operatorname{sgn}(x)$	sign of x: $\operatorname{sgn}(0) = 0$, $\operatorname{sgn}(x) = 1$ if $x > 0$, and $\operatorname{sgn}(x) = -1$ if $x < 0$.
sup	supremum.
$T_n(x)$	Chebyshev polynomial of the first kind: $T_n(x) = \cos(n \arccos x)$.
t_a, t_b, t_c	In problems about $\triangle ABC$, t_a, t_b, and t_c denote the lengths of the angle bisectors of the triangle.
$\tan\theta$	tangent of the angle θ.
$\tanh\theta$	hyperbolic tangent of the angle θ.
$\mathrm{tr}(A)$	trace of the matrix A: sum of the elements along the main diagonal.
$U_n(x)$	Chebyshev polynomial of the second kind: $U_n(x) = \sin[(n+1)\arccos x]/\sin(\arccos x)$.
$\mathbb{Z}$	the set of integers.
$Z[G]$	center of group G.

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Subject Classification Scheme

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solid geometry
thumbtacks
triangles
n-dimensional geometry
4-space
convexity
curves
inequalities
simplexes
volume
Non-Euclidean geometry
Octagons
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Euler line
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Groupoids
Groups
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alternating groups
associativity
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Subject Classification Scheme

Subject Classification Scheme

Probability

Recreational Mathematics

Set Theory

Solid Geometry

Topology

Trigonometry

SUBJECT INDEX

(Problems sorted by topic)

Use this section to

- find problems related to a given topic
- find problems similar to a given problem
- locate the text for a specific problem

In this section, we list all problems that were published during the years 1975–1979 sorted by topic. The full text of each problem is frequently given, but in many cases, only a summary of the problem will appear. The intent is to print enough of the problem so that you can locate the problem or result you are looking for without having to consult secondary references. We have taken the liberty of removing all extraneous matter from the text of the problems. Once a suitable problem has been found, you should consult the original source for more information. The proposer may have included additional information or references. We have also attempted to standardize the terminology and notation of the problems so that a consistent notation is used throughout this index. See the Glossary (page 438) or the Notation (page 3) for any unfamiliar terms or notation.

To locate the reference to where a problem originally appeared, click on the problem number to consult the Problem Chronology (page 282). The chronology will also give you references to where you can find the solution to the problem or comments about the problem. See also the Citation Index (page 423) to locate articles that reference this problem.

The Subject Classification Scheme used can be found beginning on page 7 of this index; or you can just browse around through the problems until you find the subject area you are interested in. If you are looking for a specific problem for which you remember some memorable phrase or concept, you may find it easier to locate the problem using the Keyword Index (page 440).

In many cases, a problem could be classified under more than one topic. We have chosen the one topic that we feel best represents the problem. Thus, each problem appears just once in this index. The Keyword Index (page 440) may be useful in locating a problem from secondary topics.

This index is not a problem book. The solutions to the problems are not printed here. To find the solution to a given problem, use the Problem Chronology (page 282) to locate the volume and page number of the journal in which the solution is published. Go to that journal for more information. Links to many journal archives appear in the Preface on page ix.

To the right of each problem number is the name of the person that proposed the problem. Not all journals require that published problems be original. If the journal indicated that the problem was not new, we use the phrase “submitted by” before the submitter’s name. If no author’s name is given, this means that the editor selected the problem from the mathematical folklore.

If two problems are identical or nearly the same, both problem numbers are listed in succession and then the text for the problem is printed just once.

Problems marked with an asterisk appear in the Unsolved Problems section (page 413).

When making references to a given problem in scholarly articles, you should give the original reference to the problem (or solution) in your paper. You should not reference this index. When referencing a problem or solution from a problem column in a bibliography or list of references, be sure to include the author's name and a proper reference to the journal containing the problem or solution. You should also go back to the original source and check out the exact text of the problem. Remember that the text as printed in this index may be a summary only and may contain omissions or revisions. Be sure to check the chronology of the problem (page 282) to see if there were any corrections printed for the original statement of the problem. You should also check out the comments and solutions published for the problem — there may be other notes there too concerning corrections to the statement of the problem. If you cannot locate the journal containing the problem, you can use the Journal Issue Checklist (page 401) to get the full name of the journal, as it was known at the time the problem was published, for inclusion in your bibliography. Use the Problem Chronology (page 282) to get the volume and page number where the problem was proposed. The Journal Information section (page 435 of Volume 1) may be useful to you in locating a library that holds the journal you are interested in.

JOURNALS AND CONTESTS COVERED BY THE SUBJECT INDEX

Journals:

Abbreviation	Name
AMM	The American Mathematical Monthly
CMB	Canadian Mathematical Bulletin
CRUX	Crux Mathematicorum
DELTA	Delta
FQ	The Fibonacci Quarterly
FUNCT	Function
ISMJ	Indiana State Mathematics Journal
JRM	Journal of Recreational Mathematics
MATYC	The MATYC Journal
MENEMUI	Menemui Matematik
MM	Mathematics Magazine
MSJ	The Mathematics Student Journal
NAvW	Nieuw Archief voor Wiskunde
NYSMTJ	The New York State Mathematics Teachers' Journal
OMG	Ontario Mathematics Gazette
OSSMB	Ontario Secondary School Mathematics Bulletin
PARAB	Parabola
PENT	The Pentagon
PME	The Pi Mu Epsilon Journal
SIAM	SIAM Review
SPECT	Mathematical Spectrum
SSM	School Science and Mathematics
TYCMJ	The Two-Year College Mathematics Journal

Contests:

Abbreviation	Name
AUSTRALIA	Australian Mathematical Olympiad
CANADA	Canadian Mathematics Olympiad
IMO	International Mathematical Olympiad
KURSCHAK	Kurschak Mathematical Competition of Hungary
PUTNAM	The William Lowell Putnam Mathematical Competition
USA	U.S.A. Mathematical Olympiad

Algebra

Absolute value

SSM 3664. **by Albert White**

Let a and b be real numbers. Prove that

$$|a+b|+|a-b|=|a|+|b|$$

if and only if $|a|=|b|$.

SSM 3671. **by Al White**

Show that if x, y, and z are real numbers such that

$$|x+y+z|=|x|+|y|+|z|,$$

then ($x \geq 0$, $y \geq 0$, and $z \geq 0$) or ($x \leq 0$, $y \leq 0$, and $z \leq 0$).

Age problems: different times

PME 449. **by Richard I. Hess**

A fairly young man was married at the beginning of the month. At the end of the month his wife gave him a chess set for his birthday. If he was married and received the chess set on the same day of the week he was born, how old was he when he got married?

MATYC 135. **by Frank Kocher**

Miss Cohen is in her prime. Today is her birthday and her age is (as it was last year) the product of two primes, p_1 and p_2. The difference $p_1 - p_2$ is the product of two other primes, p_3 and p_4, but $p_3 - p_4 = p_5$ where p_5 is a fifth prime.

Assuming that Miss Cohen's age is less than a century, determine her age.

OMG 17.3.5.

A man was x years old in the year x^2. How old was he in 1960?

OMG 17.3.4.

When Ernie was as old as Bert is now, Bert's age was half of Ernie's present age. When Bert will be as old as Ernie is now, the sum of their ages will be 99. Find Bert's present age.

JRM 393. **by Les Marvin**

In the year 2000, if I live that long, my age will be a perfect cube, and my son's age a perfect square. Not too many years ago the situation was reversed. How old are we?

FUNCT 3.1.6.

Hanging over a pulley is a rope with a weight at one end. At the other end, there is a monkey of equal weight. The rope weighs 250 gm per meter. The combined ages of the monkey and its father total 4 years, and the weight of the monkey is as many kilograms as his father is years old. The father is twice as old as the monkey was when the father was half as old as the monkey will be when the monkey is three times as old as the father was when he was three times as old as the monkey was. The weight of the weight plus the weight of the rope is half as much again as the difference between the weight of the weight and the weight of the weight plus the weight of the monkey. How long is the rope?

PARAB 332.

In a number of years equal to the number of times a pig's mother is as old as the pig, the pig's father will be as many times as old as the pig as the pig is years old now. The pig's mother is twice as old as the pig will be when the pig's father is twice as old as the pig will be when the pig's mother is less by the difference in ages between the father and the mother than three times as old as the pig will be when the pig's father is one year less than twelve times as old as the pig is when the pig's mother is eight times the age of the pig.

When the pig is as old as the pig's mother will be when the difference in ages between the pig's father and the pig is less than the age of the pig's mother by twice the difference in ages between the pig's father and the pig's mother, the pig's mother will be five times as old as the pig will be when the pig's father is one year more than ten times as old as the pig is when the pig is less by four years than one-seventh of the combined ages of his father and mother. Find their respective ages. (For the purposes of this problem, the pig may be considered to be immortal.)

PARAB 309.

In a family with 6 children, the five elder children are respectively 2, 6, 8, 12, and 14 years older than the youngest. The age of each is a prime number of years. How old are they? Show that their ages will never again all be prime numbers (even if they live indefinitely).

Age problems: digits

JRM 794. **by Arthur G. Bradbury**

"How old is grandfather?" David asked. His father replied, "His age, like mine, is one more than six times the sum of its digits." How old is David's grandfather?

PARAB 262.

On his birthday in 1975, John reaches an age equal to the sum of the digits in the year he was born. What year was that?

Age problems: sum and product

CRUX 329. **by Gilbert W. Kessler**

"The Product of the ages of my three children is less than 100," said Bill, "but even if I told you the exact product and even told you the sum of their ages you still couldn't figure out each child's age."

"l would have trouble if different ages are very close" said John as he looked at the children, "but tell me the product anyway."

Bill told him and John confidently told each child his age.

If you can now also tell the three ages, what are they?

JRM 699. **by L. R. Ford, Jr.**

Over the punchbowl, my host said, "Having been married on the twenty-ninth of February, we don't get to celebrate our anniversary very often: in fact this is only the fifth one. I usually ask visiting mathematicians to determine the ages of my three children given the sum and product of their ages, but since Professor Smith failed tonight, and Professor Jones also failed at our last party, I am going to let you off."

"Oh, don't do that," I replied, "I have already heard all the information I will need."

How old were the children?

JRM 659. **by David L. Silverman**

"I see that the sum of your children's ages is 36, the same as mine," said Alice to Barbara, "and the product of their ages is also the same as the product of my children's ages." "Then I know the ages of all the children, but of course I don't know which family is which," said Carol, who is known as a lightning calculator. "Well, my son is the oldest of all the children," said Barbara.

What are the ages of the children?

MSJ 437. **by Paul Weinberg**
PENT 314. **by H. Laurence Ridge**

Two mathematicians were seeing each other again for the first time in many years. One said, "Since I last saw you, I have had three children." "Well," said the other, "What are their ages?" "The product of their ages is 36, and the sum of their ages is the same as your house number," replied the first. The second thought for a moment and then said that he would need more information. "Oh, the oldest one looks like me," the first added, whereupon his friend quickly figured out their ages. What were the ages of the three children?

OSSMB 79-1.

One morning after church the verger, pointing to three departing parishioners, asked the bishop, "How old are those three people?" The bishop replied, "The product of their ages is 2450, and the sum of their ages is twice your age." The verger thought for some moments and said, "I'm afraid I still don't know." The bishop answered, "I'm older than any of them." "Aha!" said the verger. "Now I know." How old was the bishop? (Ages are in whole numbers of years and no one is over 100.)

Algorithms

SSM 3690. **by Louis J. Hall**

Establish the following generalization of a method for finding cube roots and fifth roots using the square-root key of a hand calculator: The rth root of any positive number N can be approximated by the iterative formula

$$x_{n+1} = (N x_n^i)^{1/2^m},$$

where x_n is the nth approximation, m is the smallest positive integer such that $2^m \geq r$, and $i = 2^m - r$.

PUTNAM 1977/B.3.

An ordered triple (x_1, x_2, x_3) of positive irrational numbers with $x_1 + x_2 + x_3 = 1$ is called balanced if each $x_i < 1/2$. If a triple is not balanced, say if $x_j > 1/2$, one performs the following

$$B(x_1, x_2, x_3) = (x_1', x_2', x_3'),$$

where $x_i' = 2x_i$ if $i \neq j$ and $x_j' = 2x_j - 1$. If the new triple is not balanced, repeat the procedure. Does continuation of this process always lead to a balanced triple after a finite number of repetitions?

FUNCT 3.3.3.

Give an algorithm for multiplying any two numbers (given, say, to four significant figures) using your school trigonometric tables.

PARAB 314.

Bob set himself the task of arranging all the positive rational numbers in a list. He did it as follows:

$$a_1 = 1/1,\ a_2 = 1/2,\ a_3 = 2/1,\ a_4 = 1/3,$$

$$a_5 = 2/2,\ a_6 = 3/1,\ a_7 = 1/4,\ a_8 = 2/3,$$

$$a_9 = 3/2,\ a_{10} = 4/1,\ a_{11} = 1/5, \ldots .$$

(Thus the rational number p/q precedes h/k in the list if $p + q < h + k$ or if $p + q = h + k$ and $p < h$.) His friend Joe asked him how he knew that every rational number would appear in the list. Bob answered by writing down a formula, giving the value of n when the rational number $p/q = a_n$ would appear. Joe, still unconvinced, wanted to know what the 1001st number in the list would be. After a few calculations, Bob answered him. Duplicate Bob's formula and find a_{1001}.

JRM 755. **by Friend H. Kierstead, Jr.**

Write a program which will read a decimal number and print it out as a roman numeral. Although the letters I,V,X,L,C,D, and M will have to be contained within the program as constants, there should be no constants within the program which include two or more of these letters.

Calendar problems: calendar cycles

SSM 3769. **by Kathryn W. Lynch**

There is a mathematical pattern for determining the years in which Christmas is observed on Sunday. Show that you have discovered the pattern for years since the last calendar change (1752). After December 25, 1978, state the years involved through the year 2025 A.D. How long will this pattern continue?

CRUX 231. **by Viktors Linis**

Find the period P of the Easter dates based on Gauss's algorithm, that is, the smallest positive integer P satisfying the conditions:

$$D(Y + P) = D(Y) \text{ and } M(Y + P) = M(Y)$$

for all Y, where D and M are the day and month functions of year number Y.

JRM 419. **by Sidney Kravitz**

Our calendar has 97 Leap years in every 400-year period. Every fourth year is a Leap year except that the years 2100, 2200, and 2300 will not be Leap years, but 2000 and 2400 will be. The average length of the calendar year is thus $365\frac{97}{400}$ days.

On which day in the 400-year cycle will the calendar time be the maximum behind the average true time and on which day will it be the maximum ahead? What is the variation between these extremes?

Calendar problems: day of week

ISMJ J10.1.

If a girl's 13th birthday is Tuesday, October 8, 1974, on what day of the week was she born?

FUNCT 2.1.3.

Is 2/22/2022 a Tuesday? How about 2/2/2202?

Calendar problems: Friday the 13th

FUNCT 1.1.1.
Show that the 13th day of the month is more likely to fall on a Friday than on any other day of the week.

OMG 18.1.2.
On the average, over a period of years, how frequently does Friday the 13th occur?

FUNCT 3.2.1.
Prove that every year contains at least one Friday the 13th.

PARAB 273.
What is the largest possible number of Friday the 13ths that can occur in any calendar year? What is the smallest?

Calendar problems: significant dates

JRM C9. **by Ray Lipman**
Easter is defined as the first Sunday after the full moon, on or after the vernal equinox. Using the Gregorian calendar and taking into account those known astronomical processes which will in the long run affect either the occurrence of the equinoxes or the length of the lunar period, write a program capable of listing the month and day of Easter for every year up to one million A.D.

Clock problems: chimes

OMG 18.3.1.
A clock strikes the hours and once each half hour. If you woke up at night and heard the clock strike "one", what is the longest time you would have to lie awake to be sure of the time?

Clock problems: hands

ISMJ 14.24.
Suppose the two hands of a clock are indistinguishable, that they both point exactly at minute marks, and they are 9 minutes apart. What can you deduce about the time?

ISMJ J10.2.
The hour and minute hands of a clock are each exactly on a minute mark and the angle between the hands is exactly 36°. What time is it?

ISMJ J10.9.
The hour and minute hands of a clock form a right angle. How long before they will form a right angle again?

OMG 15.3.8.
On a twelve-hour clock, how often are the minute and hour hands at right angles in 12 hours?

FUNCT 3.3.2.
The hour hand, the minute hand, and the second hand of a standard 12-hour clock are all together on the twelve at noon. If the clock keeps perfect time, they are all together again at midnight. Do they coincide at any other time? If so, when? If not, when do they most nearly coincide? When do the hands come closest to trisecting the clock-face?

OMG 18.1.8.
A clock hangs on the wall of a railway station. The wall is 71 ft 9 in. long and 10 ft 4 in. high. If the hands of the clock were pointing in opposite directions, and were parallel to one of the diagonals of the wall, what was the time?

MM 940. **by Edwin P. McCravy**
Suppose a clock has minute and hour hands of the same length and indistinguishable. Of the set of all instants in a 12-hour period, consider the partition:
A = set of all instants when the clock reading would be ambiguous;
B = set of all instants when the reading would not be ambiguous.
Which, if either, of these sets is finite?

Clock problems: stopped clock

PENT 278. **by Kenneth M. Wilke**
Dr. Knowitall noticed that his clock had stopped. So he wound it, noted the time to be 6:00 PM, and went to a friend's house to play chess. He arrived at 8:30 according to the clock in his friend's house. Dr. Knowitall left at 11:00. When he arrived home, the time, according to his clock, was 12:30 AMHe reset his clock to the correct time. Assuming that Dr. Knowitall walked at the same rate in both directions and assuming that his friend's clock kept perfect time, what was the correct time when Dr. Knowitall reset his clock?

Clock problems: time computation

OMG 17.3.3.
If at a certain instant it is 7 o'clock, what time is it 11,999,999,994 hours later?

Complex numbers: cube roots

CRUX 4. **by Léo Sauvé**
It is easy to verify that $2\sqrt{3}+i$ is a cube root of $18\sqrt{3}+35i$. What are the other two cube roots?

ISMJ 12.12.
Find a and b so that $(a+bi)^3 = i$.

Complex numbers: exponential equations

CRUX 10. **by Jacques Marion**
Does the equation $e^z = z$ have any complex roots?

Complex numbers: identities

PME 353. **by Clayton W. Dodge**
Prove that if a, b, and c are complex numbers such that $a+b+c=0$ and $|a|=|b|=|c|$, then $a^3=b^3=c^3$. Can this result be extended to more than three numbers?

Complex numbers: inequalities

AMM E2616. **by Andrew Odlyzko and Lloyd Welch**
Let a be a complex number with $|a|<1$, and let $\varepsilon>0$. Prove or disprove: There exists an algebraic integer b such that $|a-b|<\varepsilon$ and all conjugates of b lie in the annulus

$$|a|-\varepsilon<|z|<1+\varepsilon.$$

MM Q663.

Prove that $|1+z| \le |1+z|^2 + |z|$ for complex z.

PUTNAM 1979/B.6.

For $k = 1, 2, \ldots, n$ let $z_k = x_k + iy_k$, where the x_k and y_k are real and $i = \sqrt{-1}$. Let r be the absolute value of the real part of

$$\pm\sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}.$$

Prove that $r \le |x_1| + |x_2| + \cdots + |x_n|$.

Complex numbers: powers

MATYC 118. **by Lawrence Cohen**
ISMJ 12.11.

Prove that i^i is real and find its value.

Complex numbers: radicals

PME 345. **by Vladimir F. Ivanoff**

Resolve the paradox:

$$i(\sqrt{i} + \sqrt{-i}) = i\sqrt{i} + i\sqrt{-i} = \sqrt{-i} + \sqrt{i} = \sqrt{i} + \sqrt{-i}.$$

Determinants

SIAM 78-3. **by H. L. Langhaar and R. E. Miller**

A special case of a more general conjecture on determinants that has been corroborated numerically by operations with random determinants generated by a digital computer is expressed by the equation $\Omega = \Delta^{n+1}$, in which

$$\Delta = |a_1 b_2 \cdots q_{n-1} r_n|$$

is any nth order determinant, and Ω is a determinant of order $n(n+1)/2$, constructed from the elements of Δ as follows: The first row in Ω consists of all terms that occur in the expansion of $(a_1 + a_2 + \cdots + a_n)^2$. A similar construction applies for rows $2, 3, \ldots, n$. Row $n+1$ consists of expressions that occur in the expansion of

$$(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

A similar construction applies for the remaining rows. The letters in the columns in Ω are ordered in the same way as the subscripts in the rows. Prove or disprove the conjecture $\Omega = \Delta^{n+1}$.

PUTNAM 1978/A.2.

Let a, b, $p_1, p_2, \ldots, p_n$ be real numbers with $a \ne b$. Define

$$f(x) = (p_1 - x)(p_2 - x)(p_3 - x)\cdots(p_n - x).$$

Show that

$$\det\begin{pmatrix} p_1 & a & a & a & \cdots & a & a \\ b & p_2 & a & a & \cdots & a & a \\ b & b & p_3 & a & \cdots & a & a \\ b & b & b & p_4 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ b & b & b & b & \cdots & p_{n-1} & a \\ b & b & b & b & \cdots & b & p_n \end{pmatrix} = \frac{bf(a) - af(b)}{b-a}.$$

AMM E2735. **by I. P. Goulden and D. M. Jackson**

Let n be a fixed integer and define

$$f_k(x) = \sum_{r \ge 0} \frac{x^{nr+k}}{(nr+k)!}, \quad 0 \le k \le n-1.$$

For $P \subset S = \{0, 1, \ldots, n-1\}$, let $F(P;x) = F(P)$ be the square matrix whose entries are indexed by elements of P and the (i,j)-th entry is $f_{i-j}(x)$, $i, j \in P$. (We set $f_r(x) = f_k(x)$ if $r \equiv k \bmod n$.)

If n is even, show that $\det F(P) = \det F(S \backslash P)$ for all $P \subset S$. Generalize.

Discriminants

PME 414. **by Steven R. Conrad**

In discussing the discriminant of a quadratic equation, a certain textbook says, "... if a, b, and c are integers with $a \ne 0$ and if $b^2 - 4ac = 79$, the roots of $ax^2 + bx + c = 0$ will be real, irrational, and unequal." Explain why this is incorrect.

Exponential equations

MATYC 131. **by Jeffrey Goldstein**

Let $a > 1$. Prove that there exist real numbers b and c such that $a^b = 2bc$ where $-1 < b < c < 0$.

TYCMJ 114. **by Larry Hoehn**

Find all real solutions of $8^x(3x+1) = 4$.

MM 1078. **by R. P. Boas**

Describe as fully as possible the solutions of

$$xe^y + ye^x = 0.$$

MM 1081. **by Edwin P. McCravy**

Find all real t such that for all $x > y > 0$,

$$(x-y)^t(x+y)^t = (x^t - y^t)^t(x^t + y^t)^{2-t}.$$

Fair division

JRM 527. **by David L. Silverman**

It has been established that when a taxicab carries several passengers, not all picked up at the same time, the equitable way to determine each passenger's share of the total fare is to divide the trip into uninterrupted legs, i.e., legs between consecutive pickups, assign to each leg a prorated share of the total fare based on relative distance, and to each passenger during that leg a fraction of that share, prorated on the basis of the total number of passengers during that leg.

(a) Mr. and Mrs. N reside at $1/N$ on the real line for $N = 1, 2, 3, \ldots$. A taxicab drives from 1 to 0, picking up successively all the Mmes. N and depositing them at 0 to attend a Female Liberation meeting. If the total fare is a dollar, how much is owed by Mrs. One?

(b) The cab returns from 0 to 1, picking up all the Messrs. N ($N = 2, 3, 4, \ldots$) in reverse order and depositing them at Mr. One's home to attend a Male Domination meeting. The total fare is again a dollar. What is the maximum share owed by any passenger?

FUNCT 1.3.1. **by Michael Moses**

Three men go fishing and catch a certain number of fish. During the night, one man awakes and decides to go home. Without waking the others, he makes 3 equal shares and finds 2 fish left over. He takes his share and the 2 left over and goes home.

A little while later another awakes and makes 3 shares of what is left, finds 2 left over, takes these and his share, and leaves.

The last man also makes 3 shares, finds 2 left over, and takes these 2 and his share and leaves.

How many fish did they catch?

Generalize this problem so as to answer the same question but now for M men, with a remainder of N ($0 \le N < M$).

OMG 17.1.9.

Two men sold their herd of x cows at x dollars per head. With the proceeds, they bought sheep at \$10 each and a single lamb costing less than \$10. Each man received the same number of animals but the one receiving the lamb had to be compensated so as to make the division equitable. How much money did he receive from the other man?

Finite products

SSM 3675. **by Steven R. Conrad**

Simplify the product

$$\prod_{k=1}^{n}\left(x^{2^k} - a^{2^{k-1}}x^{2^{k-1}} + a^{2^k}\right).$$

AMM 6044. **by Jacques Gilles**

Show that $\prod(\alpha^4+\alpha+1) \neq 83^3$, the product being taken over all the roots of the equation $\alpha^{49}=1$ except $\alpha = 1$.

Finite sums: arithmetic progressions

OMG 16.1.7.

What is the sum of the first 30 odd natural numbers?

SSM 3585. **by Herta T. Freitag**

Consider $\{a_i\}$, an arithmetic progression of difference d and $\{b_i\}$, a geometric progression of ratio r. Let the first n terms of these sequences "intermingle" to form the series

$$\sum_{i=0}^{n} a_i b_i.$$

(a) Obtain a formula for this summation.
(b) What happens if n grows beyond bound?

SSM 3663. **by George Nichols and Robert A. Carman**

Find a formula for the sum of the following "combined arithmetic-geometric" progression:

$$a+(a+d)r+(a+2d)r^2+(a+3d)r^3+\cdots+(a+nd)r^n.$$

Finite sums: binomial coefficients

CRUX 366. **by A. Liu**

Evaluate

$$\sum_{i=n}^{2n-1}\binom{i-1}{n-1}2^{1-i}.$$

PUTNAM 1976/B.5.

Evaluate

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x-k)^n.$$

SIAM 76-14.* **by L. Carlitz**

Prove that

$$\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{i+j}{i}\binom{m-i+j}{m-i}\binom{i+n-j}{i}$$

$$\cdot\binom{m-i+n-j}{m-i} = \begin{cases}\binom{\frac12(m+n)}{\frac12 m}^2, & (m,\, n \text{ both even}),\\ 0, & \text{otherwise},\end{cases}$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\frac{\binom{m}{i}^2\binom{n}{j}^2}{\binom{m+n}{i+j}} = \delta_{mn},$$

$$\sum_{r=0}^{\min(i,j,k)}\frac{\binom{i}{r}\binom{j}{r}\binom{k}{r}}{\binom{i+j+k}{r}} = \frac{(j+k)!(k+i)!(i+j)!}{i!j!k!(i+j+k)!}.$$

AMM E2601. **by Robert Weinstock**

Prove that

$$\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{2n-k}{k}}{\binom{2n-k}{n}}\frac{2n-4k-1}{2n-2k+1}2^{n-2k} = 1.$$

AMM E2602. **by C. L. Mallows**

Prove that

$$\sum_{i=0}^{a-1}\binom{b+i-1}{b-1}\binom{2n-b-i}{n-b} = \sum_{i=b}^{n}\binom{a+i-1}{a-1}\binom{2n-a-i}{n-a}.$$

AMM E2681. **by David Burman**

If $x+y=1$, show that

$$\sum_{i=0}^{m-1}\binom{n+i-1}{i}x^i y^n + \sum_{j=0}^{n-1}\binom{m+j-1}{j}x^m y^j = 1.$$

Finite sums: exponentials

SIAM 75-3. **by U. G. Haussmann**

Let

$$f(u) = \frac{\exp(u+nu)+\exp(-nu)}{1+\exp u},$$

where

$$\cosh u = 1 + \frac{x}{2}.$$

If $y = f[u(x)]$, then show that

$$y(x) = \sum_{k=0}^{n} \binom{n+k}{2k} x^k.$$

Finite sums: fractions

CANADA 1975/1.

Simplify

$$\left(\frac{1\cdot 2\cdot 4 + 2\cdot 4\cdot 8 + \cdots + n\cdot 2n\cdot 4n}{1\cdot 3\cdot 9 + 2\cdot 6\cdot 18 + \cdots + n\cdot 3n\cdot 9n}\right)^{1/3}.$$

OSSMB G75.1-3.

Find the sum

$$\sum_{k=1}^{n} \frac{x^{k-1}}{\left(1+x^k\right)\left(1+x^{k+1}\right)}.$$

OSSMB G76.1-5.

Consider the sequence defined by

$$t_n = \frac{n}{1+n^2+n^4}, \quad n = 1,2,3,\ldots.$$

Find the sum of the first n terms of this sequence.

OSSMB G79.3-6.

Sum to n terms the series whose ith term is

$$\frac{i^4+2i^3+i^2-1}{i^2+i}.$$

MSJ 429. **by Joanne B. Rudnytsky**

For $x > 1$, find a formula for

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{x^k}.$$

FQ H-245. **by P. Bruckman**

Prove the identity

$$\sum_{k=0}^{n} \frac{x^{\frac{1}{2}k(k-1)}}{(x)_k (x)_{n-k}} = \frac{2\prod_{r=1}^{n-1}(1+x^r)}{(x)_n},$$

$(n = 1,2,\ldots)$, where

$$(x)_n = (1-x)(1-x^2)(1-x^3)\cdots(1-x^n),$$

$(n = 1,2,\ldots;\ (x)_0 = 1)$.

CRUX 393. **by Sahib Ram Mandan**

If

$$f_n(a_i) = (a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n),$$

prove that, for $k = 0,1,\ldots,n-2$,

$$\sum_{i=1}^{n} \frac{a_i^k}{f_n(a_i)} = 0.$$

Finite sums: permutations

CRUX 78. **by Jacques Sauvé**

Is there a simple formula for the sum of all the permutations

$$\sum_{r=0}^{n} P(n,r)?$$

Finite sums: radicals

CRUX 214. **by Steven R. Conrad**

Prove that if the sequence (a_i) is an arithmetic progression, then

$$\sum_{k=1}^{n-1} \frac{1}{\sqrt{a_k}+\sqrt{a_{k+1}}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}.$$

Floor function

MM 1080. **by Marlow Sholander**

Some calculators have an "int" key. The "integral part of x" is given by int $x = \lfloor |x| \rfloor \operatorname{sgn} x$.

We have $|x| = x \operatorname{sgn} x$ and $\max(x,y) = (x+y+|x-y|)/2$ as examples of familiar functions that can be expressed in terms of "sgn" together with the operations $\{+,-,\times,\div\}$. Show that these functions can be similarly expressed in terms of "int".

Functional equations: 1 parameter

CRUX 343.* **by Steven R. Conrad**

The greatest integer function satisfies the functional equation

$$f(nx) = \sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)$$

for all real x and positive integers n. Are there other functions which satisfy this equation?

AMM E2677. **by Erwin Just**

Let $n \geq 2$ be an integer. Show that there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(2x) + \cdots + f(nx) = 0$$

for all x and $f(x) = 0$ if and only if $x = 0$.

PUTNAM 1977/A.3.

Let u, f and g be functions, defined for all real numbers x, such that

$$\frac{u(x+1)+u(x-1)}{2} = f(x)$$

and

$$\frac{u(x+4)+u(x-4)}{2} = g(x).$$

Determine $u(x)$ in terms of f and g.

AMM 6106. **by D. S. Mitrinović and P. M. Vasić**

Find the general solution to the functional equation

$$\sum_{k=1}^{n} f(x^k) = \sum_{k=1}^{n} f(x^{-k}).$$

FQ B-325. **by Verner E. Hoggatt, Jr.**

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Prove that there does not exist an even, single-valued function G such that

$$x + G(x^2) = G(\alpha x) + G(\beta x)$$

on $-\alpha < x < \alpha$.

PUTNAM 1979/A.2.

Establish necessary and sufficient conditions on the constant k for the existence of a continuous real valued function $f(x)$ satisfying $f(f(x)) = kx^9$ for all real x.

MM 990. **by Harry W. Hickey**

Prove that the identity

$$f(x+1)/g(x+1) - f(x)/g(x) = h(1/x)$$

is not satisfied by any non-constant polynomials f, g, and h.

Functional equations: 2 parameters

AMM 6226. **by Marlow Sholander**

Domain D consists of the real numbers $\mathbb{R}$ from which a finite set is deleted. On domain D, the functions f, F, and G are continuous and satisfy the identity

$$f(r) - f(s) = (r - s)F(r)G(s).$$

Describe $f(x)$ on domain $\mathbb{R}$.

AMM E2575. **by David Shelupsky**

Solve the functional equation

$$f\left(\frac{x-y}{\log x - \log y}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

this to hold for all distinct $x, y \in (0,\infty)$ and $f\colon (0,\infty) \to \mathbb{R}$ to be continuous.

AMM E2583. **by C. L. Mallows**

Find all continuous $g\colon \mathbb{R} \to \mathbb{R}$ such that, for some continuous $f\colon \mathbb{R}^2 \to \mathbb{R}$, we have $g(xy) = f(x, g(y))$ for all $x, y \in \mathbb{R}$.

AMM E2661. **by Steve Galovich**

Find all functions f that satisfy the three conditions

(i) $f(x,x) = x$,
(ii) $f(x,y) = f(y,x)$,
(iii) $(x+y)f(x,y) = yf(x, x+y)$,

assuming that the variables and the values of f are positive integers.

CRUX 314. **by Michael Ecker**

Find all functions $f\colon \mathbb{R} \to \mathbb{R}$, continuous at $x = 0$, satisfying the functional relation

$$f(x) \cdot f(y) = \left[f\left(\frac{x+y}{2}\right)\right]^2$$

for all $x,\ y \in \mathbb{R}$.

CRUX PS7-1.

(a) Determine $F(x)$ if, for all real x and y,

$$F(x)F(y) - F(xy) = x + y.$$

(b) Generalize.

MM Q609. **by Julian H. Blau**

Which real functions satisfy

$$f(x+y)^2 = f(x)^2 + f(y)^2?$$

OSSMB 79-9.

In each of (a), (b) below, f denotes a real-valued function of a real variable, not identically zero and differentiable at $x = 0$.

(a) If $f(x)f(y) = f(x+y)$ for all x, y, prove that f has derivatives of all orders at all points x, and that

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{1 - f(1)} \quad \text{if} \quad f(1) < 1.$$

(b) If $f(x)f(y) = f(x-y)$ for all x, y, find f.

TYCMJ 102. **by Mangho Ahuja and Leonard Palmer**

Find all real-valued functions f on $(0,\infty)$ such that $f(x) \cdot f(y) = f(x-y)$ for all real x and y.

TYCMJ 106. **by James W. Murdock**

Let f be a real-valued function with domain $(-\infty,\infty)$ such that $f(xy) = [f(x)+f(y)]/(x+y)$ for all x and y. Does there exist a value of x for which $f(x) \neq 0$?

TYCMJ 71. **by Peter A. Lindstrom**

Let f and g be real-valued, nonconstant functions such that, for all real numbers x and y,

$$f(x+y) = f(x)g(y) + g(x)f(y)$$

and

$$g(x+y) = g(x)g(y) - f(x)f(y).$$

What are the possible values of $f(0)$ and $g(0)$?

ISMJ 13.13.

What are the continuous solutions of the functional equation $f(xy) = f(x) + f(y)$?

TYCMJ 92. **by Wm. R. Klinger**

Let f be a real-valued function defined on $(0,\infty)$ such that $f(xy) = f(x) + f(y)$ for all $x, y \in (0,\infty)$. Prove that if f is continuous at 1, then f is continuous on $(0,\infty)$.

Functional equations: 3 parameters

AMM E2607. **by E. Montana College Prob. Group**

Solve the functional equation

$$f(x,y) + f(y,z) + f(z,x) = 3f\left(\frac{1}{3}(x+y+z), \frac{1}{3}(x+y+z)\right)$$

in the class of all continuous functions $\mathbb{R}^2 \to \mathbb{R}$.

What can be said about the solutions in the class of all functions $\mathbb{R}^2 \to \mathbb{R}$?

Functional equations: derivatives

AMM 6154. **by Richard Stanley**

Define a sequence of polynomials (with rational coefficients) as follows: $p_0(x) = 1$, $p_n(0) = 0$ if $n > 0$, and $p'_{n+1} = p_n(1-x)$ if $n > 0$. Thus $p_1(x) = x$, $p_2(x) = x - \frac{1}{2}x^2$, $p_3(x) = \frac{1}{2}x - \frac{1}{6}x^3$, etc. Find $p_n(x)$. In particular, what is $p_n(1)$?

MATYC 129. **by Gino Fala**

Find all differentiable functions $f: \mathbb{R}^2 \to \mathbb{R}$, $z = f(x, y)$, such that the volumes of the tetrahedrons formed by the tangent planes and the coordinate planes remain constant.

Functional equations: fallacies

MATYC 72. **by Gene Zirkel**

Find the fallacy: One is given a function f such that $f^2 \equiv f$. Therefore

$$f^2 - f \equiv 0$$
$$f(f-1) \equiv 0.$$

Therefore f is a constant function.

Functional equations: integrals

SIAM 75-18. **by O. G. Ruehr**

Find a continuous function $F(t)$ for $t \geq 0$ satisfying

$$t\{F(t)\}^2 = \int_0^1 \frac{F(ts) - F(t-ts)}{1-2s}\, ds$$

and $F(0) = c > 0$.

Functional equations: periodic functions

OSSMB 78-1.

A real-valued function f satisfies, for all real x,

$$f(x+1) = \frac{1+f(x)}{1-f(x)}.$$

Show that f is periodic.

Functional equations: polynomials

AMM E2731. **by Bruce Reznick**

Characterize all polynomials that satisfy $P(x, y) = P(y, x)$ and $P(x, y) = P(x, x-y)$ for all x and y.

IMO 1975/6.

Find all polynomials P, in two variables, with the following properties:

(1) for a positive integer n and all real t, x and y

$$P(tx, ty) = t^n P(x, y)$$

(2) for all real a, b and c,

$$P(b+c, a) + P(c+a, b) + P(a+b, c) = 0,$$

(3) $P(1, 0) = 1$.

MATYC 100. **by Steve Kahn**

Characterize all polynomials $P(x)$ with complex coefficients such that $P(x) = P^{-1}(x)$.

MM 965. **by Bernard B. Beard**

Find all polynomials $P(x)$ satisfying the equation $P(F(x)) = F(P(x))$, $P(0) = 0$, where $F(x)$ is a given function satisfying $F(x) > x$ for all $x \geq 0$.

PME 411. **by R. S. Luthar**

Find all polynomials $P(x)$ such that

$$P(x^2+1) - [P(x)]^2 - 2xP(x) = 0$$

and $P(0) = 1$.

TYCMJ 38. **by Warren Page**

Determine all polynomials, $P(x)$, satisfying $P(0) = 0$ and $P(x) = [P(x+1) + P(x-1)]/2$.

TYCMJ 77. **by R. S. Luthar**

Determine all polynomial functions, f, such that

$$(x-1)f(x+1) - (x+2)f(x) \equiv 0.$$

Functions

NYSMTJ 51.

Consider the composition of two functions f and g: $f \circ g = g \circ f$ if $f = g$, if $f = g^{-1}$, or if either function is the identity function. Aside from these examples, composition of functions is not generally commutative.

(a) Show that, for any first-degree polynomial

$$f(x) = ax + b,$$

there are an infinite number of functions g such that

$$f \circ g = g \circ f.$$

(b) Are there other polynomial functions that, similarly, commute?

(c) How about other types of functions — trigonometric, logarithmic, exponential, etc.?

Generalized binomial theorem

CRUX 352. **by Dan Sokolowsky**

Let $x^{(0)} = 1$;

$$x^{(n)} = \prod_{k=1}^{n} [x + (k-1)c]$$

c constant, $n = 1, 2, \ldots$.

Prove that

$$(a+b)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} a^{(n-k)} b^{(k)}, \quad n = 0, 1, 2, \ldots .$$

Geometry of zeros

TYCMJ 35. **by Richard Miller**

Let $P(z)$ be a polynomial with real coefficients such that each of the zeros of $P(z)$ is pure imaginary. Prove that all but one of the zeros of $P'(z)$ are pure imaginaries.

MM 1010. **by Marius Solomon**

Prove that if the roots of a fourth degree polynomial are in arithmetic progression, then the roots of its derivative are also in arithmetic progression.

NAvW 503. **by O. Bottema**

The cubic equation with unknown u:

$$(x+8)u^3 - 3yu^2 - 3xu + y = 0$$

is mapped onto the point $P(x, y)$ of a plane V with the rectangular frame OXY. Determine in V the regions corresponding to the sets of equations with

(a) one positive and two imaginary roots,
(b) one negative and two imaginary roots,
(c) one positive and two negative roots,
(d) one negative and two positive roots.

Identities

CRUX 316. **by Hippolyte Charles**

Prove that

$$\frac{a-x}{x-b} = \frac{a-d}{b-c} \cdot \frac{c-y}{y-d}$$

implies

$$\frac{a-y}{y-b} = \frac{a-d}{b-c} \cdot \frac{c-x}{x-d}.$$

CRUX PS3-3.

If

$$\frac{a}{bc-a^2} + \frac{b}{ca-b^2} + \frac{c}{ab-c^2} = 0,$$

prove that also

$$\frac{a}{(bc-a^2)^2} + \frac{b}{(ca-b^2)^2} + \frac{c}{(ab-c^2)^2} = 0.$$

ISMJ J11.12.

Show that if $abc = 1$, then

$$\frac{a}{ab+a+1} + \frac{b}{bc+b+1} + \frac{c}{ca+c+1} = 1.$$

OSSMB G75.2-5.

Show that if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c},$$

then

$$\frac{1}{a^{2n+1}} + \frac{1}{b^{2n+1}} + \frac{1}{c^{2n+1}} = \frac{1}{(a+b+c)^{2n+1}}.$$

OSSMB G75.3-6.

Show that if a, b, c are distinct, nonzero real numbers such that $a+b+c=0$, then

$$\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-c}\right) = 9.$$

PARAB 300.

Prove that if

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{ab+bc+ca},$$

then the sum of two of the numbers a, b, and c is zero.

MSJ 469.

What general identity is exemplified by the following statements:

$$3(1^2+3^2+7^2) = 2^2+4^2+6^2+11^2,$$
$$3(2^2+11^2+16^2) = 5^2+9^2+14^2+29^2?$$

CRUX PS4-2.

If a, b, c, and d are real, prove that

$$\left\{\begin{aligned} a^2+b^2 &= 2, \\ c^2+d^2 &= 2, \\ ac &= bd, \end{aligned}\right\}$$

if and only if

$$\left\{\begin{aligned} a^2+c^2 &= 2, \\ b^2+d^2 &= 2, \\ ab &= cd. \end{aligned}\right\}$$

Inequalities: absolute value

TYCMJ 144. **by R. S. Luthar**

Does there exist a nonconstant function, f, that obeys the inequality $(f(x)-f(y))^2 \le |x-y|^3$ for all x and y?

Inequalities: degree 2

CRUX 323. **by Jack Garfunkel and M. S. Klamkin**

If $xyz = (1-x)(1-y)(1-z)$ where $0 \le x, y, z \le 1$, show that

$$x(1-z) + y(1-x) + z(1-y) \ge 3/4.$$

PARAB 290.

If x_1, x_2, x_3, x_4, and x_5 are all positive numbers, prove that

$$(x_1+x_2+x_3+x_4+x_5)^2 \ge 4\left(x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1\right).$$

PARAB 394.

Prove that, if $a^2+b^2 = x^2+y^2 = 1$, then $ax+by \le 1$.

PME 443. **by R. S. Luthar**

If x and y are any real numbers, prove that

$$x^2 + 5y^2 \ge 4xy.$$

PUTNAM 1977/B.5.

Suppose that $a_1, a_2, \ldots, a_n$ are real $(n > 1)$ and

$$A + \sum_{i=1}^n a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^n a_i\right)^2.$$

Prove that $A < 2a_ia_j$ for $1 \le i \le j \le n$.

PARAB 377.

Let x_i, y_i $(i = 1, 2, \ldots n)$ be real numbers such that $x_1 \ge x_2 \ge \cdots \ge x_n$ and $y_1 \ge y_2 \ge \cdots \ge y_n$.

Prove that if $z_1, z_2, \ldots, z_n$ is any given rearrangement of $y_1, y_2, \ldots, y_n$, then

$$\sum_{i=1}^n (x_i - y_i)^2 \le \sum_{i=1}^n (x_i - z_i)^2.$$

Inequalities: degree 3

CRUX PS1-3.

(a) If $a, b, c \geq 0$ and $(1+a)(1+b)(1+c) = 8$, prove that

$$abc \leq 1.$$

(b) If $a, b, c \geq 1$, prove that

$$4(abc+1) \geq (1+a)(1+b)(1+c).$$

CRUX PS6-3.

If $x, y, z \geq 0$, prove that

$$x^3 + y^3 + z^3 = y^2 z + z^2 x + x^2 y$$

and determine when there is equality.

SIAM 77-12. **by Peter Flor**

Establish or disprove the following inequalities where all the variables are positive:

$$a^3 + b^3 + c^3 + 3abc \geq a^2(b+c) + b^2(c+a) + c^2(a+b);$$

$$39a^3 + 15a\left(b^2+c^2\right) + 20ad^2 + 5bc(b+c+d) \geq 10a^2(b+c) + 43a^2 d + 39abc + ad(b+c);$$

$$5\left(a^4+b^4+c^4+d^4\right) + 6\left(a^2c^2+b^2d^2\right) + 12\left(a^2+c^2\right)bd + 12\left(b^2+d^2\right)ac \geq 2\left(a^3+b^3+c^3+d^2\right)(a+b+c+d) + 4(a+c)(b+d)(ac+bd) + 2\left(a^2+c^2\right)\left(b^2+d^2\right) + 8abcd.$$

Inequalities: degree 4

ISMJ 10.3.

Show that

$$\frac{x^3-1}{3} \leq \frac{x^4-1}{4}$$

for all real numbers x.

Inequalities: exponentials

MM Q658. **by M. S. Klamkin**
CMB P261. **by R. Schramm**

If $a, b > 0$, prove that $a^b + b^a > 1$.

NYSMTJ 40. **by David E. Bock**

Prove that, if a and b are positive real numbers, then

$$a^a b^b \geq (ab)^{(a+b)/2}.$$

PME 378. **by M. L. Glasser and M. S. Klamkin**

Show that

$$\left\{\frac{x^x}{(1+x)^{1+x}}\right\}^x > (1-x) + \left\{\frac{x}{1+x}\right\}^{1+x} > \frac{1}{(1+x)^{1+x}}$$

for $1 > x > 0$.

SPECT 11.7.

Show that $e^{kx} + k(1-e^x) \geq 1$ for every real number x and every integer k.

TYCMJ 123. **by V. N. Murty**

Let x and y be positive numbers. Prove that

$$x^x \cdot y^y \geq \left(\frac{x+y}{2}\right)^{x+y}$$

with equality if and only if $x = y$.

TYCMJ 149. **by V. N. Murty**

Let a, b, and α be positive with $a + b = 1$. Prove or disprove that

$$\left(a+\frac{1}{a}\right)^\alpha + \left(b+\frac{1}{b}\right)^\alpha \geq \frac{5^\alpha}{2^{\alpha-1}}.$$

AMM E2547. **by T. S. Bolis**

Let p and q be positive numbers with $p + q = 1$. Show that for all x,

$$pe^{x/p} + qe^{-x/q} \leq e^{x^2/8p^2q^2}.$$

AMM S6. **by M. S. Klamkin and A. Meir**

Let $x_i > 0$ for $i = 1, 2, \ldots, n$ with $n \geq 2$. Prove that

$$(x_1)^{x_2} + (x_2)^{x_3} + \cdots + (x_{n-1})^{x_n} + (x_n)^{x_1} \geq 1.$$

SPECT 10.3. **by T. B. Cruddis**

The positive real numbers p, q, r are such that $q \neq r$ and $2p = q + r$. Show that

$$\frac{p^{q+r}}{q^q r^r} < 1.$$

Inequalities: finite products

AMM 6254. **by Thomas E. Elsner**

For real numbers r_{ij} with $0 \leq r_{ij} \leq 1$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, prove that

$$1 - \prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} r_{ij}\right) \leq \prod_{i=1}^{m}\left[1 - \prod_{j=1}^{n}(1-r_{ij})\right].$$

CMB P270. **by M. S. Klamkin**

Prove that

$$2^n P\left\{\frac{x_1^n + x_2^n + \cdots + x_n^n}{n}\right\}^{n-1} \geq \prod_{i=1}^{n}\{x_i^n + P\}$$

where $P = x_1 x_2 \cdots x_n$, $x_i \geq 0$, and there is equality if and only if $x_i =$ constant.

AMM E2691. **by Živojin M. Mijalković and J. B. Keller**

If $x_i > 0$ $(1 \leq i \leq n)$, show that

$$\left(\prod x_i\right)^{\Sigma x_i/n} \leq \prod x_i^{x_i} \leq \left(\frac{\sum x_i^2}{\sum x_i}\right)^{\Sigma x_i}.$$

Inequalities: finite sums

AMM E2656. **by G. Tsintsifas**

Let $a_2, a_3, \ldots, a_n$ be positive real numbers and

$$s = a_2 + a_3 + \cdots + a_n.$$

Show that

$$\sum_{k=2}^{n} a_k^{1-1/k} < s + 2\sqrt{s}.$$

PARAB 390.

Let $b_1, b_2, \ldots, b_n$ be any positive numbers. Prove that

$$(b_1 + b_2 + \cdots + b_n)\left(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n}\right) \geq n^2.$$

SIAM 75-5. **by M. M. Gupta**

Suppose p and q are positive integers, $p > q$, and $Z_1, \ldots, Z_p$ are arbitrary real numbers. Define

$$\alpha = p^{-2} q^{-2} (p-q)^{-1},$$

$$\beta_p = (Z_2 - 2Z_1)^2 + \sum_{i=2}^{p-1} (Z_{i-1} - 2Z_i + Z_{i+1})^2,$$

and

$$I_{p,q} = -2q^3 \alpha Z_1 Z_p + 2p^3 \alpha Z_1 Z_q - 2\alpha Z_1^2 + (1 - 2\alpha)\beta_p.$$

Show that $I_{p,q} \geq 0$.

SPECT 9.3.

The real numbers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ $(n \geq 1)$ are such that

$$a_1 \leq \frac{1}{2}(a_1 + a_2) \leq \frac{1}{3}(a_1 + a_2 + a_3) \leq \cdots \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n),$$

$$b_1 \leq \frac{1}{2}(b_1 + b_2) \leq \frac{1}{3}(b_1 + b_2 + b_3) \leq \cdots \leq \frac{1}{n}(b_1 + b_2 + \cdots + b_n).$$

Show that

$$\left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right) \leq n \sum_{k=1}^n a_k b_k.$$

TYCMJ 45. **by Robert Sulek and Lester Suna**

Assume $a_i \geq 0$, $(i = 1, 2, \ldots, n)$, with $a_{n+1} = a_1$. Prove or disprove:

$$\sum_{i=1}^n \left(\frac{a_i}{a_{i+1}}\right)^n \geq \sum_{i=1}^n \frac{a_{i+1}}{a_i}.$$

PUTNAM 1979/A.6.

Let $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$. Show that

$$\sum_{i=1}^n \frac{1}{|x - p_i|} \leq 8n\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right)$$

for some x satisfying $0 \leq x \leq 1$.

PUTNAM 1978/B.6.

Let p and n be positive integers. Suppose that the numbers $c_{h,k}$ $(h = 1, 2, \ldots, n;\ k = 1, 2, \ldots, ph)$ satisfy $0 \leq c_{h,k} \leq 1$. Prove that

$$\left(\sum \frac{c_{h,k}}{h}\right)^2 \leq 2p \sum c_{h,k},$$

where each summation is over all admissible ordered pairs (h, k).

AMM E2744. **by H. L. Montgomery**

Let $a_n \geq 0$ and $a_{m+n} \leq a_m + a_n$ for $m, n = 1, 2, \ldots$. Show that

$$\sum_{k=1}^n k^{-2} a_k \geq \frac{1}{4} n^{-1} a_n \log n.$$

AMM E2551. **by Hugh L. Montgomery**

Let $r_1, \ldots, r_n$ be real numbers such that $-1 \leq r_i \leq 1$ for $i = 1, 2, \ldots, n$ and such that $r_1 + \cdots + r_n = 0$. It is easy to see that there is a permutation π of $\{1, \ldots, n\}$ with the property that all the partial sums

$$S_k(\pi) = \sum_{i=1}^k r_{\pi(i)}, \qquad k = 1, 2, \ldots, n$$

lie in the interval $[-1, 1]$. Strengthen this as follows: Show that there exists a permutation π such that

$$\max_k S_k(\pi) - \min_k S_k(\pi) < 2 - n^{-1}.$$

Show also that if the right-hand side of the above inequality is replaced by $2 - 4n^{-1}$, then the assertion is false for certain arbitrarily large n.

CMB P248. **by M. S. Klamkin**

Let $S = x_1 + x_2 + \cdots + x_n$, where $x_i > 0$, $T_0 = 1/S$ and

$$T_r = \sum_{\text{sym}} \{S - x_1 - x_2 - \cdots - x_r\}^{-1}, \quad 1 \leq r \leq n-1.$$

Prove that $(n-r)^2 T_r / \binom{n-1}{r}$ is monotonically increasing in r from 0 to $n-1$.

MM Q664. **by M. S. Klamkin**

Prove that

$$\sum_{k=1}^n (x_k + 1/x_k)^a \geq \frac{(n^2+1)^a}{n^{a-1}}$$

where $x_k > 0$ $(k = 1, 2, \ldots, n)$, $a > 0$ and $x_1 + x_2 + \cdots + x_n = 1$.

OSSMB G77.1-2.

Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2.$$

Inequalities: fractions

AMM E2603. **by M. S. Klamkin**

Let $x_i > 0$, $1 \leq i \leq n$. Prove that

$$r \cdot \sum_{\text{sym}} \frac{x_1 x_2 \cdots x_r}{x_1 + x_2 + \cdots + x_r} \leq \binom{n}{r}\left(\frac{x_1 + \cdots + x_n}{n}\right)^{r-1}$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

CRUX 54. **by Léo Sauvé**

If $a, b, c > 0$ and $a < b + c$, show that

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c}.$$

MM Q608. **by M. S. Klamkin**

If x, y, and z are nonnegative and are not sides of a triangle, show that

$$1+\frac{x}{y+z-x}+\frac{y}{z+x-y}+\frac{z}{x+y-z}\le 0.$$

MM Q618. **by M. S. Klamkin**

If $1\ge x,y,z\ge -1$, show that

$$\frac{1}{(1-x)(1-y)(1-z)}+\frac{1}{(1+x)(1+y)(1+z)}\ge 2$$

with equality if and only if $x=y=z=0$.

MM Q655. **by Mark Kleiman**

If a,b,c, and d are positive real numbers, prove that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\ge 2.$$

When does equality hold?

MSJ 418. **by Peter A. Lindstrom**

Find all real x such that

$$\frac{x(x-2)(x-4)}{(x-1)(x-3)(x-5)}<0.$$

PARAB 370.

Prove that, when $x>0$,

$$\frac{1+x^2+x^4}{x+x^3}\ge\frac{3}{2}\ .$$

SSM 3744. **by Bob Edwards**

If r/s and p/q are two positive fractions in lowest terms and $qr-ps=1$, prove that all fractions lying between these two must have a denominator that is not less than $q-s$.

TYCMJ 87. **by Norman Schaumberger**

Let a,b,c, and d be positive real numbers. Prove that

$$\frac{a^2+b^2+c^2}{a+b+c}+\frac{a^2+b^2+d^2}{a+b+d}+\frac{a^2+c^2+d^2}{a+c+d}+\frac{b^2+c^2+d^2}{b+c+d}\ge a+b+c+d.$$

CRUX 17. **by Viktors Linis**

Prove the inequality

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{999999}{1000000}<\frac{1}{1000}.$$

ISMJ 12.23.

Find a number n large enough so that

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}>100.$$

ISMJ 13.5.

Given that a, b, c, and d are positive numbers and $a/b<c/d$, show that

$$\frac{a}{b}<\frac{a+c}{b+d}<\frac{c}{d}.$$

ISMJ J11.2.

Prove that if a_1,a_2,a_3 and b_1,b_2,b_3 are positive numbers such that

$$\frac{a_1}{b_1}<\frac{a_2}{b_2}<\frac{a_3}{b_3}\text{ then }\frac{a_1}{b_1}<\frac{a_1+a_2+a_3}{b_1+b_2+b_3}<\frac{a_3}{b_3}.$$

CRUX 413. **by G. C. Giri**

If $a,b,c>0$, prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{a^8+b^8+c^8}{a^3b^3c^3}.$$

MSJ 484.

Let a, b, and c be positive numbers such that $a+b+c=1$. Prove that $1/a+1/b+1/c\ge 9$.

Inequalities: functional inequalities

ISMJ 12.16.

Let ϕ be a nonnegative function such that

$$\phi\left(\frac{t_1+t_2}{2}\right)<\frac{1}{3}[\phi(t_1)+\phi(t_2)]$$

for all real numbers t_1 and t_2 such that $t_1\ne t_2$. Show that

$$\phi\left(\frac{s_1+s_2+s_3+s_4}{4}\right)<\frac{1}{4}[\phi(s_1)+\phi(s_2)+\phi(s_3)+\phi(s_4)]$$

if s_1, s_2, s_3, and s_4 are real numbers, no three of which are all equal.

KURSCHAK 1979/2.

The function f satisfies the following inequalities for every pair of real numbers x and y:

$$f(x)\le x,$$
$$f(x+y)\le f(x)+f(y).$$

Show that $f(x)=x$ for every real number x.

IMO 1977/6.
PARAB 368.

Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n+1)>f(f(n))$$

for each positive integer n, then

$$f(n)=n\quad\text{for each } n.$$

Inequalities: iterated functions

OMG 16.2.6.

Let $0<u<1$ and define

$$u_1=1+u,$$
$$u_2=1/u_1+u,$$
$$\vdots$$
$$u_{n+1}=1/u_n+u,\qquad n\ge 1.$$

Show that $u_n>1$ for all values of $n=1,2,3,\ldots$.

Inequalities: logarithms

AMM E2695. **by Eliyahu Beller**
Prove or disprove the following conjecture: For $a > 1$ and $x > 0$, show that $-\log\left(1-(1-e^{-x})^a\right) < x^a$.

CRUX 98. **by Viktors Linis**
Prove that, if $0 < a < b$, then
$$\ln\frac{b^2}{a^2} < \frac{b}{a} - \frac{a}{b}.$$

PME 424. **by R. S. Luthar**
Prove that
$$\left(x^{1/n} + y^{1/n}\right)^n > \left(\frac{x-y}{\ln x - \ln y}\right)(2n+2),$$
where n is an odd integer and $n \geq 3$ and $0 < y < x$.

ISMJ 12.17.
Show that
$$|\log_a b + \log_b a| \geq 2$$
if a and b are both positive real numbers.

FQ B-357. **by Frank Higgins**
Let m be a fixed positive integer, and let k be a real number such that
$$2m \leq \frac{\log(\sqrt{5}k)}{\log\alpha} < 2m+1,$$
where $\alpha = (1+\sqrt{5})/2$. For how many positive integers n is $F_n \leq k$?

CRUX 304. **by Viktors Linis**
Prove the following inequality:
$$\frac{\ln x}{x-1} \leq \frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}, \quad x > 0,\ x \neq 1.$$

Inequalities: numerical inequalities

ISMJ 10.1.
Which is larger
$$\left(1+\frac{1}{1000}\right)^{1001} \quad \text{or} \quad \left(1+\frac{1}{1000}\right)^{1002}?$$

MM 937. **by Norman Schaumberger**
Which is greater: e^π or $(e^e \cdot \pi^e \cdot \pi^\pi)^{1/3}$?

Inequalities: polynomials

AMM E2655. **by Michael W. Chamberlain**
Prove that for integral $n \geq 2$ and $0 < x < n/(n+1)$, one has
$$(1-2x^n+x^{n+1})^n < (1-x^n)^{n+1}.$$

TYCMJ 59. **by Robert Sulek and Lester Suna**
Let n be a positive odd integer and x a positive real number. Prove that $x^n + 2 \geq 2x^{(n-1)/2} + x$, and
$$x^{36}+x^8+x^4+1 \geq x^{15}+x^{14}+x^{13}+x^6.$$

Inequalities: powers

ISMJ 10.7.
Given that x, y, m, and n are positive, prove that
$$x^m y^n + x^n y^m \leq x^{m+n} + y^{m+n}.$$

SIAM 75-19. **by K. B. Stolarsky and L. J. Yang**
If N and $m+1$ are positive integers, it is conjectured that $L_m(x) \geq R_m(x)$ for all $x \geq 0$, where
$$L_m(x) = \left\{1+\frac{1}{N^{2m-1}}\right\}\left\{x^2+\frac{2}{N}x+1\right\}^m,$$
$$R_m(x) = \left\{x+\frac{1}{N}\right\}^{2m} + \left\{1+\frac{x}{N}\right\}^{2m} + (N-1)\left\{\frac{x-1}{N}\right\}^{2m}.$$

Inequalities: radicals

CRUX 295. **by Basil C. Rennie**
If $0 < b \leq a$, prove that
$$a+b-2\sqrt{ab} \geq \frac{1}{2}\frac{(a-b)^2}{a+b}.$$

CRUX 310. **by Jack Garfunkel**
Prove that
$$\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{9a^2+b^2}} + \frac{2ab}{\sqrt{a^2+b^2}\cdot\sqrt{9a^2+b^2}} \leq \frac{3}{2}.$$
When is equality attained?

TYCMJ 154. **by V. N. Murty**
Assume that a, b, c, and d are real numbers satisfying $ad-bc=1$. Prove that $a^2+b^2+c^2+d^2+ac+bd \geq \sqrt{3}$.

USA 1977/5.
If a, b, c, d and e are positive numbers bounded by p and q, prove that
$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)$$
$$\leq 25+6\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2$$
and determine when there is equality.

AMM E2667. **by John R. Samborski**
If $\sum_{k=1}^\infty 2^{-n_k}$ is the binary expansion of $\left(\sqrt{5}-1\right)/2$, show that $n_k \leq 5\cdot 2^{k-2}-1$.

Infinite series

FQ H-251. **by Paul Bruckman**
Prove the identity:
$$\sum_{n=0}^\infty \frac{x^{n^2}}{[(x)_n]^2} = \sum_{n=0}^\infty \frac{x^n}{(x)_n},$$
where
$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n), \quad (x)_0 = 1.$$

Interest problems

FUNCT 2.1.2.

I have the following options for depositing \$100 for one year. The bank will give me \$4 interest at the end of the year. A Housing Cooperative will give me interest at the rate of 2% per half-year (compound, so it pays interest in the second half also on the first half's interest). A Credit Union will give me interest at the rate of $1/3$% per month, compound. A friend says he will give me interest equivalent to the 4% per annum rate, but compounding every instant! Which should I choose, and how much interest do I get?

TYCMJ 104. **by Roger W. Pease, Jr.**

A man wishes to purchase a new car. He finds that he can finance this car over a 36-month period for 0.8 percent interest per month on the unpaid balance. With \$1,000 in savings at 6 percent interest compounded quarterly, he wishes to decide whether to retain his savings or use it to finance the car.

He reasons that he should finance the car by borrowing the money because the difference between the sum of the 36 payments and the \$1,000 original principal is \$154.87 which is less than the interest of \$195.62 earned on the money in the bank if it is left on deposit for three years. Is this the best strategy for the car buyer to follow?

OMG 16.1.5.

A car depreciates at 20% per year for 3 years. What, at this time, is its percentage value of the original price?

Iterated functions

CANADA 1975/8.

Let k be a positive integer. Find all polynomials

$$P(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where the a_i are real, which satisfy the equation

$$P(P(x)) = \{P(x)\}^k.$$

SSM 3659. **by Brother U. Alfred**

If

$$f(x) = \sqrt{\frac{x^2+1}{x^2-1}}$$

and $f^2(x) = f(f(x))$, $f^3(x) = f\left(f^2(x)\right), \ldots$, for what values of n does $f^n(x) = f(x)$?

Logarithms

CRUX 41. **by Léo Sauvé**

Given that $\log_6 3 = p$ and $\log_3 5 = q$, express $\log_{10} 5$ and $\log_{10} 6$ as functions of p and q.

OSSMB G77.1-5.

Show that if

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z},$$

then $x^y y^x = z^y y^z = x^z z^x$.

OSSMB G79.2-6.

Prove that

$$\frac{\log_a n}{\log_{am} n} = 1 + \log_a m.$$

OSSMB G75.1-2. **by Peter Crippen**

(a) For $a, b > 0$, show that $\log_b a = 1/\log_a b$.

(b) Simplify the following without the use of tables:

$$\frac{1}{\log_{1/2} 144} + \frac{1}{\log_2 144} + \frac{1}{\log_{12} 144}.$$

MATYC 139. **by J. F. Allison**

Is there a nontrivial solution in real numbers to

$$\log(x+y+z) = \log(x)\log(y)\log(z)?$$

PENT 281. **by Kenneth M. Wilke**

An algebra student encountered the following problem on an exam: Evaluate $\frac{\log A}{\log B}$. Being pressed for time, he cancelled common factors from both numerator and denominator to obtain the correct answer.

$$\frac{\log A}{\log B} = \frac{A}{B} = \frac{3}{4}\ .$$

What are A and B?

Maxima and minima

AMM E2573. **by Murray S. Klamkin**

If n positive real numbers vary such that the sum of their reciprocals is fixed and equal to A, find the maximum value of the sum of the reciprocals of the $\binom{n}{j}$ sums of the n numbers taken j at a time.

SPECT 11.2. **by B. G. Eke**

The positive numbers x, y, z are such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that $(x-1)(y-1)(z-1) \geq 8$.

CRUX 487. **by Dan Sokolowsky**

If a, b, c, and d are positive real numbers such that $c^2 + d^2 = (a^2+b^2)^3$, prove that

$$\frac{a^3}{c} + \frac{b^3}{d} \geq 1,$$

with equality if and only if $ad = bc$.

MM 1059. **by David E. Daykin**

How should n given nonnegative real numbers be indexed to minimize (maximize)

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1?$$

Means

CANADA 1979/1.

Given: (i) $a, b > 0$; (ii) a, A_1, A_2, b is an arithmetic progression; (iii) a, G_1, G_2, b is a geometric progression. Show that $A_1A_2 \geq G_1G_2$.

CRUX 247. **by Kenneth S. Williams**

If $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, is there a constant k such that

$$k\frac{\sum_{1\leq i<j\leq n}(a_i-a_j)^2}{a_n} \leq \frac{a_i + \cdots + a_n}{n} - \sqrt[n]{a_1 \cdots a_n}$$
$$\leq k\frac{\sum_{1\leq i<j\leq n}(a_i-a_j)^2}{a_1}?$$

CRUX 362. **by Kenneth S. Williams**

In the inequality

$$\frac{1}{2n^2}\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{a_n} \le \frac{a_1+\cdots+a_n}{n} - \sqrt[n]{a_1\cdots a_n} \le \frac{1}{2n^2}\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{a_i},$$

prove that the constant $1/2n^2$ is best possible.

CRUX 395. **by Kenneth S. Williams**

The inequality

$$\frac{1}{2n^2}\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{a_n} \le A-G \le \frac{1}{2n^2}\frac{\sum_{1\le i<j\le n}(a_i-a_j)^2}{a_i}$$

is a refinement of the familiar inequality $A \ge G$ where A (resp. G) is the arithmetic (resp. geometric) mean of $a_1,\ldots,a_n$ If H denotes the harmonic mean of $a_1,\ldots,a_n$, find the corresponding refinement of the familiar inequality $G \ge H$.

SSM 3713. **by Alan Wayne**

(a) Find a necessary and sufficient condition so that the arithmetic mean of two unequal, positive numbers is closer to their geometric mean than their geometric mean is to the smaller number.

(b) Find a necessary and sufficient condition so that the arithmetic mean of two unequal, positive numbers is closer to their harmonic mean than their harmonic mean is to the smaller number.

SSM 3755. **by Alan Wayne**

If a and b are numbers such that $0 < a < b$, define their quadratic mean by $Q = \sqrt{\frac{(a^2+b^2)}{2}}$.

(a) Show that the quadratic mean lies between b and the arithmetic mean of a and b.

(b) Show that the arithmetic mean is closer to the quadratic mean than the quadratic mean is to b.

(c) Show that the arithmetic mean of a and b is the quadratic mean of their quadratic mean and their geometric mean.

TYCMJ 39. **by Norman Schaumberger**

Let $A = (x+y)/2$ and $G = \sqrt{xy}$, where x and y are unequal, positive numbers. Prove that

$$A > \frac{(x-y)^2}{8(A-G)} > G.$$

MM 1000. **by Murray S. Klamkin**

Let T denote a cyclic permutation operator acting on the indices of a sequence (a_i), that is, $T(a_1x_1+a_2x_2+\cdots+a_nx_n) = a_2x_1+a_3x_2+\cdots+a_1x_n$. If, for all i, $a_i \ge 0$ and $x_i > 0$, show that

$$\left\{\sum_{i=1}^{n}\frac{a_i}{n}\right\}^n \ge \prod_{i=1}^{n} T^i\left\{\frac{a_1x_1+a_2x_2+\cdots+a_nx_n}{x_1+x_2+\cdots+x_n}\right\} \ge \prod_{i=1}^{n} a_i.$$

Measuring problems

OMG 16.1.6.

If each volume of a twelve-book encyclopedia is 3 cm thick and the covers are 1 mm thick, what is the distance from the first page of volume 1 to the last page of volume 12 when they are stacked in order on a shelf?

PARAB 297.

A man has 3 bottles which hold exactly 8 liters, 5 liters, and 3 liters. The two smaller bottles are empty, but the largest one is full of wine which the man wishes to share with a friend. Without using any other means of measurement or any other container, how can he divide the wine into two equal amounts of 4 liters each?

NYSMTJ 96. **by Samuel A. Greenspan**

A man had an 8-gallon keg of wine and a jug. One day, he drew off a jugful of wine and filled up the keg with water. Later on, when the wine and water had been thoroughly mixed, he drew off another jugful, and again filled up the keg with water. The keg then contained equal quantities of wine and water. What was the capacity of the jug?

OMG 17.3.1.

A 16-quart radiator is filled with water. Four quarts are removed and replaced with pure antifreeze liquid. Then four quarts of the mixture are removed and replaced with pure antifreeze. This is done a third and fourth time. What part of the final mixture is water?

ISMJ 12.7.

An empty five gallon can A is filled with antifreeze. Some antifreeze is transferred from A to a second five gallon can B (originally empty). Can B is then filled with water and the contents are mixed. Enough of the mixture in B is then poured into A to fill it. Show that the mixture in A is at least 75% antifreeze.

Metric conversions

FUNCT 3.5.4.

The number of kilometers in a mile is often given as 8/5. Given only that the approximation is expressed in this form, estimate the error involved.

Money problems

JRM 735. **by Frank Rubin**

A housespouse has cents-off coupons for three different brands of detergent, all in different amounts. The regular prices, number of ounces, and number of wash loads per box are known for all three brands. If only one coupon can be used, how should one decide which?

PARAB 418.

Two classes organized a party. To meet the expenses, each pupil of class A paid \$5 and each pupil of class B paid \$3. If the pupils of class A had paid all the expenses, they would have paid \$$k$ each. At a second similar event, the pupils of class A paid \$4 each and those of class B paid \$6 each; and the total sum was the same as if each pupil in class B had paid \$$k$. Find k. Which class had more pupils?

Money problems: change

MSJ 459. **by Albert Wilansky**

I owed Mr. Smith an amount less than \$20. I offered him a \$20 bill, but he could not make change. So, I offered him a \$50 bill and he gave me the correct change. How much did I owe Mr. Smith?

Money problems: coins

PENT 290. **by Charles Trigg**

"Come on in, Bob," said Dan; "only small stakes tonight." "That's good," replied Bob, "I haven't quite three dollars in nickels, dimes, and quarters." "I haven't any pennies either," said Dan, "but I have the same number of coins that you have. That includes twice as many dimes as you have." "Correct," replied Bob, "but my number of nickels is twice yours. It also equals the number of all our quarters combined. The total value of your change is the same as mine." "Okay, let's go," said Dan; "it appears that my lucky half-dollar is the largest coin on the table."

How many coins of each type did Bob and Dan have?

Money problems: combinations

ISMJ 11.16.

In Ruritania, the basic unit of money is the farthing, however, farthings are no longer made. A forinth is worth m farthings and a schilling is worth n farthings, m and n integers, $m < n$. Schillings and forinths can be combined to make all but 35 monetary values in farthings. In particular 58 farthings can not be made from schillings and forinths. What are m and n?

JRM 447. **by Sidney Kravitz**

John has a dollar's worth of coins in his pocket, but no half dollars. He told me how many coins he had but I could not tell what those coins were because there were six different possible combinations.

How many coins does John have?

OMG 17.1.5.

Donald Corleone cashed a \$200 check at the bank and requested some one-dollar bills, 10 times as many two-dollar bills, and the balance in five-dollar bills. How did the cashier pay him?

OMG 18.2.4.

In the local Sunday School picnic, men are asked to pay 50 cents for refreshments, women are asked to pay 30 cents and children only 1 cent. At the last picnic, the total attendance was 100. If everyone paid the correct change, and the total receipts were exactly \$10, how many men, women and children attended?

Money problems: denominations

JRM 618. **by Frank Rubin**

Let S be any set of distinct positive-integer-valued coin denominations capable of making up any amount from one cent to a dollar. Let $A(S)$ be the average of the *minimal* numbers of S-type coins required to make up the hundred totals from 1 to 100. Thus A(1 cents, 5 cents, 10 cents, 25 cents, 50 cents) $= .01(1+2+3+4+1+2+\cdots+8+2) = 4.22$. Define the efficiency $E(S)$ to be $1/[A(S)N(S)]$, where $N(S)$ is the number of coin denominations in S. Thus $E(1,5,10,25,50) = 1/(4.22 \cdot 5) = .04739$.

(a) What set is most efficient?

(b) What set containing no coins of denomination greater than 100 is the least efficient?

Money problems: devaluation

FUNCT 1.1.5.

A newspaper report stated that the combined effect of Australia's 17.5% devaluation and New Zealand's 7% devaluation was to revalue the New Zealand dollar by 12.7% in comparison with the Australian dollar. Where does this figure come from? Is it correct?

Money problems: interchanged digits

PARAB 363.

An absent-minded bank clerk switched the dollars and cents when he cashed a check for Mr. Brown, giving him dollars instead of cents and cents instead of dollars. After buying a five-cent newspaper, Mr. Brown discovered that he had left exactly twice as much as his original check. What was the amount of his check?

Money problems: stamps

JRM 396. **by Ray Lipman**

The adjoining countries Angkor and Bangkor each have two denominations of postage stamps, all in the integral units of their common equivalent of the penny (the *kor*). One of Angkor's stamps is the 3-kor variety and one of Bangkor's is the 6-kor. The two types of stamps of neither country can be used to obtain *all* desired amounts of postage, but, curiously, the maximum postage unobtainable with Angkorian stamps is the same as the maximum unobtainable with Bangkorian stamps. What are the smallest possible values of the other two stamps?

Money problems: sum equals product

CRUX 297. **by Kenneth M. Wilke**

A young lady went to the store to purchase four items. In computing her bill, the nervous clerk multiplied the four amounts together and announced that the bill was \$6.75. Since the young lady had added the four amounts mentally and obtained the same total, she paid her bill and left. Assuming that the prices for each item are distinct, what are the individual prices?

Money problems: word problems

OMG 18.1.9.

A cattle dealer had 5 droves of animals consisting of oxen, pigs and sheep, with the same number of animals in each drove. He sold them all to 8 dealers. Each dealer bought the same number of animals, paying \$17 per ox, \$4 per pig and \$2 per sheep, and the dealer received \$301 in all. What was the greatest number of animals the dealer could have had and how many of each kind were there?

OSSMB 78-3.

A firm employs 350 people, some married and the rest single. It pays a total Christmas bonus of $\$B$, by giving to each single worker \$83.50 and to each married worker \$100, except that if both spouses of a married couple work for the firm, the wife gets \$100 and the husband nothing. If the total bonus can be determined when the percentage of married workers getting no bonus is known, how many male workers have wives employed by the firm?

Monotone functions

SSM 3692. **by Michael Brozinsky**

Prove that $f(x) = (1 + \frac{1}{x})^x$ is an increasing function of x where x is a positive real number.

Numerical calculations

SSM 3568. **by Alan Wayne**

In the expression $(3/2)^2 - (1/2)^2$, Lucky Larry interpreted the exponents as multipliers, obtaining

$$2(3/2) - 2(1/2) = 2,$$

a correct equivalent of the given expression. Explain his success.

MM Q619. **by Alan Wayne**

Using "the distributivity of addition over multiplication" Lucky Larry obtained the correct answer to $(0.5) + (0.2)(0.3)$ by multiplying 0.7 by 0.8. Explain his success.

NYSMTJ 62.

Here is an incorrect cancellation that produces a correct result:

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4}\ .$$

Find other such fractions.

NYSMTJ 72.

Guess by what rule the following equalities are composed. Using the rule you have found, make up one more such equality:

$$12 \times 42 = 21 \times 24$$
$$13 \times 62 = 31 \times 26.$$

CRUX 340. **by Léo Sauvé**

Find a problem whose answer is $22/7 - \pi$.

CRUX 312. **by R. Robinson Rowe**

Evaluate

$$\left\{ (\sqrt[16]{1416317954} - 2)^2 - 3 \right\}^2$$

to at least five significant figures.

Numerical inequalities

PENT 283. **by Kenneth M. Wilke**

On Professor Knowitall's College Algebra exam, the following question appeared:

Which is larger $\sqrt[6]{4}$ or $\sqrt[7]{5}$? Find the solution without using tables.

Young Percival Whizkid solved the problem easily. How did he do it?

Partial fractions

OSSMB G79.2-7.

Express $\frac{x}{1-5x+6x^2}$ as a sum of partial fractions. Then find the coefficient of x^r in the expansion of $x(1 - 5x + 6x^2)^{-1}$.

Polynomial divisibility

FUNCT 3.2.2.

Let $P(x)$, $Q(x)$, and $R(x)$ be polynomials that satisfy the identity

$$P(x^3) + xQ(x^3) = (1 + x + x^2)R(x).$$

Show that all three polynomials are exactly divisible by $x - 1$.

MSJ 486.

Let P and Q be two polynomials satisfying the equation $P(x)/Q(x) = (2x - 1)/(x + 2)$, and define $R(x) = P(x)^2 + Q(x)^2$. Prove that $x^2 + 1$ is a factor of $R(x)$.

SPECT 10.5.

The real polynomials $f_1(x), \ldots, f_{n-1}(x), g(x)$ $(n > 1)$ are such that

$$f_1(x^n) + xf_2(x^n) + \cdots + x^{n-2}f_{n-1}(x^n) = (1 + x + x^2 + \cdots + x^{n-1})g(x).$$

Show that $f_1(x), \ldots, f_{n-1}(x)$ all have $x - 1$ as a factor.

USA 1976/5.

If $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.

CANADA 1976/7.

Let $P(x, y)$ be a polynomial in two variables x and y such that $P(x, y) = P(y, x)$ for every x and y. Given that $(x - y)$ is a factor of $P(x, y)$, show that $(x - y)^2$ is a factor of $P(x, y)$.

PARAB 299.

Find all values of p, q such that $x^4 + px^2 + q$ is divisible by $x^2 + ax + b$.

PME 446. **by Clayton W. Dodge**

A teacher showing the factorization of

$$x^3 - y^3 = (x - y)\left(x^2 + xy + y^2\right)$$

emphasized that the second factor is not a square (not $(x+y)$ squared), and then chose $x = 5$ and $y = 3$ at random, obtaining $x^2 + xy + y^2 = 49$, which is a square.

(a) Explain this apparent contradiction.

(b) Show that the equation $x^2+xy+y^2 = 49$ illustrates that a 3:5:7 triangle has a 120° angle.

CRUX 7. **by H. G. Dworschak**

Find a fifth degree polynomial $P(x)$ such that $P(x)+1$ is divisible by $(x-1)^3$ and $P(x)-1$ is divisible by $(x+1)^3$.

OSSMB 76-7.

What is the remainder when $x + x^9 + x^{25} + x^{49} + x^{81}$ is divided by $x^3 - x$?

PENT 288. **by Charles Trigg**

Factor $6x^5 - 15x^4 + 20x^3 - 15x^2 + 6x - 1$.

MM 1072. **by Peter Ørno**

The professor is preparing her final exam for calculus. She wants to include the problem: "Find the relative maxima, relative minima, and points of inflection of the following function." The function should be a polynomial $P(x)$ of degree 4 with three distinct relative extrema and two distinct points of inflection. In order to solve the problem, the students must be able to factor $P'(x)$ and $P''(x)$. But the typical calculus student in her class can factor a quadratic polynomial correctly only if its roots are integers between -20 and 20, and the student can factor a cubic polynomial only if it is x times a quadratic which the student can factor. Help the professor find such a polynomial.

Polynomials: Chebyshev polynomials

FQ B-373. **by V. E. Hoggatt, Jr.**

The sequence of Chebyshev polynomials is defined by

$$T_0(x) = 1,\ T_1(x) = x,\ \text{and}$$

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

for $n = 2, 3, \ldots$. Show that $\cos\frac{\pi}{(2n+1)}$ is a root of

$$[T_{n+1}(x) + T_n(x)]/(x+1) = 0$$

and use a particular case to show that $2\cos\frac{\pi}{5}$ is a root of $x^2 - x - 1 = 0$.

Polynomials: coefficients

OSSMB 77-17.

Consider polynomials in n symbols $x_1,\ x_2, \ldots, x_n$ of the form

$$(x_1 + \delta_1)(x_2 + \delta_2)\ldots(x_n + \delta_n)$$

where each $\delta_i = 1$ or -1. If $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ are any two such polynomials, show that the sum of the products of the coefficients of corresponding terms of f and g is 0.

OSSMB G75.3-5.

Show that if a, b, c, d be any four consecutive coefficients in the expansion of $(1+x)^n$, then

$$\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}.$$

OSSMB G79.1-6.

(a) Find the number of homogeneous products of r dimensions that can be formed out of the letters a, b, c and their powers, that is, products of the form $a^x b^y c^z$ where x, y, z are nonnegative integers and $x + y + z = r$.

(b) Find the number of terms in the expansion of

$$(a+b+c)^8.$$

(c) Find the sum of the coefficients in $(a+b+c)^8$.

(d) Find the coefficient of the term $a^2b^3c^4d$ in

$$(a-b-c+d)^{10}.$$

SIAM 76-22. **by N. Liron and L. A. Rubenfeld**

Define

$$F(x) = B_m(x^2)\sin x - xA_n(x^2)\cos x,$$

where $B_m(z)$ and $A_n(z)$ are polynomials of orders m and n respectively, $B_m(0) = 1$ and where $m - n = 0$ or 1. Prove that the coefficients in the polynomials B_m and A_n can be uniquely chosen so that $F(x)$ vanishes to maximum order at $x = 0$, and the order of the zero is $2(m+n)+3$.

CRUX 198. **by Gali Salvatore**

Find the coefficient of x^8 in the expansion of the polynomial

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6)^6.$$

OSSMB G75.2-6.

Find the sum of the first n coefficients in

$$(1+x)^n(1-x)^{-2}.$$

Polynomials: complex polynomials

AMM 6136. **by H. L. Montgomery**

Let

$$P(z,w) = \sum c_{mn} z^m w^n$$

be a polynomial in $\mathbb{C}[z,w]$. Suppose that

$$Q(z,w) = P(z, w/z)$$

is also a polynomial: that is $c_{mn} = 0$ whenever $n > m$. Show that

$$\{P(z,w) : |z| < 1, |w| < 1\} = \{Q(z,w) : |z| < 1, |w| < 1\}.$$

Polynomials: degree 4

PUTNAM 1978/B.5.

Find the largest A for which there exists a polynomial

$$P(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E,$$

with real coefficients, which satisfies

$$0 \le P(x) \le 1 \quad \text{for} \quad -1 \le \text{x} \le 1.$$

Polynomials: derivatives

AMM E2550. **by I. J. Schoenberg**

Let $q > 1$, and let n be a natural number. Show that the polynomial

$$P(x) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{x^k - 1}{q^k - 1}$$

has the property that for $k = 1, 2, \ldots, n$,

$$(-1)^{k+1} P^{(k)}(x) > 0 \quad \text{if} \quad x \le q.$$

Polynomials: fixed points

PARAB 386.

Determine all polynomials $f(x) = ax^2 + bx + c$ such that

$$f(a) = a, \quad f(b) = b, \quad \text{and} \quad f(c) = c.$$

Polynomials: integer coefficients

PUTNAM 1976/A.2.

Let $P(x,y) = x^2y + xy^2$ and $Q(x,y) = x^2 + xy + y^2$. For $n = 1,2,3,\ldots$, let $F_n(x,y) = (x+y)^n - x^n - y^n$ and $G_n(x,y) = (x+y)^n + x^n + y^n$. Prove that for each n either F_n or G_n is expressible as a polynomial in P and Q with integer coefficients.

MSJ 475.

Let $g(x)$ be a fixed polynomial with integer coefficients, and let $f(x) = x^2 + xg(x^3)$. Prove that $f(x)$ can not be expressed in the form $(x^2 - x + 1) \cdot h(x)$, where $h(x)$ is a polynomial with integer coefficients.

PUTNAM 1976/A.4.

Let r be a root of $P(x) = x^3 + ax^2 + bx - 1 = 0$ and $r + 1$ be a root of $y^3 + cy^2 + dy + 1 = 0$, where a, b, c and d are integers. Also let $P(x)$ be irreducible over the rational numbers. Express another root s of $P(x) = 0$ as a function of r which does not explicitly involve a, b, c or d.

CRUX 254. **by M. S. Klamkin**

(a) If $P(x)$ denotes a polynomial with integer coefficients such that

$$P(1000) = 1000, \quad P(2000) = 2000, \quad P(3000) = 4000,$$

prove that the zeros of $P(x)$ cannot be integers.

(b) Prove that there is no such polynomial if

$$P(1000) = 1000, \quad P(2000) = 2000, \quad P(3000) = 1000.$$

JRM 589. **by Frank Rubin**

(a) Of all polynomials $f(x)$ of degree less than or equal to 3 and with integer coefficients all in the range $[-10, 10]$, which one has a zero nearest in value to π?

(b) Of all polynomials of degree less than or equal to 5 and with integer coefficients all in the range $[-100, 100]$, which has a zero nearest in value to π?

Polynomials: interpolation

USA 1975/3.

If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, 1, 2, \ldots, n$, determine $P(n+1)$.

Polynomials: number of terms

CRUX PS8-2.

Find all fourth-degree polynomials (with complex coefficients) with the property that the polynomial and its square each consist of exactly five terms.

Polynomials: roots and coefficients

CRUX 332. **by Leroy F. Meyers**

In the quadratic equation

$$A(\sqrt{3} - \sqrt{2})x^2 + \frac{B}{\sqrt{2} + \sqrt{3}}x + C = 0,$$

we are given:

$A = \sqrt[4]{49 + 20\sqrt{6}}$;

$B =$ the sum of the geometric series

$$8\sqrt{3} + (8\sqrt{6})(3^{-\frac{1}{2}}) + 16(3^{-\frac{1}{2}}) + \cdots;$$

and the difference of the roots is

$$(6\sqrt{6})^{\log 10 - 2\log\sqrt{5} + \log\sqrt{\log 18 + \log 72}},$$

where the base of the logarithms is 6. Find the value of C.

CRUX 128. **by Paul Khoury**

Find real a, b, and c given that the equation $az^2 + bz + c = 0$ has as one of its roots $v + v^2 + v^4$, where v is an imaginary root of $z^7 - 1 = 0$.

MSJ 427. **by J. Orten Gadd**

The roots of the equation

$$z^4 + az^3 + bz^2 + cz + 62500 = 0$$

are $x \pm iy$ and $y \pm ix$. Find all solutions if x and y are positive integers with $x < y$.

CRUX 335. **by Hippolyte Charles**

Find necessary and sufficient conditions for the equation $ax^2 + bx + c = 0$, $a \neq 0$, to have one of its roots equal to the square of the other.

PUTNAM 1975/A.2.

(a) For which ordered pairs of real numbers b and c do both roots of the quadratic equation

$$z^2 + bz + c = 0$$

lie inside the unit disc $\{|z| < 1\}$ in the complex plane?

(b) Draw a reasonably accurate graph of the region in the real bc-plane for which the above condition holds. Identify precisely the boundary curves of this region.

Polynomials: zeros

CRUX 425. **by Gali Salvatore**

Let $x_1, x_2, \ldots x_n$ be the zeros of the polynomial

$$P(x) = x^n + ax^{n-1} + a^{n-1}x + 1, \quad n \geq 3$$

and consider the sum

$$\sum_{k=1}^{n} \frac{x_k + 2}{x_k - 1}.$$

Find all values of a and n for which this sum is defined and equal to $n - 3$.

MATYC 128. **by Steve Kahn**

Find all real values of k such that the zeros of

$$x^4 - 2x^3 + (1 - 2k)x^2 + 2kx$$

are real, distinct, and form an arithmetic progression.

MSJ 488.

Let $f(x) = x^4 + x^3 - 1$, and $g(x) = x^4 - x^3 - 2x^2 + 1$. Prove that if $f(x) = 0$, then $g(x^2) = 0$.

MM Q659. **by Peter Ørno**

Show that for each complex number b the polynomial $P(z) = z^4 + 32z + b$ has a zero in $\{z \,|\, \mathrm{Re}(z) \geq 1\}$.

Radicals: approximations

CRUX 207. **by Ross Honsberger**

Prove that $\frac{2r+5}{r+2}$ is always a better approximation to $\sqrt{5}$ than r.

Radicals: arithmetic progressions

ISMJ 11.12.

Prove that the numbers $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ cannot be terms of a single arithmetic progression.

Radicals: irrational numbers

CRUX 104. **by H. G. Dworschak**

Prove that $\sqrt[3]{5} - \sqrt[4]{3}$ is irrational.

PARAB 287.

Show that

$$\sqrt{1976^{1977} + 1978^{1979}}$$

is irrational.

Radicals: nested radicals

OMG 14.3.2.

(a) What is the value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}$?

(b) What is the sum of $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$?

CRUX 8. **by Jacques Marion**

Investigate the convergence of the sequence (a_n) defined by

$$a_n = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}, \quad (n \text{ radicals})$$

and determine $\lim_{n\to\infty} a_n$ if it exists.

CRUX 9. **by Jacques Marion**

Investigate the convergence of the sequence (b_n) defined by

$$b_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \cdots + \sqrt{n}}}}.$$

Radicals: reciprocals

OMG 15.3.4.

Given that $a + b\sqrt{2}$, with $a, b \in \mathbb{R}$, is a closed system under multiplication, what is the reciprocal of $3 - 2\sqrt{2}$ in the form $a + b\sqrt{2}$?

Radicals: simplification

CRUX 169. **by Kenneth S. Williams**

Prove that

$$\sqrt{5} + \sqrt{22 + 2\sqrt{5}} = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}}.$$

SSM 3711. **by William D. Markel**

Simplify:

$$\sqrt[3]{\frac{5\sqrt{33}}{18} - \frac{3}{2}} - \sqrt[3]{\frac{5\sqrt{33}}{18} + \frac{3}{2}}\ .$$

Rate problems: cars

CRUX PS8-1.

At midnight, a truck starts from city A and goes to city B; at 2:40 AM a car starts along the same route from city B to city A. They pass at 4:00 AM. The car arrives at its destination 40 minutes later than the truck. Having completed their business, they start for home and pass each other on the road at 2:00 PM. Finally, they both arrive home at the same time. At what time did they arrive home?

ISMJ J10.11.

Two cars each traveling at a uniform speed set out at noon from A and travel to B. They reach B at 3 PM and 4 PM respectively. At what time was the slower car twice as far from B as the faster one?

OMG 18.2.9.

It is 52 kilometers by road from Hamilton to Toronto. At 10 AM, Peter Brown left Hamilton and traveled at a uniform pace, without stopping, to Toronto and back again. Some time later, Bill Storey left Toronto and drove his car at a uniform speed to Hamilton and back again. Storey passed Brown, on his outward journey, 15 kilometers from Toronto. He passed him again, on his return journey, 11 kilometers from Toronto. Storey was back in Toronto at 4:40 PM. What time was it when Brown arrived back in Hamilton?

OSSMB 75-3. **by Murray Klamkin and Rodney Cooper**

Al leaves at noon and drives at constant speed back and forth from town A to town B. Bob also leaves at noon, driving at 40 mph back and forth from town B to town A on the same highway as Al. Al arrives at town B twenty minutes after first passing Bob, whereas Bob arrives at town A 45 minutes after first passing Al. At what time do Al and Bob pass each other for the nth time?

OSSMB G79.1-1.

(a) An automobile traveling at a rate of 30 feet per second is approaching an intersection. When the auto is 120 feet from the intersection, a truck traveling at 40 feet per second crosses the intersection. The roads are at right angles. How fast are the truck and the auto separating 2 seconds after the truck crosses the intersection?

(b) Water is poured at the rate of 8 cubic feet per minute into a tank in the form of an inverted cone. The cone is 20 feet deep and 10 feet in diameter. If there is a leak in the bottom and the water level is rising at 1 inch per minute when the water is 16 feet deep, how fast is the water leaking?

PENT 294. **by Léo Sauvé**

Two cars leave at the same time from two towns A and B, going towards each other. When the faster car reaches the midpoint M, between A and B, the distance between them is 96 miles. They meet 45 minutes later. Finally, when the slower car reaches M, they are 160 miles apart. Find

(a) the speed of each car, and

(b) the distance between the two towns.

Rate problems: distance

FUNCT 2.2.1.

A man walks in a straight line from A to B, starting at A, at a constant speed of 5 km/hr. A fly starts at B at the same time that the man sets off from A and flies to the man's nose, then back to B, then to the man's nose, and so on. The fly flies at twice the speed that the man walks. How far has the fly flown when the man reaches B?

MSJ 445. **by Joanne B. Rudnytsky**

Some hikers start in a walk at 3 PM and return at 9 PM. If their speed is 4 mph on level land, 6 mph downhill, and 3 mph uphill, how far did they walk? If the uphill rate were x mph and the downhill rate were y mph, what must be the rate, in mph, on the flat, for such a problem to have a unique solution?

OMG 18.3.3.

Dianne goes to school cycling at 12 kph and she is 10 minutes late. Next day she goes at 15 kph and reaches school 10 minutes early. Find the distance of the school from her house. At what speed should she cycle to reach school precisely on time?

Rate problems: exponential growth

CRUX 373. **by Leroy F. Meyers**

Suppose that the human population of the Earth is increasing exponentially at a constant relative rate k, that the average volume of a person stays at V_0, and that the present population is N_0. If people are assumed packed solidly into a sphere, how long will it be until the radius of that sphere is increasing at the speed of light, c, and what will the radius of the sphere be then?

The following approximate data may be used: $N_0 = 4 \times 10^9$, $k = 1\%$/yr, 1 yr =365.25 days, 1 day $= 24 \cdot 60 \cdot 60$ sec; and $V_0 = 0.1$ m^3 and $c = 3 \times 10^8$ m/sec.

Rate problems: flow problems

OMG 17.2.7.

If x men working x hours a day for each of x days produce x articles, determine the number of articles produced by y men working y hours a day for each of y days.

NYSMTJ OBG4.

Four pipes lead into a pool. When pipes 1, 2, and 3 are open, the pool is filled in 12 minutes; when pipes 2, 3, and 4 are open, it takes 15 minutes to fill the pool; when just pipes 1 and 4 are open, it takes 20 minutes. How long will it take to fill the pool if all four pipes are open?

Rate problems: rivers

CRUX 193. **by L. F. Meyers**

A river with a steady current flows into a still-water lake at Q. A swimmer swims down the river from P to Q, and then across the lake to R, in a total of 3 hours. If the swimmer had gone from R to Q to P, the trip would have taken 6 hours. If there had been a current in the lake equal to that in the river, then the downstream trip PQR would have taken $2\,{}^1\!/_4$ hours. How long would the upstream trip RQP have taken under the same circumstances?

MATYC 123. **by Sarah Brooks**

A mathematician went home along the bank of a stream, walking upstream at a rate 1 1/2 times the flow of the stream. He held in his hands his hat and his cane. At a certain time, his hat fell unnoticed into the stream; he continued to go upstream at the same rate. After a while, he realized his mistake, threw his cane into the stream, and ran back at the rate twice as great as that at which he had been going upstream. Upon reaching the floating hat, he immediately fished it out of the water, and walked upstream at his initial rate. Ten minutes after he had fished out the hat, he met his cane floating in the stream. How much earlier would he have arrived home if he had not dropped his hat into the water?

SPECT 11.4. **by B. G. Eke**

A man rows with uniform speed v mph in a straight line against a current of c mph. After 1 hour his hat falls off; after another hour he notices, turns back, and catches up with his hat where he first started rowing. Find v/c. If now his hat falls off after 1 mile instead of 1 hour, with all the other statements the same, determine c and comment on the fact that c is independent of v in this case.

MM 1004. **by M. S. Klamkin**

A river flows with a constant speed w. A motorboat cruises with a constant speed v with respect to the river, where $v > w$. If the path traveled by the boat is a square of side L with respect to the ground, the time of the traverse will vary with the orientation of the square. Determine the maximum and minimum time for the traverse.

Rate problems: running

CRUX 356. **by R. Robinson Rowe**

Jogging daily to a landmark windmill P on the northeasterly horizon, Joe wondered how far it was. Directly (path OP), his time was 25 minutes; jogging first 2 miles due North (path ONP) took 30 minutes, and jogging first 2 miles due East (path OEP) took 35 minutes. How far was Joe's jog (path OP)?

FUNCT 3.1.4.

A man and a horse run a race, one hundred meters straight, and return. The horse leaps 3 meters at each stride and the man only 2, but then the man makes three strides to each of the horse's two. Who wins the race?

JRM 770a. **by Michael J. Messner**

Our four favorite fiends have been chasing the caped crusader all over Gotham City. As he enters a tunnel with Penguin on his heels, an alarm is sounded and at that same time the other three fiends begin to converge toward the center of the tunnel system. Batman is in top condition and able to run faster than any of the fiends. He can go 15 kph indefinitely. Since good guys always turn right, the caped crusader turns right when he reaches the center of the tunnel system and heads toward Cat Woman. When he meets her, he turns and heads back toward the center. There he turns right again and continues in this manner until one of the fiends reaches the center and cuts off his retreat. Now caught between two of them, Batman runs back and forth until all three meet and he can go no further.

Which two wicked weasels will waylay our wary worshipped wonder, and how many miles does he run before they catch him? Penguin goes 3 kph, Cat Woman and Riddler 2 kph, and Joker can go 4 kph but he doesn't – he just sits and waits.

Rate problems: sheep

CRUX 71. **by Léo Sauvé**

If ten sheep jump over a fence in ten seconds, how many would jump over the fence in ten minutes?

Rate problems: spaceships

OMG 17.2.6.

A 25-vehicle interterrestrial starship fleet took off from rebel headquarters, one ship departing every 5 earth minutes. Each ship traveled at the uniform speed of $4\frac{1}{2}$ intergalactic distance units per earth hour, attaining this velocity instantaneously, and each ship has to journey 12 earth hours before arriving at its destination. The first ship departed at 11 AM earth time. Find the total number of intergalactic distance units traveled by all these starships from 11 AM to 9 PM earth time that night.

Rate problems: traffic lights

FUNCT 3.4.1.

Five sets of traffic lights are spaced along a road at 200-meter intervals. For each set, the red signal lasts 30 sec, the green 28 sec, and the yellow 2 sec. The lights are synchronized in such a way that a car traveling at 36 kph, and just catching the first light, just catches the other four. The width of the cross-street at each light is 20 meters. Find all the speeds at which it is possible to travel without being held up at any of the lights.

JRM 730. **by Frank Rubin**

A traffic light has a one-minute cycle, divided into 25 seconds green, 5 seconds yellow, and 30 seconds red. A car approaches the light at a speed of 20 meters per second. The car can brake and accelerate at 5 meters per second. The driver has a one-second reaction time, and perfect judgment, i.e., if the light is red, he will brake at maximum rate so as to stop just at the intersection if this is possible; otherwise he will continue through the intersection at his normal speed. On the average, how much delay does the traffic light cause?

Rate problems: trains

OMG 17.3.6.

A freight train and an express train travel at constant speeds on straight parallel tracks. It takes 21 seconds for the trains to clear each other when passing in the same direction, but only 6 seconds when passing in opposite directions. Find the ratio of the speed of the freight train to the speed of the express train.

OMG 18.3.6.

Two trains start at 7 AM, one from A going to B and the other from B going to A. The first train makes the trip in 8 hours and the second in 12 hours. At what hour of the day will the two trains pass each other?

Rate problems: trips

PARAB 353.

In traveling from A to B, a distance of 100 km, a train accelerates uniformly, travels 80 km at a constant speed of 100 km/hr, and then decelerates uniformly. How long does the trip take?

PARAB 303.

Four men A, B, C, and D set out simultaneously from M to reach N, 5 kilometers away. One of them, D, owns a motorcycle. He gives A a lift for part of the way, then turns back and picks up B. When they overtake A, B alights and the unselfish D once more turns back to assist C. Eventually they all arrive at N at the same moment. If D always travels at a steady v km/hour, and A, B, and C all walk at w km/hr, how long did the trip from M to N take?

PARAB 348.

Four explorers are going to make a trip into the desert. Each man can carry enough water to last ten days. Each man can walk 24 kilometers a day. Obviously if all four stay together, they can manage a trip of only five days into the desert, leaving enough water to return. If our explorers are thinkers, how far can they manage to get into the desert before they have to return? Assume that the desert is so uninhabited that it is safe to leave water behind for the return trip, but no explorer can return to civilization to replenish his supply and then return to the desert.

Recurrences

SIAM 79-5. **by L. Erlebach and O. Ruehr**

A sequence $\{a_n\}$ is defined as follows:

$$a_n = n(n-1)a_{n-1} + \frac{1}{2}n(n-1)^2 a_{n-2},$$

$n \geq 3$; $a_1 = 0$, $a_2 = 1$. Determine how a_n behaves for large n.

AMM E2520. **by G. B. Huff**

The nontrivial sequence $a_0, a_1, \ldots$ satisfies the following recursion formula:

$$a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}^2 (n-2k)! a_k^2.$$

Find a_n.

CANADA 1977/6.

Let $0 < u < 1$ and define

$$u_1 = 1 + u, \quad u_2 = \frac{1}{u_1} + u, \ldots, u_{n+1} = \frac{1}{u_n} + u, \quad n \geq 1.$$

Show that $u_n > 1$ for all values of $n = 1, 2, 3, \ldots$.

CRUX 162. **by Viktors Linis**

If $x_0 = 5$ and $x_{n+1} = x_n + \frac{1}{x_n}$, show that

$$45 < x_{10000} < 45.1.$$

AMM E2567. **by J. H. Conway and R. L. Graham**

Define polynomials $f_m = f_m(x_1, \ldots, x_m)$ by $f_0 = 1$, $f_1 = x_1$, $f_k = x_k f_{k-1} - f_{k-2}$, $k \geq 2$. For a fixed $n \geq 3$, let $y_1, y_2, \ldots$ satisfy $f_n(y_{k+1}, \ldots, y_{k+n}) = 1$ for all $k \geq 0$. Show that $y_{n+k+2} = y_k$ for all $k \geq 1$.

IMO 1976/2.

Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \ldots$. Show that, for any positive integer n, the roots of the equation $P_n(x) = x$ are real and distinct.

CRUX 191. **by R. Robinson Rowe**

Consider the recurrence defined by

$$N_n N_{n-2} = N_{n-1} + e.$$

(a) Find sets of square integers N_0 and N_1 for which $N_5 = N_0$ when $e = 2$.

(b) Find the general relation between N_0 and N_1 for any value of e.

AMM E2737. **by Robert Ross Wilson**

Define a sequence of polynomials by $P_0 = 1$, $P_1 = x+1$, and $P_{n+1} = P_n + xP_{n-1}$ $(n \geq 1)$. Show that all roots of each P_n are real.

Roots of unity

JRM 556. **by Ray Lipman**

Consider the nth roots of unity for $n = 1, 2, \ldots, 100$. How many of these are distinct? What is the closest that two of them come to each other in the complex plane? Generalize.

OSSMB G79.2-3.

Given $w^3 = 1$, $w \neq 1$ (i.e., w is a complex cube root of unity), find the value of

$$(1-w)(1-w^2)(1-w^4)(1-w^5)(1-w^7)(1-w^8).$$

TYCMJ 82. **by Norman Schaumberger**

Choose w and r so that w is a primitive nth root of unity and $r^n \neq 1$. Prove that

$$\sum_{k=0}^{n-1} \frac{1}{1-w^k r} = \frac{n}{1-r^n} \ .$$

PUTNAM 1975/A.4.

Let $n = 2m$, where m is an odd integer greater than 1. Let $\theta = e^{2\pi i/n}$. Express $(1-\theta)^{-1}$ explicitly as a polynomial in θ,

$$a_k\theta^k + a_{k-1}\theta^{k-1} + \cdots + a_1\theta + a_0,$$

with integer coefficients a_i. [Note that θ is a primitive nth root of unity, and thus it satisfies all of the identities which hold for such roots.]

Sequences

OSSMB G78.3-1.

(a) The two middle terms of an arithmetic progression of $2n$ terms are a and b. Find the difference between the sum of the first n terms and the sum of the last n terms.

(b) Determine x such that

$$\sum_{k=0}^{n}(k+1)\left(x - \frac{k}{n}\right) = 0.$$

OSSMB G79.2-1.

Given any arithmetic progression $t_1, t_2, \ldots$ such that $t_r = 0$ for some fixed $r > 1$, show that $t_1 + \cdots + t_{2r-1} = 0$.

CANADA 1975/2.

A sequence of numbers $a_1, a_2, a_3, \ldots$ satisfies

(1) $a_1 = \frac{1}{2}$,

(2) $a_1 + a_2 + \cdots + a_n = n^2 a_n$ $(n \geq 1)$.

Determine the value of a_n $(n \geq 1)$.

Solution of equations: binomial coefficients

SSM 3736. **by William D. Markel**

Find the distinct roots of the equation

$$1 - \binom{n}{2}x^2 + \binom{n}{4}x^4 - \binom{n}{6}x^6 + \cdots + (-1)^j\binom{n}{2j}x^{2j} = 0,$$

where

$$j = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

Solution of equations: degree 2

CRUX 489. **by V. N. Murty**

Find all real numbers x, y, and z such that

$$(1-x)^2 + (x-y)^2 + (y-z)^2 + z^2 = \frac{1}{4}.$$

CRUX 51. **by H. G. Dworschak**

Solve the following equation for the positive integers x and $\mathtt{y}$:

$$(360 + 3x)^2 = \mathtt{492y04}.$$

FUNCT 3.2.7.

Let a, b, and c be real numbers that satisfy the equation

$$3a^2 + 4b^2 + 18c^2 - 4ab - 12ac = 0.$$

Prove that $a = 2b = 3c$.

NYSMTJ 87. **by Thomas Masters and Sidney Penner**

Let $t > 1$ and a, b, c, d be nonnegative. If

$$(at+b)(ct+d) = (bt+a)(dt+c)$$

and $d + c = 2(a+b)$, show that $d = 2a$ and $c = 2b$.

SSM 3725. **by Robert A. Carman**

Some students incorrectly try to solve quadratic equations by the method illustrated in the following examples:

$$\begin{array}{cc} (x+3)(4-x) = 6 & (x+1)(2-x) = 2 \\ x+3 = 6 \text{ or } 4-x = 6 & x+1 = 2 \text{ or } 2-x = 2 \\ x = 3 \text{ or } x = -2 & x = 1 \text{ or } x = 0 \end{array}$$

Notice that in each case the correct answer is obtained. Under what conditions will this approach always yield the correct result?

Solution of equations: degree 4

OMG 18.1.7.

Solve:

$$(x+1)(x+3)(x+5)(x+7) = 9.$$

Solution of equations: degree 20

OSSMB 79-18.

(a) The first two terms of a 20th degree polynomial are $x^{20} - 20x^{19}$ and the last term is 1. If all the roots are real and positive, find them.

(b) Show that if any subset of $n+1$ numbers is selected from the first $2n$ positive integers, the subset must contain two numbers that are relatively prime.

Solution of equations: determinants

CRUX 398. **by Murray S. Klamkin**

Find the roots of the $n \times n$ determinantal equation

$$\left| \frac{1}{x\delta_{rs} + k_r} \right| = 0,$$

where δ_{rs} is the Kronecker delta.

Solution of equations: exponential equations

CRUX 262. **by Steven R. Conrad**

Find the real values of x such that

$$3^{2x^2-7x+3} = 4^{x^2-x-6}.$$

JRM 653. **by J. A. H. Hunter**

Given that $k = \left(\frac{r^x-1}{r-1}\right)^x, r > 1$, determine x in terms of k and r.

Solution of equations: linear

OSSMB G76.2-5.

Solve the equation

$$\frac{a+b-x}{c} + \frac{a+c-x}{b} + \frac{b+c-x}{a} + \frac{4x}{a+b+c} = 1.$$

OSSMB G79.2-2.

Solve

$$\frac{x-ab}{a+b} + \frac{x-bc}{b+c} + \frac{x-ac}{a+c} = a+b+c$$

given that a, b, c are positive real constants.

Solution of equations: logarithms

MATYC 110. **by Louise Grinstein**

Solve for x: $x + \log_a(x) = a$.

OSSMB G76.2-7.

Solve

$$2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0.$$

Solution of equations: radicals

CRUX 116. **by Viktors Linis**

For which values of a, b, and c does the equation

$$\sqrt{x + a\sqrt{x} + b} + \sqrt{x} = c$$

have infinitely many solutions?

CRUX 287. **by M. S. Klamkin**

Determine a real value of x satisfying

$$\sqrt{2ab + 2ax + 2bx - a^2 - b^2 - x^2} = \sqrt{ax - a^2} + \sqrt{bx - b^2}$$

if $x > a$ and $b > 0$.

MSJ 457.

Solve for x:

$$\sqrt{7x} - \sqrt{3x} = 7 - 3.$$

Sports

FUNCT 3.5.1. **by Ray Bence**

Football score is calculated by adding the number of behinds to six times the number of goals. Some scores may be calculated correctly by multiplying the number of goals by the number of behinds. Give a list of all scores for which this is possible.

NYSMTJ 57. **by David Rosen**

Assume that, in a simplified version of football, there are only two types of scoring: a 3-point play and a 7-point play. What is the largest total that cannot be achieved?

JRM 624. **by Benedict Marukian**

When the Latakia State University football team played Filter Tech in the Tobacco Bowl, the lead changed hands after each tally. Moreover, following each tally, each team's score was prime. Under these conditions the final score was as large as it could be. What was it?

"Tally" here is defined to be the points awarded for any scoring play, including, in the case of a touchdown, the conversion, if successful. Thus there are five different possible tallies: 2, 3, 6, 7, and 8.

MM 1024. **by David A. Smith**

In many athletic leagues the progress of teams is reported both in terms of winning percentage and in terms of "games behind" the league leader, defined as the difference in games won minus the difference in games lost, divided by 2. Sports fans often observe, especially early in the season, that the league leader in percentage (the official standard) is behind some other team in games.

Suppose team A is the percentage leader, but team B is ahead of Team A in games. Assume no ties.

(a) Which team has played more games?

(b) What is the minimum difference in number of games played?

(c) Characterize possible won/lost records for the two teams if the difference in number of games played is minimal.

(d) Is it possible for this to occur late in the season?

Substitution

FQ B-394. **by Phil Mana**

Let $P(x) = x(x-1)(x-2)/6$. Simplify the following expression:

$$P(x+y+z) - P(y+z) - P(x+z) - P(x+y) + P(x) + P(y) + P(z).$$

OSSMB G78.2-1.

When $x = (3 + 5\sqrt{-1})/2$, find the value of

$$2x^3 + 2x^2 - 7x + 72$$

and show that it is unaltered if $(3 - 5\sqrt{-1})/2$ is substituted for x.

Sum of powers

PARAB 337.

Let x and y be real numbers such that $x + y = 1$ and $x^4 + y^4 = 7$. Find $x^2 + y^2$ and $x^3 + y^3$.

PARAB 382.

Prove or disprove: There are two numbers x, y such that $x+y=1$, $x^2+y^2=2$, and $x^3+y^3=3$.

CRUX 143. **by Léo Sauvé**

Suppose that

$$f(n)=x^n+y^n+z^n,$$

where (x,y,z) is a triple of complex numbers such that $f(n)=n$ for $n=1,2,3$. Show that the triple (x,y,z) cannot be real and calculate $f(4)$, $f(5)$, and $f(6)$.

CRUX 156. **by Léo Sauvé**

Find all integers n for which the following implication holds: For all real nonzero a, b, and c with nonzero sum,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c} \implies \frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}.$$

Systems of equations: 2 variables

CRUX 384. **by Hippolyte Charles**

Solve the following system of equations for x and y:

$$\frac{(ab+1)(x^2+1)}{x+1}=\frac{(a^2+1)(xy+1)}{y+1}$$

$$\frac{(ab+1)(y^2+1)}{y+1}=\frac{(b^2+1)(xy+1)}{x+1}.$$

ISMJ J11.3.

Solve the simultaneous equations

$$\frac{x-y+1}{x+y-1}=a$$

$$\frac{x+y+1}{x-y-1}=b.$$

Does a solution exist for all values of a and b?

SSM 3596. **by Howard L. Prouse**

Show that the system of equations

$$ax+by=c$$

$$dx+ey=f$$

always has the solution $(-1,2)$ if a, b, c, d, e, and f form a nontrivial arithmetic sequence.

TYCMJ 126. **by R. C. Buck**

If $x=1.36$ and $y=1.69$, calculation in the set of equations

$$x^2+y^2+xy=7$$

$$3x^2-y^2-y=1$$

$$-x^2+3y^2-2x=4$$

suggests you have almost found a common solution. Does there exist a common solution?

CRUX 252. **by Richard S. Field**

Discuss the solutions, if any, of the system

$$x^y=A$$

$$y^x=A+1,$$

where $A\geq 2$ is an integer.

OSSMB G77.2-1.

Find the positive solutions of the equations

$$x^{x+y}=y^a \quad\text{and}\quad y^{x+y}=x^{4a}$$

where $a>0$.

PARAB 280.

Find all solutions of the simultaneous equations:

$$y=x+\sqrt{x+\sqrt{x+\cdots+\sqrt{x+\sqrt{y}}}}$$

$$x+y=6,$$

where there are 1975 square roots in the first equation.

Systems of equations: 3 variables

CANADA 1978/3.

Determine the largest real number z such that

$$x+y+z=5$$

$$xy+yz+xz=3$$

and x and y are also real.

CRUX 438. **by Sahib Ram Mandan**

Eliminate x, y, and z from the following three equations:

$$a_ix^2+b_iy^2+cz^2+2f_iyz+2g_izx+2h_ixy=0, \quad i=1,2,3.$$

OSSMB G78.3-2.

Solve

$$x-y=1-z$$

$$3(x^2-y^2)=5(1-z^2)$$

$$7(x^3-y^3)=19(1-z^3)$$

when $x\neq y$.

OMG 15.2.3.

Find all the ordered triples (x,y,z) such that when any one of these numbers is added to the product of the other two the result is 2.

MSJ 440. **by Harry Sitomer**

Solve the system of equations:

$$x+y+z=5$$

$$x+y-z=7$$

$$(x-y)^3+(y-z)^3=(x-z)^3.$$

CRUX 272. **by Steven R. Conrad**

Solve the system of equations

$$z^x = y^{2x}$$
$$2^z = 2(4)^x$$
$$x + y + z = 16.$$

OSSMB G79.3-5.

Solve the system of equations:

$$x + y + z = 15$$
$$x^2 + y^2 + z^2 = 83$$
$$x^3 + y^3 + z^3 = 495.$$

Systems of equations: 4 variables

PUTNAM 1977/A.2.

Determine all solutions in real numbers x, y, z and w of the system

$$x + y + z = w$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}.$$

Systems of equations: 5 variables

USA 1978/1.

Given that a, b, c, d and e are real numbers such that

$$a + b + c + d + e = 8$$
$$a^2 + b^2 + c^2 + d^2 + e^2 = 16.$$

Determine the maximum value of e.

IMO 1979/5.

Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying the relations

$$\sum_{k=1}^{5} k x_k = a, \qquad \sum_{k=1}^{5} k^3 x_k = a^2, \qquad \sum_{k=1}^{5} k^5 x_k = a^3.$$

CRUX 299. **by M. S. Klamkin**

If

$$F_1 = (-r^2 + s^2 - 2t^2)(x^2 - y^2 - 2xy) - 2rs(x^2 - y^2 + 2xy) + 4rt(x^2 + y^2),$$

$$F_2 = 2rs(x^2 - y^2 + 2xy) + (r^2 + s^2 - 2t^2)(x^2 - y^2 - 2xy) + 4st(x^2 + y^2),$$

$$F_3 = -2rt(x^2 - y^2 - 2xy) - 2st(x^2 - y^2 + 2xy) + (r^2 + s^2 + 2t^2)(x^2 + y^2),$$

show that F_1, F_2, and F_3 are functionally dependent and find their functional relationship. Also, reduce the five-parameter representation of F_1, F_2, and F_3 to one of two parameters.

Systems of equations: 6 variables

CRUX 45. **by H. G. Dworschak**

Find the constants A, B, C, D, p, and q such that

$$A(x-p)^2 + B(x-q)^2 = 5x^2 + 8x + 14,$$
$$C(x-p)^2 + D(x-q)^2 = x^2 + 10x + 17.$$

Systems of equations: 13 variables

PARAB 288.

Find all solutions of the equations with 13 unknowns:

$$x_1x_2 = x_2x_3 = x_3x_4 = \cdots = x_{12}x_{13} = x_{13}x_1 = 1.$$

Solve the similar set of equations with 12 unknowns.

Systems of equations: n variables

AMM E2587. **by Bruno O. Shubert**

Consider the system of n equations

$$x_0 + x_k = \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} (a_{ijk} + x_i), \qquad k = 1, \ldots, n,$$

in $n+1$ unknowns $x_0, x_1, \ldots, x_n$, where the a_{ijk} are given constants. Show that

(a) the system always has a solution and that
(b) the first component, x_0, is unique.

MM 930. **by M. S. Klamkin**

Solve the system of equations

$$(x_i - a_{i+1})(x_{i+1} - a_{i+3}) = a_{i+2}^2,$$

$i = 1, 2, \ldots, n$, for the x_i's where $a_{n+i} = a_i$, $x_{n+i} = x_i$, and $a_1a_2\cdots a_n \neq 0$.

Systems of equations: logarithms

PARAB 349.

Solve the system of equations

$$x^{\log y} + y^{\log\sqrt{x}} = 110, \quad xy = 1000.$$

Theory of equations: constraints

MM 1074. **by Chandrakant Raju and R. Shantaram**

Suppose that all three roots of the cubic

$$x^3 - px + q = 0 \quad (p > 0, q > 0)$$

are real. Show that the numerically smallest root lies between q/p and $2q/p$.

OMG 17.1.6.

Find all real values of k so that the equation

$$x^3 + x^2 - 4kx - 4k = 0$$

has two of its three roots equal.

Theory of equations: inequalities

SPECT 7.9. **by B. G. Eke**

Let a be a positive integer and let b, c be integers. Suppose that ax^2+bx+c has two distinct roots in the range $0 < x < 1$. Show that $a \geq 5$ and find such a quadratic with $a = 5$.

Theory of equations: integer roots

CRUX 190. **by Kenneth M. Wilke**

Find all integral values of m for which the polynomial

$$P(x) = x^3 - mx^2 - mx - (m^2 + 1)$$

has an integral zero.

Theory of equations: real roots

CRUX PS7-3.

Show that the polynomial equation with real coefficients

$$P(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-3}x^3 + x^2 + x + 1 = 0$$

cannot have all real roots.

MM Q626. **by Philip Tracy**

If a, b, and c are real and $b^2 < 2ac$, prove that the cubic $x^3 + ax^2 + bx + c$ has only one distinct real root.

PARAB 282.

Let

$$f(x) = ax^2 + bx + c,$$

where a, b, c are real numbers. Prove that if the coefficients a, b, c are such that the equation $f(x) = x$ has no real roots, then also the equation $f\left(f(x)\right) = x$ has no real roots.

Theory of equations: roots

CRUX 298. **by Clayton W. Dodge**

The equation $x^2 - 9x + 18 = 4$ has the property that, if the left side is factored, so that $(x-3)(x-6) = 4$, then one of the roots, $x = 7$, is found by illegally setting one of the factors equal to the constant on the right, $x - 3 = 4$. Unfortunately, the second root cannot be similarly found; it is not $x - 6 = 4$. Find all such quadratic equations in which both roots can be obtained by equating each factor in turn to the nonzero constant on the right.

PUTNAM 1978/B.3.

The sequence $\{Q_n(x)\}$ of polynomials is defined by

$$Q_1(x) = 1 + x, \quad Q_2(x) = 1 + 2x,$$

and for $m \geq 1$, by

$$Q_{2m+1}(x) = Q_{2m}(x) + (m+1)xQ_{2m-1}(x),$$
$$Q_{2m+2}(x) = Q_{2m+1}(x) + (m+1)xQ_{2m}(x).$$

Let x_n be the largest solution of $Q_n(x) = 0$. Prove that $\{x_n\}$ is an increasing sequence and that $\lim_{n\to\infty} x_n = 0$.

USA 1977/3.

If a and b are two of the roots of $x^4 + x^3 - 1 = 0$, prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

MATYC 138. **by Mangho Ahuja**

If a, b, c, and d are in arithmetic progression, then prove that the roots of the equation

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} + \frac{1}{x-d} = 0$$

are also in arithmetic progression.

OSSMB G79.1-5.

Show that the roots of the cubic equation

$$bx^3 + a^2x^2 + a^2x + b = 0$$

are in geometric progression.

CRUX 468. **by Viktors Linis**

(a) Prove that the equation

$$a_1x^{k_1} + a_2x^{k_2} + \cdots + a_nx^{k_n} - 1 = 0,$$

where $a_1, \ldots, a_n$ are real and $k_1, \ldots, k_n$ are natural numbers, has at most n positive roots.

(b) Prove that the equation

$$ax^k(x+1)^p + bx^l(x+1)^q + cx^m(x+1)^r - 1 = 0,$$

where a, b, c are real and k, l, m, p, q, r are natural numbers, has at most 14 positive roots.

TYCMJ 55. **by Louis Rotando**

Find the set of real values of b, $b > 1$, for which $\log_b x = x$ has

(a) exactly one solution,
(b) exactly two solutions, and
(c) no solutions.

OSSMB G75.1-1.

Given that $x^3 + px + q = 0$ has 3 rational nonzero roots, α, β, γ, show that $\alpha y^2 + \beta y + \gamma = 0$ has rational roots.

CRUX 178. **by Gali Salvatore**

Prove or disprove that the equation $ax^2 + bx + c = 0$ has no rational root if a, b, and c are all odd integers.

CRUX 185. **by H. G. Dworschak**

Prove that, for any positive integer $n > 1$, the equation

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = n^2$$

has a rational root between 1 and 2.

ISMJ 11.6.

Show that if p, q, p_1, and q_1 are real numbers such that $pp_1 = 2(q + q_1)$, then at least one of the equations

$$x^2 + px + q = 0$$
$$x^2 + p_1x + q_1 = 0$$

has real roots.

Theory of equations: systems of equations

MATYC 88. **by Roger Lindley**

Show that the system

$$a_1x^2 - 2b_1xy - a_1y^2 + a_2x - b_2y + a_3 = 0$$
$$b_1x^2 + 2a_1xy - b_1y^2 + b_2x + a_2y + b_3 = 0$$

has at least one and at most two distinct real solutions.

Theory of equations: table of values

OMG 16.1.4.

Find an equation that would generate the following table of values.

n	1	2	3	4
s	0	2	6	12

Uniform growth

JRM 655. **by Friend H. Kierstead, Jr.**

It is known that most of the human beings who have ever lived are still alive today. Using the simplifying assumptions that each individual lives to the age of 70, all babies are born when the mother is 20 years old, and the population is a continuous function of time, what birth rate is necessary to guarantee that the number of living is always just equal to the number of dead?

PME 426. **by R. Robinson Rowe**

After a cold, dry snow had been falling steadily for 72 hours, a niphometer showed a depth of 340 cm, compared to a reading of 175 cm after the first 24 hours. Assuming that underlying snow had been compacted only by the weight of its snow overburden, so that the depth varied as a power of time, what would have been the depths after 12 and 48 hours?

SSM 3577. **by Max Sute**

A Boy Scout troop has enough bread to last them 11 days; if there had been 400 more boys, each boy would have received 2 oz. less per day; if there had been 600 fewer boys, each boy's daily share could have been increased by 2 oz. and the Boy Scout troop would have had enough bread to last them 12 days. How many pounds of bread did the troop have, and what was each daily share?

CRUX 402. **by R. Robinson Rowe**

An army with an initial strength of A men is exactly decimated each day of a 5-day battle and reinforced each night with R men from the reserve pool of P men, winding up on the morning of the 6th day with 60% of its initial strength. At least how large must the initial strength have been if

(a) R was a constant number each day;

(b) R was exactly half the men available in the dwindling pool?

CRUX 1. **by Léo Sauvé**

In 12 days 75 cows have grazed all the grass in a 60-acre pasture, and 81 cows have in 15 days grazed all the grass in a 72-acre pasture. How many cows can in 18 days graze all the grass in a 96-acre pasture?

JRM 476. **by Robert F. Josephson**

Seven sheep will graze my modest pasture level in six days, and eight sheep in five days. How many sheep will the pasture sustain indefinitely?

Venn diagrams

OMG 18.3.2.

In grade 9, 160 students are enrolled in Mathematics, 175 in English and 60 in French. No student is permitted to take more than two of the three subjects, but every student is required to take at least one. With 350 grade 9 students, it is known that no student taking French is taking either of the other two subjects. How many students are taking both Mathematics and English?

OMG 17.1.2.

In a survey of year IV students, the numbers studying various Sciences were found to be: Chemistry - 28, Biology - 30, Physics - 42, Chemistry and Biology - 8, Chemistry and Physics - 10, Biology and Physics - 5, all three Sciences - 3. Find the number of students studying exactly one Science.

Weights

CRUX 123. **by Walter Bluger**

By means of only three weighings on a two-pan balance, you are to find among 13 dimes the one counterfeit coin and be able to tell whether it is heavier or lighter than a true coin. You are given the 13 coins and a balance, and you may bring anything you like with you that may help you in solving the problem.

JRM 448. **by P. MacDonald**

There are 17 coins of three different weights. Light coins weigh 1 ounce, regular coins weigh 2 ounces, and heavy coins weigh 3 ounces. They are sorted by weight and placed into three boxes as shown. All the light coins are in one box, all the regular in another, and all the heavy in the third. However, each box is mismarked.

Divide the 17 coins into two groups that will balance when placed on a balance scale. No preliminary weighings or inspections are allowed.

CANADA 1976/1.

Given four weights in geometric progression and an equal arm balance, show how to find the heaviest weight using the balance only twice.

PARAB 307.

I have 5 balls, identical in appearance, of which two are unequal in weight, one heavier and one lighter than each of the other 3. Together these 2 are equal in weight to two regular balls. Show how to distinguish the balls in three comparisons using a beam balance.

AMM 6224. **by David P. Robbins**

Suppose we are given N balls that are indistinguishable except that some are heavy and some are light (the heavy balls are alike in weight, as are the light balls). Using a balance scale, find the minimum number of weighings in which it is always possible

(a) to identify one heavy and one light ball;

(b) to determine the number of heavy and light balls.

MATYC 127. **by Joseph Browne**

A set of weights is desired that may be used in various combinations to equal every multiple of 10 gm from 10 gm to the total mass of the set. Give a formula for n, the minimum number of weights needed in the set if the largest mass ever required is w gm.

PARAB 291.

Among 11 apparently identical metal spheres, 2 are radioactive. We have an instrument which detects the presence of radioactivity. Show that it is possible to determine the radioactive spheres after 7 uses of the instrument.

OMG 18.3.5.

There are ten bags, each containing ten weights, all of which look identical. In nine of the bags, each weight is 16 grams, but in one of the bags the weights are actually 17 grams each. How is it possible, in a single weighing on an accurate weighing scale, to determine which bag contains the 17 gram weights?

Word problems: counting problems

CRUX 11. **by Léo Sauvé**

A basket contains exactly 30 apples. The apples are distributed among 10 children, each child receiving n apples, where n is a positive integer. At the end of the distribution, there are n apples left in the basket. Find n.

JRM 761. **by Harry Nelson**

In 1776, thirteen colonies sent 56 representatives to Philadelphia who signed the Declaration of Independence. Pennsylvania had the most signers with nine. Four colonies had one more than the least. Four colonies had two more than the least. Two colonies had three more than Rhode Island. One colony had five more than Rhode Island. How many did Rhode Island have?

Word problems: percent problems

CRUX 28. **by Léo Sauvé**

If 7% of the population escapes getting a cold during any given year, how many days must the average inhabitant expect to wait from one cold to the next?

OMG 17.1.1.

A hockey team has won 5 out of 8 games played. With 16 games still to be played, how many more games must be won so that the team wins 75% of its season's schedule?

PENT 279. **by Kenneth M. Wilke**

In Moldavia, people pay an income tax equal to a percentage of the weekly wage based upon the number of ducats earned each week; e.g., on a weekly wage of 10 ducats, the rate is 10 percent. Assuming the maximum salary is 100 ducats per week, what is the optimum salary in Moldavia?

PENT 301. **by Kenneth M. Wilke**

If 65% of the populace have kidney trouble, 70% have diabetes, 85% have respiratory problems, and 90% have athlete's foot, what is the smallest portion of the populace who are afflicted with all four maladies?

SSM 3694. **by Charles W. Trigg**

There is a "Favorite Joke" quoted in PARADE Magazine: "Two drunks were talking about the fuel shortage. One said, 'Charlie, I installed a new carburetor, and it saved me 36 percent on gasoline. I had a new distributor put in, and it saved me 42 percent. I put new radial tires on my car, and they saved me 53 percent on gasoline. And then, by golly, I put in those new special spark plugs, and they saved me 66 percent on gasoline.' 'What happened?' asked Charlie. 'Well,' answered the first, 'I drove 426 miles, and the tank overflowed.' "

What is your reaction to this tall tale?

Word problems: population problems

SSM 3666. **by Mary S. Krimmel**

If a single cell of *E. coli*, under ideal circumstances, were to divide every twenty minutes, in a single day it could produce a colony equal in size and weight to the earth. What would the volume and mass of the original cell have to be for this to happen?

Word problems: ratios

JRM 563. **by Michael J. Messner**

Gandalf gave each of the four Hobbits one fifth of his magic biscuits, which they promptly ate, except for Frodo, who saved half of his. All the uneaten biscuits doubled overnight, and the next day Gandalf gave to each Hobbit a fifth of the biscuits he had left. Frodo ate eight of his share and put the remainder back into Gandalf's sack when he wasn't looking. Again the uneaten biscuits doubled and when Gandalf opened the sack the third day, he was amazed to find a dozen more biscuits than he expected. How many did the Hobbits eat?

MSJ 431. **by Harry Sitomer**

At a PTA affair, attended by parents and children, the number of females is 2/3 the number of males; 1/2 the males are boys; 28 of the females are girls; the husbands of 1/3 of the mothers in attendance are present, and the wives of 1/4 of the fathers in attendance are present. How many people are attending this affair?

MSJ 432. **by Don Baker**

Sam wanted to visit the fair maiden, but he had to cross six bridges to get to her house. At each bridge, the bridgekeeper took half of Sam's apples plus half an apple and let Sam continue on his journey. When he finally arrived at the fair maiden's house, he had only 13 apples left. How many apples did Sam start with?

NYSMTJ 89. **by Norman Gore**

A cookie distributor sells half of his cookies plus one-half a cookie to his first customer, half of the remaining cookies plus one-half a cookie to his second customer, and so on. If no cookies are left after n sales are made in this manner, express the distributor's original number of cookies in terms of n.

OMG 17.2.4.

A traveler sets out to cross a desert. On the first day he covers 1/10 of the journey; on the second day he goes 2/3 of the distance already traveled. He continues on in this manner, alternating the days on which he does 1/10 of the distance still to be done, with days on which he travels 2/3 of the total distance already covered. At the end of the seventh day he finds that another 22.5 kilometers will see the end of his journey. How wide is the desert?

OMG 18.1.1.

Members of a local teenage club disagreed about the way a certain outing was managed, and 15 girls withdrew. This left two boys for each girl. The boys were unhappy about the new setup and 45 moved out, leaving only one boy for each five girls. Work out how many girls there were in the club at the time of the outing.

Banach spaces

CMB P272. **by Jon Borwein**

Do there exist Banach spaces X, Y and a continuous linear operator $T: X \to Y$ with the adjoint mapping T^* having different norm and weak-star closures to its range?

AMM 6203. **by Albert Wilansky**

Let X, Y, and Z be Banach spaces, and let $T: X \to Z$ and $S: Y \to Z$ be continuous and linear functions. Show that the (equivalent) conditions

(i) $\overline{TD_1} \supset SD_\varepsilon$ for some $\varepsilon > 0$ (D_ε is the disc of radius ε),

(ii) $\|S'(f)\| \le k\|T'(f)\|$ for all $f \in Z'$ for some $k > 0$,

do not imply that $TX \supset SY$.

NAvW 549. **by W. M. Dienske**

Let X be a real-normed linear space that is not only $\{0\}$, and let Y be the set of all continuous bounded (but not necessarily linear) functions from X to $\mathbb{R}$. With the supremum norm, Y is a Banach space. For every $a \in X$, we define the function $f_a: X \to \mathbb{R}$ by

$$f_a(x) = \|x - a\| - \|x\|$$

in which $\|x\|$ is the norm of x. It follows easily that $f_a \in Y$. Let L be a line in X, and $F[L]$ its image under the mapping

$$F: X \to Y$$

where $F(a) = f_a$. Show that $F[L]$ is a curve in Y, at no point of which a tangent can be drawn.

NAvW 395. **by D. van Dulst**

A sequence $(X_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces of a Banach space X is called a finite-dimensional Schauder decomposition of X if every $x \in X$ can be uniquely written as

$$x = \sum_{n=1}^{\infty} x_n, \qquad x_n \in X_n \qquad (n = 1, 2, \ldots).$$

It is well known that in this case

$$\nu_{(X_n)_{n\in\mathbb{N}}} = \sup\{\|P_k\| : k \in \mathbb{N}\} < \infty,$$

where, for $k = 1, 2, \ldots$, P_k is the projection defined by

$$P_k(x) = \sum_{n=1}^{k} x_n, \qquad \left(x = \sum_{n=1}^{\infty} x_n \in X\right).$$

Prove that every infinite-dimensional Banach space Y contains an infinite-dimensional closed linear subspace X with the property that, for every $\varepsilon > 0$, X has a finite-dimensional Schauder decomposition $\left(X_n^{(\varepsilon)}\right)_{n\in\mathbb{N}}$ with

$$\nu_{\left(X_n^{(\varepsilon)}\right)_{n\in\mathbb{N}}} < 1 + \varepsilon.$$

Bessel functions

SIAM 75-20. **by M. L. Glasser**

Show that

$$\lim_{n\to\infty} \int_0^\infty I_n(x) J_n(x) K_n(x)\, dx = 8^{-1/2},$$

where, as usual, I_n, J_n, and K_n are Bessel functions.

SIAM 76-10.* **by L. Wijnberg and M. L. Glasser**

(a) If $\alpha > 0$, $v \ge 0$, and

$$S_v(x) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{m+n}{m} (2\alpha)^m J_{v+m+2n+1}(x),$$

show that

$$\lim_{x\to\infty} e^{-\alpha x} S_v(x) = \frac{1}{2}\left\{ \frac{\left[(1+\alpha^2)^{1/2} - \alpha\right]^v}{(1+\alpha^2)^{1/2}} \right\}.$$

(b) For $1 < \alpha$, show that

$$S_v(x) = \frac{1}{2}\left\{ \frac{e^{\alpha x}\left[(1+\alpha^2)^{1/2} - \alpha\right]^v}{(1+\alpha^2)^{1/2} - G_v(\alpha, x)} \right\},$$

where

$$G_v(\alpha, x) = \sum_{k=0}^{\infty} \alpha^{-k-1} J_v^{(k)}(x).$$

(c) Can a result corresponding to (b) be found for $0 < \alpha < 1$?

(d) Sum the series $G_v(\alpha, x)$.

SIAM 76-11. **by B. C. Berndt**

(a) Let $j_{v,n}$ denote the nth positive zero of the ordinary Bessel function $J_v(z)$, where $v > -1$. If $ai \ne 0, \pm j_{v,n}$, $1 \le n < \infty$, show that

$$\sum_{n=1}^{\infty} \frac{1}{j_{v,n}^2 + a^2} = \frac{1}{2ai} \frac{J_{v+1}(ai)}{J_v(ai)}.$$

(b) State and prove a general theorem on the summation of rational functions of zeros of Bessel functions for which the equation above is the special case corresponding to the rational function $1/\left(z^2 + a^2\right)$.

SIAM 77-6. **by J. E. Wilkins, Jr.**

To complete the solution of a certain variational problem arising in physical optics, it is necessary to verify that

$$\left[\int_0^1 J_0(vx)x\,dx\right]^2 \int_0^1 J_0''^2(vx)x^5\,dx > \left[\int_0^1 J_0''(vx)x^3\,dx\right]^2 \int_0^1 J_0^2(vx)x\,dx,$$

at least if $0 \le v \le v_0$, in which $v_0 = 2.29991$ is the smallest positive zero of

$$\int_0^1 J_0''(vx)x^3\,dx.$$

Numerical calculations indicate that the first equation is true when $v = 0.(0.1)2.9$, but not when $v = 3.0$. Establish the truth of the first equation when $0 \le v \le v_0$.

SIAM 77-8. **by M. L. Glasser**

Prove that

$$\int_0^\infty \frac{\log|J_0(x)|}{x^2}\,dx = -\frac{\pi}{2}.$$

SIAM 79-12. **by P. J. de Doelder**

(a) Evaluate in closed form:

$$S(p,q,x)=\sum_{n=1}^{\infty}\frac{J_p(nx)J_q(nx)}{n^{2m}}\ ;$$

$J_p(x)$ and $J_q(x)$ are Bessel functions of order p and q; $p+q=2l$; $p-q=2s$; $l=0,1,2,\ldots$; $s=0,1,2,\ldots$; $m=1,2,\ldots$; $0\le x\le 2\pi$.

In particular, for $p+q>2m$, show that (a) is given by

$$\sum_{n=1}^{\infty}\frac{J_p(nx)J_q(nx)}{n^{2m}}=$$

$$\frac{1}{2}\frac{\Gamma(2m)\Gamma\left(l-m+\frac{1}{2}\right)}{\Gamma\left(m+s+\frac{1}{2}\right)\Gamma\left(m-s+\frac{1}{2}\right)\Gamma\left(m+l+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{2m-1}.$$

(b) Evaluate in closed form:

$$T(p,q,x)=\sum_{n=1}^{\infty}\frac{J_p(nx)J_q(nx)}{n^{2m+1}}\ ;$$

$p+q=2l+1$; $p-q=2s+1$; $l=0,1,2,\ldots$; $s=0,1,2,\ldots$; $m=0,1,2,\ldots$; $0\le x\le 2\pi$.

In particular, for $p+q>2m+1$, show that (b) is given by

$$\sum_{n=1}^{\infty}\frac{J_p(nx)J_q(nx)}{n^{2m+1}}=$$

$$\frac{1}{2}\frac{\Gamma(2m+1)\Gamma\left(l-m+\frac{1}{2}\right)}{\Gamma\left(m+l+\frac{3}{2}\right)\Gamma\left(m+s+\frac{3}{2}\right)\Gamma\left(m-s+\frac{1}{2}\right)}\left(-\frac{x^2}{4}\right)^{m}.$$

SIAM 79-18. **by M. L. Glasser**

Show that for m a positive integer,

$$\sum_{n=1}^{\infty}(-1)^n\frac{J_{2m}(n\pi)}{a^2-n^2}=\frac{\pi J_{2m}(a\pi)}{2a\sin a\pi}\ ,$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{nJ_{2m-1}(n\pi)}{a^2-n^2}=\frac{\pi J_{2m}(a\pi)}{2\sin a\pi}\ .$$

NAvW 419. **by H. K. Kuiken**

Prove that, for $a>0$

$$\int_0^{\infty}e^{-t}\frac{K_{\nu+1}\left(\sqrt{a^2+t^2}\,\right)}{(a^2+t^2)^{\frac{1}{2}(\nu+1)}}\,dt=a^{-2\nu-1}\int_a^{\infty}x^{\nu}K_{\nu}(x)\,dx,$$

where $K_\nu(z)$ stands for the modified Bessel function of the second kind of order ν.

NAvW 557. **by N. Ortner**

Show that

$$\int_0^a J_0^2\left(b\sqrt{a^2-x^2}\,\right)\ dx=\frac{1}{2b}\int_0^{2ab}J_0(x)\ dx$$

and

$$\int_0^{\pi/2}J_0\left(c\sin\phi\right)J_1\left(c\sin\phi\right)\ d\phi=\frac{1-J_0(2c)}{2c}$$

($a>0$, $b>0$, $c>0$).

Cantor set

NYSMTJ 44.

Given line segment AB, trisect the segment, and eliminate all points in the middle third, except the points of trisection. Then trisect each of the two remaining segments, again eliminating the middle thirds (except the points of trisection); then each of the remaining four segments, etc. If the coordinates of A and B are 0 and 1, respectively, which of the coordinates 1/4, 1/5, 1/10, 1/11 belongs to a point that always remains?

Complex variables: analytic functions

AMM 6045. **by J. B. Rosser**

Let D be a domain of the complex plane. For each fixed a, let D_a be the set of z's such that both $a+z$ and $a-z$ lie in D. Choose a fixed complex α and let $f(z)$ be a function such that for each fixed a,

$$f(a+z)+\alpha f(a-z)$$

is analytic in D_a.

Can one conclude that $f(z)$ is analytic throughout D? If not, give some additional weak conditions on f from which one could infer this.

AMM 6071. **by J. G. Milcetich**

The set of analytic functions defined on the unit disc, U, with the topology of uniform convergence on compact subsets of U forms a locally convex, linear topological space. In such a space, $\overline{\text{co}}B$ denotes the closed convex hull of a subset B. Let K denote the set of analytic functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

which map U onto a convex domain. Show that for $k\ge 2$, $z+a_kz^k\in\overline{\text{co}}K$ if and only if $|a_k|\le\frac{1}{2}$.

Complex variables: conformal mappings

AMM 6047. **by C. D. Minda**

Let E_1 and E_2 be ellipses in the complex plane. Prove that there is a conformal mapping of the interior of E_1 onto the interior of E_2 that maps the foci of E_1 onto the foci of E_2 if and only if E_1 and E_2 have the same eccentricity. Moreover, show that if such a conformal mapping exists, then it must necessarily be of the form $az+b$ for some complex numbers a and b with $a\ne 0$.

Complex variables: convolutions

AMM 6145. **by Michael Barr**

Let $\mathbb{N}_0=\{0,1,2,\ldots\}$, $\mathbb{C}^*$ the nonzero complex numbers. Suppose

$$\rho\colon\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{C}^*$$

is a "kernel function" with the property that the convolution product defined on functions $\mathbb{N}_0\to\mathbb{C}^*$ by the formula

$$(f*_\rho g)\,(n)=\sum_{i+j=n}\rho(i,j)f(i)g(j)$$

is associative. Show that there is a function $\sigma\colon\mathbb{N}_0\to\mathbb{C}^*$ such that $f*_\rho g=\sigma^{-1}(\sigma f*g)$, where an unadorned $*$ denotes the usual convolution with respect to the kernel, which is identically 1. Note that this implies that $f*_\rho g=g*_\rho f$ and ultimately that ρ is symmetric.

Complex variables: harmonic functions

AMM 6198. **by Sanford S. Miller**

Let $u(z) = u(x, y)$ be harmonic in the unit disc D with $u(0) = 1$, and let $g(t)$ be a real-valued function satisfying $g(1) > 0$ and $g(0) \le 1/2$. Show that if u satisfies $g(u) + xu_x + yu_y > 0$ for $z \in D$, then $u(z) > 0$ for $z \in D$. In particular, if $g(t) = \frac{1}{2}$, or $g(t) = t + \frac{1}{2}$, we obtain, respectively,

$$xu_x + yu_y > -\frac{1}{2} \Rightarrow u > 0,$$

$$u + xu_x + yu_y > -\frac{1}{2} \Rightarrow u > 0.$$

Complex variables: inequalities

AMM 6033. **by S. S. Miller**

Let $w(z)$ be regular in the unit disc D with $w(0) = 0$, and let A be a complex number such that Re $A \ge 1$. If $z \in D$, show that

$$|w^2(z) + Aw(z) + zw'(z)| < 1$$

implies $|w(z)| < 1$.

Complex variables: number theory

AMM 6109. **by Stuart P. Lloyd**

The functions $S_1(z), S_2(z), \ldots$ are defined recursively by setting $S_1(z) = z$, $S_{n+1}(z) = \phi(S_n(z))$ for $n \ge 1$, where $\phi(s) = s + s^2$. When z is a positive integer, the series

$$\frac{1}{z} = \sum_{n=1}^{\infty} \frac{1}{S_n(z) + 1}$$

is the nonterminating Sylvester series for the rational number $1/z$. Determine the region of convergence of this series in the complex z-plane.

AMM E2778. **by David J. Allwright**

Let k and r be integers with $r \ge 1$, and let z be a complex number with $|z| < 1$. Calculate the sum of $z^{\|N\|}$ as

$$N = (n_0, n_1, \ldots, n_r)$$

ranges over all $(r + 1)$-tuples of integers such that

$$n_0 + n_1 + \cdots + n_r = k$$

and

$$\|N\| = |n_0| + |n_1| + \cdots + |n_r|.$$

Complex variables: polynomials

AMM E2808. **by P. Henrici**

Let $p(z) = a_0 + a_1 z + \cdots + a_k z^k$, where the a_i are complex numbers and $a_0 \ne 0$. Ordinary iteration applied to p in the form

$$q_{n+1} = \frac{-a_0}{a_1 + q_n\,(a_2 + q_n\,(a_3 + \cdots + q_n a_k) \cdots)}$$

may or may not produce a sequence (q_n) that converges to a zero of p. Show, however, that if the above equation is replaced by

$$q_{n+1} = \frac{-a_0}{a_1 + q_n\,(a_2 + q_{n-1}\,(a_3 + \cdots + q_{n-k+2} a_k) \cdots)},$$

then for almost all choices $(q_1, q_2, \ldots, q_{k-1})$ of starting values, the sequence (q_n) converges to the zero of smallest modulus of p, if p has a single such zero.

Complex variables: rational functions

CMB P277.* **by Allan M. Krall and D. J. Allwright**

Let $R(z)$ be a rational function of the complex variable z, and let Γ be the locus of $R(ix)$ for x real. Prove that Γ partitions the plane into finitely many regions.

CRUX 130. **by Jacques Marion**

Let A be the annulus $\{z \mid r \le |z| \le R\}$. Show that the function $f(z) = \frac{1}{z}$ is not a uniform limit of polynomials on A.

Complex variables: several variables

AMM 6091. **by H. S. Shapiro**

Let Γ denote the set of complex numbers of modulus 1, and consider for positive integers m, n the map $T\colon \Gamma^n \to \mathbb{C}^m$ defined by

$$\begin{aligned} w_1 &= z_1 + z_2 + \cdots + z_n \\ w_2 &= z_1^2 + z_2^2 + \cdots + z_n^2 \\ &\;\vdots \\ w_m &= z_1^m + z_2^m + \cdots + z_n^m, \end{aligned}$$

where each z_i ranges over Γ. Prove that for any m and any positive R, the range of T contains the ball $\|w\| \le R$ for all sufficiently large n.

Curves: curve tracing

CANADA 1978/6.

Sketch the graph of $x^3 + xy + y^3 = 3$.

Curves: inequalities

AMM S19. **by Anon**

Let C be a smooth simple arc inside the unit disc, except for its endpoints, which are on the boundary. How long must C be if it cuts off one-third of the disc's area? Generalize.

Curves: inflection points

NAvW 481. **by O. Bottema and J. T. Groenman**

With respect to a plane projective coordinate system, a cubic curve is given by the equation

$$x^2 y + y^2 z + z^2 x - 3xyz = 0.$$

Determine the coordinates of its inflection points.

Curves: normals

MM 1067. **by M. S. Klamkin**

Find the length of the shortest chord that is normal to the parabola $y^2 = 2ax$, $a > 0$, at one end. Give a completely "non-calculus" solution.

Curves: simple closed curves

AMM 6225. **by Edmund H. Anderson**

Construct a homotopically trivial mapping from the three-sphere onto the two-sphere such that the pre-images of points are simple closed curves.

Curves: space filling curves

SPECT 10.8.

Accepting as known the existence of a continuous square-filling curve, demonstrate the existence of a continuous curve that passes through every point of the entire plane.

Curves: tangents

AMM 6223. **by Harry D. Ruderman**

Let C be a convex curve. Let Q be a curve such that the two tangents to C from each point P of Q form an angle θ fixed in size. Assume that all points are in the same plane.

(a) If $\theta = 90^\circ$ and Q is a circle, must C be a circle or an ellipse?

(b) If C is an ellipse and $\theta \neq 90^\circ$, what is the nature of Q?

NYSMTJ 67.

Consider the function $y = a^x$ and its inverse $y = \log_a x$. For what value of $a > 1$ will the two graphs be tangent to each other, and what will be the point of tangency?

PUTNAM 1979/B.1.

Prove or disprove: there is at least one straight line normal to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(c, \sinh c)$.

Curves: unit square

AMM E2647. **by Daniel Gallin**

Let Γ_1 and Γ_2 be two continuous maps of the unit segment

$$I = \{x \,|\, 0 \leq x \leq 1\}$$

into the unit square I^2. Suppose that $\Gamma_1(0) = (0,0)$, $\Gamma_1(1) = (1,1)$, $\Gamma_2(0) = (0,1)$, $\Gamma_2(1) = (1,0)$. Prove by elementary means (e.g., without using the Jordan Curve Theorem) that the two curves Γ_1 and Γ_2 meet.

Derivatives: continued fractions

MATYC 103. **by Robert Carman**

Find the derivative of the continued fraction

$$y = \cfrac{1}{x + \cfrac{1}{3x + \cfrac{4}{5x + \cdots + \cfrac{n^2}{(2n+1)x}}}} .$$

MATYC 89. **by J. Kapoor**

Find the derivative of the continued fraction

$$y = 2x + \cfrac{3}{2x + \cfrac{3}{2x + \cfrac{3}{\ddots}}} .$$

Derivatives: finite products

AMM E2580. **by Clark Kimberling**

Show that

$$\frac{d}{dx}\left[\prod_{k=0}^{n-1}\left(a - 2\sqrt{x}\cos\frac{(2k+1)\pi}{2n}\right)\right] = -n\prod_{k=1}^{n-2}\left(a - 2\sqrt{x}\cos\frac{k\pi}{n-1}\right).$$

Derivatives: finite sums

MM 1053. **by Peter Ørno**

Let $f(x)$ be differentiable on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$. For each positive integer n, show that there exist distinct $x_1, x_2, \ldots, x_n$ such that

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n.$$

Derivatives: gradients

NAvW 394. **by J. J. A. M. Brands**

Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$, where $|x|$ is the Euclidean norm of x. Suppose $f \in C(\overline{B} \to \mathbb{R})$ is differentiable on B and $\max\{|f(x)| : |x| = 1\} \leq 1$. Show that there exists a point $\xi \in B$ at which $|\text{grad } f| \leq 1$.

Derivatives: higher derivatives

MATYC 83. **by Aleksandras Zujus**

Let

$$F(x) = \frac{1}{a^2 + x^2} .$$

Prove that

$$\frac{d^n}{dx^n}(F(x)) = \frac{(-1)^n \cdot n!}{a} \cdot \frac{\sin[(n+1)\alpha]}{(a^2+x^2)^{(n+1)/2}} ,$$

where

$$\alpha = \tan^{-1}\left(\frac{a}{x}\right).$$

PENT 273. **by Gary Schmidt**

Show that

$$\frac{d^2y}{dx^2} = \frac{-d^2x/dy^2}{(dx/dy)^3}$$

is an identity.

AMM E2748. **by Lance Littlejohn**

If $f(x) = x^n \log x$, find

$$\lim_{n\to\infty} \frac{f^{(n)}(1/n)}{n!}.$$

TYCMJ 122. **by Lance Littlejohn**

Let $y = x^n \ln x$ and let $y^{(n)}$ denote the nth derivative of y with respect to x. Prove that

$$\lim_{n\to\infty} y^{(n)}\left(\frac{1}{n}\right) \Big/ n! = \gamma,$$

where γ is Euler's constant.

SSM 3731. **by John Oman**

If the second derivative f'' exists at the point a, prove that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}.$$

AMM E2755. **by Michael Slater**

Let $f \in C^\infty(\mathbb{R})$ and suppose that $f(x) = o(x^n)$ as $x \to \pm\infty$ for some integer $n \geq 0$. Show that $f^{(r)}$ has a zero for every $r \geq n+1$. Is this conclusion the best possible?

Derivatives: inequalities

AMM E2759.* **by Hugh L. Montgomery**

Suppose that $a^{-1} \leq f''(x) \leq 2a^{-1}$ for $0 \leq x \leq a$, where $a \geq 8$. Prove that there exists a lattice point (m,n) such that $0 \leq m \leq a$ and $|f(m)-n| \leq 2a^{-1/2}$.

Derivatives: maxima and minima

AMM E2518. **by Derek A. Zave**

Let F be the set of real polynomials f with nonnegative coefficients for which $f(1) = 1$. Let

$$0 < x_0 < 1 \quad \text{and} \quad 0 < \alpha \leq 1$$

be fixed. Compute

$$m(x_0;\alpha) = \inf\left\{f'(1) \,|\, f \in F \quad \text{and} \quad f(x_0) \leq \alpha\right\}.$$

Derivatives: one-sided derivatives

AMM 6166. **by D. A. Gregory**

If f is a convex functional on a convex subset K of a vector space, then for all x and $x+h$ in K, the one-sided directional derivatives

$$f'_+(x,h) = \lim_{\alpha\to 0^+} \frac{f(x+\alpha h)-f(x)}{\alpha}$$

exist in the extended reals and $f(x+h) \geq f(x) + f'_+(x,h)$. Is the converse true? If so, we have an analytic characterization of convex functionals.

CMB P280. **by F. S. Cater**

Clearly any nowhere differentiable one-to-one function mapping the interval $(0,1)$ onto $(0,1)$ must be discontinuous at a dense set of points in $(0,1)$. Does such a function exist that is left continuous at every point, has a right limit at every point, but is not left or right differentiable at any point?

Derivatives: product rule

FUNCT 1.4.3.

A student believed that

$$\frac{d}{dx}[u(x)v(x)] = u'(x)v'(x).$$

Using his formula, he correctly differentiated $(x+2)^2x^{-2}$. What relation must hold between a pair of functions $u(x)$, $v(x)$ for him to get a correct answer? Give some other examples.

Derivatives: roots

MM 997. **by John Lott**

Let P be a polynomial of degree n, $n \geq 2$, with simple zeros $z_1, z_2, \ldots, z_n$. Let (g_k) be the sequence of functions defined by $g_1 = 1/P'$, and $g_{k+1} = g_k'/P'$. Prove for all k that

$$\sum_{j=1}^{n} g_k(z_j) = 0.$$

Derivatives: trigonometric functions

PME 347. **by Joe Dan Austin**

Let $f(x) = \frac{\sin x}{x} - \frac{99x}{4} + 1$ for $x > 0$. Show that $f'(x) \neq 0$ for $x > 1$.

Differential equations: Bernoulli equation

AMM E2568. **by Stroughton Bell**

Show that the Bernoulli equation

$$y' + y^2 + xy = 0$$

has exactly two solutions on the entire real line for which y'' is nowhere zero.

Differential equations: Bessel functions

SIAM 77-20. **by I. Nåsell**

Prove that the equation

$$I_\nu(x) = I_\nu'(x)$$

has exactly one positive solution $x = \xi(\nu)$ for each $\nu > 0$. Investigate the properties of the function ξ.

Differential equations: determinants

AMM E2767. **by James W. Burgmeier**

Let f be a function with sufficiently many derivatives, and let D_n be the determinant

$$D_n = \begin{vmatrix} f' & f & 0 & 0 & \ldots & 0 & 0 \\ \frac{f''}{2!} & f' & f & 0 & \ldots & 0 & 0 \\ \frac{f'''}{3!} & \frac{f''}{2!} & f' & f & \ldots & 0 & 0 \\ \vdots & & & & & & \\ \frac{f^{(n)}}{n!} & \frac{f^{(n-1)}}{(n-1)!} & & & \ldots & & f' \end{vmatrix}.$$

Show that

$$D_{n+1} = f'D_n - \frac{1}{n+1} f D_n'.$$

Differential equations: functional equations

MATYC 81. **by Steven Kerr**

Let $g(x) = x^n$, $n > 1$, where n is a positive integer. Find all nonconstant differentiable functions f such that $(f(g(x)))' = f'(x)g'(x)$ for all real numbers.

MM 1030. **by G. Edgar**

(a) Solve the following functional-differential equation for the complex-valued differentiable function f:

$$f(s+t) = f(s) + f(t) - f'(s)f'(t)$$

for all real s and t, and $f(0) = 0$.

(b) If the real part of $f(t)$ is nonpositive for all real t, but f is not identically zero, show that $f(t) = 0$ only if $t = 0$.

TYCMJ 101. **by Louis Alpert and Jerry Brantley**

Determine all functions f defined on $(-\infty, \infty)$ such that for all $a \neq b$, $f'\big(\frac{1}{2}(a+b)\big) = (f(b) - f(a))/(b-a)$.

AMM 6088. **by Nathaniel Grossman**

The functional-differential equation $f' = f^{-1}$ has a solution satisfying $f(0) = 0$ and $f'(x) > 0$ for $x > 0$, namely, $f(x) = (1/\alpha)^{1/\alpha} x^\alpha$, where $\alpha = (1+\sqrt{5})/2$. Is this the only solution satisfying the given conditions for $x \geq 0$?

Differential equations: initial value problems

SIAM 79-20. **by J. D. Love**

Derive a bounded solution of the equation

$$\frac{dy}{dt} = y(\lambda t), \qquad y(0) = 1,$$

where λ is a constant > 1.

Differential equations: Laplacian

CMB P260. **by S. Zaidman**

Let Ω be a bounded open set in $\mathbb{R}^n$ and let $d(\Omega) = \sup_{x,y \in \Omega} |x-y|$. Suppose that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and satisfies the equation $\triangle u + Au = 0$ in Ω, where $A \in C^0(\bar{\Omega})$ and $\sup_{x, \in \bar{\Omega}} A(x) < 2nd^{-2}(\Omega)$. Show that, if u satisfies the boundary condition $u(x) = 0$ on $\partial\Omega$, then $u(x) \equiv 0$ in Ω.

Differential equations: order 1

MM 950. **by Erwin Just**

Show that there is a unique real number c such that for every differentiable function f on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$, the equation $f'(x) = cx$ has a solution in $(0,1)$.

SIAM 77-16. **by I. Rubinstein**

Solve the differential equation

$$\frac{dr}{dt} + t^{-1}\sqrt{r^2 + a^2} = b, \qquad t > 1,$$

where $r(1) = r_0$ and a, b are constants.

Differential equations: order 2

MM 1050. **by W. R. Utz**

Consider the differential equation

$$y'' + P_1(x)y' + P_2(x)y = 0,$$

where $P_1$and P_2 are polynomials not both constant. Show that this equation has at most one solution of the form $x^a e^{mx}$ for real a.

PUTNAM 1975/A.5.

On some interval I of the real line, let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the differential equation

$$y'' = f(x)y,$$

where $f(x)$ is a continuous real-valued function. Suppose that $y_1(x) > 0$ and $y_2(x) > 0$ on I. Show that there exists a positive constant c such that, on I, the function

$$z(x) = c\sqrt{y_1(x)y_2(x)}$$

satisfies the equation

$$z'' + \frac{1}{z^3} = f(x)z.$$

State clearly how c depends on $y_1(x)$ and $y_2(x)$.

PUTNAM 1979/B.4.

(a) Find a solution that is not identically zero, of the homogeneous linear differential equation

$$(3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 0.$$

(b) Let $y = f(x)$ be the solution of the nonhomogeneous differential equation

$$(3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 6(6x + 1)$$

that has $f(0) = 1$ and $(f(-1) - 2)(f(1) - 6) = 1$. Find integers a, b and c such that

$$(f(-2) - a)(f(2) - b) = c.$$

SIAM 75-6.* **by P. C. T. de Boer and G. S. S. Ludford**

Show that there exists a continuous solution of

$$y'' = \left(2y^\alpha - x\right) y, \qquad \alpha > 0,$$

for $-\infty < x < \infty$ such that

$$y \sim (x/2)^{1/\alpha}\left[1 + (1-\alpha)/\alpha^3 x^3 + \cdots\right]$$

as $x \to +\infty$; and that, for some $k(\alpha)$, $y \sim k\mathrm{Ai}(-x)$ as $x \to -\infty$.

SIAM 79-11. **by D. K. Ross**

Find the general solution of the ordinary nonlinear differential equation

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) = e^{-\varepsilon y}, \qquad \text{with } x > 0$$

and where $\varepsilon = 1$ or -1.

Differential equations: order 4

MM Q631. **by M. S. Klamkin**

Solve the differential equation

$$(xD^4 - axD + 3a)y = 0.$$

Differential equations: order n

NAvW 447. **by W. R. Utz**

Assume that

$$y = (x - \alpha)^{-i}, \qquad i = 1, 2, \ldots, n, \qquad (x \neq \alpha),$$

are solutions of the differential equation

$$(x-\alpha)^n y^{(n)} + P_{1,n}(x)y^{(n-1)} + \cdots + P_{n,n}(x)y = 0.$$

It is easily seen that $P_{n,n}(x)$ depends only on n. Determine this function of n.

SIAM 76-6. **by M. S. Klamkin**

Solve the differential equation

$$\left[x^{2n}\left(D - \frac{a}{x}\right)^n - k^n\right] y = 0.$$

Differential equations: systems of equations

CRUX 498. **by G. P. Henderson**

Let $a_i(t)$, $i = 1, 2, 3$, be given functions whose Wronskian, $w(t)$, never vanishes. Let

$$u(t) = \sqrt{\sum a_i^2}$$

and

$$v(t) = (\sum a_i^2)(\sum a_i'^2) - (\sum a_i a_i')^2.$$

Prove that the general solution of the system

$$x_1'/a_1 = x_2'/a_2 = x_3'/a_3$$
$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

can be expressed in terms of

$$\int \frac{uw}{v}\, dt,$$

no other quadratures being required.

SIAM 76-12.* **by A. S. Perelson and C. Delisi**

The following system of nonlinear differential equations

$$\frac{dx_n}{dt} = 2k \sum_{m=1}^{n-1} x_{n-m}y_m - 2x_n(kS + k'n) + k' \sum_{m=n}^{\infty} (2x_m + y_m), \qquad n = 1, 2, \ldots,$$

$$\frac{dy_n}{dt} = 4k \sum_{m=1}^{n} z_{n-m}x_m + k \sum_{m=1}^{n-1} y_{n-m}y_m - y_n \left[k(S + L) + (2n - 1)k'\right] + 2k' \left[\sum_{m=n+1}^{\infty} x_m + \sum_{m=n+1}^{\infty} y_n + \sum_{m=n}^{\infty} z_m\right], \quad n = 1, 2, \ldots,$$

$$\frac{dz_n}{dt} = 2k \sum_{m=1}^{n} z_{n-m}y_m - 2z_n(kL + k'n) + k' \sum_{m=n+1}^{\infty} (2z_m + y_m), \qquad n = 0, 1, 2, \ldots,$$

where

$$S = \sum_{m=1}^{\infty} y_m + 2 \sum_{m=0}^{\infty} z_m$$

and

$$L = \sum_{m=1}^{\infty} (y_m + 2x_m),$$

subject to the initial conditions $x_1(0) = a$, $x_n(0) = 0$ ($n = 2, 3, \ldots$), $y_n(0) = 0 = z_n(0)$ ($n = 1, 2, \ldots$), $z_0 = b$, with k and k' being nonnegative constants, can be solved by a combinatorial method. Can they be solved by a direct method?

MM 1005. **by Brian Hogan**

Suppose f and g are differentiable functions for $x > 0$ and $f'(x) = -g(x)/x$ and $g'(x) = -f(x)/x$. Characterize all such f and g.

SIAM 77-17. **by L. Carlitz**

Solve the following system of differential equations:

$$F''(x) = F(x)^3 + F(x)G(x)^2,$$
$$G''(x) = 2G(x)F(x)^2,$$

where $F(0) = G'(0) = 1$, $F'(0) = G(0) = 0$.

Differential operators

SIAM 77-4. **by A. Ungar**

Let $x = (x_1, x_2, \ldots, x_n)$ be a set of n real variables and let L be the linear differential operator

$$L\{f(x)\} = \sum_{i=1}^{N} a_{p_i}(x) \frac{\partial^{p_i} f(x)}{\partial x^{p_i}} .$$

Here p_i are multi-indices of order n. For a multi-index p of order n, $p = (p_1, p_2, \ldots, p_n)$, where the entries are integers, $|p| = p_1 + p_2 + \cdots + p_n$ and

$$\frac{\partial^p}{\partial x^p} = \frac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}} .$$

The coefficients $a_{p_i}(x)$ are functions of x and N is an integer.

Prove that

$$f(x) = S_1(x) A\left[S_2(x)\right],$$

where $S_1(x)$ and $S_2(x)$ are specified functions and A is an arbitrary suitably differentiable function of $S_2(x)$, satisfies the linear partial differential equation

$$L\{F(x)\} = 0$$

in a domain, if and only if

$$g(x) = S_1(x) e^{\alpha S_2(x)}$$

is a particular solution of $L\{F(x)\} = 0$ in that domain, for every real α in some interval.

Elliptic integrals

NAvW 479. **by J. Boersma**

Show that

$$\int_0^a \frac{kK(k)}{(1 - k^2)\sqrt{a^2 - k^2}}\, dk = \frac{\pi}{4\sqrt{1 - a^2}} \log \frac{1 + a}{1 - a} , \qquad 0 \le a < 1,$$

where $K(k)$ denotes the complete elliptic integral of the first kind.

SIAM 78-10. **by A. V. Boyd**

Prove that

$$\int_0^{\pi/2} K(t \sin\phi)\, d\phi = K^2 \left\{ \frac{\sqrt{1 + t} - \sqrt{1 - t}}{2} \right\}$$

for $-1 \le t \le 1$, where $K(k)$ denotes the complete elliptic integral of the first kind.

Exponential function

PUTNAM 1975/B.5.

Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf'_n(x)$ for $n = 0, 1, 2, \ldots$. Show that

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e.$$

Fourier series

SIAM 79-9. by N. R. Pereira

For all real values of a, find the Fourier series for the function $[\mathrm{dn}(u) + ik\mathrm{sn}(u)]^a$. For integral values of a, the Fourier coefficients can be evaluated using contour integration and the results are well-known.

Functional analysis

AMM 6078. by Albert Wilansky

Is it possible for a continuous linear functional on a normed space to map every bounded closed set onto a closed set of scalars?

AMM 6278. by Stanley Wagon

Let X be the real vector space consisting of all bounded real-valued functions on the reals with bounded support. Is there a basis, B, for X that is closed under translation, i.e., if f is in B and t is real, then f_t is in B, where $f_t(x) = f(x + t)$?

NAvW 534. by S. T. M. Ackermans

Let A be a Banach algebra with identity element e, and let the elements b and c satisfy $cb = c$, $b \neq e$. The function f is analytic in a neighborhood of the spectrum of c and $f(0) = 0$. Show that $f(bc) = bf(c)$.

AMM 6021. by Charles R. Diminnie and Albert White

In l^p, $p > 2$, does

$$\|x - y\| \, \|x + y\| = \left| \|x\|^2 - \|y\|^2 \right|,$$

with $x, y \neq 0$, imply that $y = \alpha x$ for some real α?

AMM 6277. by Yasuo Watatani

If α and β are *-automorphisms of the algebra $B(H)$ of all bounded linear operators acting on a Hilbert space H such that

$$\alpha(x) + \alpha^{-1}(x) = \beta(x) + \beta^{-1}(x) \quad \text{for} \quad x \in B(H),$$

then prove that they commute.

CMB P246. by S. Zaidman

Let A be a self adjoint operator with domain $D(A)$ in Hilbert space H. Let $f\colon \mathbb{R}^1 \to H$ be almost periodic and suppose that $u\colon \mathbb{R}^1 \to D(A)$ is an almost periodic $C^1(H)$ strong solution of the equation $u' = Au + f$. Then the real number $\lambda \neq 0$ belongs to the spectrum of f if and only if it belongs to the spectrum of u.

Functions: bounded variation

AMM 6256. by A. Kussmaul and P. E. Kopp

Prove or disprove the assertion that every countably additive real-valued set function on a ring R of sets is of bounded variation. Is the assertion true if R is an algebra of sets?

Functions: C^∞

AMM 6042. by F. T. Laseau, G. M. Leibowitz, C. H. Rasmussen, and S. J. Sidney

Is every C^∞ real-valued function on the line that vanishes outside $[0, 1]$ expressible as a difference of two such functions that are nonnegative?

AMM E2756. by Michael Slater

Let $f \in C^\infty(\mathbb{R})$, $f(0)f'(0) \geq 0$, and $f(x) \to 0$ as $x \to \infty$. Show that there exists a sequence (x_n) with

$$0 \leq x_1 < x_2 < \cdots$$

such that $f^{(n)}(x_n) = 0$ for $n = 1, 2, \ldots$.

Functions: composition of functions

AMM 6244. by John Myhill

Let f_i, $i = 0, 1, 2, \ldots$, be a sequence of (everywhere defined) real functions. Prove that there exist two functions ϕ, ψ such that each of the f_i can be obtained from ϕ and ψ by composition.

Functions: continuous functions

AMM 6120. by Jack Fishburn

Let f be a continuous function from the closed unit disc into the reals. If the line integral of f over every chord is zero, must f be identically zero?

What if the continuity of f is replaced by measurability? Must $f = 0$ almost everywhere?

AMM 6250. by Harold Shapiro

Let $w = f(z)$ be a continuous complex-valued function on the closed unit disc $|z| \leq 1$, which is one-to-one on the open disc $|z| < 1$. Show that the set of boundary points of the image that have three or more distinct pre-images under the map f is at most countably infinite.

AMM E2706. by David L. Lovelady

Let

$$f(t) = g(t) \int_0^t g(s)^{-\alpha} \, ds,$$

where $\alpha > 1$ and g is a positive continuous function on $[0, \infty)$. Prove that f is unbounded. Is this true if $\alpha = 1$?

AMM E2783. by William Knight and Bruce Lund

Find all functions $\phi(z)$ such that ϕ is a one-to-one continuous map of the unit circle $\{z : |z| = 1\}$ onto itself and $[\phi(z)]^2 = \phi\left(z^2\right)$ for all z on the circle.

CRUX 20. by Jacques Marion

The function $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = x, \quad \text{if } x \text{ is irrational},$$

$$f(x) = p \sin \frac{1}{q} \quad \text{if } x = \frac{p}{q} \text{ (rational and irreducible).}$$

At which points is f continuous?

NAvW 416. **by J. J. A. M. Brands and M. L. J. Hautus**

Let g be a positive and continuous function on $(0, \infty)$ with the property that

$$\int_1^\infty (g(s))^{-1}\, ds < \infty.$$

Prove that there exists no positive and continuous function f on $(0, \infty)$ that satisfies

$$f(x+y) \geq yg(f(x)), \qquad x > 0, \quad y > 0.$$

AMM 6100. **by Eric Chandler**

For fixed integer $n > 1$, find a bijection T on the real numbers such that T^m is a contraction if and only if $m = kn$ for $k = 1, 2, \ldots$. Can T be continuous?

CRUX 184. **by Hippolyte Charles**

If $I = \{x \in R \mid a \leq x \leq b\}$ and if the function $f: I \to I$ is continuous, show that the equation $f(x) = x$ has at least one solution in I.

AMM 6133. **by J. Cano and A. Gruebler**

Let $f: [a, b] \to [a, b]$ be continuous and denote

$$P(f) = \{x: f^n(x) = x \text{ for some } n = 1, 2, \ldots\},$$

$$C(f) = \{x: f^m(x) \in P(f) \text{ for some } m = 1, 2, \ldots\},$$

and L_x the set of limit points of the sequence

$$\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}.$$

Here $f^n(x)$ is the nth iterate of $f(x)$. Prove that for each $x \in [a, b]$, $L_x \subset \overline{C(f)}$. Is this true in $\mathbb{R}^2$?

CRUX 100. **by Léo Sauvé**

Let $f(x)$ be continuous and nonnegative for all $x \geq 0$. Suppose there exists $a > 0$ such that for all $x > 0$,

$$f(x) \leq a \int_0^x f(t)\, dt.$$

Show that $f(x) = 0$ for all $x \geq 0$.

NAvW 427. **by N. G. de Bruijn**

Let u be a continuous function on $[0, \infty)$. Put

$$p(x) = \int_0^x u(t)\, dt - xu(x).$$

We assume that $x^{-1} \int_0^x u(t)\, dt \to 0$ $(x \to \infty)$, and that λ is a real number, $\lambda < 1$, with $p(x) = O\left(x^\lambda\right)$, $(x \to \infty)$.

Show that $u(x) = O\left(x^{\lambda - 1}\right)$, $(x \to \infty)$.

MM 1069. **by F. David Hammer**

Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and for every rational q there exists an n with $f^n(q) = 0$ (the nth iterate of f). Prove or disprove: For every real number t there is an n such that $f^n(t) = 0$.

TYCMJ 46. **by Louis Alpert**

Let f be a nonconstant, real-valued, continuous function defined on the real line such that its average value over any finite interval equals its value at the midpoint of that interval. Must f be a linear function?

CMB P281. **by M. S. Klamkin**

It is well known that, if $a, c \geq 0$, $b^2 \leq 4ac$ then

$$ax^2 + bxy + cy^2 \geq 0 \tag{1}$$

and

$$ax^4 + bx^2y^2 + cy^4 \geq 0 \tag{2}$$

for all real x and y. Assume a, b, and c are continuous functions of x and y.

(a) Given that $b > 0$, that $a, c \geq 0$ and that (2) is valid for all real x and y, is it necessary that $b^2 - 4ac \geq 0$?

(b) Given that $a, c \geq 0$ and that (1) is valid for all real x and y, is it necessary that $b^2 - 4ac \geq 0$?

AMM E2537. **by David Shelupsky**

Find all continuous functions f defined on $(0, \infty)$ for which

$$f(x_1 y) - f(x_2 y)$$

is independent of y.

AMM 6142. **by L. O. Chung**

Find a function $f: [0, 1] \to [0, 1]$ that is continuous everywhere except on two countable dense subsets D_1, D_2 of rationals such that on D_1, f is right-continuous but not left-continuous, and on D_2, f is left-continuous but not right-continuous.

Functions: convex functions

MM 1027. **by Daniel B. Shapiro**

Let $f(k)$ be a real-valued function on the nonnegative integers. Suppose that $f(0) = 0$ and that $f(k)$ is a convex function. That is, for $k \geq 1$, $f(k)$ is less than the average of $f(k-1)$ and $f(k+1)$. For integers k, $1 \leq k \leq n$, define

$$F_n(k) = f(k)q + f(r), \text{ for } \mathrm{n = kq + r}, \ \mathrm{0 \leq r < k}.$$

Prove that, for fixed n, $F_n(k)$ is strictly increasing for $1 \leq k \leq n$.

Functions: differentiable functions

AMM 6027. **by Philip Hanser**

Let f be a continuous real function on $\mathbb{R}$, the reals. Must there exist a strictly increasing function $g: \mathbb{R} \to \mathbb{R}$ such that $g \circ f$ is everywhere differentiable?

CRUX 129. **by Léo Sauvé**

It has been known since Weierstrass that there exist functions continuous over the whole real axis but differentiable nowhere. Describe a function which is continuous over the whole real axis but differentiable only at

(a) $x = 0$;
(b) a finite number of points;
(c) a countable number of points.

CRUX 174. **by Leroy F. Meyers**

Describe a function which is defined on $\mathbb{R}$ and is continuous and differentiable at each point in a set E (specified below), but is discontinuous at each point not in E.

(a) $E = \{O\}$;
(b) E is a finite set;
(c) E is denumerable.

CRUX 374. **by Sidney Penner**

Prove or disprove the following.

THEOREM. Let the function $f: \mathbb{R} \to \mathbb{R}$ be such that $f''(x)$ exists, is continuous and is positive for every x in R. Let P_1 and P_2 be two distinct points on the graph of f. let L_1 be the line tangent to f at P_1 and define L_2 analogously. Let Q be the intersection of L_1 and L_2 and let S be the intersection of the graph of f with the vertical line through Q. Finally, let R_1 be the region bounded by segment P_1Q, segment SQ and P_1S, and define R_2 analogously. If, for each choice of P_1 and P_2, the areas of R_1 and R_2 are equal, then the graph of f is a parabola with vertical axis.

TYCMJ 52. **by Steven Kahn**

Let f be a function that is differentiable on $[0, 1]$ such that $f(0) = 0$, $f(1) = 1$, and $f(x) \in [0, 1]$ for each $x \in [0, 1]$. Prove that there exist $a, b \in [0, 1]$ such that $a \neq b$ and $f'(a) \cdot f'(b) = 1$.

CRUX 176. **by Hippolyte Charles**

Let $f: \mathbb{R} \to \mathbb{R}$ be an even differentiable function. Show that the derivative f' is not even, unless f is a constant function.

PUTNAM 1976/A.6.

Suppose $f(x)$ is a twice continuously differentiable real valued function defined for all real numbers x and satisfying $|f(x)| \leq 1$ for all x and

$$(f(0))^2 + (f'(0))^2 = 4.$$

Prove that there exists a real number x_0, such that

$$f(x_0) + f''(x_0) = 0.$$

SIAM 79-10. **by Y. P. Sabharwal and J. Kumar**

Determine the general form of a function $F(x)$ satisfying the following conditions for $x \geq 0$:

$$0 \leq F(x) \leq 1,$$

$$\frac{d}{dx} F(x) > 0,$$

$$\frac{d}{dx} \left\{ \frac{F(x)}{x} \right\} < 0.$$

SIAM 75-16. **by J. Walter**

Let G denote a continuously differentiable positive function defined in some interval $[t_0, \infty)$ and a, b, c, x, y, z, and w be real numbers such that $0 < a \leq b$, $t_0 \leq x \leq y \leq z \leq w$. Prove the existence of a continuous function $H(a, b, c)$ of three variables such that

$$\int_y^z \frac{dt}{G(t)} = a, \qquad \int_x^w \frac{dt}{G(t)} \leq b, \qquad |G'(t)| \leq c$$

for $t \in [t_0, \infty)$ imply that

$$\int_x^w G(t)\ dt \leq H(a, b, c) \int_y^z G(t)\ dt.$$

NAvW 474. **by J. van de Lune**

Let $f, g: [0, \infty) \to \mathbb{R}$ be two given functions satisfying the following conditions:

(1) $f(x) = g(x) = 1$ for $0 \leq x \leq 1$,

(2) f and g are continuous on $[0, \infty)$,

(3) f and g are differentiable on $(1, \infty)$ such that, for $x > 1$,

$$xf'(x) = -f(x - 1)$$

and

$$xg'(x) = g(x - 1).$$

Prove that

$$\int_0^x f(x - t)g(t)\, dt = x, \qquad x \geq 0.$$

CRUX 283. **by A. W. Goodman**

The function

$$y = -\frac{2x \ln x}{1 - x^2}$$

is increasing for $0 < x < 1$ and in fact y runs from 0 to 1 in this interval. Therefore an inverse function $x = g(y)$ exists. Can this inverse function be expressed in closed form and if so what is it? If it cannot be expressed in closed form, is there some nice series expression for $g(y)$?

FQ H-292. **by F. S. Cater and J. Daily**

Find all real numbers $r \in (0, 1)$ for which there exists a one-to-one function f_r mapping $(0, 1)$ onto $(0, 1)$ such that

(a) f_r and f_r^{-1} are infinitely many times differentiable on $(0, 1)$, and

(b) the sequence of functions f_r, $f_r \circ f_r$, $f_r \circ f_r \circ f_r$, $f_r \circ f_r \circ f_r \circ f_r$, … converges pointwise to r on $(0, 1)$.

MM 987. **by Sidney Penner**

Let f be differentiable with f' continuous on $[a, b]$. Show that if there is a number c in $(a, b]$ such that $f'(c) = 0$, then we can find a number ξ in (a, b) such that

$$f'(\xi) = \frac{f(\xi) - f(a)}{b - a}.$$

AMM E2572. **by C. D. H. Cooper**

Prove or give a counterexample: If $f: \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and f' is differentiable at some point a, then f' is continuous in some neighborhood of a.

AMM E2663. **by Marius Solomon**

Let $f: (0, \infty) \to \mathbb{R}$ be differentiable, and assume that $f(x) + f'(x) \to 0$, when $x \to \infty$. Show that $f(x) \to 0$ as $x \to \infty$.

Functions: digits

AMM E2738. **by Michael W. Ecker**

Let σ be a permutation of the digits $0, 1, \ldots, 9$. Let

$$f: [0, 1] \to [0, 1]$$

be the "extension" of σ, i.e., $f(x)$ is obtained from x by applying σ to each digit in the decimal expansion of x. (For uniqueness of decimal expansions, we do not allow expansions with all but finitely many digits equal to 9.)

(a) Find the points where f is continuous (or differentiable).

(b) Show that f is Riemann integrable and compute the integral.

CMB P269. **by J. Borwein**

Let $q = (q_1, q_2, \ldots)$ be a sequence of positive real numbers with $\sum q_n = 1$, and let $T_q\colon (0,1] \to (0,1]$ be given by $T_q(a) = \sum q_n a_n$, where $0 \cdot a_1 a_2 a_3 \cdots$ is the nonterminating binary expansion of a. Are there q and r such that $T_q^{-1}\{r\}$ is a set of positive Lebesgue measure?

Functions: entire functions

AMM 6117. **by M. J. Pelling**

A well-known theorem asserts that given entire functions $f(z)$, $g(z)$ with no common zero, then there exist entire functions $a(z)$, $b(z)$ such that $af + bg = 1$ identically.

(a) Show that it is always possible to choose $a(z)$ to be zero-free.

(b) Is it always possible to choose both $a(z)$ and $b(z)$ to be zero-free?

AMM 6118. **by M. J. Pelling**

(a) Show there is no nonconstant solution to

$$e^{f(z)} + e^{g(z)} = 1$$

in entire functions $f(z)$ and $g(z)$.

(b) Is there a nonconstant solution to

$$e^{f(z)} + e^{g(z)} + e^{h(z)} = 1$$

in entire functions f, g, and h?

AMM 6279. **by Lee A. Rubel**

Let $f(z)$ be an entire function such that the maximum modulus over every closed line segment L is achieved at one of the endpoints a and b of L; that is,

$$\max\{|f(z)| : z \in L\} = \max\{|f(a)|, |f(b)|\}.$$

Prove that $f(z)$ has either the form $A(z-B)^n$ or the form $A \exp Bz$, where A and B are constants and n is a nonnegative integer.

NAvW 464. **by J. van de Lune**

Prove that the function f, defined for the real variable s $(s > 1)$ by

$$f(s) = (s-1)\sum_{n=2}^{\infty} \frac{1}{n(\log n)^s},$$

can be extended in the complex plane to an entire function.

NAvW 520. **by J. van de Lune**

Prove that all the zeros of the entire function

$$1 + 2^s + 3^s + 4^s$$

are simple.

Functions: exponentials

SPECT 9.1.

Prove that the exponential function e^x cannot be expressed in the form $f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials in x with real coefficients.

Functions: infinite series

NAvW 454. **by J. van de Lune**

Prove that the function $\lambda\colon \mathbb{R} \to \mathbb{C}$ defined by

$$\lambda(t) = \sum_{n=1}^{\infty} n^{-1}(n+1)^{-1-it}, \qquad t \in \mathbb{R}$$

has no real zeros.

Functions: iterated functions

MM 993. **by F. David Hammer**

Let g be a continuous function from $[0,1]$ to $[0,1]$ with $g(0) = 0$. If for each x in $[0,1]$ there is a positive integer $n(x)$ such that $g^{n(x)} = x$ (the $n(x)$-th iterate of g), then show that $g(x) = x$ for all x in $[0,1]$.

Functions: linear independence

AMM 6253. **by Maurice Machover**

If $0 \le \theta_1 < \theta_2 < \theta_3 < \cdots < \theta_n < 2\pi$, are the functions

$$\exp\left[i\cos(\theta - \theta_j)\right], \qquad j = 1, 2, \ldots, n$$

linearly independent over the complex numbers?

Functions: monotone functions

SIAM 77-5.* **by M. L. Glasser**

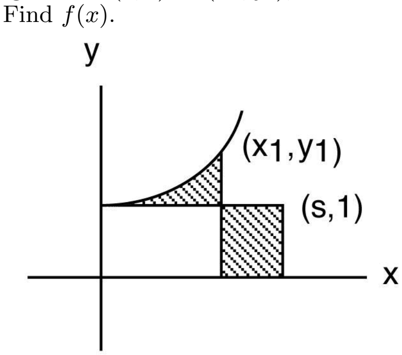
Let

$$S(r) = \sum_{k=1}^{\infty} (-1)^{k+1} \sinh y \operatorname{csch} ky \left(y = \cosh^{-1} r\right).$$

Numerical evidence suggests that $S(r)$ increases steadily between the values $S(1) = \log 2$ and $S(\infty) = 1$. Prove whether or not this is the case.

MATYC 134. **by William Stretton**

Let $y = f(x)$ be the increasing function shown, s be the arc length from $(0,1)$ to (x_1, y_1), and the shaded areas be equal. Find $f(x)$.



NAvW 434. **by J. van de Lune**

For $s > 0$, let

$$\lambda(s) = \frac{1}{\Gamma(s+1)} \int_0^s e^{-x} x^s \, dx.$$

For $n \in \mathbb{N}$, let

$$e(n) = e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Prove that $\lambda(s)$ is increasing, $e(n)$ is decreasing, and that they have the same limit.

NAvW 465. **by J. van de Lune**

Prove that the function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$f(s) = \left(\frac{s^s}{\Gamma(s+1)} \right)^{1/s}$$

is monotone increasing.

AMM S15. **by Joel L. Brenner**

Let $f(t) > 0$ for $a \le t < b$ and u be a fixed real number. Show that the functional

$$\left[\frac{\int f^{s+u}}{\int f^s} \right]^{1/u}$$

increases with s.

MATYC 97. **by Benjamin G. Klein**

Let k be a positive constant and let

$$f(x) = \frac{-\log(1-kx)}{\log(1+x)} .$$

Show that $f(x)$ is an increasing function of x for x in $(0, 1/k)$.

Functions: nearest integer function

MATYC 80. **by Gino Fala**

Define a function $f\colon \mathbb{R} \to [0, 1/2]$ verbally as follows:

For every real number x, the image of x under f is the distance from x to the nearest integer.

Find a single formula $y = f(x)$ for this function.

Functions: periodic functions

NAvW 409. **by O. P. Lossers**

Let a and b be distinct real numbers. Suppose that f is a continuous function of the real variable x, such that

$$f(x) = o\left(x^2\right), \qquad (x \to \pm\infty),$$

and

$$f(x+a) + f(x+b) = \frac{1}{2} f(2x) \qquad \text{for all } x \in \mathbb{R}.$$

Show that f must be a periodic function.

SSM 3667. **by Steven R. Conrad**

Prove that the function $\cos\sqrt{x}$ is not periodic.

CANADA 1975/7.

A function $f(x)$ is periodic if there is a positive number p such that $f(x+p) = f(x)$ for all x. Is the function $\sin(x^2)$ periodic? Prove your assertion.

SSM 3709. **by John Carpenter**

Find a nonlinear function f such that $\cos f(x)$ is periodic.

Functions: polynomials

AMM 6208. **by Gary Gundersen**

Let $p(z)$ and $q(z)$ be two polynomials with

$$\deg(q) \ge \deg(p),$$

and suppose there is a discrete real sequence $\left\{x_j\right\}_{j=1}^{\infty}$ with cluster points at $\pm\infty$. Prove that if $q(z) \in \mathbb{R}$ whenever

$$p(z) \in \left\{x_j\right\}_{j=1}^{\infty},$$

then

$$q(z) = \sum_{i=0}^{n} c_i \left(p(z)\right)^i ,$$

where $c_i \in \mathbb{R}$ $(0 \le i \le n)$.

Can the condition $\deg(p) \le \deg(q)$ be dropped?

AMM E2796. **by P. Henrici**

Prove that the polynomial p with degree less than or equal to n that agrees with a given function $f(x)$ at the Chebyshev points $x_k = \cos\phi_k$, where

$$\phi_k = \frac{(2k+1)\pi}{(2n+2)}, \qquad k = 0, 1, \ldots, n,$$

is, for x not in $\{x_0, \ldots, x_n\}$, given by $p(x) = N(x)/D(x)$ with

$$N(x) = \sum_{k=0}^{n} \frac{(-1)^k f(x_k) \sin\phi_k}{x - x_k} ,$$

$$D(x) = \sum_{k=0}^{n} \frac{(-1)^k \sin\phi_k}{x - x_k} .$$

Functions: real-valued functions

AMM E2610. **by Hugh L. Montgomery**

Let f be a real-valued function defined on the unit square $[0,1] \times [0,1]$. Suppose that $f(x,y)$ is continuous in x for each fixed y and continuous in y for each fixed x. Show that if $f^{-1}(0)$ is dense in the unit square then $f = 0$.

AMM 6132. **by Mihai Eşanu**

Find all the functions $f\colon \mathbb{R} \to \mathbb{R}$ with the Darboux property such that for some $n \ge 1$, $f^n(x) = -x$ for all x.

PUTNAM 1977/A.6.

Let $f(x,y)$ be a continuous function on the square

$$S = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}.$$

For each point (a,b) in the interior of S, let $S_{(a,b)}$ be the largest square that is contained in S, is centered at (a,b), and has sides parallel to those of S. If

$$\int\!\!\int f(x,y)\,dx\,dy = 0$$

when taken over each square $S_{(a,b)}$, must $f(x,y)$ be identically zero on S?

AMM 6273. **by K. L. Chung**

Let f be a real-valued function defined on $(-\infty, +\infty)$ and continuous from the right everywhere. Suppose also that the following is true:

$$\lim_{n\to\infty}\left[\max_{-\infty<k<\infty}\left|f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)\right|\right]=0$$

where n and k are integers, $n \geq 1$. Is f continuous in $(-\infty, +\infty)$?

AMM 6184. **by Ole Jørsboe**

Let $(\phi_n)_{n=1}^{\infty}$ be an orthonormal system of real-valued piecewise continuous functions on the interval $[0,1]$ with the property that if f is a real-valued piecewise continuous function on $[0,1]$ fulfilling

$$(f,\phi_n)=\int_0^1 f(x)\phi_n(x)\,dx=0$$

for all $n\in\mathbb{N}$, then f is 0 at all points of continuity.

Does this imply that (ϕ_n) spans the space of all real-valued piecewise continuous functions on $[0,1]$, i.e., can every piecewise continuous function f be written in the form

$$f=\lim_{N\to\infty}\sum_{n=1}^{N}a_n\phi_n?$$

AMM 6054. **by Lung Ock Chung**

Let

$$\phi\colon\{0,1,\ldots,N-1\}\to\{0,1,\ldots,N-1\}$$

be a permutation for $N\geq 2$. Then ϕ induces a function

$$\phi^*\colon(0,1)\to(0,1)$$

from the open unit interval to itself as

$$\phi^*\left(\sum_{i=1}^{\infty}\frac{m_i}{N^i}\right)=\sum\frac{\phi(m_i)}{N^i}\ ,$$

where $m_i\in\{0,1,\ldots,N-1\}$, $m_i\not\to 0$, and

$$N^i=N\cdot N\cdots N\quad(i\text{ times}).$$

Find the subgroup H of the permutation group such that ϕ^* is continuous if $\phi\in H$. Further, show that ϕ^* is differentiable for such ϕ.

Functions: transcendental functions

CRUX 300. **by Léo Sauvé**

Does there exist a dense subset E of the reals such that $\sin x$ and $\cos x$ are both algebraic for every $x\in E$?

Gamma function

AMM 6186.* **by Ronald Evans**

Let $r,k\in\mathbb{N}$, where r is fixed. Fix $\beta>1$. Let

$$F_r(k)=\sum(j_1j_2\cdots j_r)^{\beta-1}\ ,$$

where the sum is over all vectors $(j_1,j_2,\ldots,j_r)\in\mathbb{N}^r$ for which $j_1+j_2+\cdots+j_r=k$. Prove that

$$F_r(k)\sim\frac{\Gamma^r(\beta)}{\Gamma(r\beta)}k^{\beta r-1}\qquad\text{as}\qquad k\to\infty.$$

SIAM 75-11. **by D. K. Ross**

Prove that

$$\begin{vmatrix}\Gamma(\alpha)&\Gamma(\alpha+\beta)&\Gamma(\alpha+2\beta)\\\Gamma(\alpha+\beta)&\Gamma(\alpha+2\beta)&\Gamma(\alpha+3\beta)\\\Gamma(\alpha+2\beta)&\Gamma(\alpha+3\beta)&\Gamma(\alpha+4\beta)\end{vmatrix}>0,$$

provided that $\alpha,\beta>0$, and generalize the result to include higher order determinants and other classes of special functions.

SIAM 76-21.* **by P. Barrucand**

Define the polynomials $\{p_n(x,m,\gamma)\}$ by the generating function

$$\sum p_n(x,m,\gamma)t^n=\frac{\exp(xt)}{[\Gamma(1+\gamma+t)]^m}\ ,$$

m positive integer, $\gamma>-1$.

Prove that for every n, all the zeros of $p_n(x)$ are real and give an asymptotic formula for the lesser-in-modulus (i.e., the greater) negative zeros.

AMM 6067. **by Ron Evans**

Prove that for each real σ, there exist infinitely many $t>0$ for which $\Gamma(\sigma+it)<0$.

AMM 6269. **by Robert E. Shafer**

Let $F(u)=u^{-u}\Gamma\left(u+\frac{1}{2}\right)$ and

$$G(x,s,t)=\frac{1}{\left(x-s+\frac{1}{2}\right)\left(x-t+\frac{1}{2}\right)}-\frac{1}{\left(x+s+\frac{1}{2}\right)\left(x+t+\frac{1}{2}\right)}\ .$$

Prove that for $0\leq s<t\leq x$,

$$e^{(s-t)G(x,s,t)/24}<\frac{F\left(x-t+\frac{1}{2}\right)F\left(x+t+\frac{1}{2}\right)}{F\left(x-s+\frac{1}{2}\right)F\left(x+s+\frac{1}{2}\right)}<1.$$

NAvW 462. **by P. J. de Doelder**

Show that

$$\sum_{n=-\infty}^{\infty}\frac{1}{\Gamma(a+n)\Gamma(a-n)}=\frac{2^{2a-2}}{\Gamma(2a-1)}\ ,\qquad\operatorname{Re}a>\frac{1}{2}.$$

NAvW 518. **by L. Kuipers**

Let

$$\Phi(T)=\int_0^T\log\frac{\Gamma(pT+1)}{\Gamma(pt+1)\Gamma(pT-pt+1)}\,dt,\qquad(p>0,\quad T>0).$$

(a) Prove that $T^{-2}\Phi(T)\to\frac{1}{2}p$ as $T\to\infty$.

(b) If p is an integer, then prove that

$$\lim_{n\to\infty}n^{-2}\sum_{k=1}^{n}\log\binom{pn}{pk}=\frac{1}{2}p.$$

Haar functions

AMM 6013. **by J. R. Higgins**

Let $\{h_r(t)\}_{r=1}^{\infty}$ be the orthonormal Haar functions, defined by

$$h_1(t) = 1, \qquad t \in [0,1]$$
$$h_r(t) = \begin{cases} 2^{m/2}\operatorname{sgn}\sin(2^{m+1}\pi t), & t \in \left(\frac{k-1}{2^m}, \frac{k}{2^m}\right) \\ 0, & \text{elsewhere in } [0,1], \end{cases}$$

where $r = 2^m + k$, $m = 0, 1, \ldots$, and $k = 1, \ldots, 2^m$. Let p be any odd positive integer and q any positive integer such that $p < 2^q$. Set

$$I(p,q,m,k) = \int_0^{p/2^q} h_r(t)\,dt.$$

Show that

$$\sum_{m=0}^{q-1} \{I(p,q,m,k)\}^2 = \frac{p}{2^q}\left(1 - \frac{p}{2^q}\right).$$

Hankel function

NAvW 470. **by P. J. de Doelder**

Prove that

$$\int_0^\infty \frac{H_0^{(1)}\left(k\sqrt{y^2+\alpha^2}\right)}{\sqrt{y^2+1}}\,dy = \frac{\pi i}{4} H_0^{(1)}\left(\frac{1}{2}k\alpha_1\right) H_0^{(1)}\left(\frac{1}{2}k\alpha_2\right),$$

where $H_0^{(1)}$ is a Hankel function, and α_1 and α_2 are the roots of $x^2 - 2\alpha x + 1 = 0$.

Harmonic functions

AMM 6280. **by David Siegel**

Let u be a harmonic function in a regular n-gon with sides $s_1, \ldots, s_n$ and radii $r_1, \ldots, r_n$ joining the center to the vertices. Show that

$$\sum_{i=1}^n \int_{s_i} u\ ds = 2\sin\frac{\pi}{n} \sum_{i=1}^n \int_{r_i} u\ ds,$$

where the integrals are taken with respect to arc length.

Hypergeometric functions

NAvW 550. **by P. J. de Doelder**

Show that

$${}_2F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{3}{4}; \frac{3}{4}\right) = 2^{3/2} \cdot 3^{-3/4}.$$

SIAM 77-2. **by P. W. Karlsson**

Establish the identity

$$\frac{1-x}{a+c}{}_2F_1(a, 1-b; a+c+1; 1-x)\,{}_2F_1(b, 1-a; b+c; x)$$
$$+\frac{x}{b+c}{}_2F_1(a, 1-b; a+c; 1-x)\,{}_2F_1(b, 1-a; b+c+1; x)$$
$$= \frac{\Gamma(a+c)\Gamma(b+c)}{\Gamma(c+1)\Gamma(a+b+c)}.$$

SIAM 75-17. **by H. M. Srivastava**

Let

$$F\left[\begin{matrix} a : b, b', \ldots; c, c', \ldots; \\ \\ d, d', \ldots; e, e', \ldots; \end{matrix} \quad x, y, z\right]$$
$$= \sum_{l,m,n=0}^{\infty} \frac{(a)_{l+m+n}(b)_{l+m}(b')_{l+m}\cdots(c)_{l+n}(c')_{l+n}\cdots}{(d)_{l+m}(d')_{l+m}\cdots(e)_{l+n}(e')_{l+n}\cdots} \cdot \frac{x^l}{l!}\frac{y^m}{m!}\frac{z^n}{n!}$$

and

$$\phi(x,y,z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n \prod_{j=1}^p (a_j)_n \prod_{j=1}^r (\alpha_j)_n}{n! \prod_{j=1}^q (b_j)_n \prod_{j=1}^s (\beta_j)_n} \left[\frac{xyz}{(1-z)^2}\right]^n \cdot G,$$

where

$$G = {}_{p+1}F_q\left[\begin{matrix} \lambda+n, a_1+n, \ldots, a_p+n; \\ \\ b_1+n, \ldots, b_q+n; \end{matrix} \quad \frac{xz}{z-1}\right]$$
$$\times\ {}_{r+1}F_s\left[\begin{matrix} \lambda+n, \alpha_1+n, \ldots, \alpha_r+n; \\ \\ \beta_1+n, \ldots, \beta_s+n; \end{matrix} \quad \frac{yz}{z-1}\right].$$

Prove or disprove that

$$\phi(x,y,z) = F\left[\begin{matrix} \lambda : a_1, \ldots, a_p; \alpha_1, \ldots, \alpha_r; \\ \\ b_1, \ldots, b_q; \beta_1, \ldots, \beta_s; \end{matrix} \quad \frac{xyz}{1-z}, \frac{xz}{z-1}, \frac{yz}{z-1}\right].$$

SIAM 76-19. **by R. I. Joseph**

Evaluate the double integral

$$\alpha = \int_0^1 u\,du \int_0^\infty dv\ \frac{{}_2F_1\left(\frac{3}{4}, \frac{5}{4}; 2; 4u/(u+v+1)^2\right)}{(u+v+1)^{3/2}(u+v)^{5/2}},$$

where ${}_2F_1$ is Gauss' hypergeometric function.

Identities

NAvW 433. **by J. van de Lune**

Prove that

$$\left(\frac{e}{s}\right)^s \int_0^s e^{-x} x^s\,dx = s \sum_{k=1}^{\infty} \frac{k^k e^{-k}}{k!(s+k)}, \qquad s > 0.$$

Inequalities

AMM 6084. **by Theodore J. Rivlin**

Let

$$T_n(x) = t_0 + t_1 x + \cdots + t_n x^n$$

denote the Chebyshev polynomial of degree n. Suppose that

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

is real-valued. Show that if $|p(\cos(j\pi/n))| \le 1$ for $j = 0, 1, \ldots, n$, then

$$|a_{n-2m}| + |a_{n-2m-1}| \le |t_{n-2m}|,$$

$m = 0, 1, \ldots, (n-1)/2$.

AMM E2670. **by Shyam Johari and Stanley L. Sclove**

Let

$$f(x,y)=\frac{xe^{-x}-ye^{-y}}{e^{-x}-e^{-y}}\ .$$

If $0<a<b<c<\infty$ and $0<x<y<z<\infty$, prove or disprove that

$$|f(a,b)+f(b,c)-f(x,y)-f(y,z)| \le 2\max\left(|x-a|,|y-b|,|z-c|\right).$$

AMM E2782. **by Robert E. Shafer**

Prove that

$$2\arctan\frac{1}{2x-1}<\sum_{n=0}^{\infty}\frac{1}{(n+x)^2}<\frac{1}{x-\frac{1}{2}}$$

for $x>\frac{1}{2}$.

NAvW 458. **by P. J. van Albada**

Wallis' inequality

$$\frac{2}{2n+1}\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2<\pi<\frac{1}{n}\left\{\frac{(2n)!!}{(2n-1)!!}\right\}$$

restricts π to an interval of length $O\left(n^{-1}\right)$, $(n\to\infty)$. Gurland's amelioration restricts π to an interval of length $O\left(n^{-2}\right)$, $(n\to\infty)$. Find simple functions f and g such that

$$f(n)\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2<\pi<g(n)\left\{\frac{(2n)!!}{(2n-1)!!}\right\}^2$$

is sharper than the inequalities quoted above.

Infinite products

PUTNAM 1977/B.1.
OSSMB G76.3-6.

Evaluate the infinite product

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}.$$

AMM 6233. **by James Lynch and Jan Mycielski**

Prove that $\prod_{n=1}^{\infty}\left(1-a^{-n}\right)$ is irrational for every integer a with $|a|>1$.

Integral equations

SIAM 79-1.* **by I. Lux**

Let V be an arbitrary three-dimensional spatial region. Let $P=(\mathrm{r},\omega)$, a six-dimensional phase space point, where $\mathrm{r}\in V$ and ω is a directional unit vector. Define a function $M_\lambda(P)$ through the following integral equation

$$M_\lambda(P)=1-e^{-D}+\frac{\lambda}{4\pi}\int_0^D e^{-\lambda x}\,dx\int M_\lambda\left(P'\right)\,d\omega'$$

where $P'=\left(\mathrm{r}+x\omega,\omega'\right)$, λ is an arbitrary but positive parameter, D is the distance between the point r and the boundary of V along the direction ω and the integral over $d\omega'$ is a double integral over the surface of a unit sphere. Prove or disprove that

$$\frac{d}{d\lambda}M_\lambda(P)\Big]_{\lambda=1}\ge 0.$$

TYCMJ 151. **by Peter A. Lindstrom**

The function f, defined by $f(t)=1/t$, $t\in(0,\infty)$, is decreasing, has derivatives of all orders, and satisfies the equality

$$\int_1^x f(t)\,dt=\int_y^{xy} f(t)\,dt$$

for $x,y>0$. Does there exist a function defined on $(0,\infty)$ that has these three properties but which, unlike f, has a graph that is concave downward?

SIAM 75-9. **by M. L. Glasser**

Suppose

$$Z(s)=e^{-q}\int_0^1 dt\ \frac{Z(t)}{(1-st)^p}\ .$$

(a) Show that in the case $p=1$, $q=\ln\pi$, the exact solution of the above equation is

$$Z(s)=A(1-s)^{-1/2}\mathrm{K}\left(s^{1/2}\right),$$

where $\mathrm{K}(k)$ denotes the complete elliptic integral of the first kind with modulus k and A is an arbitrary constant.

(b) Are there any other exactly solvable cases?

Integral inequalities

MENEMUI 1.3.3.* **by S. L. Lee**

If f is continuously differentiable up to derivatives of 4th order and $f(-1)=f(1)=0$, find the least constant A such that

$$\left|\int_{-\sqrt{3}}^{\sqrt{3}} f(x)\,dx\right|\le A.$$

CMB P278. **by E. J. Barbeau**

Let $f(x)$ be a strictly increasing continuous function on the closed interval $[0,1]$ for which $f(0)=1$ and $f(1)=1$. Suppose that $g(x)$ is the composition inverse of $f(x)$, so that $f(g(x))=g(f(x))=x$ for $0\le x\le 1$. Is it necessarily true that

$$\int_0^1 f(x)^k g(x)^k\,dx\le\frac{1}{2k+1}$$

for each nonnegative real number k?

AMM E2622. **by S. Zaidman**

Let $f\colon[a,b]\to\mathbb{R}$ be a continuous function that is twice differentiable in (a,b) and satisfies $f(a)=f(b)=0$. Prove that

$$\int_a^b |f(x)|\ dx\le\frac{1}{12}M(b-a)^3,$$

where $M=\sup\left|f''(x)\right|$ for $x\in(a,b)$.

CRUX 79. **by John Thomas**

Show that for $x>0$,

$$\left|\int_x^{x+1}\sin(t^2)\,dt\right|<\frac{2}{x}.$$

AMM 6185. **by John Milcetich**

Let
$$f(z,\theta) = (1+e^{i\theta}z)^{\beta}(1-z)^{-\alpha},$$
where $|z|<1$, $\theta \in R$, and $\alpha \ge \beta \ge 1$. Show that for $p>0$ and $0<r<1$,
$$\int_{-\pi}^{\pi} \left| f'(re^{i\phi},\theta) \right|^p \, d\phi \le \int_{-\pi}^{\pi} \left| f'(re^{i\phi},0) \right|^p \, d\phi.$$

AMM 6075. **by H. L. Montgomery**

Prove that if $f(x) \in L^1(-\infty,\infty)$, $f(x) \ge 0$, $\hat{f}(t) \ge 0$ for all x, t where $\hat{f}$ is the Fourier transform, then for any integer $k \ge 1$,
$$\int_{-k}^{k} f(x)\, dx \le (2k+1) \int_{-1}^{1} f(x)\, dx.$$

MM Q622. **by M. S. Klamkin**

If G and F are integrable, $a>0$, $G(x) \ge F(x) \ge 0$, and
$$\int_0^1 xF(x)\, dx = \int_0^a xG(x)\, dx,$$
show that
$$\int_0^1 F(x)\, dx \le \int_0^a G(x)\, dx.$$

SIAM 78-18. **by A. Meir**

Let $F(x)$ be nonnegative and integrable on $[0,a]$ and such that
$$\left\{ \int_0^t F(x)\, dx \right\}^2 \ge \int_0^t F(x)^3\, dx$$
for every t in $[0,a]$. Prove or disprove the conjecture:
$$\frac{a^3}{3} \ge \int_0^a \{F(x)-x\}^2 \, dx.$$

Integrals: area

CRUX 380. **by G. P. Henderson**

Let P be a point on the graph of $y=f(x)$, where f is a third-degree polynomial, let the tangent at P intersect the curve again at Q, and let A be the area of the region bounded by the curve and the segment PQ. Let B be the area of the region defined in the same way by starting with Q instead of P. What is the relationship between A and B?

Integrals: asymptotic expansions

NAvW 456. **by S. L. Paveri-Fontana and D. Katz**

Find the first two terms of the asymptotic expansion of the integral
$$\omega(\lambda) = \frac{2}{\pi\lambda} \int_0^1 d\mu \cdot \mu^{-2} \left(1-\mu^2\right)^{-\frac{1}{2}} \int_0^\infty dx \sin^2(x\lambda\mu) x^{-2} f(x)$$
for real $\lambda \to +\infty$, under the assumptions:

(1) f is a real-valued continuous function on $[0,\infty)$;
(2) $f(x) = 1 + O(x)$ for $x \to 0$;
(3) $\int_0^\infty |f(x)|\, dx < +\infty$.

Integrals: evaluations

NAvW 522. **by N. Ortner**

Prove that
$$\int_0^\infty \frac{\sinh t - t}{t^3 \left(\cosh \frac{t}{2}\right)^2}\, dt = \frac{7}{2\pi^2}\zeta(3).$$

NAvW 408. **by H. K. Kuiken**

Prove that
$$\int_{-\infty}^{\infty} \frac{dx}{\pi^2 + (ye^x + y - x)^2} = \frac{1}{y+1}, \qquad y \ge 0.$$

SIAM 77-3. **by P. J. Schweitzer**

Evaluate
$$\int_0^\infty F(x) \left(F'(x) - \ln x \right) dx,$$
where
$$F(x) = \int_{-\infty}^{\infty} \frac{\cos xy \, dy}{(1+y^2)^{3/2}}.$$

AMM E2523. **by K. P. Kerney**

Evaluate
$$\int_0^1 \log(1+x) \log(1-x)\, dx.$$

TYCMJ 83. **by Joe Allison**

Evaluate
$$\int_0^1 \frac{\log(1-x)}{1+x}\, dx.$$

MATYC 125. **by Louise Grinstein**

Evaluate
$$\int \sqrt{1 + \frac{\ln x}{x}}\; dx\ .$$

SIAM 75-12. **by H. J. Oser**

Evaluate the 4-fold integral
$$F = \int_0^1 \int_{-1}^0 \int_{-1/2}^{1/2} \cdot \int_{-1/2}^{1/2} \left\{ (x_1-x_2)^2 + (y_1-y_2)^2 \right\}^{1/2} dx_1 dx_2 dy_1 dy_2$$
which gives the average distance between points in two adjacent unit squares.

CRUX 88. **by F. G. B. Maskell**

Evaluate the indefinite integral
$$I = \int \frac{dx}{\sqrt[3]{1+x^3}}.$$

NAvW 449. **by J. Boersma**

Let
$$F(t) = \int_{-\infty}^{\infty} e^{ixt} \left(1+x^6\right)^{-\frac{1}{2}} dx, \qquad t \ge 0.$$
Determine an expansion of $F(t)$ of the form
$$F(t) = A(t) + B(t) \log t$$
where $A(t)$ and $B(t)$ are power series.

CRUX 455. **by Hippolyte Charles**

Evaluate

$$I=\int_0^{\frac{\pi}{2}} \frac{x\cos x\sin x}{\cos^4 x+\sin^4 x}\,dx.$$

CRUX 477. **by Hippolyte Charles**

For $n=0,1,2,\ldots$, evaluate the integral

$$\tau_n=\int_0^{\pi} \frac{\cos nx}{5-4\cos x}\,dx.$$

SSM 3775. **by Fred A. Miller**

Evaluate the following integral:

$$\int_0^{\pi} \frac{x\,dx}{1+\cos^2 x}\ .$$

AMM E2803. **by L. R. Shenton, Frank Bowman, and H. K. Lam**

Prove that

(a) $\int_0^{\pi/4} g(\theta)d\theta=\pi^2/24$,

(b) $\int_0^{\pi/6} g(\theta)d\theta=\pi^2/32$,

where $g(\theta)=\arctan\left[(\cos 2\theta)/\left(2\cos^2\theta\right)\right]^{1/2}$.

MM 1033. **by H. Kestelman**

For given positive integers $n_1, n_2, \ldots, n_k$, when is

$$\int_0^{2\pi} \cos n_1\theta\cos n_2\theta\cdots\cos n_k\theta\,d\theta$$

different from zero and what is its value?

SIAM 78-19. **by M. L. Glasser**

Show that

$$\int_0^{\infty}\frac{\cos\left(x^2/\pi\right)\,dx}{\cosh x\cosh(x+a)\cosh(x-a)} =\frac{\pi}{2\sqrt{2}}\operatorname{sech}^2 a\operatorname{csch}^2 a\left(\cosh^2 a-\cos\frac{a^2}{\pi}-\sin\frac{a^2}{\pi}\right).$$

NAvW 537. **by J. A. van Casteren**

Prove that

$$\int_0^{\pi/2}\theta\cot\theta\log\cot\theta\ d\theta=\frac{\pi^3}{48}\ .$$

CRUX 161. **by Viktors Linis**

Evaluate

$$\int_0^{\pi/2}\frac{\sin^{25} t}{\cos^{25} t+\sin^{25} t}\,dt.$$

CRUX 432. **by Basil C. Rennie**

Evaluate

$$\int_{-\infty}^{\infty}\frac{\cos x+x\sin x}{x^2+\cos^2 x}\,dx.$$

MM 1064. **by Edward T. H. Wang**

For each positive integer n, define

$$L(n)=\int_0^{\infty}\left(\frac{\sin x}{x}\right)^n\,dx.$$

It is well known that $L(1)=L(2)=\pi/2$.

(a) Find $L(3)$, $L(4)$, and $L(5)$.

(b) Is there a formula for $L(n)$ for general n?

AMM 6206. **by Gérard Letac**

Prove that if n is a nonzero integer, then

$$\int_{-\pi/2}^{+\pi/2}\exp\left[2in(x+\tan x)\right]\ dx=0.$$

Integrals: functions

AMM E2765. **by Naoki Kimura**

Establish the two following equations:

$$\int_{-1/2}^{3/2} f\left(3x^2-2x^3\right)\ dx=2\int_0^1 f\left(3x^2-2x^3\right)\,dx,$$

$$\int_{-1/2}^{3/2} xf\left(3x^2-2x^3\right)\ dx=2\int_0^1 xf\left(3x^2-2x^3\right)\,dx,$$

for all functions f continuous on $-1/2\le x\le 3/2$.

Is there a quadratic polynomial $g(x)$ such that

$$\int_{-1/2}^{3/2} f\left(3x^2-2x^3\right)\,dx=\int_0^1 g(x)f\left(3x^2-2x^3\right)\,dx$$

for every continuous function f?

SIAM 77-7. **by L. A. Shepp**

Define

$$A(h)=A(h;T)=\int_0^T h(t)\left\{\frac{d}{dt}\frac{1}{\sqrt{h'(t)}}\right\}^2\,dt$$

where $0<T\le\infty$ and h is twice differentiable, strictly increasing on $[0,T]$ with $h(0)=0$. Show that $A(h;T)<\infty$ if and only if $A\left(h^{-1},h(T)\right)<\infty$ where h^{-1} is the inverse function of h on $[0,h(T)]$.

AMM E2658. **by W. Weston Meyer**

(a) For $0<\alpha<\pi/2$ and integral $n\ge 0$, show that

$$\int_0^{\alpha}\left(\frac{\sin\theta}{\sin\alpha}\right)^{2n}\,d\theta=\sum_{k=0}^{n} c_{nk}\int_0^{\alpha}\left(\frac{\tan\theta}{\tan\alpha}\right)^{2k}\,d\theta,$$

where the constants c_{nk} are independent of α.

(b) Find all polynomials P such that the ratio

$$\int_0^{\alpha}P\left(\frac{\sin\theta}{\sin\alpha}\right)\,d\theta\Big/\int_0^{\alpha}P\left(\frac{\tan\theta}{\tan\alpha}\right)\,d\theta$$

is independent of $\alpha\in(0,\pi/2)$.

Integrals: gamma function

AMM 6245. **by C. L. Mallows**

For $0 < a < 1$, $t \geq 0$, $b = 1 - a$, prove that

$$\frac{1}{\pi}\int_0^{\pi} \frac{(\sin u)^t}{(\sin au)^{at}(\sin bu)^{bt}}\, du = \frac{\Gamma(t+1)}{\Gamma(at+1)\Gamma(bt+1)}.$$

SIAM 77-1. **by R. A. Waller and M. S. Waterman**

If $0 < \xi_1 < \xi_2 < 1$ and $1 < b$ are fixed, consider solutions (λ, ϕ) of the system

$$f(\lambda,\phi) \equiv \int_0^{\lambda} \frac{e^{-y}y^{\phi-1}}{\Gamma(\phi)}\, dy = \xi_1,$$

$$g(\lambda,\phi) \equiv \int_0^{b\lambda} \frac{e^{-y}y^{\phi-1}}{\Gamma(\phi)}\, dy = \xi_2,$$

where $0 < \lambda$ and $0 < \phi$. Does this system always have a solution? If a solution exists, is it unique?

Integrals: improper double integrals

PUTNAM 1976/A.5.

In the (x, y)-plane, if R is the set of points inside and on a convex polygon, let $D(x, y)$ be the distance from (x, y) to the nearest point of R.

(a) Show that there exist constants a, b, and c, independent of R, such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-D(x,y)}\, dx\, dy = a + bL + cA,$$

where L is the perimeter of R and A is the area of R.

(b) Find the values of a, b, and c.

Integrals: improper integrals

NAvW 532. **by M. J. Ritter**

For $(x, y, z) \in \mathbb{R}^3$, $0 < y$, $1 < z$, we define

$$f(x,y,z) = \frac{y^x \sin x}{z^{y^x} + 1}\ .$$

For $y \neq 1$ and $z > 1$, the function F_y is defined by

$$F_y(z) = \int_{-\infty}^{\infty} f(x,y,z)\ dx.$$

Determine the values of y for which F_y is identically 0.

CRUX 58. **by Jacques Marion**

Let $f\colon \{z \mid \operatorname{Re} z = 0\} \to \mathbb{R}$ be continuous and bounded. If $\mu\colon (z \mid \operatorname{Re} z > 0\} \to \mathbb{R}$ is defined by

$$\mu(z) = \mu(x+iy) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{xf(it)}{x^2 + (y-t)^2}\, dt,$$

show that $f(ic) = \lim_{z\to ic} \mu(z)$.

PUTNAM 1978/A.3.

Let

$$p(x) = 2 + 4x + 3x^2 + 5x^3 + 3x^4 + 4x^5 + 2x^6.$$

For k with $0 < k < 5$, define

$$I_k = \int_0^{\infty} \frac{x^k}{p(x)}\, dx.$$

For which k is I_k smallest?

CRUX 273. **by M. S. Klamkin**

Prove that

$$\lim_{n\to\infty}\int_c^{\infty} \frac{(x+a)^{n-1}}{(x+b)^{n+1}}\, dx = \int_c^{\infty} \frac{(x+a)^{-1}}{x+b}\, dx, \quad (a,b,c>0),$$

without interchanging the limit with the integral.

Integrals: limits

NAvW 412. **by J. van de Lune**

Suppose that the function $f\colon \mathbb{R}^+ \to \mathbb{C}$ satisfies the following conditions:

(1) the function f is (Lebesgue) integrable over $(0, T)$ for every $T > 0$,

(2) there exists an $A \in \mathbb{R}$ such that

$$f(x) = O(e^{Ax}), \qquad (x \to \infty),$$

(3) $\lim_{x\to 1-} f(x) = L$, $\qquad \lim_{x\to 1+} f(x) = R$.

Then prove that

$$\lim_{s\to\infty}\int_0^{\infty} \frac{e^{-x}x^s}{\Gamma(s+1)} f\left(\frac{x}{s}\right)\, dx = \frac{L+R}{2}\ .$$

NAvW 442. **by L. Kuipers**

Let $h(z)$ be a real-valued function, Riemann-integrable on $[0, 1]$ such that

$$\int_0^1 h(z)\, dz = 0.$$

Let (z_n), $0 \leq z_n < 1$, $n = 1, 2, \ldots$, be a completely uniformly distributed sequence; that is, for any $k = 1, 2, \ldots$, and any set of k distinct positive integers $q_1, q_2, \ldots, q_k$, the sequence

$$(z_{n+q_1}, z_{n+q_2}, \ldots, z_{n+q_k}),$$

$n = 0, 1, \ldots$, is uniformly distributed in the k-dimensional unit cube. Let $g(t)$ be a function such that $g(t) = 0$ if $t < 0$ and $g(t) = h(z_n)$ if $0 \leq n \leq t < n + 1$.

Let $\phi(t) = g(t)g(t+\alpha)g(t+\beta)$, where α and β are real numbers.

(a) Show that the function

$$\tilde{\phi}(t) = \lim_{N\to\infty}\frac{1}{N}\sum_{q=0}^{N-1}\phi(t+q)$$

vanishes if at least two of the integers $\lfloor t \rfloor$, $\lfloor t+\alpha \rfloor$, and $\lfloor t+\beta \rfloor$ are distinct.

(b) Evaluate

$$\gamma(\tau) = \lim_{N\to\infty}\frac{1}{N}\int_0^N \phi(t)\phi(t+\tau)\, dt$$

for $\tau = \frac{3}{4}$, $\alpha = \frac{1}{4}$, and $\beta = \frac{1}{2}$.

Integrals: multiple integrals

AMM 6008. **by P. B. Gilkey**

For $\xi = (x_1, x_2, x_3) \in \mathbb{R}^3$, set

$$A = A(\xi) = x_1 A_1 + x_2 A_2 + x_3 A_3,$$

where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},$$

and let $\Gamma = \Gamma(\xi)$ be any positively oriented closed curve enclosing the eigenvalues of $A(\xi)$. Show that the integral

$$I(B) = \int_{\mathbb{R}^3} \oint_\Gamma \operatorname{tr}\left\{(\lambda - A)^{-1}\left[B(\lambda - A)^{-1}\right]^3\right\} \times \lambda \exp\left[-\lambda^2\right] d\lambda d\xi$$

vanishes for every 4×4 matrix B.

AMM 6165. **by A. G. O'Farrell**

Suppose that $f(x)$ is a real-valued function on $\mathbb{R}^n$, and define

$$M(x, r) = \frac{\int_{|x-y| \le r} f(y)\, dy}{\int_{|x-y| \le r} 1\, dy},$$

for $x \in \mathbb{R}^n$, $r > 0$. Suppose

$$\frac{M(x, r) - f(x)}{r^2} \to 0$$

as $r \downarrow 0$ for each $x \in \mathbb{R}^n$. Must $f(x)$ be harmonic?

AMM 6055. **by S. Zaidman**

Let $u_\alpha(x, t)$ be the complex-valued function defined for $x \in \mathbb{R}^n$, $t \ge 0$, through the formula

$$u_\alpha(x, t) = (2\pi)^{-n/2} \times \int_{s_1^2 + \cdots + s_n^2 \le 1} \cdots \int \exp\left(-i(x_1 s_1 + \cdots + x_n s_n)\right) \times g_\alpha(s_1, \ldots, s_n, t)\, ds_1 \cdots ds_n$$

where $g_\alpha(s_1, \ldots, s_n, t) = |s|^{-\alpha-2}(1 - e^{-|s|^2 t})$; $|s| \le 1$; $t \ge 0$; $|s| = (s_1^2 + \cdots + s_n^2)^{1/2}$; and α is a real number.

Find a number α such that

$$\lim_{t \to \infty} \int_{\mathbb{R}^n} |u_\alpha(x, t)|^2\, dx_1 \cdots dx_n = +\infty.$$

AMM 6111. **by Barthel W. Huff**

Evaluate

$$\lim_{n \to \infty} \left(-2^n \int_{-\infty}^{\infty} \left[(2\pi^3 \lambda)^{-1/2} \int_0^{|x|} \exp\left\{-\frac{y^2}{2\lambda}\right\} dy \right] \times \left[\int_{-\infty}^{\infty} \exp\left\{-\frac{|u|^\alpha}{2^n}\right\} e^{-iux}\, du \right] dx \right),$$

where $0 < \alpha < 1$ and $\lambda > 0$.

Integrals: trigonometry

MENEMUI 1.1.1.* **by T. N. T. Goodman**

For $n = 1, 2, 3, \ldots$, show that

$$\sum_{j=1}^n \int_0^\pi \left\{ \cos\left(\frac{1}{2} - \frac{2j-1}{2n}\right)(u - \pi) \cdot \sec\left(\frac{1}{2} - \frac{2j-1}{2n}\right)\pi - 1 \right\} \csc\frac{u}{2}\, du = 2n \log n.$$

Intervals

CMB P279. **by F. S. Cater**

Let F be a family of closed intervals in the real line, such that $m\left(\bigcup_{I \in F} I\right) < \infty$, where m denotes Lebesgue measure. For each number $c > 0$, prove that there exist finitely many pairwise disjoint intervals $I_1, I_2, \ldots, I_n \in F$ such that

$$m(I_1 \cup I_2 \cup \ldots \cup I_n) > \frac{1}{2} m\left(\bigcup_{I \in F} I\right) - c.$$

AMM E2733. **by Jim Fickett**

Let S_i, $i = 1, 2, \ldots, m$, be subsets of $[0, 1]$; each S_i is a finite union of disjoint intervals. Let $l(S_i)$ be the sum of the lengths of these intervals. Assume that $l(S_i) = \varepsilon$, $l\left(S_i \cap S_j\right) \le \varepsilon^2$, $i \ne j$, where $\varepsilon > 0$ is fixed. How large can m be?

Jacobians

AMM 6040. **by Jan Mycielski**

Let f be a continuously differentiable map of the unit cube I^n into the Euclidean space $\mathbb{R}^n$ that maps the boundary of I^n into one point. Let $J(f, x)$ be the Jacobian determinant of f at x. Prove that

$$\int_{I^n} J(f, x)\, dx = 0.$$

Laplace transforms

SIAM 76-3.* **by S. A. Rice**

Determine the inverse Laplace transforms, or at least asymptotic formulas for large time t, of the following three functions:

$$\frac{I_v(x)}{I_v(y)}, \quad \frac{I_v(x) I_v(z) K_v(y)}{I_v(y)}, \quad I_v(z) K_v(x).$$

Here $I_v(x)$ and $K_v(x)$ are modified Bessel functions of the first and second kind, respectively, and $v = \sqrt{as}$, where s is the Laplace transform parameter, a is a constant, and $x \ne y \ne z$.

Laurent series

MM 1087. **by Barbara Turner**

Let

$$\sum_{k=-\infty}^{k=+\infty} a_k z^k$$

be the Laurent series of $e^{z+1/z}$ for $0 < |z| < \infty$.

(a) Show that each a_k is an irrational number.

(b) Show that the set $\{a_k \mid k \geq 0\}$ is linearly dependent over the rationals.

Legendre polynomials

SIAM 79-14. **by A. K. Raina and V. Singh**

Let the successive maxima of $|P_\nu(\cos\theta)|$, considered as a function of ν ($\nu \geq 0$) for a fixed θ ($\pi/2 \geq \theta > 0$), be denoted by $m_0 = 1, m_2, m_3, \ldots$. Prove or disprove that $m_0 > m_1 > m_2 > \ldots$.

MM 941. **by Stanley Rabinowitz**

Show that each of the following expressions is equal to the nth Legendre polynomial.

$$(a) \quad \frac{1}{n!}\begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3x & 2 & 0 & \cdots & 0 \\ 0 & 2 & 5x & 3 & \cdots & 0 \\ 0 & 0 & 3 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & n-1 \\ 0 & 0 & 0 & \cdots & n-1 & (2n-1)x \end{vmatrix};$$

$$(b) \quad \frac{1}{n!}\begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3x & 1 & 0 & \cdots & 0 \\ 0 & 4 & 5x & 1 & \cdots & 0 \\ 0 & 0 & 9 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & (n-1)^2 & (2n-1)x \end{vmatrix}.$$

SIAM 79-15. **by J. D. Love**

Prove that for real $x > 0$ and nonnegative integers n,

$$\begin{aligned} \operatorname{csch} x &= P_n(\cosh x)Q_n(\cosh x) \\ &+Q_n(\cosh x)\sum_{m=0}^{n-1} P_m(\cosh x)e^{(n-m)x} \\ &+P_n(\cosh x)\sum_{m=n+1}^{\infty} Q_m(\cosh x)e^{(n-m)x} \end{aligned}$$

where $P_n(\cosh x)$ and $Q_n(\cosh x)$ are modified Legendre functions of the first and second kinds, respectively.

AMM 6227. **by D. M. Milošević**

Prove the following inequality in which $P_n(x)$ is a Legendre polynomial:

$$\int_{-1}^{+1} \frac{1-P_n(x)}{(1-x)^{5/4}}\, dx < 2^{5/4}\left(\sum_{k=1}^{n}\frac{n}{k}\right)^{1/2}.$$

Limits: arithmetic means

TYCMJ 155. **by Norman Schaumberger**

Assume that K_n, a set of n distinct real numbers, has a product equal to unity and a sum equal to S_n, ($n = 1, 2, \ldots$). Is it possible that $\lim_{n\to\infty} S_n/n = 1$?

Limits: binomial coefficients

MM 1055. **by Andreas N. Philippou**

For $0 < p < 1$, find

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{2n+1}{k}p^k(1-p)^{2n+1-k}.$$

AMM 6252. **by Ioan Tomescu**

Let

$$f(n) = \sum_{i=1}^{n}\sum_{j=1}^{n}\binom{n}{i}\binom{n}{j}i^{n-j}j^{n-i}.$$

Show that

$$\lim_{n\to\infty}\frac{[f(n)]^{1/2n}\ln n}{n} = \frac{1}{e}.$$

Limits: elementary symmetric functions

NAvW 404. **by David W. Boyd**

Given a sequence $(a_n)_{n\in\mathbb{N}}$, define

$$M_{n,k} = \left\{\binom{n}{k}^{-1}\sigma_k\left(a_1,\ldots,a_n\right)\right\}^{1/k},$$

where σ_k denotes the kth elementary symmetric function. If $(a_n)_{n\in\mathbb{N}}$ is the sequence that alternates between the two nonnegative numbers a and b, determine $\lim_{n\to\infty} M_{2n,n}$.

Limits: exponentials

CRUX 124. **by Bernard Vanbrugghe**

Evaluate:

$$\lim_{x\to\infty} x\int_0^x e^{t^2-x^2}\,dt.$$

Limits: factorials

JRM 645. **by Richard S. Field, Jr.**

Evaluate $\lim_{n\to\infty}(n!)^{1/n} - ((n-1)!)^{1/(n-1)}$.

SSM 3791. **by John Oman**

Find

$$\lim_{n\to+\infty}\frac{(n^2+n-1)!}{n^{2n}(n^2-1)!}.$$

FQ B-401. **by Gary L. Mullen**

Show that

$$\lim_{n\to\infty}\left[\frac{(n!)^{2n}}{(n^2)!}\right] = 0.$$

Limits: finite products

MM 933. **by Norman Schaumberger**

Show that

$$\lim_{n\to\infty} \frac{n^2}{(1\cdot 2^2\cdot 3^3\cdots n^n)^{4/n^2}} = e.$$

TYCMJ 76. **by Peter A. Lindstrom**

Prove that

$$\lim_{n\to\infty} \prod_{i=1}^{n} \sqrt[n]{(a_i+1)^{a_i+1}} = 4e^{-3/4},$$

where

$$\frac{i-1}{n} < a_i < \frac{i}{n}\ , \qquad (i = 1, 2, \ldots, n).$$

Limits: finite sums

FUNCT 1.1.8.

The expressions

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{and} \quad \ln n$$

are not equal. But for large natural numbers n, the difference between them is quite small. Use a calculator or computer to investigate how the difference between them varies as n increases.

In fact,

$$\lim_{n\to\infty} (H_n - \ln n)$$

exists. Roughly, what is the limiting value?

AMM E2723. **by Allen Moy**

For a fixed $t > 0$, find

$$\lim_{n\to\infty} \left(e^{-nt} \sum_{k=0}^{n-1} \frac{(nt)^k}{k!} \right).$$

FQ H-303. **by Paul Bruckman**

If $0 < s < 1$, and n is any positive integer, let

$$H_n(s) = \sum_{k=1}^{n} k^{-s}, \quad \text{and}$$

$$\theta_n(s) = \frac{n^{1-s}}{1-s} - H_n(s).$$

Prove that $\lim_{n\to\infty} \theta_n(s)$ exists, and find this limit.

CRUX 258. **by Peter A. Lindstrom**

For any rational k other than 0 and -1, find the value of the following limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^{1/k}(n^{k-1/k} + i^{k-1/k})}{n^{k+1}}.$$

AMM 6056. **by Simeon Reich**

Let (a_n) be an increasing sequence of real numbers tending to infinity, and set

$$p_n(t) = \sum_{k=0}^{n} a_{n-k} \frac{t^k}{k!}\ .$$

Is it true that

$$\lim_{n\to\infty} e^{-a_n} \frac{p_n(a_n)}{a_n} = 0?$$

MM 928. **by Norman Schaumberger**

If k is a positive integer, prove that

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{j=1}^{n} \cot^k(\frac{1}{j}) = \frac{1}{k+1}.$$

Limits: floor function

PUTNAM 1976/B.1.

Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\left\lfloor \frac{2n}{k} \right\rfloor - 2\left\lfloor \frac{n}{k} \right\rfloor \right)$$

and express your answer in the form $\log a - b$, with a and b positive integers.

Limits: functional inequalities

NAvW 426. **by J. J. A. M. Brands and M. L. J. Hautus**

Prove that if $f\colon (0,\infty) \to \mathbb{R}$ satisfies

$$f(xy) \le y^{-1} f(x) + f(y), \qquad x > 0, \quad y > 0,$$

then $\lim_{x\to\infty} f(x)$ exists.

Limits: functions

AMM 6167. **by Charles R. Williams and Joseph C. Warndof**

Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$, and for each point $a \in \mathbb{R}^n$, the limit

$$\lim_{x\to a} \frac{|f(x) - f(a)|}{|x-a|}$$

exists. Is f necessarily a constant function?

Limits: infinite series

NAvW 423. **by J. van de Lune**

Let

$$Q_x(n) = n \sum_{k=1}^{\infty} x^k \left(1 - x^k\right)^{n-1}, \qquad 0 < x < 1, \quad n \in \mathbb{N}.$$

Prove that

(a) $\limsup_{n\to\infty} Q_x(n) \le x^{-1} \sum_{k=-\infty}^{\infty} x^k e^{-x^k}$,

and

(b) $\liminf_{n\to\infty} Q_x(n) \ge x \sum_{k=-\infty}^{\infty} x^k e^{-x^k}$.

Also show that the sequence $(Q_x(n))_{n\in\mathbb{N}}$ does not converge for any $x \in (0,1)$.

Limits: integrals

PUTNAM 1979/B.2.

Let $0 < a < b$. Evaluate

$$\lim_{t\to 0}\left\{\int_0^1 [bx + a(1-x)]^t\,dx\right\}^{1/t}.$$

SIAM 78-1.* **by J. S. Lew**

Let (x, y) be an arbitrary point of the Euclidean unit disc D, let $a(p; x, y)$ denote the average l^p distance to a random disc point (u, v), and let $b(p; r)$ denote the rotational average of this function $a(p; x, y)$:

$$D = \left\{(x, y) : x^2 + y^2 \le 1\right\},$$

$$a(p; x, y) = \int\int_D \left\{|x-u|^p + |y-v|^p\right\}^{1/p}\,du\,dv/\pi,$$

$$b(p; r) = \int_0^{2\pi} a(p; r\cos\theta, r\sin\theta)\,d\theta/(2\pi).$$

To measure the deviation from this average, we introduce the ratio of these quantities and we consider its extrema on the disc:

$$c(p; x, y) = a(p; x, y)/\left[b\left(p; \sqrt{x^2+y^2}\right)\right],$$

$$\lambda(p) = \inf\left\{c(p; x, y) : (x, y) \in D\right\},$$

$$\mu(p) = \sup\left\{c(p; x, y) : (x, y) \in D\right\}.$$

Conjecture. $\lambda(p) \uparrow 1$ and $\mu(p) \downarrow 1$ as either $p \uparrow 2$ or $p \downarrow 2$.

Limits: logarithms

TYCMJ 51. **by Joseph Rothschild**

Let (a_n) and (b_n) be sequences of positive, real numbers for which

$$a = \lim_{n\to\infty}\frac{1}{n}\log a_n \ge \lim_{n\to\infty}\frac{1}{n}\log b_n > 0.$$

Prove or disprove that

$$\lim_{n\to\infty}\frac{1}{n}\log(a_n + b_n) = a.$$

Limits: sequences

AMM 6265. **by John H. Cook and David Sanders**

Prove or disprove the following assertion: If $x = s_n$ is the solution to the equation

$$e^{-x}\left(1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n\right) = \frac{1}{2},$$

then $s_n - n \to 2/3$ as $n \to \infty$.

AMM E2692. **by Donald R. Woods**

Show that the sequence of increasingly complex fractions

$$\frac{1}{2}, \left(\frac{1}{2}\right)\Big/\left(\frac{3}{4}\right),$$

$$\frac{\left(\frac{1}{2}\right)/\left(\frac{3}{4}\right)}{\left(\frac{5}{6}\right)/\left(\frac{7}{8}\right)}, \frac{\left(\frac{1}{2}\right)/\left(\frac{3}{4}\right)}{\left(\frac{5}{6}\right)/\left(\frac{7}{8}\right)}\Big/\frac{\left(\frac{9}{10}\right)/\left(\frac{11}{12}\right)}{\left(\frac{13}{14}\right)/\left(\frac{15}{16}\right)}, \ldots$$

approaches a limit, and find that limit.

What can be said about the more general sequence

$$\frac{x}{x+1}, \left(\frac{x}{x+1}\right)\Big/\left(\frac{x+2}{x+3}\right), \frac{\left(\frac{x}{x+1}\right)\Big/\left(\frac{x+2}{x+3}\right)}{\left(\frac{x+4}{x+5}\right)\Big/\left(\frac{x+6}{x+7}\right)}, \ldots ?$$

CRUX 48. **by Léo Sauvé**

Let the function $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2 + \sin x \cos\frac{1}{x}, \quad \text{if } x \ne 0,$$
$$f(0) = 2.$$

For each $n \ge 1$, consider the integral

$$I_n = \int_{-\frac{2}{n}}^{\frac{2}{n}} \left(n + (\frac{1}{n} - n)X_n(x)\right) f(x)\,dx,$$

where X_n is the characteristic function of the interval $\left[-\frac{1}{n}, \frac{1}{n}\right]$. Express I_n as a function of n and find $\lim_{n\to\infty} I_n$.

MM 958. **by Murray S. Klamkin**

Give direct proofs of the following two results:

(a) If $\mathrm{Re}(z_0) > 0$ and the sequence (z_n) is defined for $n \ge 1$ by

$$z_n = \frac{1}{2}\left(z_{n-1} + \frac{A}{z_{n-1}}\right),$$

where A is real and positive, then $\lim_{n\to\infty} z_n = \sqrt{A}$.

(b) Suppose (x_n) is a real sequence defined for $n \ge 1$ by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{A}{x_{n-1}}\right),$$

where A is positive. Show that if p is a given integer greater than 1, then the initial term x_0 can be chosen so that (x_n) is periodic with period p.

SSM 3698. **by Michael Brozinsky**

Let a, b, c, and d, with $c < d$, be positive real numbers. It is known that $x = \frac{ad+bc}{a+b}$ divides the interval $[c, d]$ in the ratio a/b (that is, $(x-c)/(d-x) = a/b$). Consider the sequence $\{x_n\}$ defined by the following: $x_1 = c$, $x_2 = d$, and $x_n = \frac{ax_{n-1}+bx_{n-2}}{a+b}$ for $n = 3, 4, 5, \ldots$. Find $\lim_{n\to+\infty} x_n$.

SSM 3760. **by N. J. Kuenzi**

It is known that if a_i is an arbitrary positive number and $a_n = \sqrt{a_{n-1}}$, $n = 2, 3, \ldots$ then $\lim_{n\to+\infty} a_n = 1$. Suppose a_1 and m are arbitrary, positive numbers. Define $a_n = m\sqrt{a_{n-1}}$, $n = 2, 3, \ldots$. Find $\lim_{n\to+\infty} a_n$.

Limits: trigonometry

AMM E2699. **by Emile Haddad and Peter Johnson**

Suppose that $1 = \theta_0 > \theta_1 > \cdots > \theta_k > 0$ and that

$$\sum_{i=0}^{k} a_i \cos n\theta_i\pi \to 0$$

as $n \to \infty$ through the integers.

Does it follow that $a_i = 0$ for all i?

PME 376. **by Solomon W. Golomb**

Let the sequence $\{a_n\}$ be defined inductively by $a_1 = 1$ and $a_{n+1} = \sin(\arctan a_n)$ for $n \geq 1$. Let the sequence $\{b_n\}$ be defined inductively by $b_1 = 1$ and $b_{n+1} = \cos(\arctan b_n)$ for $n \geq 1$. Give explicit expressions for a_n and b_n, and find $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$.

Location of zeros: complex polynomials

AMM 6191. **by Harry D. Ruderman**

Let $P(z)$ be a monic polynomial with complex coefficients, in the complex variable z. Let $P(z_1)$ and $P(z_2)$ be in opposite quadrants I and III or II and IV. Let $z_3 = (z_1 + z_2)/2$. What is an upper bound (least, if possible) on r that will guarantee that a zero of $P(z)$ will be within a distance r from z_3?

AMM 6237. **by Emeric Deutsch**

Show that every zero z of the complex polynomial

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$$

satisfies $-\beta \leq \mathrm{Re}(z) \leq \alpha$, where α and β are the unique positive roots of the equations

$$x^n + \mathrm{Re}(a_1)x^{n-1} - |a_2|x^{n-2} - |a_3|x^{n-3} - \cdots - |a_{n-1}|x - |a_n| = 0$$

and

$$x^n - \mathrm{Re}(a_1)x^{n-1} - |a_2|x^{n-2} - |a_3|x^{n-3} - \cdots - |a_{n-1}|x - |a_n| = 0,$$

respectively.

AMM E2761. **by Ron Adin**

Let $P(z)$ be a polynomial of degree at least 2 with complex coefficients, not all of them real. Prove that the equation

$$P(z)P(-z) = P(z)$$

has roots in both the upper and lower open half-planes, $\mathrm{Im}(z) > 0$ and $\mathrm{Im}(z) < 0$.

AMM E2801. **by Louis Nirenberg, D. Kinderlehrer, and J. Spruck**

Let $P_1(z)$ and $P_2(z)$ be monic polynomials with complex coefficients of degree $m + k$, $0 \leq k < m$, such that $z_1, \ldots, z_m$ in the upper half-plane are zeros of P_1 while $\overline{z}_1, \ldots, \overline{z}_m$ are zeros of P_2. Show that $P_1 - P_2$ has degree greater than $m - k - 2$.

CRUX 138. **by Jacques Marion**

Let

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

be a nonconstant polynomial such that $|p(z)| < 1$ on the circle $|z| = 1$. Show that $p(z)$ has a zero on $|z| = 1$.

CRUX 237. **by Basil C. Rennie**

Suppose a closed set E in the complex plane has the property that if a polynomial has all its zeros in E then the derivative also has all its zeros in E. Must E be convex?

SPECT 8.9.

The polynomial f has complex coefficients, and all its roots have positive real parts. Show that all the roots of the derivative of f have positive real parts.

Location of zeros: complex variables

CRUX 60. **by Jacques Marion**

Let f be an analytic function on the closed disc $B(0, R)$ such that $|f(z)| < M$, and $|f(0)| = a > 0$. Show that the number of zeros of f in $B(0, \frac{R}{3})$ does not exceed $\frac{1}{\log 2} \log \frac{M}{a}$.

CRUX 196. **by Hippolyte Charles**

Show that if $|a_i| < 2$ for $1 \leq i \leq n$, then the equation

$$1 + a_1 z + \cdots + a_n z^n = 0$$

has no roots in the disc $|z| \leq \frac{1}{3}$. Is the converse true?

CRUX 152. **by Jacques Marion**

If $a > e$, show that the equation $e^z = az^m$ has m solutions inside the circle $|z| = 1$.

Location of zeros: entire functions

NAvW 498. **by J. van de Lune**

For any positive integer m, we define

$$f_m(z) = \sum_{n=m+1}^{\infty} \frac{z^{n-m-1}}{n!} \qquad z \in \mathbb{C}.$$

From the theory of entire functions, it follows that $f_m(z)$ has infinitely many zeros.

Prove that none of these zeros are real and that all of them have positive real parts.

Location of zeros: limits

AMM E2787. **by James V. Whittaker**

Show that if $k \geq 3$, then the equation $(\log x)^k = x$ for $x \geq 1$ has just two solutions r_k and s_k, where $r_k \to e$ and $s_k \to \infty$ as $k \to \infty$.

Maclaurin series

FQ H-249. **by F. D. Parker**

Find an explicit formula for the coefficients of the Maclaurin series for

$$\frac{b_0 + b_1 x + \cdots + b_k x^k}{1 + \alpha x + \beta x^2}.$$

AMM E2688.* **by David Jackson**

Let $\{f_i\}$ and $\{g_i\}$, $i = 0, 1, 2, \ldots$, be the solutions of the recurrence equation

$$u_{m+1} = -u_m - m(m+1)xu_{m-1}$$

satisfying the initial conditions $f_0 = 0$, $f_1 = 1$, $g_0 = 1$, and $g_1 = -1$, respectively. Show that the coefficient of x^{n-1} in the Maclaurin expansion of $-f_n/g_n$ is t_{2n-1}, where

$$\tan x = \sum_{n\geq 1} t_{2n-1} \frac{x^{2n-1}}{(2n-1)!} .$$

Maxima and minima: bounds

AMM E2519. **by H. L. Montgomery**

Let P be a complex polynomial of degree n with $P(1) = 0$ and $P(0) = 1$. Show that

$$\max\{|P(z)| : |z| \leq 1\} \geq 1 + \frac{1}{3n} .$$

Maxima and minima: complex numbers

AMM E2600. **by Ron Evans**

Fix $r \geq 2$ and suppose that z_1, z_2, z_3, and z_4 are complex numbers of modulus $\geq r$. Find the point at which

$$2 - (z_1 + z_2)(z_3 + z_4) + z_1 z_2 z_3 z_4$$

attains its minimum modulus.

Maxima and minima: constraints

MM 942. **by M. S. Klamkin**

Determine the maximum value of

$$S = \sum_{1\leq i<j\leq n} \left(\frac{x_i x_j}{1-x_i} + \frac{x_i x_j}{1-x_j} \right)$$

where $x_i \geq 0$ and $x_1 + x_2 + \cdots + x_n = 1$.

AMM 6076. **by Robert L. Anderson**

Given n real numbers $p_1, p_2, \ldots, p_n$, find a continuous function $x(t)$ with piecewise continuous derivative $x'(t)$ on $[0, n]$ such that $x(t)$ minimizes

$$L(x) = \int_0^n \sqrt{1 + [x'(t)]^2}\, dt$$

subject to the n constraints

$$\int_{i-1}^{i} x(t)\, dt = p_i, \qquad i = 1, 2, \ldots, n.$$

Is the solution unique?

PUTNAM 1975/A.3.

Let a, b and c be constants with $0 < a < b < c$. At what points of the set

$$\{x^b + y^b + z^b = 1, x \geq 0, y \geq 0, z \geq 0\}$$

in three-dimensional space $\mathbb{R}^3$ does the function $f(x, y, z) = x^a + y^b + z^c$ assume its maximum and minimum values?

Maxima and minima: derivatives

AMM 6173. **by Otomar Hájek**

For C^2 functions $f \neq 0$ vanishing at 0 and π, consider the functional $\inf_{(0,\pi)} f''/f$ (ignore undefined values). Show that its maximum -1 is attained only by $\sin x$ and its multiples.

Maxima and minima: integrals

AMM E2707. **by Leonard Shapiro**

Find $\sup \sigma(f)$ where

$$\sigma(f) = \inf_{x>0} \left\{ \frac{f(x)}{x} \int_0^x (1 - f(t))\ dt \right\}$$

and f ranges over continuous functions on $[0, \infty)$. For which f (if any) is this supremum achieved?

AMM 6140. **by F. S. Cater**

Let f be a continuous real-valued function on $[0, 1]$, and let E_f denote the (possibly void) set

$$\left\{x \in [0, 1] : f'(x) \text{ exists and is finite}\right\} .$$

Let $a(f)$ be the Lebesgue outer measure of $f\left([0, 1]\backslash E_f\right)$,

$$m(t) = \begin{cases} f'(t), & \text{for } t \in E_f, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$b(f) = a(f) + \int_0^1 m(t)\, dt$$

and

$$c(f) = a(f) + \int_0^1 \left[1 + m(t)^2\right]^{1/2} dt.$$

Find $\max c(f)$ and $\min c(f)$ over all f such that $b(f) = 1$. Describe functions for which $c(f)$ takes one of these values.

Maxima and minima: limits

SIAM 77-13. **by Ilia Kaufman**

For $x \geq 1$, $c \geq 0$ let

$$f_c(x) = (x + c) \left[B(x + c \,|\, x - 1) - B(x + c \,|\, x)\right],$$

where

$$B(y \,|\, x) = \frac{e^{-y} y^x}{\int_y^\infty e^{-t} t^x\, dt} .$$

The function B, or its restriction to integral values of x,

$$B(y \,|\, n) = \frac{y^n/n!}{\sum_{k=0}^{n} y^k/k!} ,$$

is called the first Erlang function. It is easy to prove that for any fixed value of c, $\lim_{x\to\infty} f_c(x) = 2/\pi$. Determine or numerically estimate

$$\Delta = \inf_{c\geq 0} \sup_{x\geq 1} \left| f_c(x) - \frac{2}{\pi} \right|.$$

Maxima and minima: polynomials

PUTNAM 1975/B.3.

Let $s_k(a_1, \ldots, a_n)$ denote the kth elementary symmetric function of $a_1, \ldots, a_n$. With k held fixed, find the supremum (or least upper bound) M_k of

$$s_k(a_1, \ldots, a_n)/[s_1(a_1, \ldots, a_n)]^k$$

for arbitrary $n \geq k$ and arbitrary n-tuples $a_1, \ldots, a_n$ of positive real numbers.

Maxima and minima: radicals

MM Q610. **by C. F. Pinzka**

Maximize $(7+x)(11-3x)^{1/3}$.

CRUX 358. **by Murray S. Klamkin**

Determine the maximum of x^2y, subject to the constraints

$$x + y + \sqrt{2x^2 + 2xy + 3y^2} = k \text{ (constant)}, \quad x, y \geq 0.$$

CRUX 347. **by M. S. Klamkin**

Determine the maximum value of

$$\sqrt[3]{4 - 3x + \sqrt{16 - 24x + 9x^2 - x^3}} + \sqrt[3]{4 - 3x - \sqrt{16 - 24x + 9x^2 - x^3}}$$

in the interval $-1 \leq x \leq 1$.

Maxima and minima: unit circle

MM Q662. **by M. S. Klamkin**

Determine the maximum of

$$R = \frac{|z_1z_2 + z_2z_3 + z_3z_4 + z_4z_5 + z_5z_1|^3}{|z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_5 + z_4z_5z_1 + z_5z_1z_2|^2}$$

where z_1, z_2, z_3, z_4, and z_5 are complex numbers of unit length.

Measure theory: arcs

AMM 6007. **by Rollin Sandberg**

Let f be a nondecreasing, continuous function from $[0, a]$ onto $[0, b]$ such that f' vanishes almost everywhere. Determine the length of this arc.

AMM 6074. **by H. L. Montgomery**

Let f be a weakly increasing continuous function defined on $[0, 1]$, with $f(0) = 0$, $f(1) = 1$, and let l denote the arc length of the curve $(x, f(x))$, $0 \leq x \leq 1$. Prove that $l \leq 2$, with equality if and only if $f'(x) = 0$ almost everywhere.

Measure theory: Borel sets

AMM 6242. **by Jan Mycielski**

Let I be the interval $[0, 1]$, λ the Lebesgue measure in I, and μ a Borel measure in I. Suppose that $\lambda(A) = \frac{1}{2}$ implies $\mu(A) = \frac{1}{2}$ for every Borel set $A \subseteq I$. Prove that $\mu(B) = \lambda(B)$ for every Borel set $B \subseteq I$.

Measure theory: function spaces

AMM 6131. **by Lee A. Rubel**

Suppose $\phi \geq 0$ is in $L^1(-\infty, \infty)$, ϕ vanishes outside of $[a, b]$, and ϕ is strictly decreasing on $[a, b]$. Prove that the span of the translates of ϕ is dense in $L^1(-\infty, \infty)$.

Measure theory: geometry

AMM 6231. **by Terry R. McConnell**

Let A be a subset of $\mathbb{R}^2$ with nonzero Lebesgue measure. Prove that A contains the vertices of a square.

Measure theory: integrals

CMB P256. **by T. Zaidman**

Let $(S, \mathcal{B}, m)$ be a measure space, $m(S) < \infty$, and let f be a bounded, measurable, real-valued function of S. For any real number t let $E_t = \{s \in S : 0 \leq f(s) + t < 1\}$. Prove, without using Fubini's theorem, that $\int_{-\infty}^{\infty} m(E_t)\, dt = m(S)$.

NAvW 443. **by J. van de Lune**

Let X be a set equipped with a (nonnegative) measure μ. Let $\phi\colon X \to [0, \infty]$ be μ-measurable. Find a necessary and sufficient condition on ϕ that guarantees the existence of a μ-measurable function $\psi\colon X \to [0, \infty]$ satisfying

(1) $\int_X \psi\, d\mu < \infty$

and

(2) $\int_X \frac{\phi}{\psi}\, d\mu < \infty$.

(For convenience let $\frac{0}{0} = 0$ and $\frac{\infty}{\infty} = \infty$.)

Measure theory: Lebesgue outer measure

AMM E2710. **by J. A. Andrews**

Call two real numbers equivalent if their difference is rational. Call $S \subset \mathbb{R}$ a choice set if S is a set of representatives of the equivalence classes in $\mathbb{R}$. Let $\mathcal{F}$ be the family of all choice sets contained in $[0, 1]$. Show that the numbers $m^*(S)$ $(S \in \mathcal{F})$ are dense in $[0, 1]$. (m^* is the usual outer measure.)

Measure theory: monotone functions

AMM 6218. **by M. J. Pelling**

Let S be a subset of the real line $\mathbb{R}$ having cardinality of the continuum. Is there always a monotonic $f\colon \mathbb{R} \to \mathbb{R}$ such that $m^* f(S) > 0$ where m^* is outer Lebesgue measure?

AMM 6073. **by George Crofts**

Let f be an increasing real-valued function from $[a, b]$ onto $[c, d]$ and let m denote Lebesgue measure. If there is a set $E \subset [a, b]$, with $m(E) = 0$, for which $m(f(E)) = d - c$, must f be singular (i.e., $f' = 0$ almost everywhere)?

Measure theory: probability measures

AMM 6143. **by A. L. Macdonald**

Let $\pi_1, \pi_2, \ldots, \pi_n$ be nonatomic probability measures on a set X. Prove that there are pairwise disjoint sets $B_1, B_2, \ldots, B_n$ with $\pi_i(B_i) \geq 1/n$.

Measure theory: uniform integrability

AMM 6085. **by William J. Sánchez**

Call a family F of functions uniformly integrable if there exists $k(\varepsilon)$ such that

$$\int \{|f|\,d\mu : |f| > k\} < \varepsilon$$

for all $f \in F$. If there exists integrable h such that $|f| \le h$ (almost everywhere) for all $f \in F$, then F is uniformly integrable. Is the converse true?

Numerical analysis

SIAM 78-2. **by J. C. Cavendish and W. W. Meyer**

For p a positive integer, let $\Phi_k(x)$ denote a $(2p+1)$-degree basis polynomial for $(2p+1)$-Hermite interpolation on $0 \le x \le 1$. That is, for $n, k = 0, 1, \ldots, p$,

$$\left.\frac{d^n\Phi_k}{dx^n}\right\}_{x=0} = \begin{cases} 0, & \text{if } n \ne k, \\ 1, & \text{if } n = k, \end{cases}$$

$$\left.\frac{d^n\Phi_k}{dx^n}\right\}_{x=1} = 0.$$

Establish the following two recurrence relations for any $t \in [0,1]$:

$$t\Phi_{k-1}(t) - k\Phi_k(t) = \frac{(2p-k+1)!}{p!(k-1)!(p-k+1)!}\, t^{p+1}(1-t)^{p+1}, \quad (0 < k \le p),$$

$$\Phi_{k-1}(t) - \Phi'_k(t) = \frac{(2p-k+1)!}{p!k!(p-k+1)!}\, t^{p}(1-t)^{p}(p+1-kt), \quad (0 < k \le p).$$

Numerical approximations

SPECT 7.8.

Use the identity

$$\frac{4}{1+t^2} = 4 - 4t^2 + 5t^4 - 4t^5 + t^6 - \frac{t^4(1-t)^4}{1+t^2}$$

to show that

$$\frac{22}{7} - \frac{1}{1260} > \pi > \frac{22}{7} - \frac{1}{630}\ .$$

Partial derivatives

AMM 6018. **by Antonio Marquina**

Does there exist a real-valued function $f(x,y)$ defined at every point of $\mathbb{R}^2$, satisfying the following properties?

(i) For every point (x,y), $f(x,y)$ is continuous.

(ii) For every point (x,y), the two partial derivatives $D_x f$ and $D_y f$ exist.

(iii) The function $f(x,y)$ is not differentiable in (x,y) for every point of $\mathbb{R}^2$.

Point sets

AMM E2598. **by Erwin Just**

Does there exist a set of rational points that is dense in the plane such that the distance between each pair of points in the set is irrational?

Power series

MM 978. **by L. Carlitz**

For $\lambda > 0$, let

$$(1 - x - y + axy)^{-\lambda} = \sum_{m,n=0}^{\infty} c_{m,n}^{(\lambda)} x^m y^n.$$

Show that $c_{m,n}^{(\lambda)} \ge 0$ for all m and n if and only if $a \le 1$.

AMM 6080. **by R. N. Hevener, Jr.**

A theorem of Abel states that if

$$\sum_{n=0}^{\infty} a_n z^n$$

converges on the closed interval A, then

(i) convergence is uniform on A, whence

(ii) it determines a continuous function on A.

Is either part of this theorem true if A denotes a closed disc instead of an interval? If we impose the additional hypothesis, trivially satisfied in Abel's theorem, that the function be continuous on the boundary of A, is either part true?

CRUX 259. **by Jacques Sauvé**

The function

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2$$

is defined for all real x. Can one express $f(x)$ in closed form in terms of known (not necessarily elementary) functions?

AMM 6038. **by Oto Strauch**

Let

$$f(x) = \sum a_i x^i$$

and

$$s_n(x) = \sum_{i \le n} a_i x^i.$$

Let $r \ne 0$ be an interior point of the interval of convergence of the power series $\sum a_i x^i$. Prove that if $s_n(r) < f(r)$ for every $n = 0, 1, 2, \ldots$, then the derivative $f'(r) \ne 0$.

NAvW 519. **by J. van de Lune**

Let

$$z\frac{e^z+1}{e^z-1} = \sum_{n=0}^{\infty} \beta_n \frac{z^{2n}}{(2n)!}\ .$$

Prove that $\beta_n^2 \le \beta_{n-1}\beta_{n+1}$ for $n \ge 2$.

FQ H-293. **by Leonard Carlitz**

Show that if a set of polynomials $(f_n(x))_{n=0}^{\infty}$ satisfies

$$\sum_{n=0}^{\infty} f_{n+k}(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty} f_n(x)\frac{z^n}{n!} f_k(x-z)$$

for $k \ge 0$, $f_0(x) = 1$, and $f_1(x) = 2x$, then

$$f_n(x) = H_n(x),\ n = 0, 1, 2, \ldots\ .$$

FQ B-399. **by V. E. Hoggatt, Jr.**

Let

$$f(x) = u_1 + u_2 x + u_3 x^2 + \cdots$$

and

$$g(x) = v_1 + v_2 x + v_3 x^2 + \cdots ,$$

where $u_1 = u_2 = 1$, $u_3 = 2$, $u_{n+3} = u_{n+2} + u_{n+1} + u_n$, and $v_{n+3} = v_{n+2} + v_{n+1} + v_n$. Find initial values v_1, v_2, and v_3 so that $e^{g(x)} = f(x)$.

Pursuit problems

JRM 534. **by David L. Silverman**

A farmer carrying two chicks on a narrow North-South road inadvertently drops them. The slower chick runs North, the other South. The farmer, who is faster than either, wants to catch both in minimum time.

(a) Solve the farmer's problem.

(b) Using any tools you wish, determine which chick the farmer should chase first if his objective is to deliver them in minimum time to a town located on the road. Consider all four relative locations of the town.

(c) Generalize (b) to the case in which the town is not situated on the road. Consider it confined to the plane of the road first, then generalize to 3-space.

JRM C5. **by Travis Fletcher**

Let A, B, and C denote three point-like entities, capable of motion in the plane with respective velocities in the ratio 1:2:3. At the start of a game of tag in which C is "it" A and B are together, and C is displaced from them at a distance d, which happens to be the common distance required for each of the three players to accelerate from zero to his maximum velocity. The game ends only after C has tagged both of his opponents, so it behooves A and B to separate.

Determine the evasion and pursuit history in which A and B maximize and C minimizes the time necessary to make the two tags.

PME 357. **by David L. Silverman**

Able, Baker, and Charlie, with respective speeds $a > b > c$, start at point P with Able designated "it" in a game of Tag, which terminates when Able has tagged both Baker and Charlie. At time $-T$, Baker heads north and Charlie south. After a count taking time T, Able starts chasing one of the two quarries. Assuming that Baker and Charlie will maintain their speeds and directions, whom should Able chase first in order to minimize the time required to make the second and final tag?

CANADA 1979/4.

A dog standing at the center of a circular arena sees a rabbit at the wall. The rabbit runs around the wall and the dog pursues it along a unique path which is determined by running at the same speed and staying on the radial line joining the center of the arena to the rabbit. Show that the dog overtakes the rabbit just as it reaches a point one-quarter of the way around the arena.

PME 401. **by Zelda Katz**

From a point 250 yards due north of Tom, a pig runs due east. Starting at the same time, Tom pursues the pig at a speed 4/3 that of the pig, and changes his direction so as to run toward the pig at each instant. With each running at uniform speed, how far does the pig run before being caught?

Rate problems

IMO 1979/3.

Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point traveling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that, at any time. the distances from P to the moving points are equal.

OSSMB 78-6. **by J. Levitt**

A man always drives his automobile at a constant speed. The points A, B, C, D are such that $BC = CD = 10$ km, and $\angle BCD = \pi/2$. Point A is inside angle BCD. If he travels from A to C directly in 30 min, A to C via B in 35 min, and A to C via D in 40 min, at what constant speed does he drive?

FUNCT 3.5.2.

A camera at O tracks a horse running along PQ with $OP \perp PQ$. Let s be the distance from P to Q. Let θ be the measure of $\angle POQ$. Find the value of s for which $\dot{\theta}$ is maximized, given that its velocity at P is u, and that its uniform acceleration is a.

MM 926. **by Melvin F. Gardner**

A swimmer can swim with speed v in still water. He is required to swim for a given length of time T in a stream whose speed is r with $r < v$. If he is also required to start and finish at the same point, what is the longest path (total arc length) that he can complete? Assume the path is continuous with piecewise continuous first derivatives.

JRM 796. **by Peter MacDonald**

A. J. Gunnet is poised at one end of Lookout Avenue, anxious to try out his super-charged racer. Three traffic lights divide the distance from A.J. to the end of the Lookout Avenue into four equal parts. Each traffic light is red for one minute and green for two minutes (no yellow light). No two lights are ever red at the same time. A.J. notices that the light farthest from him has just turned green, and the light nearest him has just turned red. He decides to wait until such time as the round trip can be made in the fastest possible time. He must maintain a constant speed throughout, and each light must be green as he goes through it. How long should he wait before embarking on his journey, and how fast can he make the round trip? Assume instantaneous acceleration at the start and instantaneous reversal at the end of the street without any loss of speed.

Riemann zeta function

AMM 6127. **by M. J. Pelling**

Sum the series

$$\sum_{n=2}^{\infty} \zeta(n) \left(\frac{a}{b}\right)^n ,$$

where $0 < a/b < 1$ is rational.

NAvW 524. **by Mihály Bencze**

Let ζ be the Riemann zeta-function, and let $d(n)$ denote the number of divisors of n.

Show that

$$\sum_{k=1}^{\infty}\left(k^{-1}d(k)\right)^{s} > \left(\frac{2}{\zeta\left(1+\frac{1}{s-1}\right)}\right)^{s-1}, \qquad (s>1).$$

CRUX 440. **by Kenneth S. Williams**

Find a simple elementary proof of

$$\zeta(2) = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}.$$

NAvW 444. **by J. van de Lune**

Let $\zeta(s)$ denote Riemann's zeta-function. For $t>0$, let $I(t)$ be the imaginary part of $\zeta(1+it)$. Prove that $I(t)$ has infinitely many real zeros.

Sequences: cluster points

NAvW 542. **by A. A. Jagers and H. Th. Jongen**

Let s be a sequence of elements of ℓ^2 of the form

$$s = (\alpha_n e_n)_{n=1}^{\infty},$$

where $e_1, e_2, e_3, \ldots$ is an orthonormal basis of ℓ^2 and $\alpha_n > 0$ for all n. Prove that 0 is a weak cluster point of s if and only if $\left(\alpha_n^{-1}\right)_{n=1} \notin \ell^2$. Compare:

(1) 0 is the weak limit of a subsequence of s if and only if

$$\left(\alpha_n^{-1}\right)_{n=1}^{\infty} \notin c_0.$$

(2) 0 is a strong cluster point of s if and only if

$$\left(\alpha_n^{-1}\right)_{n=1} \notin \ell^{\infty}.$$

Sequences: complex numbers

SPECT 9.6. **by I. J. Maddox**

(a) Let (b_n) be a sequence of complex numbers such that $b_{n+1}-b_n \to l$ as $n\to\infty$. Show that $b_n/n \to l$ and that $|b_{n+1}|-|b_n| \to |l|$ as $n\to\infty$.

(b) Let (a_n) be a sequence of nonzero real numbers, and put $b_n = a_{n+1}/a_n$ for $n=1,2,3,\ldots$. Put

$$c_n = b_{n+1}-b_n,$$
$$c_n' = |b_{n+1}|-|b_n|.$$

Show that it is possible for c_n' to tend to zero as $n\to\infty$ but for the sequence (c_n) to diverge.

Sequences: convergence

AMM 6090. **by T. Šalát and O. Strauch**

Define ϕ-convergence of a sequence $\{\gamma_n\}$ of real numbers in this way: $\phi\text{-}\lim \gamma_n = \lambda$ if and only if $\lim s_n = \lambda$, where $s_n = n^{-1}\sum_{d/n}\phi(d)\gamma_d$. ($\phi$ denotes Euler's totient function.) Find a sequence that is ϕ-convergent but is not convergent.

SIAM 75-14.* **by M. W. Green, A. J. Korsak, and M. C. Pease**

It has been found in practice that the following very simple (but very effective) procedure always converges for any n starting trial roots:

$$x_i' = \frac{x_i - P(x_i)}{\prod_{j\neq i}(x_i-x_j)}, \qquad i=1,2,\ldots,n,$$

where $P(x)$ is an arbitrary (complex coefficient) monic polynomial in x of degree n. In fact, even when $P(x)$ has multiple roots, the above procedure still converges, but only linearly (as opposed to quadratically in the distinct root case). Show that this procedure is globally convergent outside of a set of measure zero in the starting space and describe this set for $n=2$. Show the same result, if possible, for arbitrary n.

Sequences: inequalities

NAvW 492. **by J. J. A. M. Brands**

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers with the property that

$$\frac{a_{n+1}-a_{n+2}}{a_n} \geq \frac{1}{4}, \qquad n=1,2,\ldots.$$

Prove that there is a number C, such that

$$\sum_{k=1}^{n}\frac{a_{k+1}-a_{k+2}}{a_k} = \frac{1}{4}n + C + O\left(n^{-1}\right), \qquad (n\to\infty).$$

SPECT 7.3. **by B. G. Eke**

The real numbers $a_1, a_2, a_3, \ldots$ are positive, less than 1, and such that

$$a_n < \frac{1}{2}(a_{n-1}+a_{n+1})$$

for $n=2,3,\ldots$. Show that a_n tends to a limit as n tends to infinity.

Sequences: monotone sequences

NAvW 510. **by J. van de Lune**

Determine all constants $c>-1$ for which the sequence

$$(H_n - \log(n+c))_{n\in\mathbb{N}}$$

is strictly monotonic, where $H_n = \sum_{k=1}^{n} k^{-1}$.

TYCMJ 112. **by Richard Johnsonbaugh**

Find the least positive integer N for which

$$\frac{(n+1)^{1/(n+1)}}{n^{1/n}},$$

$(n = N, N+1, N+2, \ldots)$, is monotonic increasing.

SPECT 9.9.

Let

$$u_n = \left(1+\frac{1}{n}\right)^{n}, \qquad v_n = \left(1+\frac{1}{n}\right)^{n+1}.$$

Show that the sequence (u_n) is strictly increasing, whereas (v_n) is strictly decreasing.

CMB P247. **by P. Erdős and R. E. Bixby**

Let $a_1 < a_2 < a_3 \cdots$ be an increasing sequence such that $a_n = o(n^{1+\varepsilon})$ and $0 < c < d_n = a_{n+1} - a_n = o(n^{\varepsilon})$ for every $\varepsilon > 0$. Show that there exist sequences of integers n_i, m_i such that $a_{n_i}/a_{m_i} \to \infty$ and $d_{n_i}/d_{m_i} \to 1$. Show also that, if $a_n/n^k \to \infty$ for every k, then there exist integers n_i, m_i such that $a_{n_i}/a_{m_i} \to 1$ and $d_{n_i}/d_{m_i} \to \infty$.

NAvW 446. **by R. J. Stroeker**

For each $n = 2, 3, \ldots$, let x_n be the unique solution of the equation

$$n = \frac{x^n + x^{-n}}{x + x^{-1}}$$

in the interval $(0, 1)$. Show that the sequence $(x_n)_{n\geq 2}$ is increasing and determine

$$\lim_{n\to\infty} x_n.$$

NAvW 399. **by J. van de Lune**

For $n \in \mathbb{N}$ and $s \in \mathbb{R}$, let

$$\sigma_n(s) = \sum_{k=1}^{n} k^s,$$

$$U_n(s) = n^{-s-1}\sigma_n(s),$$

$$L_n(s) = n^{-s-1}\sigma_{n-1}(s),$$

where $\sigma_0(s) = 0$.

Prove that if s is positive, $U_n(s)$ is decreasing in n and $L_n(s)$ is increasing in n.

NAvW 400. **by J. van de Lune**

For $n \in \mathbb{N}$ and $s \in \mathbb{R}$, let

$$\sigma_n(s) = \sum_{k=1}^{n} k^s,$$

$$U_n(s) = n^{-s-1}\sigma_n(s),$$

$$L_n(s) = n^{-s-1}\sigma_{n-1}(s),$$

where $\sigma_0(s) = 0$.

We define

$$T_n(s) = \frac{1}{2}\left\{U_n(s) + L_n(s)\right\}.$$

Prove that $T_n(s)$ is increasing in n if $0 < s < 1$ and decreasing in n if $s > 1$.

Sequences: pairs of sequences

TYCMJ 60. **by Richard Johnsonbaugh**

Assume that (x_n) and (y_n) are sequences satisfying $y_n = x_n + x_{n+1}$ and that (y_n) converges. For which values of $\varepsilon \in (0, 1]$ must (x_n/n^{ε}) converge?

TYCMJ 133. **by Barbara Turner**

Let $a, m, n > 0$ and $m^2 = an^2$. Define M_k and N_k inductively as follows: $M_1 = an - m$, $M_{k+1} = aN_k - M_k$, $N_1 = m - n$, and $N_{k+1} = M_k - N_k$. Prove that the sequences (M_i) and (N_i) diverge if and only if $a > 4$.

Sequences: rearrangements

MM 1021.* **by Peter Ørno**

Prove or disprove that a countably infinite set of positive real numbers with a finite nonzero cluster point can be arranged in a sequence, (a_n), so that $((a_n)^{1/n})$ is convergent.

MM 972. **by Marius Solomon**

Prove or disprove that the set of all positive rational numbers can be arranged in an infinite sequence, (a_n), such that $((a_n)^{1/n})$ is convergent.

Sequences: recurrences

NAvW 407. **by M. L. J. Hautus**

Let $\alpha > 0$. Consider the sequence $(x_n)_{n=0}^{N+1}$ defined by

$$x_0 = 1,$$

$$x_{n+1} = x_n - \frac{\sqrt{x_n}}{n+\alpha}, \qquad (n = 0, 1, \ldots, N),$$

where N is determined by the condition

$$x_{N+1} < 0 \leq x_N.$$

Show that such N exists and that

$$N \sim \left(e^2 - 1\right)\alpha, \qquad (\alpha \to \infty).$$

TYCMJ 62. **by N. J. Kuenzi**

Let (x_n) be a sequence defined by the recurrence relation $x_{n+1} = x_n/(1 + \frac{1}{2}x_n)$ for $n \geq 0$. For which initial values x_0 will the sequence converge to zero?

MM 1085. **by Bert Waits**

Consider the polynomial

$$P(x) = x^4 - 14x^2 + x + 38.$$

Find a function $g = g(x; \varepsilon_1, \varepsilon_2)$, where ε_1, and ε_2 are ± 1, such that the recursive sequence $x_{n+1} = g(x_n)$ converges to a different zero of $P(x)$ for each of the four distinct values of $(\varepsilon_1, \varepsilon_2)$.

CRUX 194. **by Steven R. Conrad**

A sequence $\{a_n\}$ is defined by

$$a_1 = X, \quad a_n = X^{a_{n-1}}, \quad n = 2, 3, \ldots$$

where $X = \left(\frac{4}{3}\right)^{3/4}$. Discuss the convergence of the sequence and find the value of the limit, if any.

AMM E2721. **by Allen Emerson**

Let $a_0, a_1 > 0$ and define a_n, $n \geq 2$, recursively by

$$a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}.$$

Show that (a_n) is convergent, and compute its limit.

NAvW 529. **by D. Furth**

For $\alpha \in \mathbb{R}$, $x_0 \in \mathbb{R}$, let $S(\alpha, x_0)$ denote the sequence $(x_n)_{n=0}^{\infty}$, defined by

$$x_{n+1} = (\alpha - x_n)^{-1}, \qquad (n \geq 0).$$

Show that, for every $k \geq 2$, there exists an $\alpha \in \mathbb{R}$ such that $S(\alpha, x_0)$ has period k for every x_0 (except for a finite number of values of x_0).

Sequences: tetration

TYCMJ 41. **by Harry Schor**

Let $b > 0$, $b_1 = b$, and $b_{k+1} = b^{b^{b_k}}$ for $k = 1, 2, \ldots$. Prove that there exists a number B, such that (b_n) converges if and only if $b \leq B$.

Sequences: trigonometry

AMM E2788. **by Kwang-Nan Chow and David Protas**

Let (u_n) be any sequence of real numbers such that $u_n \to \infty$ and $(\cos u_n)$ converges. Does there always exist a real number c such that $(\cos cu_n)$ diverges?

CRUX 80. **by Jacques Marion**

Does there exist a sequence of integers (a_n) such that $\lim_{n\to\infty} a_n = \infty$ and the sequence $\{\sin a_n x\}$ converges for all $x \in [0, 2\pi]$?

Series: arrays

FUNCT 1.5.2.

Let

$$S = X_{1,1} + X_{1,2} + \cdots + X_{1,n} + \cdots$$

be a convergent series with sum S. Construct an array where the entry $X_{i,j}$, for $i \geq 2$, and $j = 1, 2, \ldots$, in row i and column j is given by the formula $X_{i,j} = (X_{i,j-1} + X_{i-1,j})/2$. Also, $X_{i,0} = 0$ and $X_{0,j} = 0$, for $i, j = 1, 2, \ldots$. Show that each row in this array has sum S.

Series: binomial coefficients

AMM 6083. **by Emil Grosswald**

Prove that for real $p > 0$, the following identity holds:

$$\sum_{r=1}^{\infty} \frac{p}{r(p+r)} = \sum_{r=1}^{\infty} (-1)^{r-1} \binom{p}{r} \frac{1}{r} .$$

What is the function represented by both sides of this identity?

Series: closed form expressions

PUTNAM 1977/A.4.

For $0 < x < 1$, express

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

as a rational function of x.

FQ B-361. **by L. Carlitz**

Show that

$$\sum_{r,s=0}^{\infty} x^r y^s u^{\min(r,s)} v^{\max(r,s)}$$

is a rational function of x, y, u, and v when these four variables are less than one in absolute value.

Series: complex numbers

CRUX 40. **by Jacques Marion**

Let (a_n) be a sequence of nonzero complex numbers such that for some $r > 0$,

$$m \neq n \Longrightarrow |a_m - a_n| \geq r.$$

If $u_n = \frac{1}{|a_n|^\alpha}$, where $\alpha > 2$, show that the series $\sum_{n=1}^{\infty} u_n$ converges. What if $\alpha = 2$?

Series: continuous functions

AMM E2626. **by Richard Johnsonbaugh**

Is there a positive continuous function f on $[1, \infty)$ such that

$$\sum_{n=1}^{\infty} f(n) = \infty$$

but

$$\sum_{n=1}^{\infty} a^n f(a^n) < \infty$$

for all $a > 1$?

Series: cubes

AMM E2791. **by John W. Vogel**

If the series of real numbers $\sum_{n=1}^{\infty} a_n$ converges, does $\sum_{n=1}^{\infty} a_n^3$ converge?

Series: differentiable functions

MM 1060. **by Peter Ørno**

Prove or disprove: There exists a function f defined on $[-1, 1]$ with f'' continuous such that $\sum_{n=1}^{\infty} f(1/n)$ converges but $\sum_{n=1}^{\infty} |f(1/n)|$ diverges.

AMM 6112. **by Jan Mycielski**

Let $f(x)$ be a differentiable function such that $f(0) = 0$, $0 < f(x) < x$ for $x > 0$, and $f'(0) = 1$. Set $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for $n = 0, 1, \ldots$. Find conditions under which the series $\sum_{n=0}^{\infty} f^n(1)$ converges (diverges).

Series: divergent series

MM 938. **by S. C. Geller and W. C. Waterhouse**

Let $\sum a_n$ be an infinite series, and set $s_n = a_1 + a_2 + \cdots + a_n$. A familiar theorem of Abel says that if the a_n are positive and $\sum a_n$ diverges, then $\sum (a_n/s_n)$ also diverges. If we allow arbitrary signs, can we make $\sum a_n$ diverge to $+\infty$ while $\sum (a_n/s_n)$ converges?

AMM E2558. **by A. Torchinsky**

Suppose that $\sum a_n$ is a divergent series of positive terms, and let $s_n = a_1 + \cdots + a_n$ for $n = 1, 2, \ldots$. For which values of p does the series $\sum a_n/s_n^p$ converge?

MATYC 112. **by Gino Fala**

Prove: For all uncountable subsets $X \subset (0, +\infty)$, there exists a denumerable subset $A \subset X$, $A = \{a_1, a_2, a_3, \ldots\}$ such that $\sum_{i=1}^{\infty} a_i$ diverges.

Series: evaluations

AMM 6243. **by Emil Grosswald**

Find in closed form the sum S of the conditionally convergent series

$$\sum_{n=2}^{\infty}(-1)^n n^{-1}\log n.$$

CRUX 47. **by Jacques Sauvé**

For $a>1$, evaluate

$$\sum_{k=1}^{\infty}\frac{k^2}{a^k}.$$

Series: exponential function

FQ H-267. **by V. E. Hoggatt, Jr.**

Show that

$$S(x)=\sum_{n=0}^{\infty}\frac{(kn+1)^{n-1}x^n}{n!}$$

satisfies $S(x)=e^{xS^k(x)}$.

Series: hyperbolic functions

SIAM 76-2. **by Murray Geller**

Show that

$$\sum_{n=1}^{\infty}\left(\cosh^6 n\pi\sqrt{2}\,\right)^{-1}=$$

$$\frac{2\sqrt{2}}{15\pi}+\frac{\sqrt{2}}{15}A+\frac{\left(\sqrt{2}-1\right)}{12}A^2+\frac{\left(5-3\sqrt{2}\,\right)}{120}A^3-\frac{1}{2}\ ,$$

where

$$A=\frac{\sqrt{2}\Gamma^4(1/8)}{16\pi^2\Gamma^2(1/4)}\ .$$

SIAM 79-8. **by Chih-Bing Ling**

Show that, for $a>0$,

$$\sum_{n=0}^{\infty}\frac{1}{\cosh(2n+1)a}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sinh(2n+1)a}\ .$$

CRUX 448. **by G. Ramanaiah**

A function f is said to be an inverse point function if $f(k)=f(1/k)$ for all $k>0$. Show that the functions g and h defined below are inverse point functions:

$$g(k)=\frac{1}{k}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(1-\operatorname{sech}\lambda_n k)}{\lambda_n^3},$$

$$h(k)=\frac{1}{k^2}\sum_{n=1}^{\infty}\frac{\lambda_n k-\tanh\lambda_n k}{\lambda_n^5},$$

where $\lambda_n=(2n-1)\pi/2$.

Series: inequalities

MM 922. **by Alan Schwartz**

Let (x_n) be a sequence of nonnegative numbers satisfying

$$\sum_{n=0}^{\infty}x_n x_{n+k}\le Cx_k$$

for some constant C and $k=0,1,2,\ldots$. Prove that $\sum x_n$ converges. Is the result still true if $k=0,1,2,\ldots$ is replaced with $k=1,2,\ldots$?

Series: integrals

NAvW 406. **by P. J. de Doelder**

If

$$\operatorname{Ci}(x)=-\int_x^{\infty}t^{-1}\cos t\,dt,$$

then show that

$$\sum_{n=0}^{\infty}\operatorname{Ci}\left(\left(n+\tfrac{1}{2}\right)a\right)=$$

$$\begin{cases}\frac{1}{2}\log 2+\frac{1}{2}\sum_{s=1}^{k}(-1)^s s^{-1}, & \\ \qquad 2k\pi<a<2(k+1)\pi,\quad k\ge 0, & \\ \frac{1}{2}\log 2+\frac{1}{2}\sum_{s=1}^{k-1}(-1)^s s^{-1}+(-1)^k(4k)^{-1}, & \\ \qquad a=2k\pi,\quad k=1,2,\ldots\ . & \end{cases}$$

Series: iterated functions

DELTA 6.2-2. **by Jan Mycielski**

Considering $f^{(0)}(x)=x$ and $f^{(n+1)}(x)=f(f^{(n)}(x))$ for $n=0,1,\ldots$, prove the following:

(a) If $f(x)=\ln(1+x)$, then $\sum_0^\infty f^{(n)}(1)=\infty$.

(b) If $f(x)=\frac{x}{1+x}$, then $\sum_0^\infty f^{(n)}(1)=\infty$.

(c) If $f(x)=\frac{x}{1+\sqrt{x}}$, then $\sum_0^\infty f^{(n)}(1)<\infty$.

Series: iterated logarithms

MM 1032. **by R. P. Boas**

Let $l_1(x)=\log x$, $l_2(x)=\log\log x$, and $l_k(x)=\log l_{k-1}(x)$. Let $N(k)$ be the first integer n such that $l_k(n)>1$. When k is fixed, the integral test shows that the series

$$\sum_{n=N(k)}^{\infty}\frac{1}{nl_1(n)l_2(n)\cdots(l_k(n))^p}$$

diverges for $p=1$ and converges for $p>1$. It is known that this equation is very slowly divergent if $p=1$ and k (the number of logarithmic factors in the equation) is no longer fixed but depends on n, being taken as large as possible so that all the logarithms exceed 1, i.e., so that $l_k(n)>1$ but $l_{k+1}(n)<1$. With this choice of $k=k(n)$, how large can $p=p(k)$ be before the series becomes convergent? Will $p=2$ or $p=k$ suffice?

Analysis

Series: monotone sequences

CMB P265. **by P. Erdős**

Let $0 < a_1 < a_2 < \cdots, \sum \frac{1}{a_n} < \infty$. Show that

$$\sum \left| \frac{n}{a_n} - \frac{n+1}{a_{n+1}} \right| < \infty.$$

Series: pairs of sequences

CRUX 209. **by L. F. Meyers**

Suppose that the sequence $(a_n)_{n=1}^{\infty}$ of nonnegative real numbers converges to 0. Show that there exists a sequence $(e_n)_{n=1}^{\infty}$ each of whose terms is 1 or -1 such that

$$\sum_{n=1}^{\infty} e_n a_n$$

converges.

AMM E2591. **by Jan Mycielski**

Prove that for every sequence $a_1, a_2, \ldots$ with $\lim a_n = 0$, there exists a sequence $b_1, b_2, \ldots$ with $b_1 \geq b_2 \geq \cdots \geq 0$ such that $\sum b_n$ diverges and $\sum a_n b_n$ converges absolutely.

Series: pairs of series

SPECT 8.3.

The real series $\sum a_n$, $\sum b_n$ are such that $\sum a_n$ is convergent, no a_n is zero, and $b_n/a_n \to 1$ as $n \to \infty$. Does the series $\sum b_n$ have to be convergent?

Series: tail series

JRM 602. **by Travis Fletcher**

The sequence $a_1, a_2, \ldots$ satisfies the equation

$$a_n = \sum_{k=n+1}^{\infty} a_k$$

for each n. Find a_9.

Sets

NAvW 558. **by I. J. Schoenberg**

Let S be the set of all $(x_1, x_2) \in \mathbb{R}^2$ such that x_1, x_2, and 1 are arithmetically independent. Let $\tilde{S}$ be the set of all $X = (x_1, x_2)$ such that $|X - A| \neq |X - B|$ if $A, B \in \mathbb{Z}^2$ and $A \neq B$. Prove:

(a) The set $\mathbb{R}^2 \backslash \tilde{S}$ has measure zero.

(b) $S \subset \tilde{S}$.

(c) The set $\tilde{S}$ contains no continuous arc.

Weierstrass zeta function

SIAM 78-5. **by Chih-Bing Ling**

Show that

$$\zeta\left(\frac{1}{2} \,|\, 1, i\right) = \frac{\pi}{2},$$

$$\zeta\left(\frac{1}{2} \,|\, 1, e^{\pi i/3}\right) = \frac{\pi}{\sqrt{3}},$$

$$\zeta\left(\frac{1}{2} \,|\, 1, \frac{e^{\pi i/6}}{\sqrt{3}}\right) = \pi\sqrt{3},$$

where $\zeta(z \,|\, 2\omega_1, 2\omega_2)$ is a Weierstrass zeta function of z with double pseudo-periods $2\omega_1$ and $2\omega_2$.

Acoustics

FUNCT 1.3.6. **by Andrew Fortune**

If a record is played at $33\,1/3$ rpm, and three musical notes are heard, namely middle C, E, and G, what will the three notes be if

(a) the same record is played at 45 rpm?
(b) it is played at 78 rpm?

Astronomy

NYSMTJ 50.

An astronaut is in circular orbit around a spherical planet. If the radius of the planet is r miles, and the altitude of the orbit is a miles, express in terms of r and a the fraction of the total surface area of the planet that the astronaut can see during one complete orbit.

Demographics

MSJ 436. **by Steven R. Conrad**

What was the population of the United States when a man, by going from New York City to San Francisco, a distance of 3100 miles, would shift the center of population of the United States $1\,1/4$ inches?

Electrical networks

JRM 529. **by D. C. Morley**

Current is made to flow between two opposite vertices of a tesseract, each of whose 32 edges is a 1-ohm resistor. What is the resistance across the tesseract?

SIAM 79-16.* **by D. Singmaster**

Determine the resistances $R(n, i)$ between two nodes a distance i apart in an n-cubical network if all of the edges are of unit resistance.

AMM E2620. **by Albert Mullin and Derek Zave**

Let Γ be the graph consisting of the vertices and edges of one of the five regular polyhedra. Suppose all edges of Γ are one-ohm resistors. Compute the resistance between any two of the most remote vertices of Γ.

Answer the same question when Γ is the graph of the n-dimensional cube.

FUNCT 3.4.2.

Many hallways have light switches at either end, allowing the light to be operated from each. How can the wiring be arranged to achieve this?

Engineering

ISMJ 11.1.

Suppose that a brick will support the weight of 999 bricks but will be crushed by the weight of 1000 bricks. We will build a tower whose top is a column one brick wide and $K_1 (= 999)$ bricks high. Supporting it is a column two bricks wide. Supposing that the weight of the 1-brick column is evenly distributed, the top bricks of the 2-brick column will not be crushed. The 2-brick column has K_2 courses, the largest possible, so the bottom bricks will not be crushed. Then we start a 3-brick column and make it as long as possible, etc. Show that for any j, K_j is within one whole number of $1000/j$.

Geography

FUNCT 2.3.2.

What point on the earth's surface is farthest from the earth's center?

Meteorology

MM 1056. **by Daniel A. Moran**

"Oh, drat!" exclaimed the meteorologist stormily. "I've just anchored my new rain gauge onto a cement post, and it seems to be crooked."

"What does your rain gauge look like?" asked his friend, the math student.

"It's in the shape of a circular cylinder 8 centimeters in diameter with height-markings all around its sides. Its axis is only 3 degrees off-vertical, but this will affect the amount of rain entering the top, and besides, which height-marking should I use? The water level will look tilted. I'm very discouraged about this whole business."

"Do you have any interest in measuring extremely light rains?" asked his friend.

"Not really. Anything less than a half-centimeter is too hard to measure accurately anyway, so I just record it as being a 'trace of precipitation'."

"I think I can help you," said the math student.

Tell the meteorologist how to correct the readings on his crooked rain gauge.

Navigation

JRM 375. **by R. Robinson Rowe**

In World War II a destroyer miraculously survived a straddling salvo of three near misses — two fore and aft to port and the third amidship to starboard, twisting its keel and hull so that it veered to starboard, even with full left rudder. Its identity being still classified, it became known as the USS Sidewinder.

When its skipper determined that with full left rudder it circled to starboard on a long radius R, or that with full right rudder it circled to starboard on a short radius r, he computed the quickest way to reach a repair base due north. With full left rudder he sailed until he was headed NE, then switched to full right rudder, turning through a 270° loop until he was headed NW, then switched back to full left rudder. Repeating such cycles, he cruised the Sidewinder along a trochoid-like sidewinding path to safety.

Now, if r was 1 mile, what was R? And, relatively, how much longer was this path than the beeline distance to the repair base?

JRM 478. **by Ray Lipman**

A swimmer is suddenly blanketed by fog in a river with straight banks. Devise a program that will teach itself a swimming procedure that assures the swimmer of attaining a bank within some fixed time specified as an input parameter.

Operations research

SIAM 76-7.* **by R. D. Spinetto**

Suppose a company wants to locate k service centers that will service n communities and suppose that the company wants to locate these k centers in k of the communities so that the total population distance traveled by the people in the $n-k$ communities without service centers to those communities with service centers is minimized. This problem can be set up as a 0-1 integer programming problem as follows. Let

$$x_{jj} = \begin{cases} 1, & \text{if community } j \text{ gets a service center,} \\ 0, & \text{otherwise,} \end{cases}$$

and let

$$x_{ij} = \begin{cases} 1, & \text{if community } i \text{ is to be serviced by a center} \\ & \text{in community } j, \\ 0, & \text{otherwise.} \end{cases}$$

Let p_i be the population of community i and let d_{ij} be the distance from community i to community j. The problem then is to minimize

$$\sum_{i=1}^{n}\sum_{j=1}^{n} p_i d_{ij} x_{ij},$$

subject to constraints

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, 3, \ldots, n;$$

$$x_{ij} - x_{jj} \le 0 \qquad \text{for } i = 1, 2, 3, \ldots, n, \text{ and for } j = 1, 2, 3, \ldots, n;$$

$$\sum_{j=1}^{n} x_{jj} = k,$$

and with the added condition that each of the variables x_{ii} and x_{ij} takes on only the values of 0 or 1.

(a) What are the smallest n and k for which there exists a linear programming problem of the above form which will have only non-0-1 optimal extreme point solutions?

(b) Can the non-0-1 extreme points of polyhedrons determined by the constraints shown above be characterized in any set theoretic way that would be useful in developing more efficient algorithms for solving this facility location problem?

Optics

CRUX 291. **by Gilbert W. Kessler**

Using soap, on a mirror, please trace
The apparent outline of your face;
Now explain (if you're wise)
Why it turns out "half size",
Using geometry as your base.

MENEMUI 1.3.2. **by S. L. Lee**

A certain diagram shows a cross section of a symmetrical trough with side mirrors whose base is of a fixed length a. If all the light hitting the mirror is concentrated on the base of the trough, we shall say that the system has a concentration factor of x/a. Find the minimum value of l so that the system has a concentration factor of 4. Find also the angle of inclination of the mirror for this value of l.

PARAB 304.

Prove that a ray of light, having been reflected from three mutually perpendicular mirrors in turn, becomes parallel to its original direction but in the opposite sense.

CRUX 289. **by L. F. Meyers**

Let L be a straight line, and let A and B be points not on L. Let the speed of light on the side of L on which A lies be c_1, and let the speed of light on the other side of L be c_2. Characterize the points C on L for which the time taken for the route ACB is smallest, if

(a) A and B are on the same side of L, (reflection);

(b) A and B are on opposite sides of L (refraction).

Physics: cars

FUNCT 1.3.2.

A road sign shows a car with skid marks behind. The skid marks are "S" shaped but cross each other. How could a car make the skid marks as indicated on the sign?

Physics: center of gravity

NYSMTJ 53. **by Walter van B. Roberts**
ISMJ 10.15.

Assume the center of gravity of a can full of beer is at the center of the can. As the beer is consumed, the center of gravity of the can and remaining contents begin to drop; but by the time the can is empty, the center of gravity has returned to its original position. When does the center of gravity reach its lowest point?

Physics: equilibrium

CRUX 424. **by J. Walter Lynch**

Is it possible to make a convex object out of homogeneous material that will be at rest in exactly one position?

JRM 541. **by Horace W. Hinkle**

After the King had spitefully cut off his daughter Rapunzel's hair, her lover braided it into a rope, spliced one end to form an eyeloop, drew the other end through to form a lariat, and lassoed the conical spire of Rapunzel's tower, which offered no friction to the lariat and was just steep enough to prevent the rope from rolling or slipping. The lariat found an equilibrium and supported his weight while he climbed to Rapunzel's window. How steep was the roof?

Physics: falling bodies

FUNCT 2.4.4.

From the roof of a 300-meter building in New York, two marbles are dropped, one being released when the other has already fallen 1 mm. How far apart will they be when the first hits the ground?

Physics: fluids

MM 971. **by Sidney Kravitz**

In designing pipes and other conduits it is usually desirable to enclose the maximum cross-sectional area for a given weight of pipe. Mathematically, this may be simplified by enclosing the maximum area for a given perimeter.

Dual ducts are often used to convey fluids in two directions. They have a portion of their perimeter in common. For example, two equal squares, each of side s are placed to share a common side. The total perimeter is $7s$ and the total cross-sectional area is $2s^2$. Thus, the ratio of the area to the square of the perimeter is $2/49$. Assume equal cross-sectional area of the two ducts.

(a) Which regular polygon is the most efficient for use as a dual duct?

(b) Which contour is the most efficient for use as a dual duct?

Physics: force fields

NAvW 393. **by O. Bottema**

In a 4-dimensional Euclidian space with orthogonal coordinate system $OX_1X_2X_3X_4$ a force field is given such that the force $\overline{F}$ per unit mass depends on the velocity $\overline{v}$ as follows:

$$\overline{F} = A\overline{v},$$

where $\overline{F}$ is the row matrix of the force components and A is the matrix

$$\begin{pmatrix} 0 & p & q & r \\ -p & 0 & -r & q \\ -q & r & 0 & -p \\ -r & -q & p & 0 \end{pmatrix}$$

for some constants p, q and r.

Determine the motion of a mass point released at a given initial point with a given initial velocity.

NAvW 403. **by O. Bottema**

In a plane with an orthogonal coordinate system OXY, a force field is given. The X- and Y-components of the field strength at point (x, y) are

$$F_x = p^2 y, \qquad F_y = q^2 x,$$

where p and q are positive constants. A mass point P is to be released with initial velocity zero; for which release points A will the the curve of P have an inflection point at A?

Physics: gravity

SIAM 78-17. **by J. S. Lew**

It is well known that if a uniform thin flexible cord is suspended freely from its endpoints in a uniform gravitational field, then the shape of the cord will be an arc of a catenary. Determine the shape of the cord if we use a very long one which requires the replacement of the uniform gravitational field approximation by the inverse square field.

Physics: particles

JRM 564. **by Sherry Nolan**

Three perfectly elastic balls A, B, and C, considered as equal point-masses, are moving at constant velocities along the x-axis. At $t = 0$, A is at $x = 0$, C is at $x = 1$, and B is somewhere between them. At $t = 1$, A and B collide. At $t = 2$, a second collision occurs and at $t = 3$ a third. Where was B at $t = 0$?

NAvW 461. **by O. Bottema**

A particle P of mass m moves on the surface of a sphere (center O, radius R) under influence of a force $F = mk\overline{AP}$; A is a given point in space ($OA = d \neq 0$) and k is a constant unequal to zero. Determine the motion of P.

NAvW 437. **by O. Bottema**

The force on a unit of mass at the point A of a plane field is directed towards the center O and equal to cr^n, where $OA = r$ and c and n are constants. Two mass points P_1 and P_2 move, in the same direction, on different circles with center O. Has the motion of P_2, as seen from P_1, a permanent direction?

Physics: projectiles

AMM E2535. **by M. S. Klamkin**

A body is projected in a uniform gravitational field and is subject to a resistance that is a function of its speed $|v|$. If the acceleration a of the body always has a constant direction, no matter what the initial velocity v_0, show that

$$a = a_0 e^{-kt}$$

for some constant k.

CRUX 348. **by Gilbert W. Kessler**

I launched a missile, airward bound;
Velocity — the speed of sound;
Its angle-30. Can you tell
How far from here that missile fell?

PARAB 295. **by J. Scott**

A man is able to throw a cricket ball 30 meters vertically upwards. What is the furthest distance he can throw it horizontally? (Ignore any air resistance.)

SPECT 7.1.

Two projectiles are fired from a point O at the same time. Describe how the direction and length of the straight line joining the projectiles vary with time during the subsequent flight. (Air resistance can be neglected.)

SPECT 7.5.

Two men stand on the edges of two cliffs, the heights of the cliffs above sea level being the same. The cliffs are separated by a deep chasm. The men point loaded pistols directly at each other (the pistols may not be of the same make), and each fires at the same moment. Show that the bullets collide.

SPECT 8.2.

A projectile is fired upwards from a cliff 45 meters high at an angle of 45° to the horizontal, and lands in the sea at a distance 360 meters from the foot of the cliff. The operation is then repeated, but this time a wind of speed 2 meters/sec is blowing on shore. How does this affect the range of the projectile? With the wind blowing, could the range of the projectile be increased by altering the angle of inclination? (You may take the acceleration due to gravity to be 10 meters/sec^2.)

Physics: rods

NAvW 450. **by O. Bottema**

A train travels from A to B along a horizontal straight track OX. On the floor of one of the cars, a rod PQ is pivoted so that it may move about the fixed point P on the floor in the vertical plane OXY. The rod is under the influence of gravity. The motion of the train is arbitrary but known beforehand. It is known that there exists at least one initial position of the rod ($\angle QPX = \alpha$) such that the rod will not fall to the floor during the entire journey. Suppose that the train is initially at rest at A; it gets an impact such that it leaves A with velocity V; it moves uniformly to B. Determine α such that the rod does not fall to the floor for any position of B.

The mass center of the rod is G, $PG = \ell$, its mass is m, its moment of inertia with respect to P is $m\rho^2$, the acceleration of gravity is g.

Physics: rolling objects

FUNCT 1.5.1. **by Elijah Glenn Merlo**

If you are given a hoop, a disc, and a sphere, each of uniform density and each of radius r units, and you roll them simultaneously down the slope of steepest descent of an inclined plane, which ones arrive first and last at the inclined plane's foot?

FUNCT 1.4.1. **by Alisdair McAndrew**

Imagine a circle rolling, without slipping, on a flat surface. At the same time, a plank rolls (without slipping) along the top of the circle. What is the ratio of the speed of the plank to the speed of the center of the circle?

Physics: solid geometry

NAvW 468. **by O. Bottema**

On a Cartesian frame $OXYZ$, with OZ vertical and upward, the paraboloid S has the equation

$$x^2 + y^2 = 2pz, \qquad p > 0.$$

A particle P moves under gravity (with acceleration g) on the smooth inner surface of S.

(a) Show that P moves between two parallel circles C_i on S, given by $z = z_1$ and $z = z_2$.

If P is on C_i, let its angular velocity about OZ be ω_i ($i = 1, 2$).

(b) Show that $\omega_1 z_1 = \omega_2 z_2$ and $\omega_1\omega_2 = g/p$.

Physics: systems of differential equations

SIAM 79-7. **by O. Hájek**

The controlled harmonic oscillator $\mathring{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, has the curious property that it is controllable for every real vector $\mathbf{b} \neq \mathbf{0}$. Determine which real square matrices A have this "super-controllability" property.

Physics: temperature

AMM S11. **by R. C. Buck and E. F. Buck**

A solid tetrahedron carries a continuous temperature distribution. What is the maximum number of points having the same temperature one can be sure of finding on the edges of the tetrahedron?

Physics: tunnels

PME 343. **by R. Robinson Rowe**

There is some interest in a fall-through tunnel under the Bering Strait. From Cape Prince of Wales on Alaska's Seward Peninsula to Mys Dezhneva (East Cape) on Siberia's Chukchi Peninsula is 51 miles. A straight tunnel 58 miles long could be driven in earth below the bed of the Strait, which is 20 fathoms deep near each shore and 24 fathoms near mid-Strait. A frictionless vehicle could "fall" through such a tunnel without motive power. How long would it take? (At latitude 66° North, the earth's radius is 3954 miles and the acceleration of gravity, $g = 32.23$ ft/sec^2.)

Algorithms

JRM 513. **by P. J. Flores**

As their boat sinks, N crewmen are arranged in a circle and counted off by k's until only one remains, who is given the sole lifejacket. As each man is eliminated, the circle is closed up and the count is resumed. Devise an algorithm that produces the position $S(k, N)$ of the survivor.

The diagonal sequence $S(N, N)$ begins 1, 1, 2, 2, 2, 4, 5. Devise an algorithm to determine the smallest N such that $S(N, N) = r$, for any given r.

SIAM 79-17.* **by W. R. Utz**

Determine an algorithm, better than complete enumeration, for the following problem: Given a nonnegative integer matrix, permute the entries in each column independently so as to minimize the largest row sum.

JRM C8. **by Marshall Willheit**

What is the least number of BASIC commands that will print out every possible way of changing a dollar?

Arrays: 0-1 matrices

AMM E2794.* **by Robert A. Leslie**

Let m, n, r, and c be positive integers with $rm = cn$. How many $m \times n$ matrices are there with each entry either 0 or 1 and where every row sum is r and every column sum is c?

Arrays: binary arrays

FUNCT 2.3.1.

Two students keep a calendar of the weather as follows: Days with good weather are marked $+$, while days with bad weather are marked $-$. Each student makes three observations daily, at the same times. The first student writes $-$ if it rains at the time of any of these observations, but otherwise he writes $+$. The second student writes $+$ if the weather is fair at any of these times and $-$ otherwise. Thus it would seem that the weather on any given day might be described as $++$, $+-$, $-+$, or $--$ (the first symbol made by the first student, the second symbol by the second student). Are these four cases all actually possible?

SSM 3676. **by Charles W. Trigg**

In the following square array, select 10 elements (two from each row and each column) so as to include a nonadjacent a and b in each column and row.

$$\begin{matrix} a & b & a & b & b \\ a & b & b & a & a \\ a & b & b & a & a \\ b & a & a & b & b \\ b & a & b & a & a \end{matrix}$$

How many distinct solutions are there?

Arrays: circular arrays

OSSMB 76-14.

Let any number of 0's and 1's be arranged around a circle in any order. Let a concentric copy of this arrangement be spun around on top of it through any number of positions to pair off the numbers one above the other. In some pairs a 0 faces a 0, or a 1 faces a 1. No matter what the situation, however, prove that the number of pairs in which a 0 faces a 1 is even.

FUNCT 2.2.3.

Fifty knights of King Arthur sit at a round table. Each has a goblet of red or white wine in front of him. At midnight, each passes his goblet to his right-hand neighbor if he has red wine, to his left-hand neighbor if he has white wine. Assuming that both red and white wine were at the table, prove that someone at the table will be left without wine after midnight. Is the conclusion still true if the King was also at the table?

Arrays: distinct rows

KURSCHAK 1979/3.

Letters are arranged in an $n \times n$ array so that no two rows of the array are identical. Prove that it is possible to delete one of the columns of the array so that the remaining rows will remain distinct.

Arrays: inequalities

OSSMB 77-6.

Let A denote an $m \times n$ matrix of distinct real numbers. Prove that there exists a real number x such that either each row of A contains a pair of elements that straddle x or each column contains a pair of elements that straddle x.

Arrays: Latin rectangles

AMM E2577. **by F. W. Light, Jr.**

Given the $2 \times n$ Latin rectangle

$$\begin{matrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1, \end{matrix}$$

find the number of ways $f(k)$ in which a $3 \times n$ Latin rectangle can be built up from it by adding a third row starting with k, where k is one of the numbers $3, 4, \ldots, n$.

Arrays: maxima and minima

CRUX 2. **by Léo Sauvé**

A rectangular array of m rows and n columns contains mn distinct real numbers. For $i = 1, 2, \ldots, m$, let s_i denote the smallest number of the ith row; and for $j = 1, 2, \ldots, n$, let l_j denote the largest number of the jth column. Let $A = \max(s_i)$ and $B = \min(l_j)$. Compare A and B.

MM 1061. **by Edward T. H. Wang**

In how many ways can n^2 distinct real numbers be arranged into an $n \times n$ array (a_{ij}) such that

$$\max_j \min_i a_{ij} = \min_i \max_j a_{ij}?$$

OMG 18.1.3.

The positive integers $1, 2, 3, \ldots, 25$ have been arranged very carefully into the table below:

11	17	25	19	16
24	10	13	15	3
12	5	14	2	18
23	4	1	8	22
6	20	7	21	9

In 120 different ways a set of 5 numbers from this table can be chosen so that a number is taken from each row and from each column. In each set of 5 there occurs a minimum number. Find the largest number which occurs as one of these minima.

PARAB 263.

Three hundred soldiers are positioned in 15 rows, each containing 20 soldiers. From each of the 20 columns thus formed, the shortest soldier falls out and the tallest of these 20 men proves to be Private Jones. They then resume their places on the parade ground. Next the tallest soldier in each row falls out, and the shortest of these 15 soldiers is Private Smith. Who is taller, Jones or Smith?

SIAM 75-2. **by G. J. Simmons**

Is it possible to form a marching column of two's with $n-1$ members from each of n regiments in such a way that every regiment is paired with every other regiment and no two members of the same regiment have fewer than the obvious maximum-minimum of $\lfloor (n-3)/2 \rfloor$ ranks separating them?

Arrays: symmetric arrays

AMM E2717.* **by E. Ehrhart**

Find the number of symmetric 4×4 matrices whose entries are all the integers from 1 to 10 and whose row sums are all equal.

Arrays: transformations

PARAB 311.

In a classroom, there are 25 seats in a square array, each occupied by a pupil. Each pupil moves to an adjacent seat to his right, left, front or rear, or stays in his seat. Prove that at least one pupil must in fact have stayed in his seat.

PARAB 326.

Suppose that mn boys are standing in a rectangular formation of m rows and n columns. Suppose that the boys in each row get shorter going from left to right. Suppose someone rearranges each column, independently of one another, so that going from front to back the boys get shorter. Show that the boys in each row still get shorter going left to right.

Arrays: triangular arrays

AMM E2541. **by E. T. H. Wang**

A Steinhaus triangle is formed as follows: Start with a row of n plus and minus signs. Under each pair of like signs, a plus sign is written and under each pair of unlike signs, a minus sign is written. Continuing, one finally obtains a triangle of $n(n+1)/2$ plus and minus signs.

Prove that if the first row pattern of a Steinhaus triangle is $--+--+\cdots$ (i.e., two minuses followed by a plus), then the same pattern repeats itself when one traverses all the entries in a clockwise spiral fashion.

Card shuffles

CRUX PS5-1.

A pack of 13 distinct cards is shuffled in some particular manner and then repeatedly in exactly the same manner. What is the maximum number of shuffles required for the cards to return to their original position?

OSSMB 77-14.

A perfect shuffle of a deck of $2n$ cards, ordered as 1, 2, $3, \ldots, 2n$, yields the order 1, $n+1$, 2, $n+2$, $3, \ldots, 2n-2$, $2n$. How many perfect shuffles will restore the deck to its original order?

PARAB 327.

If a pack of playing cards is shuffled systematically and the operation of shuffling repeated exactly, then after a certain number of repetitions of the operation, the original order of the pack will be restored. Suppose the pack is shuffled as follows: Hold the pack face down in the left hand; in the right hand, take the top half of the pack and insert it into the lower half so that each right-hand card is above the corresponding left-hand card.

(a) After how many shuffles is a 52-card pack returned to order?

(b) After how many shuffles is a 26-card pack returned to order?

PARAB 343.

We define a "shuffle" of a deck of N cards numbered $1, 2, \ldots, N$ to be a specific procedure for arranging them in a different order. If one systematically repeats the same shuffle of the deck enough times, it returns to its original order. What shuffle of a deck of 28 cards requires the largest number of repetitions before returning to the original order?

SPECT 11.6. **by A. K. Austin**

A number of cards are dealt into m not necessarily equal piles. They are then collected together and redealt into $m+k$ piles, where $k>0$. Show that there are at least $k+1$ cards that are in smaller piles in the second dealing than in the first.

FUNCT 2.1.1.

We have a pack of cards, an even number c of them. By a "shuffle" we shall mean that we divide the pack into a top half and a bottom half, then put the pack back together again by alternately taking one card from each half, starting with the bottom half. How many shuffles does it take for the cards to return to their original position?

Cards

TYCMJ 89. **by Warren Page**

Let n be a positive integer. Mark any one card in a deck of $3n$ playing cards. Deal the cards to the positions in an $n \times 3$ array proceeding across the first row from left to right, then similarly across the second row, and so on until the nth row of cards has been dealt. Assemble the n cards in each column into a vertical stack such that the top to bottom order in the stack corresponds to the top to bottom order in the column. Combine the stacks by sandwiching the stack containing the marked card between the other two stacks. Repeat this dealing and stacking procedure twice more. For which values of n is the final position of the marked card independent of its initial position?

PARAB 427.

The four aces, kings, queens, and jacks are taken from a pack of cards and dealt to four players. Thereupon, the bank pays \$1 for every jack held, \$3 for every queen, \$5 for every king, and \$7 for every ace. In how many ways can it happen that all four players receive equal payments (namely \$16)?

Coloring problems: arcs

FQ B-415. **by V. E. Hoggatt, Jr.**

The circumference of a circle is partitioned into n arcs of equal length. In how many ways can one color these arcs if each arc must be red, white, or blue? Colorings that can be rotated into one another should be considered to be the same.

Coloring problems: concyclic points

CANADA 1976/8.

Each of the 36 line segments joining 9 distinct points on a circle is colored either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the 6 segments connecting them are all red.

Coloring problems: graphs

AMM 6157.* **by C. C. Chen and D. E. Daykin**

(a) Find integers Δ and p with the following property: Whenever the lines of the complete graph K_p are colored so that every vertex is on not more than Δ lines of each color, there is a triangle whose lines have different colors.

(b) Find integers δ, p, and n with the following property: Whenever the lines of a complete graph K_p are colored with n colors so that every vertex is on at least δ lines of each color, there is a triangle whose lines have different colors.

Coloring problems: hexagons

TYCMJ 42. **by Bernard Eisenberg**

Each pair of vertices of a convex hexagon is connected with a straight line segment that is either blue or red. Among the 20 triangles, each of which is determined by three vertices, prove that at least two of the triangles consist entirely of blue segments, two consist entirely of red segments, or one triangle consists of blue segments and one consists of red segments.

Coloring problems: pennies

AMM E2527. **by F. D. Hammer**

(a) A finite number of pennies are placed flat in the plane. Prove that these (nonoverlapping) pennies can be painted with at most four colors so that touching pennies bear different colors.

(b) Prove the same result for an infinite collection of pennies in the plane.

(c) What is the minimum number of pennies that requires four colors?

AMM E2651. **by P. Erdős**
PARAB 387.

A finite number of pennies are placed flat on the plane so that no two overlap and no three touch each other. Prove that these pennies can be painted with at most three colors so that touching pennies bear different colors.

AMM E2745. **by David Hammer**

Can every collection of nonoverlapping pennies in the plane be colored with three colors so that no penny touches more than one penny with the same color?

Coloring problems: pentagons

ISMJ 11.13.

Let $ABCDE$ be a convex pentagon. In how many ways is it possible to color the edges and diagonals red or blue so that no triangle determined by three vertices has all its sides the same color?

Coloring problems: points in plane

PUTNAM 1979/A.4.

Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are colored red and the remaining n blue. Prove or disprove: there are n closed straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

Coloring problems: sets

TYCMJ 113. **by Sidney Penner**

Let S be a set of $n(n+1)/2$ elements and let $k = \lfloor n(n+1)/6 \rfloor$. Assume that k of the elements of S are colored red, k are colored white and k are colored blue, with one remaining element (if there is one) colored red. Show that, for $n > 3$, it is possible to partition S into n subsets T_m ($m = 1, 2, \ldots, n$) such that for each m,

(a) T_m has m elements, and

(b) the elements of T_m are all the same color.

Coloring problems: tournaments

SIAM 78-11. **by N. Megiddo**

We define an edge k-coloring of a tournament (i.e., a directed graph with a unique edge between every pair of vertices) to be that of coloring the edges in k colors such that every directed cycle of length n contains at least $\min(k, n)$ edges of distinct colors. Does every tournament have a three-coloring?

Coloring problems: triangles

PARAB 362.

(a) In the morning, a working man leaves his cat in the house. The house has one door which has been left open. When the man returns in the evening, the cat is outside. Prove that the cat crossed the threshold an odd number of times.

(b) A triangle ABC is the union of a finite family, F, of triangles. If two different triangles in F intersect, they intersect in a vertex of both or an edge of both. Color each of the vertices of the triangles in F red, blue, or yellow. Color A red, B blue, and C yellow. If a vertex V lies on AB, color it red or blue; if V lies on BC, color it blue or yellow; and if V lies on CA, color it red or yellow. Prove that the number of triangles in F which have one red, one blue, and one yellow vertex is odd.

Compositions

PARAB 408.

The number 3 can be expressed as the sum of one or more positive integers in 4 ways: 3, 2+1, 1+2, and 1+1+1. Note that the ordering of the summands is significant; $1+2$ is counted as well as $2+1$. Find a formula for the number of ways in which an arbitrary positive integer n can be so expressed as a sum of positive integers.

AMM S20. **by A. P. Hillman**

Let n be a nonnegative integer, and let S consist of all ordered quintuples $Q = (x_1, x_2, x_3, x_4, x_5)$ of nonnegative integers x_i with $x_1 + x_2 + x_3 + x_4 + x_5 = n$. Prove or disprove that there are exactly the same number of Q in S with $x_2 \leq x_3 \leq x_4 \leq x_5$ as there are satisfying the simultaneous conditions

$$x_1 \leq x_2 \leq x_4,$$
$$x_1 \leq x_3 \leq x_4,$$
$$x_3 \leq x_5.$$

MM 1026. **by Michael Capobianco**

A decomposition of a positive integer n is an ordered tuple $(n_1, n_2, \ldots, n_k)$ of positive integers such that $\sum_{i=1}^{k} n_i = n$. Find the total number of decompositions of n that are palindromes.

Configurations: chains

PARAB 267.

A chain has 2047 links in it. It is to be separated into a number of pieces by cutting and disengaging appropriate links, in such a way that any number of links (from 1 to 2047) may be gathered together from the parts of chain thus produced. What is the smallest number of links which must be cut to achieve this?

Configurations: circular arrays

JRM 729. **by Frank Rubin**

A blind man keeps his keys on a circular key ring. There are s distinct handle shapes that he can tell apart by feel, and he can purchase any key with any desired handle shape. Assume that all keys are symmetrical so that a rotation of the key ring about an axis in its plane is undetectable from examination of a single key. How many keys can he keep on the ring and still be able to select the proper key by feel?

PARAB 266.

At the mad hatter's afternoon tea party, there are twenty seats numbered consecutively clockwise around a circular table with 4 neighboring ones with red cushions (1, 2, 3, and 4) being initially occupied by Alice, the mad hatter, the march hare, and the dormouse respectively. Instead of all moving round one seat at a time (as in the classical story), the members of the party move quite independently as the fancy takes them, but always to an unoccupied seat 7 places away in either direction. Even the dormouse proves to be wakeful enough to carry out this complicated maneuver several times.

At a later time, it turns out that they are again sitting next to one another on the same red-upholstered chairs (1, 2, 3, and 4), though none is in the same place as initially. How many possible seating arrangements are there at the finish and what are they?

PARAB 406.

Given n beads numbered $1, 2, 3, \ldots, n$, show how you can make a single-strand closed necklace from them with the property that the numbers on adjacent beads always differ by either 1 or 2.

CANADA 1975/6.
OSSMB 77-7.

(a) Fifteen chairs are equally placed around a circular table on which are name cards for 15 guests. The guests fail to notice these cards until after they have sat down and it turns out that no one is sitting in front of his own card. Prove that the table can be rotated so that at least two of the guests are simultaneously correctly seated.

(b) Give an example of an arrangement in which just one of the 15 guests is correctly seated and for which no rotation correctly places more than one person.

Configurations: committees

USA 1979/5.

A certain organization has n members, and it has $n+1$ three-member committees, no two of which have identical membership. Prove that there are two committees which share exactly one member.

JRM C4. **by David L. Silverman**

A certain corporation issues shares only in integer amounts, and every shareholder is a director. On every "yea-nay" question that comes before the Board, each director's vote is weighted according to the number of shares he holds. Among the corporate bylaws are two that govern the various numbers of shares held by the directors.

(1) No tie vote must be possible (unless all directors abstain).

(2) No group of directors must be capable of being outvoted by a smaller group.

Given N directors, let $S(N)$ represent the minimum total number of shares consistent with bylaws (1) and (2). Listed below are the values for S for $N = 1$ through 5, together with the share allocations that result in these values of S:

N	$S(N)$	*Share Allocation*
1	1	1
2	3	$1, 2$
3	9	$2, 3, 4$
4	21	$3, 5, 6, 7$
5	51	$6, 9, 11, 12, 13$

The allocations are unique, though they may not be so for larger values of N. By the time one gets to the case $N = 6$, however, one is likely to find pencil and paper analysis formidable. Write a program that will list $S(N)$ for N up to 10 as well as all share allocations that total $S(N)$ without violating either of the two bylaws.

SIAM 78-9.* **by W. Aiello and T. V. Narayana**

Suppose we assign positive integer weights $x_1, \ldots, x_n$ to the vote of each member of a board of directors that consists of n members so that the following conditions apply:

(1) Different subsets of the board always have different total weights so that there are no ties in voting (tie-avoiding).

(2) Any subset of size k will always have more weight than any subset of size $k-1$ ($k = 1, \ldots, n$) so that any majority carries the vote, abstentions allowed (nondistorting).

Find a solution $(x_1, \ldots, x_n)$ such that no other solution $(y_1, \ldots, y_n)$ exists with $x_i \geq y_i$ for $i = 1, \ldots, n$.

Configurations: concyclic points

CRUX 354. **by Sidney Penner**
NYSMTJ 81. **by Sidney Penner**

Along a circular road there are n identical parked automobiles. The total amount of gas in all of the vehicles is enough for only one of them to travel the whole circular road. Prove that at least one of these cars could travel the entire road, taking on gas along the way from the other $n-1$ vehicles.

Configurations: couples

OMG 18.2.1.

At a dance there are 50 men and 38 women. How many different couples could appear on the dance floor? How many couples could appear if there are three men such that two certain women refuse to dance with any in that set of three men?

Configurations: digital displays

PENT 302. **by Randall J. Covill**

Consider the following digital display problem. A character is a set of parallel and/or perpendicular non-intersecting line segments of constant length. If a character has height, the height is equal to a constant whole number of line segments. If a character has width, the width is equal to a different constant whole number of line segments. If any segment or subset of segments can be either displayed or not displayed, what is the minimum number of segments necessary to represent all ten digits 0 to 9?

Configurations: maxima and minima

ISMJ 13.22.

A box is locked with several padlocks, all of which must be opened to open the box and all of which have different keys. Five people each have keys to some of the locks. No two of the five can open the box but any three of them can. What is the smallest number of locks with which this can be done?

PARAB 372.

After the first day of classes, each of 5 different students knows a different bit of gossip about the teachers in their school. When they get to their separate homes, the telephoning begins. Assume that whenever anyone calls anyone else, each tells the other all the gossip he knows. What is the smallest number of calls after which it is possible for every student to know all 5 bits of gossip?

Configurations: money problems

OMG 15.3.2.

In how many ways is it possible to make up 28 cents using coins worth 1 cent, 5 cents, 10 cents, and 15 cents?

Configurations: people

SSM 3579. **by Stanley E. Payne**

Fifty-six graduate assistants are to be split into eight seminars, with seven assistants in each seminar and with each grouping to be maintained for one month. Although the school year consists of nine months, a little time lost here and there during the year renders eight distinct groupings sufficient. The problem is, then, to group the assistants in eight distinct ways so as never to have two assistants in a seminar together more than once. Show that this is possible.

CRUX 263. **by Sahib Ram Mandan**

Ten friends, identified by the digits $0, 1, \ldots, 9$, form a lunch club. Each day four of them meet and have lunch together. Describe minimal sets of lunches $ijkl$ such that

(i) every two of the friends lunch together an equal number of times;

(ii) every three of them lunch together just once;

(iii) every four of them lunch together just once.

USA 1978/5.

Nine mathematicians meet at an international conference and discover that among any three of them, at least two speak a common language. If each of the mathematicians can speak at most three languages, prove that there are at least three of the mathematicians who can speak the same language.

PARAB 313.

The King's men have captured a band of outlaws with an odd number of men. The rangers demand to know which ones shot the King's deer. The outlaws in panic each point to the nearest man. Prove that at least one man will not be accused. (Assume that no two pairs of outlaws are the same distance apart.)

Counting problems: geometric figures

AMM 6179. **by E. Ehrhart**

Find all cubes in a cubic lattice whose vertices are lattice points.

MM 939. **by Richard A. Gibbs**

Consider an $n \times n \times n$ cube consisting of n^3 unit cubes. Using only the unit cubes, determine, in terms of n:

(a) the number of possible sizes of rectangular parallelepipeds "imbedded" in the cube,

(b) the number of cubes of all sizes "imbedded" in the cube, and

(c) the number of rectangular parallelepipeds of all sizes "imbedded" in the cube.

PARAB 296.

A parallelepiped is a solid figure with six faces, each of which is a parallelogram. You are given four points, A, B, C, D, in space not all lying in the same plane. How many parallelepipeds exist with A, B, C, D included amongst the eight vertices?

CRUX 286. **by Richard A. Gibbs**

Find, for positive integers $W \le L \le H$:

(a) the number of rectangular parallelepipeds,

(b) the number of cubes,

(c) the number of different sizes of rectangular parallelepipeds imbedded in a $W \times L \times H$ rectangular parallelepiped made up of WLH unit cubes.

CRUX 204. **by R. Robinson Rowe**

A sheet of coordinate paper is 80 spaces wide by 100 spaces long with 8,000 small squares.

(a) Including larger ones, how many squares are there?

(b) How many oblongs (nonsquare rectangles) are there?

SSM 3655. **by Herta T. Freitag**

The total sum of the areas of all squares of an $n \times n$ checkerboard is 1 for $n = 1$, 8 for $n = 2$, and 34 for $n = 3$. Obtain a formula for the total area of all possible squares on an $n \times n$ checkerboard.

CRUX 19. **by H. G. Dworschak**

How many different triangles can be formed from n straight rods of lengths $1, 2, 3, \ldots, n$ units?

ISMJ J11.20.

How many triangles are there whose vertices are vertices of a given cube?

MM 1001. **by Edward T. Wang**

Find a formula for the number of parallelograms contained in an equilateral triangular lattice of side n.

SSM 3704. **by Herta T. Freitag**

Find a formula for the number of rhombuses contained in an equilateral triangular lattice of side n.

MM 975. **by Charles L. Hamberg and Thomas M. Green**

SSM 3746. **by Michael Brozinsky**

Find a formula for the number of regular hexagons contained in an equilateral triangular lattice of side n.

Counting problems: jukeboxes

CRUX 280. **by L. F. Meyers**

A jukebox has N buttons.

(a) If the set of N buttons is subdivided into disjoint subsets, and a customer is required to press exactly one button from each subset in order to make a selection, what is the distribution of buttons which gives the maximum possible number of different selections?

(b) What choice of n will allow the greatest number of selections if a customer, in making a selection, may press any n distinct buttons out of the N? How many selections are possible then?

Counting problems: ordered pairs

FQ B-332. **by Philip Mana**

Let $a(n)$ be the number of ordered pairs of integers (r, s) with both $0 \leq r \leq s$ and $2r + s = n$. Find the generating function

$$A(x) = a(0) + xa(1) + x^2 a(2) + \cdots.$$

Counting problems: paths

CANADA 1977/7.
OMG 16.2.7.

A rectangular city is exactly m blocks long and n blocks wide. A woman lives in the southwest corner of the city and works in the northeast corner. She walks to work each day but, on any given trip, she makes sure that her path does not include any intersection twice. Show that the number $f(m, n)$ of different paths she can take to work satisfies $f(m, n) \leq 2^{mn}$.

CANADA 1979/5.

A walk consists of a sequence of steps of length 1 taken in directions north, south, east or west. A walk is self-avoiding if it never passes through the same point twice. Let $f(n)$ denote the number of n-step self-avoiding walks which begin at the origin. Compute $f(1)$, $f(2)$, $f(3)$, $f(4)$ and show that

$$2^n < f(n) \leq 4 \cdot 3^{n-1}.$$

AUSTRALIA 1979/3.
IMO 1979/6.

Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that $a_{2n-1} = 0$, $a_{2n} = \frac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$, $n = 1, 2, 3, \ldots$, where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

OMG 16.1.1.

If movement is allowed only in the direction of the arrows in a certain diagram, find the number of paths from A to B.

SIAM 75-1. **by R. W. Allen**

An optical fiber carries power in two modes represented by 0 and 1. The path of one photon is represented by an N-bit binary number. The sequence 0 1 or 1 0 is counted as one transition. Thus the path 1 0 0 0 1 1 1 contains two transitions and three zeros. Determine the number of paths $S(N, T, M)$ that contain T transitions and M zeros. Prove whether or not the following formula is valid for all N:

$$S(N, T, M) = 2H(N, T)\binom{M-1}{U}\binom{N-M-1}{U},$$

where

$$H(2N, T) = \frac{\binom{N-1}{V}}{\binom{N-1}{U}},$$

$$H(2N+1, T) = \frac{\binom{2N}{2V}}{\binom{2N}{T-1}},$$

and

$$U = \left\lfloor \frac{T-1}{2} \right\rfloor, \qquad V = \left\lfloor \frac{T}{2} \right\rfloor.$$

AMM E2608. **by Judith Q. Longyear**

A child is riding in a train n cars long and wishes to go exploring. An exploration may be described by listing in order the cars traversed; each exploration must end in the same car in which it began. How many explorations of length k can it make?

Suppose we regard the exploration

$$(e_1, e_2, \ldots, e_k)$$

to be equivalent with all explorations

$$(e_r, e_{r+1}, \ldots, e_k, e_1, \ldots, e_{r-1})$$

$(r = 2, \ldots, k)$. How many nonequivalent explorations can it make?

Counting problems: sequences

MM 989. **by L. Carlitz and Richard Scoville**

Let $r \geq 0$, $s \geq 0$, and $r + s \leq n$. Find the number of sequences of positive integers $(a_1, a_2, \ldots, a_n)$ such that for $1 \leq k \leq n$, $a_k \leq k$ where $a_k = 1$ for r values of k, and $a_k = k$ for s values of k.

OSSMB 76-3.

The numbers $1, 2, \ldots, n$ are placed in a row so that, except for an arbitrary choice of first number, the number k can be placed in the row only if it is preceded either by $k-1$ or $k+1$ (not necessarily immediately). How many such arrangements are there for the numbers $1, 2, \ldots, n$?

Counting problems: subsets

OSSMB 75-9.

Counting the empty set, how many subsets of the set $\{1, 2, \ldots, n\}$ do not contain a pair of consecutive numbers?

AMM E2521.* **by John A. Cross**

An instructor has a file of p questions of equal diagnostic value in testing students on a certain topic. He gives q-question tests repeatedly ($q < p$). How many test forms can he compose if any n-size subset, $1 \leq n < q$, of the p questions may appear on at most two tests, and no subset of size $m > n$ may appear on more than one test? Determine an algorithm for composing the set of possible tests, for any allowable p, q, and n.

OSSMB 78-5.

The Fibonacci sequence $\{F_n\}$ satisfies

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for all n. A sequence $\{b_n\}$ is defined by requiring that b_n is the number of subsets of $\{1, 2, \ldots, n\}$ having the property that any two different elements of the subset differ by more than 1. Find a similar formula for the sequence $\{b_n\}$.

AMM E2764. **by Ioan Tomescu**

Let X be a finite set. Prove that

$$\sum |A_1 \cup A_2 \cup \cdots \cup A_k| = \left(2^k - 1\right) \sum |A_1 \cap A_2 \cap \cdots \cap A_k|,$$

where the sums are over all choices of $A_1, \ldots, A_k \subseteq X$.

Counting problems: tournaments

PARAB 264.

In the 1974 cricket XI, there were 7 boys who had been in the 1973 XI; and in the 1973 XI, there were 8 boys who had been in the 1972 XI. What is the least number who have been in all three XI's?

Answer the same question with x instead of 7 and y instead of 8. For what values of x and y is it possible that there were no boys in all three XI's?

Counting problems: words

CRUX 433. **by Dan Sokolowsky**

An exam question asked: How many distinct 5-letter words can be formed using the letters A, A, A, B, B, B?

A student misread the question and determined instead the number of distinct 6-letter words using these same letters, yet obtained the correct answer. Was this accidental or is it a special case of a more general pattern?

PARAB 341.

A certain tribe of early men had an alphabet consisting of two letters `A` and `B`. They also had the rule that, in any word, `ABA` was equivalent to `B` (that is, each could replace the other in the word and the word was considered to be the same); and the rule that `BAB` was equivalent to `A`.

(a) How many different words could be represented?

(b) Find two other ways of writing down the name of the Swedish pop group `ABBA`.

Distribution problems

PARAB 322.

Suppose there were 250,000 people in Sydney in 1968 who made between \$8,000 and \$9,000. Show that there were at least 3 people who made the same salary down to the last cent.

Geometry: coloring problems

IMO 1979/2.
AUSTRALIA 1979/1.

Consider a given prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as top and bottom faces. Each side of the two pentagons and each of the line segments A_iB_j, for all $i, j = 1, \ldots, 5$, is colored either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Show that all 10 sides of the top and bottom faces are the same color.

Geometry: concyclic points

TYCMJ 33. **by Norman Schaumberger**

Let P be any point on a circle. Prove that the four distances from P to the vertices of a square inscribed in the circle cannot all be rational.

TYCMJ 105. **by Norman Schaumberger**

Let $n > 1$ be odd and $\{A_1, A_2, \ldots, A_n\}$ be a set of n points on a circle such that the lengths of the chords A_iA_{i+1} ($i = 1, 2, \ldots, n; A_{n+1} = A_1$) are all equal. Is it possible that three of these points are rational?

ISMJ 13.27.

Show that if 5 points are located on a circle so that every 3 of them lie on a semicircle, then all 5 of them are on a semicircle.

Geometry: dissection problems

OSSMB 79-14.

A convex n-gon is a plane figure with n sides such that a straight line joining any two points on different sides lies inside the figure. For what values of n can the figure be divided into black and white triangles so that all of the sides are edges of black triangles and no two triangles of the same color share an edge? (Note that points are added in the interior.)

Geometry: points in plane

ISMJ 10.14.

Suppose that n points are located in the plane so that the maximum distance apart of any two of them is 1. Prove that there are not more than n pairs of points whose distance apart is 1 and that the n points can be located so that there are n pairs whose distance apart is 1.

AMM 6130. **by Erwin Just and Eugene Levine**

Prove that there exists a partition of the rational points of the plane into an infinite number of everywhere dense subsets such that each straight line containing two rational points will have a nonempty intersection with each of the subsets.

Geometry: points in space

PARAB 402.

Consider 5 points in space such that each pair is not more than 1 cm apart. What is the greatest number of pairs which can be exactly 1 cm apart?

AMM E2593. **by Jeanne W. Kerr and John E. Wetzel**

Three points are given on each of three parallel lines, the three lines not all lying in the same plane. These points by threes, one on each line, determine 27 triangular plates, and these triangular plates could, on the face of it, meet to determine as many as $\binom{27}{3} = 2925$ points, though it is clear that not that many can actually occur. At most how many points can the 27 plates determine?

Graph theory: bipartite graphs

AMM E2565. **by T. Nemetz**

Given a bipartite graph on n and $2n$ vertices that is regular on either set (of degree $2k$ and k, respectively), can one necessarily find n vertices of the second kind such that upon their removal along with all arcs containing them the remaining graph is regular of degree k?

CMB P268.* **by P. Erdős and E. C. Milner**

A graph $G = (V, E)$ is said to be realized if there is a family of sets $\{A_x \,|\, x \in V\}$ associated with the vertices of G such that $A_x \subset \{0, 1, 2 \ldots\}$ and such that $\{x, y\}$ is an edge of G if and only if $A_x \cap A_y = \varnothing$. It is easy to see that any realizable graph has chromatic number that is not larger than $\aleph_0$. Is it true that any bipartite graph on $2^{\aleph_0}$ vertices is realizable?

AMM 6079. **by D. J. Kleitman**

Given a bipartite graph connecting n vertices with n others. If the symmetry group of the graph is transitive on both parts of the graph, must it be transitive on the whole graph?

Graph theory: complete graphs

AMM E2562. **by N. C. K. Phillips**

Each of the $\binom{m}{2}$ edges of the complete graph on m vertices is assigned a direction and one of n colors in such a way that there is no monochromatic directed path $\overrightarrow{AB}$, $\overrightarrow{BC}$ of length 2. How large can m be in terms of n?

AMM E2672. **by Marianne Gardner**

Each of the $\binom{m}{2}$ edges of the complete graph K_m is assigned a direction, and each vertex is assigned one of n colors in such a way that there is no directed path of length k, $k < m$, whose vertices are all of the same color. How large can m be in terms of n and k?

AMM 6034. **by Fred Galvin**

Suppose the edges of the complete graph on n vertices are colored so that no color is used more than k times.

(a) If $n \geq k + 2$, show that there is a triangle no two of whose edges are the same color.

(b) Show that this is not necessarily so if $n = k + 1$.

Graph theory: counting problems

JRM 421. **by Mary Youngquist**

Find all connected, topologically distinct, spatial arrangements of three C's and six O's, in which exactly four bonds (kinks, paths) emanate from each C and two from each O.

Graph theory: covering problems

AMM E2549. **by David Singmaster**

Let G be a connected graph with $2k$ vertices of odd degree. It is well known that G can be covered by a k-part Euler path, i.e., a union of k edge-disjoint paths having no repeated edges. When can G be covered by a single path with at most $k - 1$ repeated edges?

AMM E2564. **by R. L. Graham**

Can one cover the vertices of any regular graph of degree four (every vertex in it has degree four) by disjoint arcs and stars?

Graph theory: directed graphs

ISMJ 13.18.

We are given a finite set S of points. From each point of S an arrow is drawn connecting it to some other point of S. Show that the points of S can be colored with three colors so that no two points of the same color are joined by an arrow.

PARAB 308.

Seven towns $T_1, T_2, \ldots, T_7$ are connected by a network of 21 one-way roads such that exactly one road runs directly between any 2 towns. Given any pair of towns T_i, T_j $(1 \leq i < j \leq 7)$, there is a third town, T_k, such that T_k can be reached by a direct route from both T_i and T_j.

(a) Prove that, of the 6 roads with an end at any town T_i, the number in which traffic is directed away from T_i is at least 3. Hence prove that it is exactly 3.

(b) Let the towns that can be reached directly from T_1 be numbered T_2, T_3, T_4. Show that the roads between T_2, T_3, T_4 form a circuit.

(c) Display on a sketch a possible orientation of traffic on the 21 roads.

NAvW 453. **by J. H. van Lint**

Let us call a directed graph "of type k" if, for any two (not necessarily distinct) vertices P and Q of the graph, there is exactly one path of length at most k from P to Q. Prove that if $k > 2$, a graph of type k is a circuit with k points.

ISMJ 13.23.

Let $P_1, P_2, P_3, \ldots$ be an infinite set of distinct points. From some of these points P_n, two arrows go out and join P_n to P_m and P_ℓ where $n < m < \ell$. From others of the P_n, no arrows go out. (For example, we could have arrows from P_n to P_{2n} and P_{3n} when n is odd and no arrows otherwise.) A point P_n is called reachable if there is a path starting from P_1 that consists of arrows and gets to P_n.

(a) How many points with subscripts not exceeding 100 are reachable in this example?

(b) Assume that there are infinitely many reachable points. Show that there exists an infinitely long path that starts from P_1 and that consists entirely of arrows.

Graph theory: family trees

SSM 3630. **by Charles W. Trigg**

A man announced that he was one-third Cherokee. How do you arrange the branches on his family tree?

Graph theory: friends and strangers

PARAB 439.

Given any two people, we may classify them as friends, enemies, or strangers. Prove that at a gathering of seventeen people, there must be either (a) three mutual friends, (b) three mutual enemies, or (c) three mutual strangers.

PARAB 391.

Each of three classes has n students. Each student knows altogether $(n+1)$ students in the other two classes. Prove that it is possible to select one student from each class so that all three know one another. (Acquaintances are always mutual.)

PENT 315. **by H. Laurence Ridge**

Four married couples meet for dinner. There is some shaking of hands. No one shakes hands more than once with the same person. Spouses do not shake hands.

When the hand shaking is finished, one husband asks all of the other people how many times they shook hands. Everyone gives a different answer. How many times did the questioner's wife shake hands?

FUNCT 3.2.3.

Prove that, of all the teenagers in the world, at least two have the same number of teenage friends.

PARAB 278.

At a party, the guests are lined up so that each person (with the exception of the two at the ends) is acquainted with exactly as many people to his right as to his left. Show that the first and last person have the same number of acquaintances.

Graph theory: isomorphic graphs

NAvW 527. **by A. M. Cohen and A. A. Jagers**

Let G and H be graphs. Choose two vertices i and j of G, and for each vertex k adjacent to j $(k \neq i)$, delete the edge between k and i (resp. place an edge between k and i) whenever k and i are adjacent (resp. nonadjacent) in G. The result is a new graph denoted by $\pi_{ij}(G)$. If a graph isomorphic to H can be obtained from G by repeated application of operations of the form π_{ij}, then H is called a conjugate of G; notation $H \sim G$. Clearly $\sim$ is an equivalence relation. Now for any graph K, let $n(K)$ be the number of vertices of K and let $m(K)$ be the number of edges in a maximal matching of K. Denote by $r(K)$ the rank of the adjacency matrix of K over $\mathbb{Z}_2$. Then prove that

(a) $(H \sim G) \Longleftrightarrow (n(H) = N(G)$ and $r(H) = r(G))$,

(b) $r(T) = 2m(T)$ if T is a tree.

AMM 6037. **by Jim Lawrence**

Show that any graph H is isomorphic to an induced subgraph of some finite graph H' which has a group of automorphisms that acts transitively on its vertices.

NAvW 459. **by M. R. Best**

Determine all graphs (without loops or multiple edges) whose complement and line graph are isomorphic.

NAvW 495. **by J. I. Hall**

Determine all finite graphs (loops and multiple edges allowed) that are isomorphic to their line graphs.

Graph theory: map problems

AMM 6182. **by A. K. Austin**

Prove or disprove that any finite planar graph can be represented by a map in which all the regions are L-shaped with sides horizontal and vertical.

Graph theory: maxima and minima

NAvW 487. **by H. C. A. van Tilborg**

Let Γ be a De Bruijn graph on 2^n points, i.e., a directed graph with vertices labeled by elements of $\{0,1\}^n$ with a directed edge from $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$ if and only if $(a_2, a_3, \ldots, a_n) = (b_1, b_2, \ldots, b_{n-1})$. Determine the maximal k such that every path of length k in Γ starting in $(0, 0, \ldots, 0)$ is the initial part of an Euler path in Γ.

AMM 6159. **by Thomas E. Elsner**

It is well known that for a graph on k vertices with no triangles, the maximum number of edges is $L(k) = mn$, where $m = \lfloor k/2 \rfloor$ and $n = \lfloor (k+1)/2 \rfloor$ and that this value occurs for the complete bigraph $K_{m,n}$. Express the maximum number of edges in case we add the restriction that the graph be

(a) Hamiltonian;

(b) Eulerian.

PME 441. **by Richard A. Gibbs**

Prove that a self-complementary graph with an even number of vertices has no more than $2i$ vertices of degree i, and that the number of them is even.

Graph theory: trees

AMM 6262. **by A. Blass, F. Harary, and W. T. Trotter, Jr.**

What is the probability that a tree selected at random has a fixed point? More specifically, let t_n be the number of (nonisomorphic) trees with n points, and let f_n be the number of such trees T with at least one point fixed under all automorphisms of T. Calculate $\lim_{n\to\infty} f_n/t_n$.

AMM E2671. **by Ibrahim Cahit**
SIAM 77-15. **by I. Cahit**

Let $T = (V, E)$ be a k-level complete binary tree with vertex set V and edge set E. Thus $|V| = 2^k - 1$, and we set $N = \{1, 2, 3, \ldots, 2^k - 1\}$. For every bijection $f\colon V \to N$ define

$$W(f) = \sum_{\{i,j\}\in E} |f(i) - f(j)|.$$

Prove or disprove that $\min_f W(f) = (k-1)2^{k-1}$ $(k \geq 2)$.

Josephus problem

MM 1031. **by Richard A. Gibbs**

There are n people, numbered consecutively, standing in a circle. First 2 sits down, then 4, 6, etc., continuing around the circle with every other standing person sitting down until just one person is left standing. What is his number?

Latin squares

NAvW 439. **by R. H. F. Denniston**

By taking two rows of a Latin square as "equivalent" when one is an even permutation of the other, we can define an equivalence relation on the set of rows. Let us say that a Latin square of even order is "row-odd" when there are two equivalence classes with odd cardinalities. Let L_1 be any Latin square of order $2m$; let L_2 be the transpose of L_1, and L_3 the square obtained from L_1 by interchanging the parts played by the rows and the symbols used as entries. Prove that the number of row-odd squares among these three L_i has the same parity as m.

Lattice points

AMM 6192. **by Harry D. Ruderman**

Let R be a rectangular array of lattice points having at least two rows and two columns. Let each lattice point of R be labeled by one of the numbers 1, 2, 3, or 4. Suppose that the boundary points of R contain at least one of each of the four numbers and the boundary is oriented, say counterclockwise, with repetitions permitted, and with possibly more than one cycle (1 is allowed to follow 4). Call two lattice points adjacent if they are vertices of a common small square. Call two lattice points opposite if they are labeled either 1 and 3 or 2 and 4. Prove that for every such R, there is a square containing two lattice points that are both opposite and adjacent.

Paths

MENEMUI 1.1.2. **by S. L. Lee**
MENEMUI 1.2.2. **by S. L. Lee**

A certain figure shows a network consisting of 49 points. What is the minimum number of turnings one has to make to travel from S to T, passing through all 49 points at least once?

Permutations

SIAM 76-17. **by David Berman and M. S. Klamkin**

A deck of n cards is numbered 1 to n in random order. Perform the following operations on the deck. Whatever the number of the top card is, count down that many in the deck and turn the whole block over on top of the remaining cards. Then, whatever the number of the (new) top card, count down that many cards in the deck and turn this whole block over on top of the remaining cards. Repeat the process. Show that the number 1 will eventually reach the top.

Consider the following set of related and more difficult problems:

(a) Determine the number $N(k)$ of initial card permutations, so that the 1 first appears on top after k steps of the process. In particular, show that $N(0) = N(1) = N(2) = (n-1)!$ and that

$$N(3) = \begin{cases} (n-1)! - \frac{1}{2}(n-1)(n-3)(n-4)!, & n \text{ odd}, \\ (n-1)! - \frac{1}{2}(n-2)^2(n-4)!, & n \text{ even}. \end{cases}$$

(b) Estimate the maximum number of steps it takes to get the 1 to the top.

(c) For what n is there a unique permutation giving the maximum number of steps?

(d) Does the last step of a maximum step permutation leave the cards in order (i.e., $1, 2, \ldots, n$)?

AMM 6214.* **by Leonard Carlitz**

Let k and t be fixed integers, $k \geq 2$, $t \geq 0$, and let $A_k(kn+t)$ denote the number of permutations of

$$Z_{kn+t} = \{1, 2, 3, \ldots, kn+t\}$$

such that

$$\begin{aligned} &a_{kj+1} < a_{kj+2} < \cdots < a_{kj+k}, \\ &a_{kj+k} > a_{kj+k+1} \quad j = 0, 1, \ldots, n-1, \\ &a_{kn+1} < a_{kn+2} < \cdots < a_{kn+t}. \end{aligned}$$

It has recently been proved as a corollary of a general result that $A_4(2n+1) = 2^{-n}A_2(2n+1)$. Prove this identity by a direct combinatorial argument.

AMM E2702.* **by David Jackson**

Let $a = (a_1, a_2, \ldots, a_{2m})$ be a nondecreasing sequence of positive integers. Let S denote the set of sequences obtained from a by permuting its terms. Let A, B, C be the subsets of S consisting of those sequences $s = (s_1, s_2, \ldots, s_{2m})$ that satisfy

$$s_1 < s_2 \geq s_3 < s_4 \geq \cdots \geq s_{2m-1} < s_{2m},$$

$$\prod_{i=1}^{2m}(s_i - a_i) > 0, \qquad \prod_{i=1}^{2m}(s_i - a_i) < 0,$$

respectively. Show that $|A|$ is equal to the absolute value of $|B| - |C|$.

NAvW 543. **by H. W. Lenstra, Jr.**

Let n and m be integers ($n > 1$, $m > 1$), and let σ be the permutation of $\{1, 2, 3, \ldots, nm\}$ suggested by the following picture:

$$\begin{matrix} 1 & 2 & 3 & \ldots & n \\ n+1 & n+2 & n+3 & \ldots & 2n \\ \vdots & & & & \vdots \\ (m-1)n+1 & & & & mn \end{matrix} \quad \rightarrow \quad \begin{matrix} 1 & m+1 & 2m+1 & \ldots & (n-1)m+1 \\ 2 & m+2 & 2m+2 & \ldots & (n-1)m+2 \\ \vdots & \vdots & \vdots & & \vdots \\ m & 2m & 3m & \ldots & nm \end{matrix}.$$

Clearly $\sigma(1) = 1$ and $\sigma(nm) = nm$, so the cycle decomposition of σ contains two cycles of length one. Suppose that there is only one other cycle, of length $nm - 2$, in this decomposition; i.e., that the numbers $2, 3, 4, \ldots, nm-1$ are cyclically permuted by σ, in a suitable order.

Prove that each n, m is 2 or 3 (mod 4), and they are not both 3 (mod 4).

MM Q639. **by Frank Gillespie**
ISMJ 13.11.

Let $k_1, k_2, \ldots, k_n$ be any given set of n integers and let $m_1, m_2, \ldots, m_n$ be any permutation of this set. Prove that

$$|k_1 - m_1| + |k_2 - m_2| + \cdots + |k_n - m_n|$$

is even.

AMM S14. **by C. L. Mallows**

Let $n(m,f,r)$ represent the total number of arrangements $(a_1,a_2,\ldots,a_m)$ of $(1,2,\ldots,m)$ that have f fixed points $(a_i = i)$ and r rises $(a_i < a_{i+1})$. Prove that $n(m,0,r) = n(m,1,r)$ for $1 \le r \le m-1$, and that

$$n(m,f,r) = \sum_{j=2}^{m-r} (-1)^{m-r-j}(j-1)^j j^{m-f-j} \times \binom{f+j-1}{j-1}\binom{m+1}{m-r-j} + (-1)^{m-f}\frac{\delta_{r+1-f}}{m-f+1}\binom{m}{f}$$

for $0 \le f \le r+1 \le m$, $0 \le r$, where

$$\delta_k = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \ne 0. \end{cases}$$

NAvW 430. **by H.S.M. Kruijer**

Given is an engine with n cylinders $(n \ge 1)$ numbered 1 to n going from left to right in a line.

The ignition sequence can be characterized using the cylinder numbers as a permutation of the numbers 1 to n, assuming that cylinder 1 is ignited first. Determine the number of ignition sequences for which this characterization does not change when the cylinders are renumbered from right to left.

Selection problems

PARAB 376.

Given are n sacks each holding the same number of apples. On the first day, an apple is removed from one sack. On the second day, an apple is removed from each of 2 sacks and so on, until the nth day when one apple is removed from each of the n sacks.

The sacks are now all empty. For which n is this possible, and how is it to be done?

Sequences

JRM 757. **by Michio Matsuda**

Deal out nine cards face-up in a row from a well-shuffled deck. You will find that there are always at least three cards of the same color at equal spacing.

(a) Now deal out thirteen cards face-down. What is the minimum number of cards which you must turn face-up in order to determine the locations of three cards of the same color at equal spacing?

(b) Same question, but with n cards, where $n \ge 9$.

AMM E2795. **by Doug Wiedemann**

Let S be a nonempty subset of

$$\{0,1\}^n = \{0,1\} \times \cdots \times \{0,1\}$$

such that each member of S is adjacent to exactly k other members of S, where "adjacent" means differing in one coordinate position. Show that the size of S is even and at least 2^k. Furthermore, if the graph of the adjacency relation of S is connected, show that it will still be connected after removal of any point.

NAvW 511. **by I. H. Smit**

For an integer $n \ge 3$, let S_n be the set of finite sequences $(x_i)_{i=1}^n$ of length n with

$$x_i \in \{-1,+1\},$$

i.e., $S_n = \{-1,+1\}^n$. If $x \in S_n$, then $\beta(x)$ denotes the number of alternating subsequences $\left(x_{j_k}\right)_{k=1}^3$ of length 3 $(j_1 < j_2 < j_3)$, i.e., subsequences such that $x_{j_k} + x_{j_{k+1}} = 0$ $(k = 1,2)$.

(a) Determine

$$f(n) = \max\left\{\beta(x) \;\middle|\; x \in S_n\right\}.$$

(b) Determine the cardinality of the set

$$\left\{x \in S_n \;\middle|\; \beta(x) = f(n)\right\}.$$

OSSMB 78-7.

Consider sequences of length n with elements drawn from the set $\{1,2,\ldots,9\}$. Let E_n be the number of such sequences whose entries sum to an even number and O_n the number of sequences whose entries sum to an odd number.

(a) Show that $E_n - O_n = (-1)^n$.

(b) Find E_n and O_n in terms of n.

Sets: cardinality

AMM 6060.* **by Daniel Sokolowsky**

For fixed $k \ge 2$, A_i, B_i $(i = 1,2,\ldots,k)$ are $2k$ subsets of a finite set S. What is the largest possible value of $n = |S|$ such that the following three conditions can hold simultaneously for $i = 1,2,\ldots,k$?

(i) $A_i \cap B_i = \varnothing$,

(ii) $|A_i \cup B_i| = n-1$,

(iii) For each $x \in S$, $\{x\}$ is the intersection of an appropriate subcollection of the $2k$ sets A_i, B_i $(i = 1,2,\ldots,k)$.

SSM 3738. **by Philip Smith**

Consider a collection of n nonempty sets of positive integers such that

(1) no two distinct sets in the collection have the same cardinal number, and

(2) no set in the collection is a subset of any other set in the collection. What is the minimum possible cardinal number of the union of the n sets?

Sets: determinants

AMM E2690. **by Anthony J. Quinzi**

Let $S_1, S_2, \ldots, S_k$ be a list of all non-empty subsets of $\{1,2,\ldots,n\}$. Thus $k = 2^n - 1$. Let $a_{ij} = 0$ if $S_i \cap S_j = \varnothing$ and $a_{ij} = 1$ otherwise. Show that the matrix $A = (a_{ij})$ is nonsingular.

Sets: differences

MM 1041. **by Richard A. Gibbs**

For $0 < m < n$, find $N(m,n)$, the minimum positive integer such that any subset of $\{1,2,\ldots,n\}$ of $N(m,n)$ elements contains numbers differing by m.

MSJ 476.

Prove that any subset of 55 numbers chosen from the set $\{1,2,3,4,\ldots,100\}$ must contain numbers differing by 9, 10, 12, and 13, but need not contain a pair differing by 11.

AMM S5. **by R. L. Graham**

For a finite set X of integers, let $|X|$ denote the cardinality of X, and let $X - X$ denote $\{x - x' \mid x, x' \in X\}$. Show that if $A, B \subseteq \{1, 2, \ldots, n\}$ with $|A||B| \geq 2n-1$, then

$$(A - A) \cap (B - B)$$

contains a positive element. Here $n > 1$.

PARAB 419.

Write on a large blackboard the numbers

$$1, 2, 3, \ldots, 1979.$$

Erase any two of the numbers and replace them by their difference. Repeat this process until only a single number is left on the board. Prove that this number is even.

Sets: family of subsets

AMM E2654. **by D. E. Daykin**

Let $A = \{0, 1, 2, \ldots, n-1\}$. For $m \in A$, let $f(m, n)$ be the least integer k with the following property: If F is a family of subsets of A such that every $i \in A$ belongs to more than k members of F, then A can be covered by $n-m$ members of F. Evaluate $f(m, n)$ for $2m \leq n$.

Sets: partitions

AMM E2582. **by Ioan Tomescu**

Let $\{A_i; 1 \leq i \leq n\}$, $\{B_i; 1 \leq i \leq n\}$, and $\{C_i; 1 \leq i \leq n\}$ be three partitions of a finite set M. If for every i, j, and k we have

$$\left|A_i \cap B_j\right| + |A_i \cap C_k| + \left|B_j \cap C_k\right| \geq n,$$

prove that $|M| \geq n^3/3$ and that this inequality cannot be improved when n is divisible by 3.

Sets: sums

IMO 1978/6.

An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

CRUX 404. **by A. Liu**

Let A be a set of n distinct positive numbers. Prove that

(a) the number of distinct sums of subsets of A is at least $\frac{1}{2}n(n+1)+1$;

(b) the number of distinct subsets of A with sum equal to half the sum of A is at most $2^n/(n+1)$.

CRUX 344. **by Viktors Linis**

Given is a set S of n positive real numbers. With each nonempty subset P of S, we associate the number

$$\sigma(P) = \text{Sum of all its elements}.$$

Show that the set $\{\sigma(P) | P \subseteq S\}$ can be partitioned into n subsets such that in each subset the ratio of the largest element to the smallest is at most 2.

Sorting

AMM E2569.* **by Harry Dweighter**

The chef in our place is sloppy, and when he prepares a stack of pancakes, they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?

JRM 736. **by Frank Rubin**

An automated warehouse contains a large collection of numbered cartons stored in unnumbered bins, and no two of the cartons have the same number. In order to improve the efficiency of the warehouse, it is decided to sort the cartons into numerical order. What is the least number of moves required when:

(a) Two automated selectors perform pairwise interchanges of cartons, and all of the bins are filled.

(b) A single selector can move one carton at a time, and there is only one empty bin.

Tournaments: chess tournaments

CANADA 1976/3.

Two grade seven students were allowed to enter a chess tournament otherwise composed of grade eight students. Each contestant played once with each other contestant and received one point for a win, one half point for a tie and zero for a loss. The two grade seven students together gained a total of eight points and each grade eight student scored the same number of points as his classmates. How many students from grade eight participated in the chess tournament? Is the solution unique?

OMG 17.2.5.

Twenty-four players competed in a recent chess tournament. The committee divided them into two sections. In each section, each player played one game against every other competitor. There were 69 more games in section B than in section A. Mr. Gambit, unbeaten in Section A, scored $5\frac{1}{2}$ points (win = 1 point; draw = $\frac{1}{2}$ point). Determine how many of Mr. Gambit's games were drawn.

PARAB 323.

Twenty-six entrants with names $A, B, C, \ldots, Z$ play in a chess tournament, each against all others. Score 2 points for a win, 1 for a draw, and 0 for a loss. No one's total was odd, there were no ties, and they ended in the order $A, B, C, \ldots, Z$. What was the result of the match between M and N?

PARAB 357.

Chess players from two schools competed. Each player played one game with every other player. There were 66 games among players from one school, and in all there were 136 games. How many players from each school entered the tournament?

Tournaments: elimination tournaments

OMG 14.2.1.

How many games are needed to produce a winner in a knock-out tournament with (a) 8, (b) 27, (c) 47, and (d) n teams?

OMG 17.1.4.

If 94 players enter a knockout tennis tournament for a singles championship, how many matches must be played to determine the winner? For a 95-player tournament, how many matches must be played?

Tournaments: incomplete information

JRM 715. **by Peter J. Green**

Partway through a round-robin soccer tournament involving five teams, all official match records were accidentally destroyed. The only parts of the standings that could be established definitely from memory are shown. The scoring is two points for a win, one point for a draw, and zero points for a loss. Each team was supposed to play each of the others once.

Has C played D yet, and if so what were their respective scores?

					Goals		
Team	Played	Won	Lost	Drawn	For	Against	Points
A						1	4
B	1						
C					5	0	6
D						4	
E	4			2		2	2

Tournaments: maxima and minima

PARAB 420.

King Arthur's knights arrange a tournament. After it is all over, the King notices that to every two knights, there is a third one who has vanquished both. How many knights (at least) must have taken part in the tournament?

Tournaments: soccer

OMG 18.2.6.

Four high school soccer teams each played one game against each of the others. The scoring was:
MACDONALD: Goals For - 13, Against - 17, Points - 4.
LAURIER: Goals For - 17, Against - 13, Points - 3.
CLARK: Goals For - 17, Against - 13, Points - 3.
WESTVIEW: Goals For - 13 , Against - 17, Points - 2.

Two points were scored for a win and one for a tie. Each game produced the same number of goals but no two matches produced the same score. Of their 13 goals, Westview scored two against Clark. What was the result of the match between Westview and Laurier?

Tournaments: tennis

FUNCT 3.1.5.

There are $2n$ participants in a tennis tournament. In the first round of the tournament each participant plays just once, so there are n games each occupying a pair of players. Show that the pairings for the first round can be arranged in exactly $1 \times 3 \times 5 \times \cdots \times (2n-1)$ different ways.

Tournaments: triangular matches

SSM 3617. **by James F. Ulrich**

There are n athletic teams that should meet each other exactly once in a given season. How can the teams be matched in a league that allows only dual and triangular meets but requires that a maximum number of triangular meets be held? Assume that n is a positive integer between 3 and 20.

Tower of Hanoi

AMM E2713.* **by Saul Singer**

A stack of x rings is given, decreasing in size from the bottom up. In addition, y empty stacks are provided ($y \geq 2$). Let $N(x,y)$ be the minimum number of moves necessary to transfer the rings to one of the empty stacks subject to the following two rules:

(i) Move just one ring at a time.

(ii) At no time can a larger ring be placed atop a smaller.

It is conjectured that

$$N(x,y) = \sum_{k=1}^{m} 2^{k-1}\binom{k+y-3}{y-2} + 2^m\left[x - \binom{m+y-2}{y-1}\right],$$

where m is the largest integer such that the expression in the brackets is ≥ 0.

Urns

MSJ 426. **by Ira Ewen**

Fifteen balls, numbered 1 through 15, are placed in a hat. They are then withdrawn, one at a time, until all the balls have been removed from the hat. In how many ways is it possible to empty the hat under the following restriction: at any time after two or more balls have been removed, it should be possible to arrange these removed balls so that the numbers on them form a set of consecutive integers.

OMG 18.2.7.

I have two little bags, of which the contents are identical. Each has in it four blue marbles, four red ones, and four yellow ones. I close my eyes and remove from Bag No. 1 enough marbles (but just enough) to ensure that my selection includes two marbles at least of any one color, and one marble at least of either of the other colors. These marbles I transfer to Bag No. 2. Now (again closing my eyes), I transfer from Bag No. 2 to Bag No. 1 enough marbles to ensure that in Bag No. 1, there will at least be three marbles of each of the three colors. How many marbles will be left in Bag No. 2?

Game Theory

Betting games

PME 350. **by R. Robinson Rowe**

In the game of *ELDOS*, an acronym for *Each Loser Doubles Opponents' Stacks*, each of n players starts with his "bank" (B) and at any point in the play holds his "stack" (S), which he bets on the next round. For each round, there is just one loser; in paying the $n-1$ winners, he doubles their stacks. Consider here a unique game when, after n rounds, each player has lost once and all players end with equal stacks.

(a) For $n = 5$, what was the minimum bank, B, for each player?

(b) How many players were there if the least initial B was 11 cents?

(c) Find a general formula for B_m, the initial B of the mth player to lose, as a function of m and n.

(c) What was the initial bank B of the 9th of 13 players to lose?

Board games

SIAM 76-1. **by D. N. Berman**

The board used here consists of a single row of positions $(1, 2, \ldots, n)$, ordered from left to right, in which a given number of pieces are placed in some fashion among the positions. Only one piece may ever occupy a given position. Alternating play between two players is made by moving any one of the pieces as far to the left as desired but still remaining to the right of the piece immediately on its left. The winner is the player who leaves his opponent no possible move.

Another variation of the game allows the players to move as far to the left as desired to an unoccupied position. Determine a winning strategy for the game.

MM 1084. **by William A. McWorter, Jr.**

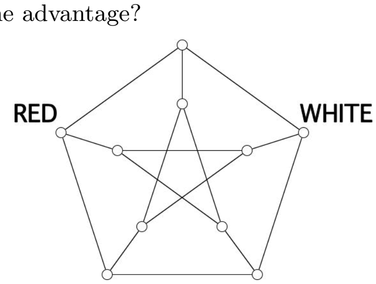
In the game of Kriegspiel Hex, two players sit back to back, each with his own Hex board. An umpire with a master board directs the game as each player attempts to make a legal move without seeing his opponent's move. The umpire's duties are: (1) Advise each player of his turn, following a legal move by his opponent. (2) Declare an illegal move so that the offending player can try a different move. (3) State when a player has won.

(a) Show that there is a winning strategy for the first player in Kriegspiel Hex played on a 3×3 board.

(b) Prove that there is no winning strategy for the first player in Kriegspiel Hex played on an $n \times n$ board, $n \geq 4$.

JRM 501. **by Makoto Arisawa**

In the new game of Yashima played on a Petersen graph, two players move alternately, starting at the marked positions, until one (the loser) no longer has a move. A move consists of transferring one's counter to an adjacent, unoccupied vertex and, as in Hackenbush, erasing the edge just traversed, which cannot then be used as a thoroughfare. Who has the advantage?



PARAB 281.

Two people play the following game on an 8×8 chessboard:

A pawn is placed on the lower-left corner square and moved alternately by the players to a neighboring square either up, to the right, or diagonally up and right. The game stops when the pawn reaches the upper-right corner square, the player making the final move being the winner.

Which player has a winning strategy, and what is it?

JRM 475. **by Ray Lipman**

Two opponents play on an infinite 3-dimensional chessboard. One has a king, the other a nondescript-looking piece that can move to any unoccupied cell. The king may not move to any cell that has once been occupied by the other. It is conjectured that the king can be trapped in a finite number of moves regardless of how he moves. Prove or disprove.

Bridge

JRM 597. **by Les Marvin**

Against South's 3 No Trump contract, West leads the five of spades and East follows with the nine. South's task is obviously to set up clubs without letting in the dangerous opponent to lead spades through South's tenace. He must hope that clubs are not split 3 and 0. When he leads the seven of clubs, how should he respond to the next player's play of the jack? The king? The queen?

MM 944. **by Richard Johnsonbaugh and R. Rangarajan**

Compute the total number of distinct auctions in contract bridge.

JRM 560. **by Sherry Nolan**

How many calls (pass, double, redouble, or one to seven of a suit or no trump) can be made during a single contract bridge auction? How many nonpassing calls can be made by one player? By one partnership?

JRM 442. **by John Selfridge**

In the last hand of a rubber of bridge, each of the four players had (A, B, C, D) distribution (without, of course, specifying the order of the suits). Does it follow that each of the four suits was distributed $(A; B; C, D)$ among the players (again not specifying the order of the players)?

JRM 536. **by Sherry Nolan**

Between two tricks in a hand of contract bridge, the Kibitzer came on the scene and quickly looked at all four hands, each of which contained n cards. With no other information, he made a correct deduction and ostentatiously announced it: "One of you clowns has revoked!" What is the maximum possible value of n? The minimum?

Card games

JRM 462. **by Fred Foldvary**

In a game of Mental Heck with four suits, thirteen tricks and a bid of six (no jokers) prove that the first player should always win.

Game Theory

JRM 601. **by David L. Silverman**

In the card game Concentration, the 52 playing cards are laid face down on a table top. Cyclically the players, in turn, turn over two cards simultaneously. If the cards do not match in rank, they are returned to their positions face down and the turn is complete. If they match, they are removed, a point is added to the player's total, and he is permitted to attempt another match. After the final match the player with the most points wins.

(a) Consider Mini-Concentration involving two players and only six cards: two kings, two queens, and two jacks. In order to assure that the game terminates, a rule is added that a player is not permitted to turn up the same pair of cards as the previous player. Who has the advantage, the first or the second player?

(b) Whom does the game favor if the mini-deck has two aces, two kings, two queens and two jacks?

JRM 647. **by David L. Silverman**

In Chili Poker, as played in northern Italy, each player has received ten cards by the time the betting is over, and from them he is required to make up two five-card hands. When the time comes to compare hands, only the poorer of the two hands is permitted to compete. It thus behooves each player to make the poorer hand as good as possible. The best possible second-best hand is called the "chili hand."

(a) Among all 10-card deals which has the worst chili hand?

(b) In the 15- and 20-card variants, what are the worst possible chili deals? In these variants, the chili hands are respectively third- and fourth-best hands.

Chess problems

JRM 540. **by David L. Silverman**

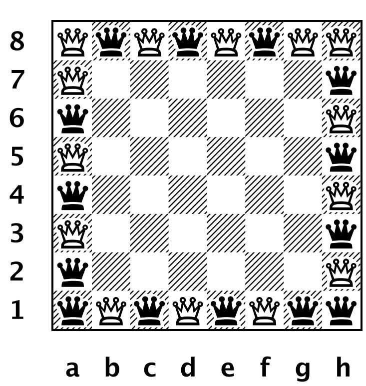
A rook and a knight play a private game on an $n \times n$ chessboard, their object being to capture the other. They start at opposite corners and rook has first move.

(a) Demonstrate rook wins in no more than 3 moves on the 3×3 and 4×4 board, in no more than 4 moves on the 5×5 board, and in no more than 5 on the 6×6 board.

(b) How about boards of larger dimension?

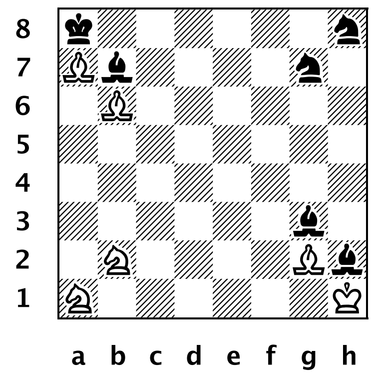
JRM 468. **by Frank Rubin**

Depicted is an arrangement of fourteen black and fourteen white queens, with a total number of 412 available moves. Is there another arrangement with the number of black and white pieces arbitrary in which the number of available moves is larger?



JRM 587. **by Les Marvin and Sherry Nolan**

White to play in the adjoining diagram. If both players play optimally, will White win, lose, or draw?



JRM 561. **by Emil Prochaska**

Is it possible to create a legal chess position with fewer than eight pieces such that the game is stalemated and such that it is impossible to deduce whose move it is?

JRM 758. **by Karl Scherer**

Does there exist a legal chess position with more than 30 pieces, in which the game is stalemated and in which it is impossible to deduce whose move it is?

JRM 424. **by Paul Morphy IV**

White and Black start with an empty chessboard and two pawns and a king apiece. In turn, beginning with White, they place their three men, in any order, but subject to these restrictions: A king cannot be placed next to opponent's king or in such a way to be attacked by opponent's previously placed pawn, and a pawn cannot be placed on the first or the eighth rank. After the six men have been placed, White has the first move in the endgame thus generated.

White's advantage in playing first seems to be more than offset by the disadvantage of having to begin the placement sequence. If both players play optimally, what is the result?

JRM 446. **by Michael Keith**

In a game of chess, what is the minimum number of moves required, after which White will be legally entitled to a draw by virtue of a perpetual check, the first move of the first cycle of which would take place following White's claim for the draw, if:

(a) No captures are made and the Black King does not move prior to the perpetual checking cycle?

(b) No captures are made but the restriction against moving the Black King prior to the perpetual checking cycle is removed?

(c) No restrictions on movement of the Black King or against captures are made?

JRM 680. **by Sidney J. Rubin**

While doubled pawns (two pawns of the same color on the same file) occur frequently in chess games, tripled or quadruples pawns are rare. It is possible, however, to have sextupled pawns on any file in a legal game. It is even possible to have *momentarily* septupled pawns on the king, queen, or either bishop file.

The minimum number of moves necessary in a legal game to achieve n-tupledness on the various files, known to the proposer, is presented in the table shown. Fill in the gaps, reduce the known minimum numbers, and/or offer proofs that no smaller numbers are possible.

JRM 493. **by Emil Prochaska**

Call a sequence of consecutive chess moves a "straitjacket sequence" if there is no "choice" of moves available; that is, if one and only one legal move can be made by both White and Black at each stage in the sequence. What is the maximum possible length of a straitjacket sequence?

Cribbage

JRM 510. **by Marshall Willheit**

Two cribbage players are tied at 120 points. Since one point will win the game, and at least one point in the pegging is inevitable, the pegging will determine the winner. Cribber, who plays second, clearly has the advantage. Taking into account the expectation of various 6-card deals, possible 4-card selection policies for each deal, and possible pegging strategies, devise a program that will estimate this advantage.

Dots and Pairs

ISMJ 12.3.

Work out the solution of the 3×3 Dots and Pairs game.

Mastermind

JRM 772. **by Ronald J. Lancaster**

Code Pegs: `WWGR` + `WWOO` + `YYBB` + `WOBB` = `OWBB`

Key Pegs: `B` + `WB` + `BB` + `BBWW` + `BBBB`

In a recent game of Mastermind between two cunning opponents, the codebreaker broke the code in five logical moves. Interestingly enough, the code pegs form an alphametic which has a unique solution! Can you find it?

Nim variants: 1 pile

FUNCT 2.3.3.

Two players in turn take matches from a pile of 21 matches. At each turn, a player must take at most 5 matches and at least 1 match. The player who takes the last match wins. Devise a winning strategy for playing this game. Generalize.

JRM 682. **by David L. Silverman**

Sulucrus is a one-pile countdown game for two players. Alternately they remove chips from an n-chip pile, the winner being the player who takes the last chip or chips. One player has the option at each turn of removing 1, 3, or 6 chips; the other player may remove 2, 4, or 5 chips. (Note that the latter player loses if his opponent leaves him with no chips or a single chip.)

On an ocean cruise a well-dressed stranger invites you to play a game of Sulucrus for high stakes. He offers to let you pick any 3-digit number for the initial pile number, and he also gives you either the choice of position (first or second play) or of role (1,3,6- or 2,4,5-player), reserving to himself whichever of those two choices you pass up. Which initial conditions should you select?

MATYC 116. **by Richard Gibbs**

In a game with two players, A and B, A goes first and chooses an integer between 1 and 10 inclusive. Player B then selects an integer from the same range and adds his choice to A's. Then A selects and adds his to the sum, etc. The winner is the player whose selection makes the total equal to 100. What is the winning strategy? Generalize.

PARAB 371.

A game is played by two players with matchsticks, as follows. To start, 36 matches are equally spaced in a row. Each player picks up, in turn, either one, two, or three matches. The player who picks up the last match wins the game.

(a) Prove that the second player can always win.

(b) The rules are changed to require that the one, two, or three matches must be neighboring matches from one group. Can the second player still always win?

PME 379. **by David L. Silverman**

You play in a nonsymmetric, two-man subtractive game in which the players alternately remove counters from a single pile, the winner being the player who removes the last counter(s). At a stage when the pile contains k counters, if it is your opponent's move, he may remove $1, 2, \ldots,$ up to $\lfloor\sqrt{k}\rfloor$ counters. If it is your move, you may remove $1, 2, \ldots,$ up to $\phi(k)$ counters, where ϕ is the Euler totient function. If you play first on a pile of 1776 counters, can you assure yourself of a win against best play by your opponent?

Nim variants: 3 piles

JRM 648. **by David L. Silverman**

Two persons play alternately on several piles of chips. On each play a number of chips equal to the current number of piles must be subtracted from a single pile having at least that many chips. The winner is the player who is last able to make a legal move.

If the game starts with two piles of six and one pile of seven, who has the advantage and what is his winning strategy?

OSSMB 79-15.

Consider the following two player game. Three piles are given containing x, y, and z pennies. Players alternately select a pile, then choose 1, 2, or 3 pennies from that pile. The player who is forced to take the last penny loses. Determine a winning strategy for one of the players.

Nim variants: opponent decrees

JRM 372. **by Jesse Croach, Jr.**

In this Nim variant two players, as in Nim, are confronted with several piles of varying number of counters and alternately remove one or more (up to all) counters from one pile, the winner being the player who removes the last counter(s). Unlike Nim, however, one's opponent has an important say in one's decision at each play. Specifically, on your play you announce the number of counters you intend to play — a positive integer that does not exceed the current size of the largest pile. Your opponent can then require you to remove that number of counters from any pile that contains at least that number. Naturally you have the same privilege on your opponent's plays.

Determine the optimal strategy in this game, which is equivalent to finding a practical technique for recognizing which pile arrays constitute "safe leaves".

JRM 373. **by David L. Silverman**

Same as JRM 372 (above) except at his turn, each player announces the pile he intends to reduce, and his opponent decrees the number of counters removed, from one to the entire pile. Determine the optimal pile choice if you are confronted with the array (1.2.3.4.5). What criterion distinguishes safe from unsafe leaves?

Nim variants: stars

ISMJ 12.1.

Two players play Star Nim on a 10 point star. Can you describe the winning strategy?

ISMJ 12.2.

Which of two players has the winning strategy on a nine point star in a game of Star Nim?

Nim variants: Target Nim

JRM 539. **by Jesse Croach, Jr.**

Roughly speaking, Target Nim is played like standard Nim, but in reverse fashion. Several positive integers are written down and, from a common supply of counters equal in number to the total of the integers, two players alternately build up piles of counters, the winner being the player attaining pile numbers equal to the target, i.e., to the initial set of integers. A play is never allowed that would make it impossible for either player to attain the target, e.g., if the target is $(2, 3, 7)$ and the current pile sizes $(1, 2, 4)$ neither of the two smaller piles may be made larger than 3, nor can a fourth pile be started.

If the piles and the target were associated by some ordering, then this game would be equivalent to standard Nim. But there is no such ordering, and in the above example, *either* of the two smaller piles may be raised to 3. By what criterion can safe leaves be identified?

Selection games: arrays

OSSMB 75-2.

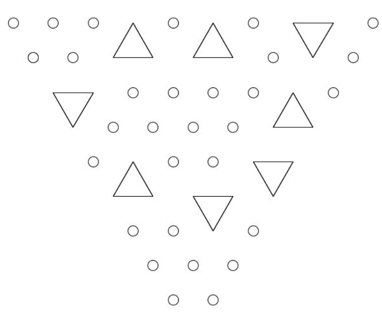
A penny is placed at each vertex of a regular n-gon. The pennies are removed alternately by two players, each move consisting of the withdrawal of a single penny or of two pennies that occupy adjacent vertices. The player to take the last penny wins the game. Determine a winning strategy for the second player.

JRM 709. **by Ronald E. Ruemmler**

An equilateral triangle of 55 dots is first drawn as shown. Players alternate drawing equilateral triangles by connecting three adjacent unused dots. The winner is the player who is last able to draw a triangle.

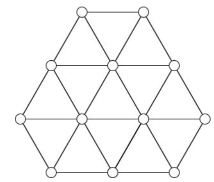
(a) A partially completed game is shown. Which player has the advantage?

(b) Develop a general strategy for the complete game.



JRM 533. **by Karl Scherer**

The popular German game of Nimbi is played on a truncated hexagonal field with twelve stones arranged in twelve rows. A play consists of removing one or more stones from the same horizontal, 45°, or 135° row. Determine who has the advantage and the winning strategies in the two versions: Last stone wins and last stone loses.



Selection games: dates

PME 342. **by David L. Silverman**

In *The Game of the Century*, two players alternately select dates of the Twentieth Century (1 January 1901 – 31 December 2000) subject to the following restrictions:

(1) The first date chosen must be in 1901.

(2) Following the first play, each player, on his turn, must advance his opponent's last date by changing exactly one of the three "components" (day, month, year).

(3) Impossible dates such as 31 April or 29 February of a non-leap year are prohibited.

The player able to announce 31 December 2000 is the winner.

(a) What are the optimal responses by the second player to first player openings of 4 July 1901 and 25 December 1901?

(b) Who has the advantage and what is the optimal strategy?

(c) What is the maximum number of moves that can occur if both players play optimally?

Selection games: players select digits

DELTA 6.1-4. **by Philip Miles**

Two players take turns choosing digits for an infinite decimal expansion beginning from the decimal point. Player A wins if the result of this infinite game is an irrational number; player B wins if the result is rational. Which player can win and what is his winning strategy?

Selection games: players select integers

PME 388. **by David L. Silverman**

In the game of "Larger, But Not That Large", two players each write down a positive integer. The numbers are then disclosed and the winner (who is paid a dollar by the loser) is the player who wrote the larger number, unless the ratio of larger number to smaller is three or more, in which case the player with the smaller number wins. If the same number is picked by both players, no payment is made.

(a) What is the optimal strategy?

(b) Suppose, instead, that the players are not restricted to integers but to the set $[1, \infty)$ and that the larger number wins provided the larger-to-smaller ratio is less than r (for some $r > 1$); otherwise the larger number loses. Find an optimal strategy.

NAvW 405. **by N. G. de Bruijn**

Players P and Q play a game, of which the rules are determined by positive integers k, ℓ, and m. There is a countable set of markers labeled $1, 2, 3, \ldots$. Players P and Q move alternately; P moves first. Each move of P consists of taking k markers, and each move of Q consists of taking ℓ markers. Player P has a win as soon as his set of markers contains a sequence of m consecutive integers. Determine all cases (k, ℓ, m) where P has a winning strategy.

CRUX 418. **by James Gary Propp**

Given a sequence S consisting of n consecutive natural numbers with $n \geq 3$, two players take turns striking terms from S until only two terms a, b remain. If a and b are relatively prime, then the player with the first move wins; otherwise, his opponent does. For what values of n does the first player have a winning strategy, regardless of S?

JRM 658. **by Harry Nelson**

You are allowed to choose any integer y in the range $2 \leq y < b$, and then a random integer x is chosen in the same range. If $\gcd(x, y) = 1$, you lose; if $\gcd(x, y) > 1$, you win. Assuming you apply your best strategy:

(a) For what value of $b, 3 < b \leq 200$, do you have the lowest probability of winning?

(b) For what value of b do you have the highest probability of winning?

(c) Same questions for $3 < b \leq 2000000$.

JRM 558. **by Les Marvin**

Two players alternate in selecting integers from the set $1, 2, \ldots, n$ until all have been taken. (First player gets the last integer if n is odd.) First player wins if either player's total is prime. Otherwise the second player wins. For what n does the first player have the advantage? Same question for the *misère* version in which first player wins if both totals are composite.

Selection games: polynomials

CRUX 396. **by Viktors Linis**

Given is the following polynomial with some undetermined coefficients denoted by stars:

$$x^{10} + *x^9 + *x^8 + \cdots + *x^2 + x + 1.$$

Two players, in turn, replace one star by a real number until all stars are replaced. The first player wins if all zeros of the polynomial are imaginary, the second if at least one zero is real. Is there a winning strategy for the second player?

Tic-tac-toe variants

JRM 599. **by Les Marvin**

At this point in the incomplete game shown, two Tic-Tac-Toe players agreed to a draw. Only later did they discover that both were experts. (A Tic-Tac-Toe expert always exploits but never affords an opportunity to win.) Reconstruct the first and last moves.

X	O	O
		X
	X	O

CANADA 1978/5.

Eve and Odette play a game on a 3×3 checkerboard, with black checkers and white checkers. The rules are as follows:

1. They play alternately.
2. A turn consists of placing one checker on an unoccupied square of the board.
3. In her turn, a player may select either a white checker or a black checker and need not always use the same color.
4. When the board is full, Eve obtains one point for every row, column or diagonal that has an even number of black checkers, and Odette obtains one point for every row, column or diagonal that has an odd number of black checkers.
5. The player obtaining at least five of the eight points wins.

(a) Is a $4 - 4$ tie possible?

(b) Describe a winning strategy for the girl who is first to play.

JRM 508. **by David L. Silverman**

Felix and Rover play a variant of Tic-Tac-Toe on a 4×4 board. Rover wins if either player gets four of his marks on any of the four rows, four columns, or two main diagonals. Felix wins if neither player appropriates any of the ten lines. Does the player who moves first have the winning advantage?

AMM S10. **by Richard K. Guy and J. L. Selfridge**

When n-in-a-row (the generalization of tic-tac-toe) is played on a large enough board, it is easy to see that the first player has a winning strategy if $n = 1, 2, 3,$ or 4. There is a folk theorem that Go Moku ($n = 5$) is also a first-player win, but nothing has been proved for $5 \leq n \leq 8$. Show that the second player can force a draw if $n \geq 9$, no matter how large the board is.

JRM 572. **by David L. Silverman**

In the game of Go Moku, two players alternate in placing their marks on an infinite grid, the winner being the first player to get five of his marks adjacent in a vertical, horizontal, or diagonal row. Demonstrate a first-player win against any defense.

JRM 465. **by David L. Silverman**

In Kriegspiel Tic-Tac-Toe, the two players sit back to back, each with his own board. An umpire announces "No move" when a player attempts to occupy a cell already taken by his opponent and advises each player when his turn comes up. To offset opener's great advantage, he is penalized with a loss of turn when he receives a "no move" call. Second player is allowed to play at each turn until he makes a valid move. In one game the second player, O, received 3 straight "no move" calls, pinpointing X's position as shown:

–	–	X
–	X	–
O	X	O

Rating win, tie, and loss 1, 0, and -1 respectively and speaking game-theoretically, how should O continue?

JRM 389. **by Azriel Rosenfeld**

Tic-tac-toe can be regarded as played with integers, for example 1's and 0's rather than X's and O's; the player using 1's tries to fill some row, column, or diagonal so that it sums to 3, while the player using 0's tries to achieve sum 0. Consider the alternative versions of the game in which:

(a) The 1 and 0 players try to achieve sums 2 and 1, respectively.

(b) They try to achieve sums 2 and 0, respectively.

Prove that in version (a) the first player should always win, and in version (b), whichever player goes first, the 1 player should always win.

Yes or no questions

PENT 300. **by Kenneth M. Wilke**

Let A and B play a game according to the following rules:

Player A selects a positive integer. Player B then must determine the number chosen by A by asking not more than thirty questions, each of which can be answered by only no or yes.

What is the largest number that A can choose which can be determined by B in thirty questions? Generalize to n questions.

Affine transformations

AMM 6158.* **by M. J. Pelling**

Prove that if R is a bounded convex region of the plane of area 1, then there is a $d > 0$ independent of R such that R is equivalent under an area preserving affine transformation to a region of diameter $\leq d$. What is the best possible value of d?

Analytic geometry: circles

CRUX 315. **by Orlando Ramos**

Prove that, if two points are conjugate with respect to a circle, then the sum of their powers is equal to the square of the distance between them.

OSSMB G76.2-1.

Find the equations of two circles each of which passes through $(3, 1)$ and $(3, -1)$ and touches the line $x = y$.

OSSMB G76.3-1.

A circle is tangent to line l_1, $4x - 3y + 10 = 0$, at $B(-12)$ and also tangent to line l_2, $3x + 4y - 30 = 0$. Use vector methods to find the equation of the circle.

OSSMB G79.1-3.

Two circles touch the y-axis and intersect in the points $(1, 0)$ and $(2, -1)$. Find their radii and find the second common tangent.

OSSMB G79.2-8.

Two equal rectangles, both inscribed in the circle

$$x^2 + y^2 = 1$$

with their axes of symmetry along the x-axis and y-axis, respectively, cross each other forming a square $ABCD$ which is common to both rectangles.

(a) If θ is the acute angle between the diagonal and its major axis of symmetry, find, in terms of θ, the total area of the four rectilinear figures exterior to square $ABCD$.

(b) Find the value of $\tan\theta$ when this area is a maximum.

OSSMB G76.3-2.

Find the equation of the circle that cuts orthogonally each of the three circles

$$x^2 + y^2 + 2x + 17y + 4 = 0,$$
$$x^2 + y^2 + 7x + 6y + 11 = 0, \quad \text{and}$$
$$x^2 + y^2 - x + 22y + 3 = 0.$$

PENT 305. **by John A. Winterink**

If $(x - h)^2 + (y - g)^2 = r^2$ represents a circle tangent to three given circles, then (h, g, r) is called an Apollonian triple. Given the three circles

$$(x + 3)^2 + (y - 3)^2 = 6^2$$
$$(x - 1)^2 + (y + 5)^2 = 2^2$$
$$(x - 2)^2 + (y + 2)^2 = 1^2,$$

find all Apollonian triples (h, g, r) for the given circles such that h, g, and r are rational and such that $r > 0$.

AMM E2669. **by I. J. Schoenberg**

Let $a > b > 0$. For a given r, $0 < r < b$, there is a unique $R > 0$ such that the circle

$$(x - a + r)^2 + y^2 = r^2$$

lies inside and touches the circle

$$x^2 + (y - b + R)^2 = R^2.$$

For which r is R/r minimal?

OSSMB G75.2-4.

The circle $x^2 + y^2 - ax - ay = 0$ passes through the origin and also intersects the x and y axes at A and B respectively. From any point P on the circle, perpendiculars are drawn to meet the x-axis at L, the y-axis at M and AB at N. Prove that L, M, and N are collinear.

CRUX 109. **by Léo Sauvé**

(a) Prove that rational points (i.e. both coordinates rational) are dense on any circle with rational center and rational radius.

(b) Prove that if the radius is irrational the circle may have infinitely many rational points.

(c) Prove that if even one coordinate of the center is irrational, the circle has at most two rational points.

NYSMTJ 45. **by Sidney Penner and H. Ian Whitlock**

A point of a plane is rational if both of its coordinates are rational numbers.

(a) Show that there are three concentric circles on which there are exactly zero, one, and two rational points.

(b) Is there a circle on which there are exactly three rational points?

Analytic geometry: concyclic points

AMM E2697. **by William Anderson and William Simons**

Is there a dense subset S of the unit circle such that each point in S has rational coordinates and the (Euclidean) distance between any pair of points in S is also rational?

IMO 1975/5.

Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.

Analytic geometry: conics

CRUX 442. **by Sahib Ram Mandan**

Prove that the equation of any quartic may, in an infinity of ways, be thrown into the form

$$aU^2 + bV^2 + cW^2 + 2fVW + 2gWU + 2hUV = 0,$$

where $U = 0$, $V = 0$, and $W = 0$ represent three conics.

CRUX 469. **by Gali Salvatore**

Of the conics represented by the equations

$$\pm x^2 \pm 2xy \pm y^2 \pm 2x \pm 2y \pm 1 = 0,$$

how many are proper (nondegenerate)?

OSSMB G75.2-2.
A straight line inclined at an angle θ touches both curves $y^2 = 8x$ and $x^2+y^2 = 9$. Find, by analytic geometry, the values of θ and the x-intercept of the required lines.

MATYC 114. **by Dean Jordan**
Show that unless both of the equations

$$a_1x^2 - 2b_1xy - a_1y^2 + a_2x - b_2y + a_3 = 0$$
$$b_1x^2 + 2a_1xy - b_1y^2 + b_2x + a_2y + b_3 = 0$$

represent degenerate conics, the curves they describe intersect perpendicularly.

Analytic geometry: curves

MATYC 137. **by Aaron Seligman and Larry Cohen**
Let $y = f(x)$ be differentiable everywhere with $A = (a, b)$ and $f(a) \neq b$. Prove or disprove the following theorem and its converse: If $|AM|$ is the minimum distance from A to $f(x)$, then $AM \perp TM$ where TM is the line tangent to $f(x)$ at M.

Analytic geometry: ellipses

MM 1062. **by G. A. Edgar**
(a) Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be three points in the Cartesian plane. Assume the points and their negatives are all distinct. Show that there is an ellipse, centered at the origin, passing through the three points if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 & -1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & -1 \\ x_3 & y_3 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & -1 \end{vmatrix} > 0.$$

Interpret this condition geometrically.
(b) Find a necessary and sufficient condition for the existence of an ellipsoid, centered at the origin, passing through four given points in 3-space.

Analytic geometry: Euclidean geometry

OMG 17.1.8.
Prove, by the methods of analytic geometry, that if two medians of a triangle are equal, then the triangle is isosceles.

OMG 18.2.8.
Prove, using the methods of analytic geometry, that the diagonals of a rhombus are perpendicular.

OMG 18.3.8.
Prove, using the methods of analytic geometry, that a triangle is right-angled if the square on the hypotenuse equals the sum of the squares on the other two sides.

Analytic geometry: exponentials

CRUX 293. **by David R. Stone**
For which b is the exponential function $y = b^x$ tangent to the line $y = mx$? Conversely, given $y = b^x$, for which m is $y = mx$ tangent to $y = b^x$?

Analytic geometry: family of lines

PUTNAM 1977/A.1.
Consider all lines which meet the graph of

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points, say (x_i, y_i), $i = 1, 2, 3, 4$. Show that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line and find its value.

Analytic geometry: floor function

CANADA 1975/3.
Indicate on the (x, y)-plane the set of all points (x, y) for which $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 4$.

Analytic geometry: folium of Descartes

CRUX 417. **by John A. Tierney**
It is easy to guess from the graph of the Folium of Descartes,

$$x^3 + y^3 - 3axy = 0, \quad a > 0$$

that the point of maximum curvature is $(3a/2, 3a/2)$. Prove it.

Analytic geometry: lines

CRUX 480. **by Kenneth S. Williams**
In a Cartesian plane let l_1 and l_2 be two nonparallel lines intersecting in a point P and $Q(x_1, y_1)$ a point distinct from P. Let l be a line which does not pass through either P or Q, is not parallel to PQ, and intersects PQ at the point $R(x_2, y_2)$.
If $ax + by = c$, $a_1x + b_1y = c_1$, and $a_2x + b_2y = c_2$ are equations for l, l_1, and l_2, respectively, find, as simply as possible, the coordinates of R in terms of

$$a, b, c; \quad a_1, b_1, c_1; \quad a_2, b_2, c_2; \quad \text{and} \quad x_1, y_1.$$

Analytic geometry: locus

OSSMB 78-11.
The "taxicab" distance between 2 points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the cartesian plane is defined by

$$d(A, B) = |a_1 - b_1| + |a_2 - b_2|.$$

If $A = (-2, -2)$ and $B = (2, 2)$ find all points X on the "taxicab ellipses"
(a) $d(A, X) + d(B, X) = 8$,
(b) $d(A, X) + d(B, X) = 10$.

OSSMB 78-12.
Let $d(A, B) = |a_1 - b_1| + |a_2 - b_2|$. Then if $A = (-2, -2)$, $B = (2, 2)$, $C = (0, 3)$, and $D = (3, 7)$, find all points on the "taxicab bisectors"
(a) $d(C, X) = d(D, X)$,
(b) $d(A, X) = d(B, X)$.

Analytic geometry: polar curves

MATYC 104. **by Hung C. Li**

Let $\theta > 0$. Let the reciprocal spiral $r = 1/\theta$ intersect the lines PQ (passing through the pole P and perpendicular to the polar axis PT) and PT at $C_1, C_2, C_3, C_4, \ldots$ consecutively. Construct triangles PC_1C_2, PC_2C_3, PC_3C_4, $\ldots$. Find the sum of the areas of these infinitely many triangles.

TYCMJ 108. **by Arnold Lapidus**

Let Γ be the circle with center at the origin and radius $\sqrt{3}/2$. Let T and P be the points with polar coordinates $(\sqrt{3}/2, \theta)$ and $(\sqrt{3}/2, \pi/6)$, respectively, where $0 \le \theta < \pi/6$. Let A be the point on the line tangent to Γ at T such that $\angle TAP = \pi/3$. Define $S(\theta) = \frac{1}{2} - TA, 0 \le \theta < \pi/6$. Prove or disprove that $S(\theta) = \sin\theta$.

Analytic geometry: tangents

SSM 3756. **by Gregory Wulczyn**

Show that if a third-degree polynomial function is symmetric with respect to the origin, then there are infinitely many intervals $[a, b]$ such that the line joining $(a, f(a))$ to $(b, f(b))$ is a tangent line to the graph of the polynomial.

FUNCT 1.2.1.

(a) A curve has equation $y = 3x^4 - 4x^3 - 6ax^2 + 12ax$, where a is a positive constant. For what values of x does the curve have a horizontal tangent? Determine the nature of all stationary points if $0 < a < 1$, and if $a = 1$.

Sketch the curve when $a = 1$. State the coordinates of all stationary points but make no attempt to determine exactly the x-coordinates of any points (other than the origin) at which the curve crosses the x-axis.

(b) Extend the discussion to cover $a < 0$, $a = 0$, and $a > 1$.

Analytic geometry: triangles

CRUX 119. **by John A. Tierney**

A line through the first quadrant point (a, b) forms a right triangle with the positive coordinate axes. Find analytically the minimum perimeter of the triangle.

MATYC 106. **by Gino Fala**

Let T be the triangle in the plane whose vertices are $(-1, -1)$, $(1, -1)$, and $(2, 5)$. Find an equation $E(x, y) = 0$ for T.

Angle measures

OSSMB G75.1-5.

A river flows due north, and a vertical tower, CD, stands on its left bank. From a point A upstream and on the same bank as the tower, the elevation of the tower is 60°; and from a point B just opposite A on the other bank, the angle of elevation of the tower is 45°. If the tower is 150 feet high, find the width of the river.

Billiards

CRUX 137. **by Viktors Linis**

On a rectangular billiard table $ABCD$, where $AB = a$ and $BC = b$, one ball is at a distance p from AB and at a distance q from BC, and another ball is at the center of the table. Under what angle α (from AB) must the first ball be hit so that after the rebounds from AD, DC, and CB it will hit the other ball?

NAvW 475. **by I. J. Schoenberg**

Let E be an ellipse and n be an integer greater than or equal to 3. We think of E as the rim of a billiard table, the objective being to determine all closed billiard ball paths Π_n that are closed convex n-gons. This requires that, at each vertex of Π_n, the angle of incidence with E be equal to the angle of reflection. Prove the following:

(a) There is a 1-parameter family F_n of n-gons Π_n inscribed in E with the reflection property, the initial vertex of Π_n being chosen arbitrarily on E.

(b) All these Π_n are circumscribed to a fixed ellipse E_n confocal to E.

(c) All n-gons of the family F_n have the same (maximal) perimeter.

NAvW 476. **by I. J. Schoenberg**

Let E be an ellipse that we think of as the rim of a billiard table, the objective being to determine all convex quadrilaterals $Q = A_1A_2A_3A_4$ that are closed billiard ball paths. Equivalently, Q should have equal incidence and reflection angles at each A_i, and we call this "the reflection property."

Prove the following statements:

(a) Circumscribe to E an arbitrary rectangle

$$B_1B_2B_3B_4,$$

and let B_iB_{i+1} be tangent to E at A_i $(B_5 = B_1)$. Then

$$Q = A_1A_2A_3A_4$$

is a parallelogram having the reflection property, and the perimeter of Q is constant and equals $4\left(a^2 + b^2\right)^{1/2}$.

(b) The Q are circumscribed to an ellipse E_4, confocal to E, and having the semi-axes

$$a_4 = \frac{a^2}{\left(a^2 + b^2\right)^{1/2}},$$

$$b_4 = \frac{b^2}{\left(a^2 + b^2\right)^{1/2}},$$

where a and b are the semi-axes of E.

(c) The parallelograms Q give all convex quadrilateral billiard ball paths.

MM 1003. **by Richard Crandall and Peter Ørno**

Let P and Q be two distinct points in the interior of a circular disc with neither point at the center. With the boundary of the disc acting as a mirror, a ray of light from point P determines, by the successive reflections from the boundary, a polygonal path in the disc. This path is dependent on the initial direction of the ray of light. Given a positive integer k, show that there is such a path with the kth reflection of the ray intersecting Q.

With k, P, and Q given, can the number of such distinct paths be determined?

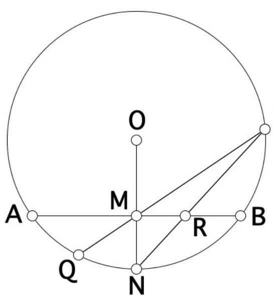
OSSMB 77-4.

Let P and Q be points inside $\triangle ABC$. Determine how to aim a ray from P so that, upon reflection by each of the sides of $\triangle ABC$, the ray goes through Q.

Butterfly problem

OSSMB 75-5. **by P. Erdős**
CRUX 75. **by R. Duff Butterill**
MM 949. **by P. Erdős and M. S. Klamkin**

Let AB be a chord of a circle, center O. Let ON be the radius perpendicular to AB, meeting AB at M. Let P be any point in the major arc AB, not diametrically opposite N. Let PM and PN determine Q and R, respectively, on the circle and AB. Prove that $RN > MQ$.



Cake cutting

PARAB 381.

A square cake has frosting on its top and on all four sides. Show how to cut it in order to serve nine people so that each one gets exactly the same amount of cake and exactly the same amount of frosting.

Circles: 2 circles

CRUX PS1-2.

If two circles pass through the vertex and a point on the bisector of an angle, prove that they intercept equal segments on the sides of the angle.

PME 338. **by Hung C. Li**

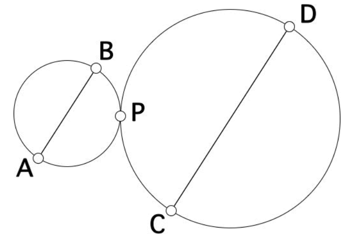
Let (O) be a circle centered at O with radius a. Let P, any point on the circumference of (O), be the center of circle (P). What is the radius of (P) such that it divides the area of (O) into two regions whose areas are in the ratio $s{:}t$?

SSM 3730. **by Fred A. Miller**

Let C_1 and C_2 be two concentric circles with radii r_1 and r_2 respectively, $r_1 > r_2$. Under what conditions is it possible to draw a line cutting both C_1 and C_2 so that the length of the chord intercepted by C_1 is twice the length of the chord intercepted by C_2? If this is possible, describe how it can be done.

ISMJ 11.3.
PARAB 401.

Show that if AB and CD are parallel diameters of two circles that are tangent at P then AD and BC intersect at P.



CRUX 62. **by F. G. B. Maskell**

Prove that if two circles touch externally, their common tangent is a mean proportional between their diameters.

NYSMTJ OBG5.

Two unit circles are drawn with centers O and O_1. One of the points of intersection is D. Let B and C be the points of tangency on the common tangent nearer to D. Segment OO_1 meets the two circles at points A and E. Find the length of OO_1 if mixtilinear triangles BCD and AED have the same area.

AUSTRALIA 1979/2.

Two circles in a plane intersect. Let A and B be the two points of intersection. Starting simultaneously from A two points P and Q move with constant speeds around different circles, each point traveling along its own circle in the same sense as the other point. The two points return to A simultaneously after one revolution. Prove

(a) P, B and Q are always collinear;

(b) that there is a fixed point S in the plane such that, at any time, the distances from S to the moving points are equal.

CRUX 63. **by H. G. Dworschak**

Given are two nonintersecting circles C_1 and C_2. From the center of C_1 both tangents are drawn to C_2. These tangents intersect C_1 at points P and Q. Points R and S on C_2 are obtained similarly. Prove that the chords PQ and RS are equal in length.

Circles: 3 circles

OMG 17.1.7.

Three circles are on the same side of a straight line and are tangent to the line. One of the circles has radius 4 and each of the three circles is tangent to the other two. Draw a diagram and then determine the radius of the two equal circles.

FUNCT 3.1.3.

Prove that the points of intersection of all common tangents to three circles are collinear.

PME 344. **by J. A. H. Hunter**

Three circles whose radii are a, b, and c are tangent externally in pairs and are enclosed by a triangle, each side of which is an extended tangent of two of the circles. Find the sides of the triangle.

Circles: 4 circles

SSM 3684. **by Donald L. Chambers**

Without using calculus, find the area of the region which is the intersection of the four circular regions which have, as their centers, the vertices of a square and the side of the square as radius.

CRUX 248. **by Dan Sokolowsky**

Circles (M) and (N) are externally tangent at point P and mutually circumscribed by circle (O). Point Q is the center of the circle inscribed in the mixtilinear triangle bounded by (M), (N), and (O). The diameter of (Q) parallel to the line containing points M, N, O, and P is given by FG. Point W is the radical center of circles (M), (N), and (O). Prove that WQ is equal to the circumradius of $\triangle PFG$.

Circles: arcs

SSM 3695. **by Steven R. Conrad**

An equilateral Gothic arch ABC is made by drawing line segment AC, circular arc AB with center C, and circular arc BC with center A. A circle is inscribed in this Gothic arch, tangent to arcs AB and BC and also to line segment AC. If $AC = 24$, find the area of this inscribed circle.

SSM 3724. **by Alan Wayne**

A plane figure $ABCD$ consists of two parallel, circular arcs AD and BC, together with two line segments AB and DC, each of length a. If the arcs AD and BC have lengths s and t respectively, find a formula for the area K in terms of s, t, and a.

OMG 17.2.2.

A section of railway track 5000 meters long was laid in the desert. Because of the heat during the day, the workmen put the track down during the cool of the night and securely fastened each end. In the heat of the following day the section of track expanded by 1 meter in length. If the track bowed upwards, how high would the center of the track be above the ground level?

Circles: area

MATYC 93. **by Elliott Hartman**

Three circles A, B, and C have radii equal to 6, 4, and 2, respectively. Circles B and C are externally tangent to one another and both are tangent to A internally. Find the area of the largest possible circle that is interior to A and exterior to B and to C.

Circles: chords

CANADA 1975/5.

Let A, B, C and D be four "consecutive" points on the circumference of a circle and P, Q, R and S be points on the circumference which are respectively the midpoints of the arcs AB, BC, CD and DA. Prove that PR is perpendicular to QS.

CRUX 466. **by Roger Fischler**

Let AB and BC be arcs on a circle such that arc $AB >$ arc BC and let D be the midpoint of arc AC. If $DE \perp AB$, show that $AE = EB + BC$.

CRUX 225. **by Dan Sokolowsky**

Let C be a point on the diameter AB of a circle. A chord through C, perpendicular to AB, meets the circle at D. Two chords through B meet CD at T_1, T_2 and arc AD at U_1, U_2 respectively. It is known that there are circles C_1 and C_2 tangent to CD at T_1 and T_2 and arc AD at U_1 and U_2 respectively. Prove that the radical axis of C_1 and C_2 passes through B.

CRUX 110. **by H. G. Dworschak**

(a) Let AB and PR be two chords of a circle intersecting at Q. If A, B, and P are kept fixed, characterize geometrically the position of R for which the length of QR is maximal.

(b) Give a Euclidean construction for the point R which maximizes the length of QR, or show that no such construction is possible.

PARAB 289.

In a circle of radius 5, we have two parallel chords CB and ED of lengths 8 and 6, respectively. Let CD and EB be extended to meet at A. Let AF be an altitude of the triangle ABC. Calculate the length of AF.

CRUX 220. **by Dan Sokolowsky**

Let C be a point on the diameter AB of a circle. A chord through C, perpendicular to AB, meets the circle at D. A chord through B meets CD at T and arc AD at U. Prove that there is a circle tangent to CD at T and to arc AD at U.

SSM 3688. **by Fred A. Miller**

Prove that if two chords of a circle intersect at right angles, then the sum of the squares of the lengths of the four segments formed is equal to the square of the length of the diameter.

Circles: circumference and diameter

OMG 16.1.2.

A string is stretched tightly around the equator of a perfect sphere the size of the earth, i.e., 6400 km radius. Six meters more string is added, and the whole circle of string is raised equally above the surface. What approximately will the height of the string above the surface be?

Circles: inscribed rectangles

MSJ 447. **by Michael Massimilla**

Tom, Dick, and Harry faced the problem of creating a baseball-like diamond within a circular field. Tom decided that it would be a good idea to inscribe a rectangle in the field. Dick decided to place one base at the midpoint of each side of the rectangle. Finally, Harry decided to locate the pitcher's mound at the very center of the field. In this makeshift diamond, the distance from the pitcher's mound to first base was 15 meters and the distance from first base to the edge of the field was 12 meters. What was the 'distance around the bases' in this diamond?

Circles: interior point

CANADA 1977/2.
OMG 16.2.2.

Let O be the center of a circle and A a fixed interior point of the circle different from O. Determine all points P on the circumference of the circle such that the angle OPA is a maximum.

JRM 535. **by Sherry Nolan**

Points A, B, and C are selected on a circle and point P inside the circle so that the perimeter of the quadrilateral $ABCP$ is equal to the circumference of the circle.

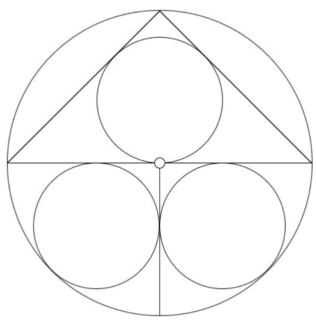
(a) Prove that P cannot be the center of the circle.

(b) If B is fixed, determine the positions of A, P, and C that maximize the area of $ABCP$.

Circles: isosceles right triangles

JRM 370. **by Leon Bankoff**
OSSMB 78-13.

An isosceles right triangle is inscribed in a semicircle, and the radius bisecting the other semicircle is drawn. Circles are inscribed in the triangle and in the two quadrants as shown. Prove that these three smaller circles are equal.



Circles: line segments

OMG 18.1.4.

Given is a circle with center O and radius OD. Points A, B, and C are selected such that B is on the circumference of the circle, C is on OD, OA and BC are perpendicular to OD, and AB is parallel to OD. If $OC = 5$ and $CD = 1$, find the length of AC.

Circles: mixtilinear triangles

PME 362. **by Zelda Katz**

A diameter AB of a circle (O) passes through C, the midpoint of a chord DE. Let M be the midpoint of arc AB, and let MC meet the circle again at P. The radius OP cuts the chord DE at Q. Point O_1 is the center of the circle on AC as diameter. Point O_2 is the center of the circle on BC as diameter. Point W_1 is the center of the circle inscribed in the mixtilinear triangle bounded by (O), (O_1), and CE. Point W_2 is the center of the circle inscribed in the mixtilinear triangle bounded by (O), (O_2), and CE.

Show that $DQ = W_1W_2$.

Circles: orthogonal circles

MM 1020. **by Leon Gerber**

For $i = 1, 2$, and 3, let the circle C_i have center (h_i, k_i) and radius r_i. Find a determinant equation for the circle orthogonal to these three given circles which generalizes the well-known result for the circle through three points.

Circles: surrounding chains

SPECT 9.7. **by J. G. Brennan**

A chain of six unit circles are each externally tangent to a central unit circle, and tangent to the preceding and following one of the chain. A chain of six circles each of radius r are such that each is externally tangent to two of the unit circles and each member of the chain is tangent to the preceding and following one of the chain. Find a quadratic equation, one of whose roots is r. What is the geometrical significance of the other root of the quadratic equation?

PME 428. **by Solomon W. Golomb**

One circle of radius a may be "exactly surrounded" by 6 circles of radius a. It may also be exactly surrounded by n circles of radius t, for any $n \geq 3$, where

$$t = a(\csc \frac{\pi}{n} - 1)^{-1}.$$

Suppose instead that we surround it with $n{+}1$ circles, one of radius a and n of radius b (again $n \geq 3$). Find an expression for b/a as a function of n.

Circles: tangents

AMM E2625. **by Hüseyin Demir**

Let A_i, $i = 0, 1, 2, 3 \pmod 4$ be four points on a circle Γ. Let t_i be the tangent to Γ at A_i, and let p_i and q_i be the lines parallel to t_i passing through the points A_{i-1} and A_{i+1}, respectively. If

$$B_i = t_i \cap t_{i+1},$$
$$C_i = p_i \cap q_{i+1},$$

show that the four lines B_iC_i have a common point.

SSM 3710. **by Steven R. Conrad**

Tangents TA and TB are drawn to points A and B of a circle, and an arbitrary point P is selected on arc AB. Prove that the perpendicular from P to AB is the mean proportional between the perpendiculars drawn from P to TA and TB.

Combinatorial geometry: concyclic points

AMM E2789. **by Doug Hensley**

Suppose $\gcd(n, 30) = 1$ and $n \geq 13$. Let S_n be a set of n points equally spaced around a circle. Show that there are $\left(n^2 - 1\right)/12$ incongruent triangles with vertices in S_n. Show further that their areas are distinct when n is a prime.

Combinatorial geometry: counting problems

ISMJ 14.21.

In the plane, n circles are drawn so that every two distinct circles meet in exactly two points and no three of the circles have a common point. Give a formula for the number of regions into which the circles partition the plane.

OMG 14.3.3.

Into how many regions do n planes divide space if no two planes are parallel and no four intersect at a point?

PARAB 412.

Consider a convex polygon with n vertices, and suppose that no three of its diagonals meet at the same point inside the polygon. Determine

(a) the total number of line segments into which the diagonals are divided by their points of intersection, and

(b) the total number of regions into which the figure is divided by all its diagonals.

ISMJ J11.15.

Every room of a house has an even number of doors. Show that the number of doors leading directly to the outside must be even.

Combinatorial geometry: equilateral triangles

FQ B-413. **by Herta T. Freitag**

For every positive integer n, let U_n consist of the points $j+ke^{2\pi i/3}$ in the Argand plane with $j \in \{0, 1, 2, \ldots, n\}$ and $k \in \{0, 1, \ldots, j\}$. Let $T(n)$ be the number of equilateral triangles whose vertices are subsets of U_n.

(a) Obtain a formula for $T(n)$;

(b) Find all n for which $T(n)$ is an integral multiple of $2n + 1$.

Combinatorial geometry: intervals

PARAB 284.

You are given 50 intervals on a line. Prove that at least one of the following statements about those intervals is true:

(a) There are 8 intervals, all of which have at least one point in common.

(b) There are 8 intervals so that no two of them have a common point.

Combinatorial geometry: lines in plane

AMM E2754. **by Jim Fickett**

Given n arbitrary lines $k_1, \ldots, k_n$ in the plane, need there exist another n lines $h_1, \ldots, h_n$ having the same intersection pattern but with all intersection points rational? The first condition means that for every subset S of $\{1, \ldots, n\}$, we have

$$\bigcap_{i\in S} k_i \neq \varnothing \iff \bigcap_{i\in S} h_i \neq \varnothing.$$

Combinatorial geometry: packing problems

AMM E2612. **by Sidney Penner**

How many diamonds can be packed in a Chinese checkerboard? This board consists of two order 13 triangular arrays of holes, overlapping in an order 5 hexagon, 121 holes in all. A diamond consists of four marbles that fill four adjacent holes.

Combinatorial geometry: planes

OMG 15.3.1.

What is the number of intersection points of 4 planes if no two are parallel and no three intersect in a straight line?

OMG 15.3.10.

What is the number of intersection points of 5 planes if no two are parallel and no three intersect in a straight line?

Combinatorial geometry: points in space

PARAB 437.

Two hundred points are distributed in space so that no three are collinear and no four are coplanar. Prove that it is possible to draw 10,000 line segments joining them without completing a single triangle.

Combinatorial geometry: polygons

NYSMTJ 38. **by Richard Bury**

Find the maximum number of points of intersection of the diagonals of an n-gon.

Combinatorial geometry: triangles

AMM E2736. **by E. Ehrhart**

Let Δ be a closed triangle and $P_0, A_0, P_1, A_1, \ldots$ an infinite sequence of points in a plane. Assume that $P_i \neq P_{i+1}$, $A_i \neq A_{i+1}$, each A_i is a vertex of Δ and the midpoint of the segment $[P_i, P_{i+1}]$, and $[P_i, P_{i+1}] \cap \Delta = \{A_i\}$.

Prove that $P_n = P_0$ for some positive n.

Combinatorial geometry: triangulations

PARAB 395.

A polygon is said to be triangulated when diagonals, no two of which cross, are drawn cutting the polygon into triangles. A polygon other than a triangle can be triangulated in more than one way.

(a) Show that a triangulated n-gon is always cut into $n - 2$ triangles by $n - 3$ diagonals.

(b) Show that there are at least two vertices of a triangulated polygon, each of which lies in a single triangle.

Concyclic points

CRUX 173. **by Dan Eustice**

For each choice of n points on the unit circle ($n \geq 2$), there exists a point on the unit circle such that the product of the distances to the chosen points is greater than or equal to 2. Moreover, the product is less than or equal to 2 for all points on the unit circle if and only if the n points are the vertices of a regular polygon.

Conics

CRUX 279. **by F. G. B. Maskell**

Three collinear points A, O, and B are given such that O is between A and B, and $AO \neq OB$. Show that the three conics having two focii and one vertex at the three given points intersect in two points.

NAvW 484. **by J. T. Groenman**

Let A_i ($i = 1, 2, 3, 4$) be four points on a given conic K. Let B_{ij} be the midpoints of A_iA_j and ℓ_{ij} the line through B_{ij} conjugated with respect to K, to the line A_kA_ℓ opposite A_iA_j.

Prove that the six lines ℓ_{ij} have one common point S and specify the position of this point S.

Geometry

NAvW 490. **by O. Bottema**

The rectangular coordinates (x, y) of the vertices of the triangle $A_1A_2A_3$ are given: $A_1 = (-a, 0)$, $A_2 = (a, 0)$, $A_3 = (p, h)$, $a > 0$, $0 \le p < a$. The circumscribed Steiner ellipse K of a triangle is defined as the conic passing through the vertices, the tangents at these points being parallel to the opposite sides. The fourth intersection S of K and the circumcircle C is called Steiner's point. Determine the limiting position of S if a and p are constants and $h \to 0$.

AMM E2751. **by Paul Monsky**

Let X be a conic section. Through what points in space do there pass three mutually perpendicular lines, all meeting X?

Constructions: angle bisectors

NAvW 553. **by J. T. Groenman**

Construct a scalene triangle ABC such that the external bisectors of angles A and B are of equal length, given the measurements: $\angle C = \gamma$ and $AB = c$. Show that this construction is only possible if $\gamma < 60^\circ$.

Constructions: angles

CRUX 96. **by Viktors Linis**

By Euclidean methods divide a 13° angle into thirteen equal parts.

CRUX 420. **by J. A. Spencer**

Given an angle AOB, find an economical Euclidean construction that will quadrisect the angle. "Economical" means here using the smallest possible number of Euclidean operations: setting a compass, striking an arc, drawing a line.

ISMJ 14.22.

A piece of cardboard is cut in a certain shape, where $PQ = TS = 1$, $QR = 2$, and the curve is a circular arc centered at Q. The angles at T, R, and S are right angles. To use this device to trisect an angle AOB, place it so that Q lies on TU, R lies on OB, and OA is tangent to the circle. Prove that TU trisects $\angle AOB$.

PME 412. **by Solomon W. Golomb**

Are there examples of angles which are trisectible but not constructible? That is, can you find an angle α which is not constructible with straightedge and compass, but such that, when α is given, $\alpha/3$ can be constructed from it with straightedge and compass?

TYCMJ 119. **by Thomas E. Elsner**

The following construction is well known as a false trisection of an angle. For a given angle $Z \le \pi$, construct a circle with center on the vertex of Z and label as A and B the intersections of the circle with the rays of the angle. Label as M and N, respectively, the diametric points opposite to A and B. Construct diameter EF as bisector of angle Z with $F \in AB$, and bisect each of these half angles with radii ending at G and H, $G \in AF$, $H \in BF$. Now label K as the intersection of lines EG and MF and L as the intersection of lines EH and NF. Then ZK and ZL approximately trisect angle Z. What is the greatest error in any of the trisection angles for Z?

PME 341. **by Jack Garfunkel**

Prove that the following construction trisects an angle of 60°. Triangle ABC is a $30^\circ - 60^\circ - 90^\circ$ right triangle inscribed in a circle. Median CM is drawn to side AB and extended to M' on the circle. Using a marked straightedge, point N on AB is located such that CN extended to N' on the circle makes $NN' = MM'$. Then CN trisects the 60° angle ACM.

TYCMJ 75. **by Norman Schaumberger**

Find an integer-sided right triangle such that each of its angles can be trisected with straightedge and compasses.

Constructions: chords

NYSMTJ 73. **by John J. Sullivan**

In a given circle, construct a chord of given length which is part of a line passing through a point exterior to the given circle.

USA 1979/4.

Show how to construct a chord BPC of a given angle A through a given point P such that $1/BP + 1/PC$ is a maximum.

PENT 321. **by Fred A. Miller**

In a circle whose center is at O, radii OA and OB are drawn. Construct a chord that will be trisected by radii OA and OB.

Constructions: circles

ISMJ 11.11.
ISMJ 12.5.

Given two circles, show how to construct with straightedge and compass a circle whose area is the sum of the areas of the two given circles.

USA 1975/4.

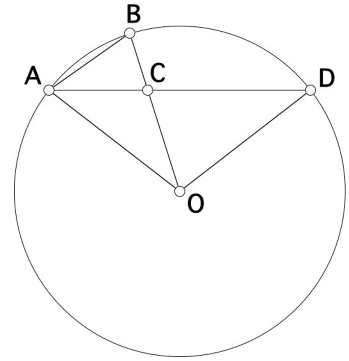
Two given circles intersect in two points P and Q. Show how to construct a segment AB passing through P and terminating on the two circles such that $AP \cdot PB$ is a maximum.

MSJ 466.

Let C be a given circle and A a point outside of C. Construct a line through A intersecting C at points P and Q so that $PQ = 2(AP)$.

CRUX 284. **by W. A. McWorter, Jr.**

Given a sector AOD of a circle with B on arc AD, can a straightedge and compass construct the line OB so that AB = AC?



ISMJ J10.12.

A circle is to be inscribed in a quadrant of a circle of radius R so that it touches all three sides of the quadrant. Find its radius and show how to construct the circle using straightedge and compass.

OSSMB 76-6.

Establish the following method, known as Swale's method, to determine the radius of a circle given just the circumference:

With any point O on the circumference C construct a circle D to cut C at P and Q. With center Q and the same radius, cut off the point R on D inside C. Let PR meet C at L. Then QL (and also LR) is the radius of C.

PARAB 423.

Given two intersecting straight lines a and b and a point P on b, show how to construct a circle whose center is on b and which passes through P and touches a.

Constructions: compass only

CRUX 125. **by Bernard Vanbrugghe**

Using compass only, determine the center of a given circle.

Constructions: conics

CRUX 325. **by Basil C. Rennie**

It is well known that if you put two thumbtacks in a drawing board and a loop of string around them you can draw an ellipse by pulling the string tight with a pencil. Now suppose that instead of the two thumbtacks, you use an ellipse cut out from plywood. Will the pencil in the loop of string trace out another ellipse?

CRUX 242. **by Bruce McColl**

Give a geometrical construction for determining the focus of a parabola when two tangents and their points of contact are given.

Constructions: equilateral triangles

CRUX 463. **by Jack Garfunkel**

Construct an equilateral triangle so that one vertex is at a given point, a second vertex is on a given line, and the third vertex is on a given circle.

NYSMTJ 54.
NYSMTJ OBG6. **by Aaron L. Buchman**

(a) Given three coplanar parallel lines, construct an equilateral triangle having one vertex on each line.

(b) Suppose the parallel lines are not coplanar; is the construction still possible?

Constructions: line segments

MATYC 85. **by Robert Forster**

Given is a linear distance ℓ. Find an equation or algorithm that will divide ℓ into a given number of segments p such that the segments are in geometric proportion.

MM Q637. **by Bertram Ross**

Bisect a line segment with a straightedge given only a line parallel to it.

Constructions: lines

CRUX 488. **by Kesiraju Satyanarayana**

Given a point P within a given angle, construct a line through P such that the segment intercepted by the sides of the angle has minimum length.

Constructions: parallel lines

ISMJ 12.10.

Let A, B, and C be three given points in the plane. Determine whether it is possible to draw equidistant parallel lines through these points and show how such lines might be found.

Constructions: pentagons

CRUX 428. **by J. A. Spencer**

Let AOB be a right-angled triangle with legs $OA = 2OB$. Use it to find an economical Euclidean construction of a regular pentagon whose side is not equal to any side of $\triangle AOB$. "Economical" means here using the smallest possible number of Euclidean operations: setting a compass, striking an arc, drawing a line.

Constructions: points

ISMJ 11.14.

Suppose you are given that somewhere on the side AB of the pentagon $ABCDE$ there is a point M such that DM divides the pentagon into two quadrilaterals of equal area. Show how to construct DM.

JRM 538. **by Harold Wyatt**

A quadrilateral $ABCD$ is drawn on a sheet of paper. Let E be the intersection of the diagonals, P the intersection of AB and CD, R the intersection of PE and AD, and Q the intersection of AD and BC.

(a) How can R be obtained by Euclidean construction when P does not lie on the sheet of paper?

(b) Assume that both P and Q lie off the sheet, but PQ intersects the sheet in the segment MN. Show how to obtain MN by Euclidean construction.

Constructions: quadrilaterals

ISMJ J10.5.

Show how to construct a quadrilateral if you are given the four angles and a pair of opposite sides.

Constructions: rectangles

ISMJ 13.24.

The point P is on one side of a parallelogram $ABCD$. Show how to construct (with compass and straightedge) a rectangle with P as one vertex and the other vertices on the other three sides of the parallelogram.

ISMJ 11.10.

Show how to construct a rectangle whose area is equal to that of a given pentagon (not necessarily regular).

Constructions: right triangles

MSJ 480.

Construct a right triangle with hypotenuse of length 12, if it is given that two of its medians are perpendicular.

Constructions: rulers

PARAB 265.

If you are required to make an exact copy of an irregular hexagon given a ruler and a protractor, what is the least number of measurements you would have to make?

If you had no protractor could you still do it? If so, would a greater number of measurements be needed?

What would be the least number of measurements required to copy an irregular polygon with n sides?

NAvW 402. **by O. Bottema**

Show that any construction in the plane with ruler and compass can also be performed by means of the ruler only, if a triangle, its circumcircle, and one of the following points are given:

(1) its centroid,
(2) its orthocenter,
(3) its incenter.

Prove that the statement does not hold if a triangle, its circumcircle, and its symmedian point are given.

FUNCT 2.5.1. **by Gordon C. Smith**

Let $\angle BAC$ be any angle. Construct BB' parallel to AC, and BP perpendicular to BB'. Mark a length equal to twice BA on a ruler. Place your ruler on the point A, turn it and slide it until the marked length has its ends on BP and BB', with G on BP and D on BB'.

Prove that $\angle DAC$ is 1/3 of $\angle BAC$.

Constructions: rusty compass

CRUX 492. **by Dan Pedoe**

(a) A segment AB and a rusty compass of span $r \geq \frac{1}{2}AB$ are given. Show how to find the vertex C of an equilateral triangle ABC using, as few times as possible, the rusty compass only.

(b) Is the construction possible when $r < \frac{1}{2}AB$?

JRM 505. **by Sherry Nolan**

Given a point P on a line L, construct a perpendicular through P using straightedge and rusty compass. In how few applications of the rusty compass can the task be done?

Constructions: squares

CRUX 127. **by Viktors Linis**

Let A, B, C, and D be four distinct points on a line. Construct a square by drawing two pairs of parallel lines through the four points.

CRUX 32. **by Viktors Linis**

Construct a square given a vertex and a midpoint of one side.

CRUX 44. **by Viktors Linis**

Construct a square $ABCD$ given its center and any two points M and N on its two sides BC and CD, respectively.

PME 453. **by Jack Garfunkel**

Given two intersecting lines and a circle tangent to each of them, construct a square having two of its vertices on the circumference of the circle and the other two on the intersecting lines.

JRM 466. **by Vincent J. Sealy**

Given is a triangle ABC. Construct a square with two sides meeting at A and with the other two sides containing B and C, respectively.

Constructions: straightedge only

CRUX 257. **by W. A. McWorter, Jr.**

Can one draw a line joining two distant points with a BankAmericard?

CRUX 338. **by W. A. McWorter, Jr.**

Can one locate the center of a circle with a VISA card?

ISMJ 13.20.
ISMJ 13.14.

Given a line ℓ and a point P not on ℓ on a piece of lined paper, show how to construct the line parallel to ℓ through P using a straightedge alone. Do not assume P is on one of the printed lines.

Constructions: trapezoids

NYSMTJ 59.

Construct a trapezoid, given both bases and both diagonals.

Constructions: triangles

CRUX 415. **by A. Liu**

Is there a Euclidean construction of a triangle given two sides and the radius of the incircle?

ISMJ J10.4.

Show how to locate the vertices B and C of a triangle ABC if you are given the point A, the circumcenter of $\triangle ABC$, and the centroid of $\triangle ABC$.

MM 1054. **by Jerome C. Cherry**

(a) Show how to construct triangle ABC by straightedge and compass, given side a, the median m_a to side a, and the angle bisector t_a to side a.

(b) Show how to construct triangle ABC by straightedge and compass, given angle A, m_a, and t_a.

SSM 3642. **by Ed Silver and Philip Smith**

Construct triangle ABC given angle A, side a, and a segment $b + c$ equal in length to the sum of the triangle's other two sides.

JRM 562. **by Michael J. Messner**

Watson was busily engaged in constructing the three altitudes of a triangle. He had just swung three intersecting semicircular arcs from the three vertices, using the same radius, when he got an emergency call. "I say, Holmes, can you finish the job for me?" he asked. "Certainly, my dear fellow, and using the compass only twice more." How did the great detective plan to do it?

CRUX 379. **by Peter Arends**

Construct a triangle ABC, given angle A and the lengths of side a and t_a (the internal bisector of angle A).

CRUX 454. **by Ram Rekha Tiwari**

(a) Is there a Euclidean construction for a triangle ABC given the lengths of its internal angle bisectors t_a, t_b, and t_c?

(b) Find formulas for the sides a, b, and c in terms of t_a, t_b, and t_c.

CRUX 288. **by W. J. Blundon**

Show how to construct (with compass and straightedge) a triangle given the circumcenter, the incenter and one vertex.

CRUX 472. **by Jordi Dou**

Construct a triangle given side b and circumradius R such that the line joining circumcenter and incenter is parallel to side a.

CRUX 476. **by Jack Garfunkel**

Construct an isosceles right triangle such that the three vertices lie each on one of three concurrent lines, the vertex of the right angle being on the inside line.

CRUX 120. **by John A. Tierney**

Given a point P inside an arbitrary angle, give a Euclidean construction of the line through P that determines with the sides of the angle a triangle

(a) of minimum area;

(b) of minimum perimeter.

ISMJ 12.6.

Construct a triangle given the lengths of its three medians. Can any three numbers be the lengths of the medians of a triangle?

MATYC 99. **by Aleksandras Zujus**

Using only straightedge and compass, construct triangle ABC, given the measure of $\angle A$ and the medians m_b and m_c.

Convexity

MATYC 126. **by Gino Fala**

Let G represent a convex polygon in the plane with perimeter $|G|$ and enclosed area $\|G\|$. Encircle G with a smooth curve C in the plane of G such that C is at a constant distance r from G. Denote the perimeter of C and the area enclosed by C by $|C|$ and $\|C\|$, respectively. Prove that:

$$|C| = |G| + 2\pi r$$

and

$$\|C\| = \|G\| + r|G| + \pi r^2.$$

AMM E2714. **by M. J. Pelling**

Let G_1 and G_2 be two bounded convex regions in $\mathbb{R}^2$, and suppose G_1 is translated to $G_1(t)$ by the transformation

$$x \to x + ta,$$

where a is a fixed unit vector. Consider the area $A(t)$ of

$$G_1(t) \cap G_2$$

as a function of t. Is it always true that there is a constant c such that $A(t)$ is monotonic increasing for $t \le c$ and monotonic decreasing for $t \ge c$?

What happens in $\mathbb{R}^n$?

PUTNAM 1979/B.5.

In the plane, let C be a closed convex set that contains $(0,0)$ but no other point with integer coordinates. Suppose that $A(C)$, the area of C, is equally distributed among the four quadrants. Prove that $A(C) \le 4$.

PARAB 416.

Let S be a convex area which is symmetric about the point O. Show that the area of any triangle drawn in S is less than or equal to half the area of S.

OSSMB 75-10.

Consider a plane convex set K that has a center of symmetry. Prove that a circumscribing parallelogram P of minimum area contacts K at the midpoints of its four sides.

AMM 6089.* **by E. Ehrhart**

Let K be a convex body in $\mathbb{R}^n$ of Jordan content

$$V(K) > \frac{(n+1)^n}{n!}$$

and with centroid at the origin. Does $K \cup (-K)$ contain a convex body C, symmetric about the origin, for which $V(C) > 2^n$?

Covering problems

PARAB 279.
MSJ 502.

On each side of a convex quadrilateral, a circle is drawn having that side as diameter. Prove that every point inside the quadrilateral lies inside at least one of the 4 circles.

ISMJ J10.6.

A square one unit on each side is to be covered by two circular discs of the same size (overlapping is permitted). How small can the discs be?

AMM E2790. **by Mark D. Meyerson**

Suppose we have a collection of squares, one each of area $1/n$ for $n = 1, 2, 3, \ldots$, and any open set, G, in the plane. Show that we can cover all of G except a set of area 0 by placing some of the squares inside G without overlap. (The edges of the squares are allowed to touch.)

NAvW 411. **by J. van de Lune**

Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of (closed) squares with corresponding areas $(a_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} a_n$ diverges.

Prove that it is possible to cover the plane by means of the given "pile of tiles" (overlap permitted).

Cyclic polygons

MSJ 416. **by Albert Wilansky**

A polygon inscribed in a circle has congruent angles. Must it also have congruent sides?

ISMJ 12.29.

If a polygon inscribed in a circle has equal angles, must its sides all be equal?

Cyclic quadrilaterals

OSSMB G75.2-3.

Given is a cyclic quadrilateral with sides a, b, c, d and perimeter $2s$. Show that the total area of this quadrilateral is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

CRUX PS7-2. **by Jan van de Craats**

Let $A_1A_2A_3A_4$ be a kite (i.e., $A_1A_2 = A_1A_4$ and $A_3A_2 = A_3A_4$) inscribed in a circle. Show that the incenters I_1, I_2, I_3, and I_4 of the respective triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$ are the vertices of a square.

AMM E2553. **by V. B. Sarma**

Suppose that A, B, C, and D are cyclic points and that the Simson line of A with respect to triangle BCD is perpendicular to the Euler line of triangle BCD. Show that the Simson line of B will be perpendicular to the Euler line of triangle CDA. Is this true if we replace "perpendicular" by "parallel"?

CRUX 483. **by Stanley Collings**

Let $ABCD$ be a convex quadrilateral; let $AB \cap DC = F$ and $AD \cap BC = G$; and let I_A, I_B, I_C, and I_D, be the incenters of triangles BCD, CDA, DAB, and ABC, respectively. Prove that:

(a) $ABCD$ is a cyclic quadrilateral if and only if the internal bisectors of the angles at F and G are perpendicular.

(b) If $ABCD$ is cyclic, then $I_AI_BI_Cl_D$ is a rectangle. (*) Is the converse true?

Cycloids

NAvW 438. **by O. Bottema**

The circles $C = (M; R)$ and $c = (m; r)$ are given in the coinciding planes U and u respectively. The plane u moves with respect to U in the following way: c remains tangent to C at a point that moves along C with velocity V and along c with velocity v, such that $V = \lambda v$, λ being a constant. Show that, except for some special values of λ, the motion is cycloidal.

Discs

PARAB 328.

Six circular discs are lying in the plane so that no one of them covers the center of another. Show that there is no point in common to all six discs.

Dissection problems: angles

PENT 299. **by Kenneth M. Wilke**

Devise a method for dividing a 17° angle into seventeen equal parts.

Dissection problems: equilateral triangles

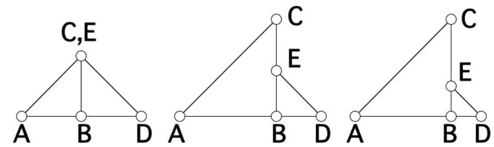
CRUX 256. **by Harry L. Nelson**

Prove that an equilateral triangle can be dissected into five isosceles triangles, n of which are equilateral, if and only if $0 \le n \le 2$.

Dissection problems: isosceles right triangles

PME 416. **by Scott Kim**

Each of the three figures shown is composed of two isosceles right triangles, $\triangle ABC$ and $\triangle DBE$, where $\angle ABC$ and $\angle DBE$ are right angles, and B is between points A and D. Points C and E coincide so that $CB/EB = 1$ in the first figure. In the second figure, we are given that $CB/EB = 2$, and in the third figure that $CB/EB = 3$. Consider each pair of triangles as a single shape, and suppose that the areas of the three shapes are equal. For each pair of figures, find the minimum number of pieces into which the first figure must be cut so that the pieces may be reassembled to form the second figure. Pieces may not overlap, and all pieces must be used in each assembly.



Dissection problems: line segments

CRUX 158. **by André Bourbeau**

Devise a Euclidean construction to divide a given line segment into two parts such that the sum of the squares on the whole segment and on one of its parts is equal to twice the square on the other part.

Dissection problems: partitions of the plane

CRUX 170. **by Leroy F. Meyers**

Is it possible to partition the plane into three sets A, B, and C (so that each point of the plane belongs to exactly one of the sets) in such a way that

(i) a counterclockwise rotation of 120° about some point P takes A onto B, and

(ii) a counterclockwise rotation of 120° about some point Q takes B onto C?

Dissection problems: polygons

PARAB 330.

Certain convex polygons can be dissected into squares and equilateral triangles all having the same length of sides. If a convex polygon can be dissected in this way, how many sides did it have originally?

Dissection problems: rectangles

AMM 6178. **by Robert Kowalski**

Define the shape of a rectangle to be the ratio of the longer side to the shorter side. Suppose one has an unlimited number of congruent squares at one's disposal. Given shape α and an error ε, what is the least number of squares one needs to construct a rectangle whose shape differs from α by less than ε?

ISMJ J10.8.

Show that any triangle can be cut into three pieces that can be rearranged to form a rectangle whose area is the same as that of the triangle.

Dissection problems: regular pentagons

MM 1057. **by D. M. Collison**

Dissect a regular pentagon into six pieces and reassemble the pieces to form three regular pentagons whose sides are in the ratio 2:2:1.

Dissection problems: regular polygons

CRUX 308. **by W. A. McWorter, Jr.**

Some restaurants give only one pat of butter with two rolls. To get equal shares of butter on each roll, one can cut the butter square along a diagonal with a knife.

(a) What regular n-gons can be cut in half with only a straightedge?

(b) What convex n-gons can be cut in half with a straightedge and compass?

Dissection problems: right triangles

MSJ 460. **by Zalman Usiskin**
MSJ 461. **by Zalman Usiskin**

Any right triangle can be partitioned into three triangles similar to it. Prove that no other triangles can be partitioned into three triangles similar to it.

MSJ 428. **by Robert Lam**

Suppose that ABC is a right triangle, with right angle at C. Construct a line perpendicular to AB that divides triangle ABC into two regions of equal area.

Dissection problems: squares

PARAB 286.

Show how to cut up and reassemble five squares of side length 1 into a single square.

PARAB 310.

A man had a square window with sides of length 1 meter. However, the window let in too much light and so he blocked up one-half of it. How did he do this in such a way as to still have a square window which was 1 meter high and 1 meter wide?

PARAB 339.

Given is a square made up of 100 squares arranged in 10 rows and 10 columns. The first, second, third, and fourth squares in the first, second, third, and fourth rows, respectively, are colored black. Show how to dissect the square into 4 congruent pieces, each containing one of the black squares.

PME 380. **by V. F. Ivanoff**

Form a square from a quadrangle by bisecting segments and the angles.

PARAB 320.

A large square is divided into one small square (with sides of length s) and four rectangles A, B, C, and D which are not squares. No side of any rectangle is the same length as a side of another nor the side of the big square. The sides of A are $4s$ and $2s$. Rectangle B has the largest area of any of the rectangles. Rectangle C has sides in the ratio 3:1 and its area is 300. Find the area of D.

ISMJ 11.15.

Given five squares each of side length 1. Show how to cut them up and reassemble them to form a single square.

MSJ 499.

Prove that if n is a positive integer greater than 5, then it is possible to subdivide a square into n smaller squares whose sides are parallel to the sides of the original square.

PARAB 334.

Find all positive integers n such that it is impossible to dissect a square into n squares.

PARAB 356.

A suitor asking for the hand of the king's daughter is given the following task:

Divide the square wall of the princess's room into ten smaller squares, a different way for each day of the week. No square should have the same 4 vertices as any square used on previous days. Is it possible for the suitor to marry the princess, or will he end up on the chopping block?

CRUX 29. **by Viktors Linis**

Cut a square into a minimal number of triangles with all angles acute.

Dissection problems: triangles

PARAB 435.

Show how to cut a square piece of paper into acute triangles.

OSSMB 79-4. **by M. Poirier**

Let ABC be a triangle with an obtuse angle at A. Show that it is possible to partition ABC into smaller triangles all having only acute angles. What is the least number of line segments required to obtain such a partition?

CRUX 24. **by Viktors Linis**

A paper triangle has base 6 cm and height 2 cm. Show that by three or fewer cuts the sides can cover a cube of edge 1 cm.

JRM 593. **by Nobuyuki Yoshigahara**

Nine matches are arranged to form a $2 \times 3 \times 4$ triangle. Place two additional matches in such a way as to divide the triangle into two equal areas.

CRUX 200. **by Léo Sauvé**

(a) Prove that there exist triangles that cannot be dissected into two or three isosceles triangles.

(b) Prove or disprove that, for $n \geq 4$, every triangle can be dissected into n isosceles triangles.

ISMJ 10.2.

What triangles can be partitioned into 3 congruent triangles?

PME 448. **by R. Robinson Rowe**

Analogous to the median, call a line from a vertex of a triangle to a point of trisection of the opposite side a "tredian". Then, if both tredians are drawn from each vertex, the 6 lines will intersect at 12 interior points and divide the area into 19 subareas, each a rational part of the area of the triangle. Find two triangles for which each subarea is an integer, one being a Pythagorean right triangle and the other with consecutive integers for its three sides.

Geometry

Ellipses

CRUX 419. **by G. Ramanaiah**

A variable point P describes the ellipse $x^2/a^2+y^2/b^2 = 1$. Does it make sense to speak of "the mean distance of P from a focus S"? If so, what is this mean distance?

CRUX 180. **by Kenneth S. Williams**

Through O, the midpoint of a chord AB of an ellipse, is drawn any chord POQ. The tangents to the ellipse at P and Q meet AB at S and T, respectively. Prove that $AS = BT$.

OSSMB G79.3-4.

Let P be any point on the ellipse $b^2a^2 + a^2y^2 = a^2b^2$ having foci F_1, F_2. Show that the circles drawn with diameters PF_1 and PF_2 are tangent to the circle having center at the origin, and diameter the major axis.

PUTNAM 1976/B.4.

For a point P on an ellipse, let d be the distance from the center of the ellipse to the line tangent to the ellipse at P. Prove that $(PF_1)(PF_2)d^2$ is constant as P varies on the ellipse, where PF_1 and PF_2 are the distances from P to the foci F_1 and F_2 of the ellipse.

OSSMB G77.2-5.

An ellipse and a hyperbola have the same foci. Show that the two curves intersect at right angles.

MM Q660. **by Alan Wayne**

Find the ratio of the area of an ellipse to the area of the largest inscribed rectangle.

CRUX 132. **by Léo Sauvé**

If $\cos\theta \neq 0$ and $\sin\theta \neq 0$ for $\theta = \alpha,\ \beta,\ \gamma$, prove that the normals to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points of eccentric angles $\alpha,\ \beta,\ \gamma$ are concurrent if and only if

$$\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0.$$

OSSMB G78.3-3.

Let any tangent to an ellipse meet the tangents at the ends of the major axis at P_1 and P_2. Show that the circle having P_1P_2 as diameter passes through the foci.

OSSMB G76.2-2.

Show that the portion of any tangent to the ellipse $2x^2 + y^2 = 1$ intercepted between the lines $x = 1$ and $x = -1$ is divided (by the point of tangency) into two parts that subtend equal angles at the origin.

Envelopes

MM 1068.* **by James Propp**

Given a simple closed curve S, let the "navel" of S denote the envelope of the family of lines that bisect the area within S.

(a) If S is a triangle, find sharp upper and lower bounds for the ratio of the area within the navel of S to the area within S.

(b) If S bounds a convex set, find a sharp upper bound for this ratio.

(c) If S is arbitrary, find a sharp upper bound for this ratio.

Equilateral triangles: exterior point

SSM 3714. **by Charles W. Trigg**

From a point in the exterior of an equilateral triangle, the distances to the vertices of the triangle are 5, 4, and 3 respectively. Determine the length of a side of the triangle.

Equilateral triangles: interior point

OSSMB 75-7. **by Maurice Poirier**
CRUX 39. **by Maurice Poirier**
SSM 3682. **by Alan Wayne**

Let P be a point inside an equilateral triangle ABC such that $PA = 3$, $PB = 4$, and $PC = 5$. Determine the length of the side of the triangle.

Equilateral triangles: isosceles triangles

OMG 17.2.9.

Triangle ABC is drawn inside an equilateral $\triangle ADE$ so that $AB = AC = \sqrt{7}$, $BC = 1$, and $DB = CE = 2$. Find the length of one side of $\triangle ADE$.

Equilateral triangles: midpoints

SPECT 11.5.

A sum and product are defined on the points of the plane as follows: $A + B$ is the unique point such that A, B, and $A + B$ form an equilateral triangle, described in a counterclockwise direction, and $A \times B$ is the midpoint of the straight line joining A and B. Show that

$$A \times (B + C) = (B + A) \times (A + C).$$

Equilateral triangles: orthogonal projection

MM 988. **by Murray S. Klamkin**

A given equilateral triangle ABC is projected orthogonally from a given plane P to another plane P'. Show that the sum of the squares of the sides of triangle $A'B'C'$ is independent of the orientation of triangle ABC in plane P.

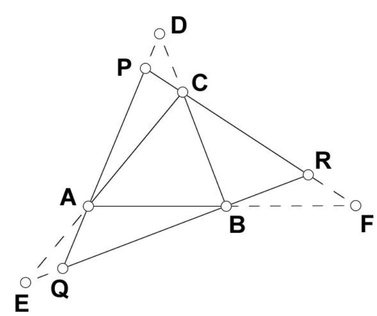
Equilateral triangles: sides

MM 1014. **by K. R. S. Sastry**

Given a triangle, $\triangle ABC$, points D, E, and F are on the lines determined by BC, CA, and AB, respectively. The lines AD, BE, and CF intersect to form triangle $\triangle PQR$, and satisfy $AD = BE = CF$.

(a) Show that $\triangle PQR$ is equilateral iff $\triangle ABC$ is.

(b) Express the area of $\triangle PQR$ in terms of that of $\triangle ABC$.



CRUX 412. **by Kesiraju Satyanarayana**

The sides BC, CA, and AB of $\triangle ABC$ are produced to D, E, and F, respectively so that $CD = AE = BF$. Show that $\triangle ABC$ is equilateral if $\triangle DEF$ is equilateral.

Equilateral triangles: similar triangles

PME 387. **by Jack Garfunkel**

On the sides AB and AC of an equilateral triangle ABC, mark the points D and E respectively, such that $AD = AE$. Erect directly similar equilateral triangles CDP, AEQ, BAR on CD, AE, and AB respectively. Show that triangle PQR is equilateral. Also show that the midpoints of PE, AQ, and RD are vertices of an equilateral triangle.

Fallacies

CRUX 141. **by Leon Bankoff**

What is wrong with the following "proof" of the Steiner-Lehmus Theorem?

If in a triangle two angle bisectors are equal, then the triangle is isosceles.

At the midpoints of the angle bisectors, I erect two perpendiculars which meet in O; with O as center and AO as radius, I describe a circle which will evidently pass through the points A, M, N, C.

Now the angles MAN, MCN are equal since the measure of each is arc $\frac{MN}{2}$; hence $BAC = ACB$, and triangle ABC is isosceles.

Family of lines

MSJ 419. **by Sidney Penner**

A point in the plane is called a rational point if its coordinates are rational. Let L be the set of lines determined by lattice points of the plane. Let Q be the set of lines determined by the rational points of the plane.

(a) Show that L is a proper subset of Q and characterize all equations in the xy plane that define lines that are elements of Q but which are not elements of L.

(b) Does there exist a point in the plane that is not on a line of L but is on a line of Q?

Grazing goat

JRM 395. **by R. S. Johnson**

A farmer has a circular fenced field, of radius 200 feet, in which two goats are grazing. The goats are not friendly; consequently they are tethered by long ropes, each rope being secured to diametrically opposite fence posts. One tether permits a grazing area of one-half of the field; the other permits grazing of one-third of the field.

One day the farmer discovered that the goats were fighting furiously, and hastily shortened the tethers.

What was the area of the inadvertent "battle arena"?

JRM 710. **by Bruce E. Bushman**

A farmer ties his cow to a pole in a grassy field with a ten-foot rope. After the cow has grazed all of the grass within reach, the farmer moves the pole to the edge of the grazed area. He then lengthens the rope just enough to allow the cow an ungrazed area equal to what it had originally. How long is the rope?

PENT 282. **by Kenneth M. Wilke**

A farmer has a circular plot of radius 50 feet. At a point on the circumference of the plot, he places a stake to which a goat is connected by a rope. How long is the rope if the goat can graze on exactly one-half of the area of the plot?

PME 382. **by R. Robinson Rowe**

Two cows, Lulu and Mumu, are tethered at opposite ends of a 120-foot rope threaded through a knothole in a post of a straight fence separating two uniform pastures. How much area can they graze, presuming they eat, nap, and ruminate on identical schedules, and the rope length is also the extreme reach from muzzle to muzzle of Lulu and Mumu? If Mumu is replaced by the heifer Nunu with half the appetite, what is the area accessible to Lulu and Nunu?

CRUX 89. **by Vince Bradley and Christine Robertson**

A goat is tethered to a point on the circumference of a circular field of radius r by a rope of length l. For what value of l will it be able to graze over exactly half of the field?

Heptagons

PARAB 422.

The heptagon $ABCDEFG$ is inscribed in a circle and three of its angles are 120°. Prove that the heptagon has two equal sides.

Hexagons

PARAB 340.

Let O be the center of a circle C of radius r. Let A be the vertex of a regular hexagon inscribed in C. Using A and the other vertices of the hexagon as centers, arcs of radius r are drawn. The result is a six-petaled "flower". Next are drawn the largest circles that will fit between petals, for example C_1. Then the next largest, C_2, is drawn, and so on. What are the radii of the circles C_1, C_2, C_3, and so on?

PME 438. **by Ernst Straus**

Prove that the sum of the lengths of alternate sides of a hexagon with concurrent major diagonals inscribed in the unit circle is less than 4.

MATYC 107. **by Roger Debelak**

Hexagon $ABCDEF$ is inscribed in a circle. Triangle ACE is equilateral. Show that the sum of the lengths of the three diagonals AD, BE, and CF is equal to its perimeter.

MATYC 121. **by F. David Hammer**

A hexagon with three sides of length a and three sides of length b is inscribed in a circle. What is the radius?

ISMJ 12.28.

A hexagon is inscribed in a convex decagon so that the area A is a maximum. Show that there is a hexagon of area A in that decagon whose vertices are vertices of the decagon.

MM 992. **by Kenneth Fogarty, Erwin Just, and Norman Schaumberger**

Call a vertex of a convex hexagon ordinary if it is the intersection of at least three diagonals or sides of different lengths. Otherwise, let the vertex be called exceptional.

(a) Prove that at least one vertex of a convex hexagon is ordinary.

(b) What is the maximum number of exceptional vertices that a convex hexagon can have?

Hyperbolas

OSSMB G78.3-4.

A circle is described with a focus of the hyperbola $9x^2 - 16y^2 = 144$ as center, and with radius 1/4 of the length of the latus rectum. Show that the lines joining the points of intersection of the circle and the hyperbola to the focus are parallel to the asymptotes.

CRUX 15. **by H. G. Dworschak**

Let A, B, and C be three distinct points on a rectangular hyperbola. Prove that the orthocenter of $\triangle ABC$ lies on the hyperbola.

OSSMB G79.3-3.

The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at any point P, meets the asymptotes at Q and R. Show that the area of the triangle OQR, where O is the origin, is constant for all positions of the point P.

Inequalities: area

SPECT 10.7.

Let S be any finite system of similarly-oriented squares of equal size in the plane. Denote by $A(S)$ the total area covered by S. Show that it is always possible to find a discrete subsystem T of S such that $A(T) \geq \frac{1}{6}A(S)$.

Inequalities: cyclic quadrilaterals

DELTA 5.1-2. **by R. S. Luthar**

Let $ABCD$ be a cyclic quadrilateral with AC and BD as its diagonals. Prove that

$$(AD - BC)^2 + (AB - DC)^2 \geq (AC - BD)^2.$$

Inequalities: points in plane

PARAB 302.

Let A, B, C, and D be four points, in that order, on a straight line.

(1) If $AB' = CD'$, show that for any point P in the plane, $PA' + PD' \geq PB' + PC'$.

(2) Conversely, if $PA' + PD' \geq PB' + PC'$ for every position of P, show that $AB' = CD'$.

Inequalities: polygons

ISMJ 10.9.

For a polygon P_n of n sides let p be its perimeter and let d be the maximum distance between two points of the polygon. Let $\pi(P_n) = p/d$.

(a) Show that $\pi(P_n) > 2$ for any polygon with n sides.
(b) Show that for any triangle P_3, $\pi(P_3) \leq 3$.
(c) Find a triangle P_3 such that $\pi(P_3) = 3$.
(d) Find the rectangle R with largest possible $\pi(R)$.

Inequalities: quadrilaterals

CRUX 106. **by Viktors Linis**

Prove that, for any quadrilateral with sides a, b, c, d,

$$a^2 + b^2 + c^2 > \frac{1}{3}d^2.$$

NAvW 488. **by W. J. Blundon and R. H. Eddy**

Let $ABCD$ be a quadrilateral inscribed in a circle of radius R and circumscribed about a circle of radius r. If s is the semiperimeter of the quadrilateral, prove the inequalities

$$s \leq \sqrt{4R^2 + r^2} + r$$

and

$$s^2 \geq 8r\left(\sqrt{4R^2 + r^2} - r\right),$$

and find when equality holds in each case. Hence, derive the inequalities

$$s \leq 2R + \left(4 - 2\sqrt{2}\,\right) r$$

and

$$s^2 \geq \frac{32\sqrt{2}}{3}Rr - \frac{16}{3}r^2,$$

again stating when equality holds.

SPECT 10.1. **by B. G. Eke**

Show that the sum of the lengths of the diagonals of a plane quadrilateral exceeds the sum of the lengths of two opposite sides.

Inequalities: rectangles

ISMJ 13.16.

Let $ABCD$ be a rectangle with point P in its interior. Let the distances from P to A, B, C, and D be a, b, c, and d respectively, and let α be the area of the rectangle.

(a) Show that $a^2 + b^2 + c^2 + d^2 \geq 2\alpha$.
(b) Can equality ever occur? If so, when?

Inequalities: right triangles

PME 431. **by Jack Garfunkel**

In a right triangle ABC, with sides a, b, and hypotenuse c, show that $4(ac + b^2) \leq 5c^2$.

Inequalities: squares

PARAB 350.

Let $ABCD$ be a square of side 1. Suppose P lies on BC, Q lies on DC, and that $AP = AQ$. Show that the perimeter of the triangle APQ is not more than $2 + \sqrt{2}$.

Inequalities: triangles

AMM E2634. **by Jack Garfunkel**

Let A_i, $i = 0, 1, 2 \pmod 3$ be the vertices of a triangle, and let Γ be its inscribed circle with center O. Let B_i be the intersection of the segment A_iO with Γ and let C_i be the intersection of the line A_iO with the side $A_{i-1}A_{i+1}$.

Prove that

$$\sum A_iC_i \leq 3 \sum A_iB_i.$$

CRUX 397. **by Jack Garfunkel**

Given is $\triangle ABC$ with incenter I. Lines AI, BI, and CI are drawn to meet the incircle (I) for the first time in D, E, and F, respectively. Prove that

$$(AB + DE + CF)\sqrt{3}$$

is not less than the perimeter of the triangle of maximum perimeter that can be inscribed in the incircle.

PME 368. **by Jack Garfunkel**

Given is a triangle ABC with its inscribed circle (I). Lines AI, BI, CI cut the circle in points D, E, F respectively. Prove that

$$AD + BE + CF \geq \frac{\partial DEF}{\sqrt{3}} .$$

AMM E2716. **by Jack Garfunkel**

Let ABC be a triangle with P an interior point. Let A', B', and C' be the points where the perpendiculars drawn from P meet the sides of ABC. Let A'', B'', and C'' be the points where the lines joining P to A, B, and C meet the corresponding sides of ABC. Prove or disprove that

$$A'B' + B'C' + C'A' \leq A''B'' + B''C'' + C''A''.$$

OSSMB 76-8.

Suppose BC is the longest side of $\triangle ABC$. Let a point O be chosen anywhere inside the triangle and let AO, BO, CO cut the opposite sides in A', B', C' respectively. Prove that

$$OA' + OB' + OC' < BC.$$

AMM E2517. **by Alex G. Ferrer**

Let P be a point interior to the triangle ABC, and let r_1, r_2, and r_3 be the distances of P from the sides of the triangle. If p denotes the perimeter of the pedal triangle, show that

$$\sum (r_1 + r_2) \cos \frac{1}{2} C \leq p.$$

When does equality occur?

MM Q651. **by Geoffrey Kandall**

Given any triangle ABC. Divide BC (respectively, AC, AB) into n equal segments by means of points A_i (respectively, B_i, C_i), $i = 1, 2, \ldots, n-1$. Prove that

$$\sum_{i=1}^{n-1} \left\{ (AA_i)^2 + (BB_i)^2 + (CC_i)^2 \right\} = \frac{(n-1)(5n-1)}{6n}(a^2 + b^2 + c^2).$$

SPECT 9.5. **by B. G. Eke**

The triangle T_1 lies inside the triangle T_2. Show that the perimeter of T_1 is shorter than that of T_2.

ISMJ J11.6.

Prove that the sum of the lengths of the legs of a right triangle does not exceed the length of the diagonal of the square on the hypotenuse.

PME 450. **by Clayton W. Dodge**

In $\triangle ABC$, let $\angle A \leq \angle B \leq \angle C$. Prove that

$$s \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} (R + r)\sqrt{3} \text{ if and only if } \angle B \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \pi/3,$$

where s is the semiperimeter, r the inradius, and R the circumradius of $\triangle ABC$.

PARAB 274.

A triangle has area 1 and sides of length a, b, c, where $a \geq b \geq c$. Prove that $b \geq \sqrt{2}$.

FUNCT 2.3.5.

If a side of a triangle is of length less than the average length of the other two sides, show that its opposite angle is less, in magnitude, than the average of the other two angle magnitudes.

TYCMJ 94. **by Martin Berman**

Given a, $b+c$ and angle A $(0 < A < \pi)$, prove that there exists a triangle ABC if and only if $a < b+c \leq a/\sin(A/2)$.

PME 435. **by David R. Simonds**

Two noncongruent triangles are "almost congruent" if two sides and three angles of one triangle are congruent to two sides and three angles of the other triangle. Clearly two such triangles are similar. Show that the ratio of similarity k is such that $\phi^{-1} < k < \phi$, where $\phi = (1 + \sqrt{5})/2$, the golden ratio.

Isosceles right triangles

CRUX 33. **by Viktors Linis**

On the sides CA and CB of an isosceles right-angled triangle ABC, points D and E are chosen such that $CD = CE$. The perpendiculars from D and C on AE intersect the hypotenuse AB in K and L respectively. Prove that $KL = LB$.

OMG 17.3.7.

The isosceles right triangle EFG in a certain diagram has a vertex at the center of the square $ABCD$. Determine the area of the common quadrilateral given $BC = 7$, $FG = 8$, $HD = 2$.

Ladders

CRUX 122. **by Jeremy Wheeler**

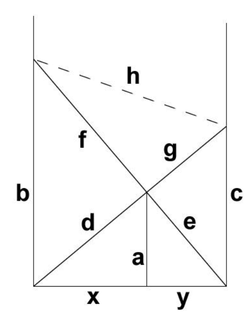
I had leant my ladder up against the side of the house to paint my bedroom window and found that it just reached the bottom of the window. My son was pushing a box around and was just able to get it under the ladder. The box was a 1-meter cube and the ladder was 4 meters long. How high was the bedroom window off the ground?

JRM 793. **by Harry L. Nelson**

There is an alley between two buildings with ladders extending across the alley from the base of each building to the side of the other. The two ladders are not the same length.

(a) Find all solutions such that all of the labeled lengths except h are integers and the lengths of the ladders are each less than 200 units.

(b) Find the solution with the smallest length for the longer ladder such that all lengths, including h, are integers.



PME 413. **by R. Robinson Rowe**

The new tall building on one side of the alley was vertical, but on the other side the old low building, having settled, leaned toward the alley. Projected, its face would have met the top of the tall building and would have been one foot longer than the height of the tall building. The ladders, unequal in length, rested against the buildings 21 feet above the ground and crossed 12 feet above the ground. How high was the tall building and how wide was the alley?

OSSMB G76.3-3.

The angular elevation of a tower CD at a place A due south of it is 30° and at a place B 100 feet due west of A is 18°. Without the use of tables, find the heights of the tower to the nearest 1/10 foot.

Lattice points: circles

TYCMJ 53. **by Sidney Penner**

Let h, k, and n be integers and assume that the circle defined by $(x-h)^2+(y-k)^2=n$ contains a point with rational coordinates. Prove or disprove that the circle must also contain a lattice point.

Lattice points: collinear points

CRUX 408. **by Michael W. Ecker**

A zigzag is an infinite connected path in a Cartesian plane formed by starting at the origin and moving successively one unit right or up. Prove or disprove that for every zigzag and for every positive integer k, there exist (at least) k collinear lattice points on the zigzag.

PARAB 342.

Let S be the set of all points in the Cartesian plane whose coordinates (x, y) are both integers such that $0 \le x \le 100$, $0 \le y \le 100$. Show that however one chooses 5 points P_1, P_2, P_3, P_4, P_5 from S, at least one pair of these points has the property that the straight line through them contains a third point of S (possibly, but not necessarily, another of the chosen points). Does the statement remain true if 5 is replaced by 4?

PARAB 375.

A cornfield has 1000 cornstalks. When the farmer stands at a cornstalk at the corner of the field, he notices that some of the cornstalks line up with the one he is standing at. On closer examination, it turns out that the number of these lines which contain an odd number of other cornstalks is odd. Is this true no matter which cornstalk he stands at?

Lattice points: convexity

CRUX 495. **by J. L. Brenner**

Let S be the set of lattice points (points having integral coordinates) contained in a bounded convex set in the plane. Denote by N the minimum of two measurements of S: the greatest number of points of S on any line of slope 1, -1. Two lattice points are adjoining if they are exactly one unit apart. Let the n points of S be numbered by the integers from 1 to n in such a way that the largest difference of the assigned integers of adjoining points is minimal. This minimal largest difference we call the discrepancy of S.

(a) Show that the discrepancy of S is no greater than $N+1$.

(b) Give such a set S whose discrepancy is $N+1$.

(c)* Show that the discrepancy of S is no less than N.

Lattice points: counting problems

CRUX 275. **by Gilbert W. Kessler**

Given are the points $P(a, b)$ and $Q(c, d)$, where a, b, c, and d are all rational. Find a formula for the number of lattice points on segment PQ.

Lattice points: ellipses

AMM E2682. **by Douglas Hensley**

Let E be an ellipse in the plane whose interior area $A \ge 1$. Prove that the number n of integer points on E satisfies $n < 6A^{1/3}$.

Lattice points: equilateral triangles

PARAB 398.

Show that there is no equilateral triangle whose vertices are lattice points in the plane.

Lattice points: mappings

AMM E2633. **by Benjamin G. Klein**

Two points x and y in $\mathbb{Z}^n$ are said to be neighbors if

$$y - x = \pm e_i$$

for some $i = 1, \ldots, n$ ($e_1, \ldots, e_n$ is the canonical basis of $\mathbb{Z}^n$). A subset $S \subset \mathbb{Z}^n$ is said to be permutable if there is a bijection $T\colon S \to S$ such that for each $x \in S$, Tx and x are neighbors. Show that if a finite subset $S \subset \mathbb{Z}^n$ is permutable, then card(S) is even.

Find necessary and sufficient conditions for a subset $S \subset \mathbb{Z}^2$ to be permutable.

Lattice points: maxima and minima

OMG 15.1.2.

What is the greatest number of noncollinear points you can select such that the midpoint of any line joining any pair of selected points is not a lattice point?

Lattice points: n-dimensional geometry

TYCMJ 129. **by Warren Page**

For any n^m+1 ($n \ge 2$) lattice points in m-space, prove there is at least one pair of points $\{P, Q\}$ such that $(P-Q)/n$ is a lattice point.

Lattice points: squares

PARAB 397.

The smallest square on a pegboard has unit area.

(a) Show how to construct squares of areas 8 and 10.

(b) Prove that it is not possible to construct a square of area $4n+3$, where n is an integer.

Lattice points: triangles

PARAB 392.

Prove that, out of any 9 lattice points, it is always possible to choose 3 with the property that the center of gravity of the triangle formed by them is also a lattice point.

Limiting figures

CRUX 422. **by Dan Pedoe**

The lines l and m are the parallel edges of a strip of paper and P_1, Q_1, are points on l and m, respectively. Fold P_1Q_1 along l and crease, obtaining P_1Q_2 as the crease. Fold P_1Q_2 along m and crease, obtaining P_2Q_2. Fold P_2Q_2 along l and crease, obtaining P_2Q_3. If the process is continued indefinitely, show that the triangle $P_nP_{n+1}Q_{n+1}$ tends towards an equilateral triangle.

AMM 6062. **by B. H. Voorhees**

Consider an infinite sequence of regular n-gons such that each $(n+1)$-gon is contained within the preceding n-gon and is of maximal area consistent with this constraint. Take the first element of this sequence as an equilateral triangle having unit area. Is the limit of this sequence a point or a circle? If it is a circle, determine its area.

CRUX 416. **by W. A. McWorter Jr.**

Let A_0BC be a triangle and a a positive number less than 1. Construct P_1 on A_0B so that $A_0P_1/A_0B = a$. Construct A_1 on P_1C so that $P_1A_1/P_1C = a$. Inductively construct P_{n+1} on A_nB so that $A_nP_{n+1}/A_nB = a$ and construct A_{n+1} on $P_{n+1}C$ so that $P_{n+1}A_{n+1}/P_{n+1}C = a$. Show that all the P_i are on a line and all the A_i are on a line, the two lines being parallel.

Locus: angles

OSSMB G76.3-4.

Two straight lines meet at a fixed point A so that the angle formed is a fixed angle, θ. The lines at A are intersected by a third line at K and L such that KL is of fixed length. Describe the locus of the center of the circumcircle to $\triangle AKL$.

SSM 3781. **by Michael Brozinsky**

An immortal ant starts at A, crawls along a perpendicular to radius OB, then along a perpendicular to radius OA, then along a perpendicular to OB again, and so on ad infinitum. Find the distance covered by the ant if $\angle AOB = 30°$ and OA has length 1 inch.

Locus: circles

CANADA 1976/4.

Let AB be a diameter of a circle, C be any fixed point between A and B on this diameter, and Q be a variable point on the circumference of the circle. Let P be the point on the line determined by Q and C for which $\frac{AC}{CB} = \frac{QC}{CP}$. Describe, with proof, the locus of the point P.

OSSMB G75.3-3.

The circle $x^2 + y^2 = r^2$ and points $B(m, 0)$, $C(n, 0)$, with $m + n \neq 0$, are given. Let Q and R be the ends of an arbitrary diameter of the circle and let QB and RC intersect at P. Determine the locus of P.

CRUX 177. **by Kenneth S. Williams**

Let P be a point on the diameter AB of a circle whose center is C. On AP and BP as diameters, circles are drawn. The point Q is the center of a circle that touches these three circles. What is the locus of Q as P varies?

USA 1976/2.

If A and B are fixed points on a given circle and XY is a variable diameter of the same circle, determine the locus of the point of intersection of lines AX and BY. You may assume that AB is not a diameter.

PME 447. **by Zelda Katz**

A variable circle touches the circumferences of two internally tangent circles.

(a) Show that the center of the variable circle lies on an ellipse whose foci are the centers of the fixed circles.

(b) Show that the radius of the variable circle bears a constant ratio to the distance from its center to the common tangent of the fixed circles.

(c) Show that this constant ratio is equal to the eccentricity of the ellipse.

OSSMB G78.1-4.

A wheel of radius R with its center at the origin rotates in the xy-plane. A rod of length $2R$ has one end pivoted at the rim of the wheel and the other end is free to move along the positive x-axis. Find the equation of the locus traced by the midpoint of the rod.

SPECT 7.7.

Distinct points L and M are given in the plane, and k is a real number such that $0 < k < 1$. Then the locus of all points X in the plane, such that $LX/MX = k$, is a circle (Apollonius' Circle). A tangent is drawn through M to touch the circle at T. Show that $\angle TLM = 90°$.

Locus: conics

CRUX 370. **by O. Bottema**

If K is an inscribed or escribed conic of the given triangle $A_1A_2A_3$, and if the points of contact on A_2A_1, A_3A_1, and A_1, A_2 are T_1, T_2, and T_3, respectively, then it is well-known that A_1T_1, A_2T_2, and A_3T_3 are concurrent in a point S. Determine the locus of S if K is a parabola.

Locus: ellipses

SSM 3777. **by Irwin K. Feinstein**

Consider the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ with foci F_1 and F_2. Let P be a point on the ellipse, ℓ be the tangent to the ellipse at P, and Q be the foot of the perpendicular from F_2 to ℓ. As P moves around the ellipse, describe the motion of Q.

NYSMTJ 60.

When a carpenter's square is rotated around a circle in such a way that the two sides remain tangent to the circle, a pen held at the vertex of the right angle would trace a circle concentric with the original circle. When the same process is completed, starting with an ellipse, what is traced out?

Locus: equal distances

OSSMB G76.1-2.

Find the equation of the locus of a point that moves so that it is always equidistant from the line $x + 3 = 0$, and the circle $x^2 + y^2 = 25$.

Locus: equilateral triangles

OSSMB G77.2-6.

The sum of the squares of the distances of a moving point from the sides of a fixed equilateral triangle is a constant. Identify the locus of the point and find its equation.

Locus: lines

SSM 3788. **by Michael Brozinsky**

Describe the locus of points in the plane the sum of the squares of whose distances from n distinct straight lines is a constant k (where k is such that the locus is real).

Locus: linkages

CRUX 479. **by G. P. Henderson**

A car, of wheelbase L makes a left turn in such a way that the locus of A, the point of contact of the left front wheel, is a circle of radius $R > L$. B is the point of contact of the left rear wheel. Before the turn, the car was traveling in a straight line with A moving toward the circle along a tangent. Find the locus of B.

JRM 472. **by Friend H. Kierstead, Jr.**

A garage door is divided into two sections: A is hinged to the garage at one end and to section B at the other. The other end of B slides in tracks at top and bottom of the garage opening. Describe mathematically the curve traced by the bottom of the garage door when opened after a snowstorm.

Locus: midpoint

OSSMB G75.1-4.

Let P and Q be any 2 points on the lines represented by $2x - 3y = 0$ and $2x + 3y = 0$ respectively. Let O be the origin. Find the locus of the midpoint of PQ, given that the area of $\triangle POQ$ is 5.

Locus: rotating lines

PME 436. **by Carl Spangler and Richard A. Gibbs**

Let P_1 and P_2 be distinct points on lines L_1 and L_2, respectively. Let L_1 and L_2 rotate about P_1 and P_2, respectively, with equal angular velocities. Describe the locus of their intersection.

Locus: triangles

OSSMB G76.1-1.

Triangle ABC has a base AB of length k and C is such that $\angle CAB = 2\angle CBA < 120^\circ$. Find the equation of the locus of C.

NAvW 415. **by O. Bottema**

A focal curve is defined as a plane cubic curve c passing through the isotropic points J_1, J_2 and such that the intersection of the tangents at J_1 and J_2 (the principal focus of c) is on c.

Let P_1 and P_2 be two points, isogonally conjugate with respect to a given triangle such that their midpoint M is on a given line ℓ. Prove that the locus of P_1 and P_2 is a focal curve.

NAvW 504. **by O. Bottema**

In the plane U, a triangle $A_1A_2A_3$ and a point M are given, such that M does not lie on the line through A_i parallel to the opposite side ($i = 1, 2, 3$); k_1, k_2, and k_3 are three given real numbers ($k_i \neq 0, k_i \neq 1, i = 1, 2, 3$).

Each point P in U is associated with three points P_1, P_2, and P_3 in the following way: If the line ℓ_i through P parallel to A_iU intersects the opposite side of A_i at S_i, then P_i lies on ℓ_i such that $P_iS_i = k_iPS_i$ ($i = 1, 2, 3$).

(a) Determine the locus of P if P_1, P_2, and P_3 are collinear.

(b) Determine the locus of P if the six points A_i, P_i are on a conic.

NAvW 436. **by O. Bottema and M. C. van Hoorn**

Let P be a point in the plane of a given triangle ABC, P' the isogonal conjugate of P with respect to ABC, L_1 the line PP', L_2 the trilinear polar (or harmonical) of P with respect to ABC. Show that the locus of the points P such that L_1 and L_2 are perpendicular is a quintic curve, with nodes at the vertices of ABC, passing through the isotropic points, through the incenter and the three excenters, through the centroid, through the orthocenter, and through the vertices of the pedal triangle of the orthocenter.

NAvW 535. **by O. Bottema**

In a Euclidean plane, a triangle ABC and a line ℓ are given. The points P and P' are isogonal conjugates with respect to the triangle. Determine the locus of the point P such that the line PP' is parallel to ℓ.

CRUX 450. **by A. Liu**

Triangle ABC has a fixed base BC and a fixed inradius. Describe the locus of A as the incircle rolls along BC. When is AB of minimal length?

PARAB 424.

A triangle ABC is given in the xy-plane. Now, O is the origin, the point P moves along the line $x = 1$, and the point Q is determined so that the triangles ABC and OPQ are similar (that is, $\angle QOP = \angle CAB$ and $\angle QPO = \angle CBA$). Describe the motion of Q as P moves.

Map problems

USA 1978/2.

Given are $ABCD$ and $A'B'C'D'$, square maps of the same region, drawn to different scales and superimposed. Prove that there is only one point O on the small map which lies directly over point O' of the large map such that O and O' each represent the same place of the country. Also, give a Euclidean construction for O.

Maxima and minima: angles

PUTNAM 1976/A.1.

Given an interior point P of the angle whose sides are the rays OA and OB. Locate X on OA and Y on OB so that the line segment $\overline{XY}$ contains P and so that the product of distances $(PX)(PY)$ is a minimum.

JRM 504. **by Robert Walsh**

A spherical planet of radius r has a satellite ring in the plane of its equator extending from altitude h_1 to h_2. To an observer on the planet, at what latitude will the ring appear widest?

Maxima and minima: circular arcs

ISMJ J10.14.

An equilateral triangle and a square are inscribed in the same circle in such a way that no vertices of the triangle and the square coincide. Show that among the seven circular arcs thus obtained, there will always be at least one that is not longer than 1/24 of the circumference of the circle. How many such arcs can there be?

MM 976. **by Miller Puckette and Steven Tschantz**

A road is to be built connecting two towns separated by a river whose banks are concentric circular arcs. If the road must bridge the river banks orthogonally, describe the minimum length road (assuming coplanarity).

Maxima and minima: collinear points

OSSMB 76-10.

Given n points $x_1, x_2, \ldots, x_n$ on a line, find the point x on the line at which the sum S of the distances from the n given points is a minimum.

Maxima and minima: convex hull

JRM 427. **by Susan Laird**

How should five circles with radii 1, 2, 3, 4, and 5 be arranged with respect to each other so as to minimize the area of their convex hull?

Maxima and minima: equilateral triangles

DELTA 5.2-2. **by Walter Rudin**
DELTA 6.1-2. **by Walter Rudin**

Let A, B, and C be the vertices of an equilateral triangle. Denote the triangle together with its interior by $\triangle$. Define

$$f(P) = AP \cdot BP \cdot CP, \quad P \in \triangle.$$

The compactness of $\triangle$ shows that f attains its maximum at some point $P_0 \in \triangle$. "By symmetry", P_0 is the center of $\triangle$. Is it true or false? Find the largest value of f on $\triangle$.

Maxima and minima: isosceles triangles

PENT 284. **by Kenneth M. Wilke**

Given two sides of an isosceles triangle, what is the length of the third side which produces the maximum area?

NYSMTJ 85. **by Alan Wayne**

Let ABC be an isosceles triangle ($\angle B = \angle C$) with an inscribed square having one of its sides on segment BC. Find the measure of $\angle A$ for which the ratio of the area of the inscribed square to that of $\triangle ABC$ is a maximum.

Maxima and minima: line segments

JRM 464. **by C. F. White and N. R. White**

Find the maximum area definable by the outer extremities of four line segments of lengths 1, 2, 3, and 4 units radiating from a common point.

Maxima and minima: paths

PENT 276. **by Kenneth M. Wilke**

A class of school children were to run an unusual race. In the school yard there were two flagpoles, one located 60 feet due south of the wall of the building and the other located 90 feet due southeast from the first pole. Each child starts at the first pole, runs to any point in the wall, makes a chalk mark on the wall, and then runs to the other pole. One child's time was much better than any other's. Assuming that all the children ran equally fast, what path did the winner take?

Maxima and minima: quadrilaterals

PENT 291. **by Leigh James**

Prove that the quadrilateral having sides a, b, c, and d has maximum area when the quadrilateral is cyclic.

Maxima and minima: rectangles

CRUX 427. **by G. P. Henderson**

A corridor of width a intersects a corridor of width b to form an "L". A rectangular plate is to be taken along one corridor, around the corner and along the other corridor with the plate being kept in a horizontal plane. Among all the plates for which this is possible, find those of maximum area.

ISMJ 13.6.

A man has 100 feet of fence with which he wants to enclose a rectangular garden plot of as great an area as possible. What is the greatest area?

JRM 500. **by Sherry Nolan**

(a) A man died at age 80, leaving his land to his four sons. His will stated the following: "My sons are to receive nonoverlapping rectangular plots of land with the following characteristic: Each plot will contain the same odd number of square units of area, and its units in length shall exceed its units in width by the age of the son who receives it." If it is known that each son is of a different age, and if all ages and edges are to be measured in whole numbers, what is the smallest rectangular area in which the plots can be contained?

(b) Can the four plots be contained in a square 48.5 units on a side?

JRM 731. **by Frank Rubin**

The high priests of Heterodoxy have ordered the building of a new temple. It will be rectangular on a single level, and will have several (two or more) rectangular interior rooms. To maximize heterogeneity, no room may have any dimension in common with any other room; i.e., if there are k rooms, the dimensions of the rooms must be $2k$ distinct integers.

You have been hired to build the temple for a fixed fee. To maximize your profit, you wish to minimize the total floor area of the temple. What floor plan should you adopt?

OSSMB 79-11.

A chord of length $\sqrt{3}$ divides a circle of unit radius into two regions. Find the rectangle of maximum area that can be inscribed in the smaller region.

Maxima and minima: regular polygons

AMM E2632. **by Azriel Rosenfeld**

Define the discrepancy $d(A,B)$ between two plane geometric figures to be the area of their symmetric difference. Let A be a circle of radius r. Determine the inradius of the regular n-gon B for which $d(A,B)$ is minimal.

Maxima and minima: right triangles

MM 947. **by Steve Moore and Mike Chamberlain**

A line through the point (a,b) which is in the first quadrant forms a right triangle with the positive coordinate axes. Find the equation of the line that forms the triangle with minimum perimeter.

Maxima and minima: semicircles

OSSMB 76-4.

A semicircle is drawn outwardly on chord AB of the unit circle with center O. Prove that the point C on this semicircle that sticks out of the given circle the farthest is on the radius OD that is perpendicular to AB.

The farther AB is moved from the center O, the smaller it gets, accordingly yielding a smaller semicircle. Determine the chord AB that makes OC a maximum.

Maxima and minima: shortest paths

JRM 603. **by Fred Walbrook**

While driving in the first Quadrant, A. Point allowed his engine to overheat and found himself at $(5,2)$ without water in his radiator or oil in his crankcase. The nearest water was in the x-axis river and the nearest oil in the y-axis pipeline. Toting a couple of containers he took the shortest hike necessary to replenish his crankcase and radiator. What was his route?

PARAB 407.

Let ℓ be a given line and let A and B be two points on the same side of ℓ. Find the point P on ℓ with the property that the sum of the distances AP and PB is as small as possible.

MM 1083. **by M. S. Klamkin and A. Liu**

Given an equilateral point lattice with n points on a side, it is easy to draw a polygonal path of n segments passing through all the $n(n+1)/2$ lattice points. Show that it cannot be done with less than n segments.

MSJ 501.

Mr. Geo. Metric walks diagonally from one corner of a rectangular parking lot to the opposite corner. Due to the angular parking of cars in two strips of width 3 m, he can walk in these strips only in the SW direction. The lot is 48 m wide and 60 m long. Find the minimal distance he must walk.

Maxima and minima: solid geometry

IMO 1979/4.

Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $(QP+PR)/QR$ is a maximum.

SSM 3683. **by Herta T. Freitag**

A familiar elementary calculus problem requires determination of the open-top, square prism of largest volume which can be obtained by cutting congruent squares from each corner of a square cardboard and bending up the remaining flaps. Generalize this problem by letting the cardboard be any regular n-sided polygon, $n \geq 3$.

Maxima and minima: thumbtacks

MM 996. **by Richard A. Gibbs**

Suppose thumbtacks are used to tack congruent square sheets of paper to a large bulletin board subject to the following conditions:

(i) the sides of the sheet are parallel to the sides of the bulletin board;

(ii) each sheet has exactly four thumbtacks, one in each corner; and

(iii) the sheets may overlap slightly so that one thumbtack could secure a corner of from one to four sheets.

(a) Find, in terms of n, the minimum number of thumbtacks required to tack n such sheets.

(b) For a given n, find the number of distinct minimal arrangements.

(c) Can the problem be generalized to hypercubes and hyperthumbtacks in three or more dimensions?

Maxima and minima: triangles

MM 955. **by Charles F. White**

For three line segments of unequal lengths a, b, and c drawn on a plane from a common point, characterize the proper angular positions such that the outer endpoints of the line segments define the maximum-area triangle. Show how to approximate the exact values of the angles for $a=3$, $b=4$, and $c=5$.

FUNCT 3.2.8.

Prove that amongst all the triangles of a given perimeter, the equilateral triangle has the largest area.

PME 405. **by Norman Schaumberger**

Locate a point P in the interior of a triangle such that the product of the three distances from P to the sides of the triangle is a maximum.

TYCMJ 140. **by Norman Schaumberger**

Locate a point P in the interior of a triangle such that the sum of the squares of the distances from P to the sides of the triangle is a minimum.

JRM 565. **by Archimedes O'Toole**

(a) Given a triangle with sides 3, 4, and 5, what is the smallest perimeter a triangle can have and not fit within it?

(b) What if the sides are 4, 4, and 4?

(c) Given a triangle with sides a, b, and c, what is the smallest perimeter a triangle can have and not fit within it?

NYSMTJ OBG8. **by Alan Wayne**

In what type of triangle is the ratio of the area of the inscribed square to that of the triangle a maximum?

n-dimensional geometry: 4-space

ISMJ 12.19.

Show that a plane in 4-space does not have two sides by constructing a square whose edges surround the plane.

ISMJ 12.20.

Describe or make a picture of the three dimensional map of a 4-cube.

ISMJ 12.22.

The four dimensional "volume" or content of a 4-cube is the fourth power of its side. Can you find the content of a regular 4-simplex?

n-dimensional geometry: convexity

AMM 6098. **by Peter L. Renz**

Let A be the group of affine bijections from $\mathbb{R}^n$ to $\mathbb{R}^n$. For any subset S of $\mathbb{R}^n$, define $A(S)$ to be the subgroup of A that takes S onto itself. A convex body is a compact convex set with nonempty interior. We say a convex body K in $\mathbb{R}^n$ is maximally symmetric if $A(K)$ is not properly contained in $A(L)$ for any convex body L in $\mathbb{R}^n$. Characterize the maximally symmetric convex bodies in $\mathbb{R}^n$.

n-dimensional geometry: curves

SIAM 75-21. **by I. J. Schoenberg**

In $\mathbb{R}^n$, we consider the curve

$$\Gamma : x_i = \cos(\lambda_i t + a_i),$$

$i = 1, \ldots, n$, $-\infty < t < \infty$, which represents an n-dimensional simple harmonic motion entirely contained within the cube $U : -1 \le x_i \le 1$, $i = 1, \ldots, n$. We want Γ to be truly n-dimensional and will therefore assume without loss of generality that $\lambda_i > 0$ for all i. We consider the open sphere

$$S : \sum_{i=1}^{n} x_i^2 < r^2$$

and want the motion of the first equation to take place entirely outside of S, hence contained in the closed set $U - S$. What is the largest sphere S such that there exist motions Γ entirely contained in $U - S$? Show that the largest such sphere S_0 has the radius $r_0 = \sqrt{n/2}$, and that the only motions Γ within $U - S_0$ lie entirely on the boundary $\sum x_i^2 = r_0^2$ of S_0.

n-dimensional geometry: inequalities

CMB P244. **by P. Erdős and M. S. Klamkin**

Let P denote any point within or on a given n-dimensional simplex $A_1, A_2, \ldots, A_{n+1}$. The point P is "reflected" across each face of the simplex along rays parallel to the respective medians to each face producing an associated simplex $A'_1, A'_2, \ldots, A'_{n+1}$ (PA'_i is parallel to the median from A_i and is bisected by the face opposite A_i). Show that

$$n^n \text{Volume}(A'_1, A'_2 \ldots, A'_{n+1}) \le 2^n \text{Volume}(A_1, A_2, \ldots A_{n+1})$$

with equality if and only if P is the centroid of the given simplex.

SIAM 78-20. **by M. S. Klamkin**

The lines joining the vertices $\{V_i\}$, $i = 0, 1, \ldots, n$ of a simplex S to its centroid G meets the circumsphere of S again in points $\{V'_i\}$, $i = 0, 1, \ldots, n$. Prove that the volume of simplex S' with vertices V'_i is $\ge$ the volume of S.

n-dimensional geometry: simplexes

AMM E2548. **by Murray S. Klamkin**

Let $A_0, A_1, \ldots, A_n$ be distinct points of n-space that lie within a hyperplane. Suppose that these points are parallel projected into another hyperplane and that their images are $B_0, B_1, \ldots, B_n$, respectively. Prove that for any $r = 0, 1, \ldots, n$, the volumes of the simplexes spanned by A_0, $A_1, \ldots, A_r$, B_{r+1}, $B_{r+2}, \ldots, B_n$ and by B_0, $B_1, \ldots, B_r$, A_{r+1}, $A_{r+2}, \ldots, A_n$ are equal.

CRUX 224. **by M. S. Klamkin**

Let P be an interior point of a given n-dimensional simplex with vertices $A_1, A_2, \ldots A_{n+1}$. Let $P_i (i = 1, 2, \ldots, n+1)$ denote points on A_iP such that $A_iP_i/A_iP = 1/n_i$. Finally, let V_i denote the volume of the simplex cut off from the given simplex by a hyperplane through P_i parallel to the face of the given simplex opposite A_i. Determine the minimum value of $\sum V_i$ and the location of the corresponding point P.

AMM E2674. **by G. Tsintsifas**

Let

$$S = \{A_0, A_1, \ldots, A_n\}$$

and

$$S' = \{A'_0, A'_1, \ldots, A'_n\}$$

be regular n-simplices such that A'_i lies on the face

$$\{A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n\}$$

of S, $0 \le i \le n$. Is it true that the centroids of S and S' coincide?

AMM E2657. **by G. Tsintsifas**

Let $\mathcal{A} = A_0A_1 \ldots A_n$ and $\mathcal{B} = B_0B_1 \ldots B_n$ be regular simplices in $\mathbb{R}^n$. Assume that the ith vertex of $\mathcal{B}$ lies on the ith face of $\mathcal{A}$, $0 \le i \le n$. What is the minimal value of their similarity ratio λ ($\lambda\mathcal{A}$ congruent to $\mathcal{B}$, $\lambda > 0$)?

n-dimensional geometry: volume

NAvW 531. **by W. A. J. Luxemburg**

Determine the volume of the body S in $\mathbb{R}^n$ ($n \ge 2$) determined by the set of points $y = (y_1, y_2, \ldots, y_n)$, satisfying

$$y_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

($0 \le x_1 < x_2 < \cdots < x_n \le 1$ and $k = 1, 2, \ldots, n$).

AMM E2701. **by Richard Stanley**

Find the volume of the convex polytope determined by

$$x_i \ge 0, \qquad 1 \le i \le n,$$

and

$$x_i + x_{i+1} \le 1, \qquad 1 \le i \le n-1.$$

Non-Euclidean geometry

AMM S2. **by H. S. M. Coxeter**

In the hyperbolic plane, the locus of a point at constant distance δ from a fixed line (on one side of the line) is one branch of an "equidistant curve" (or "hypercycle"). In Poincaré's half-plane model, this curve can be represented by a ray making a certain angle with the bounding line of the half-plane. Show that this angle is equal to $\prod(\delta)$, the angle of parallelism for the distance δ.

AMM E2680. **by Jerrold W. Grossman**

Let $ABCD$ be a convex quadrilateral in the hyperbolic plane. Assume that $AD = BC$ and that

$$\angle A + \angle B = \angle C + \angle D.$$

Does $AB = CD$ follow from these hypotheses?

Octagons

PUTNAM 1978/B.1.

Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$ with r, s and t positive integers.

MSJ 448. **by Steven R. Conrad**

Find the area of an equiangular octagon, the lengths of whose sides are alternately 1 and $\sqrt{2}$.

MM 925. **by Julius G. Baron and Thomas E. Elsner**

(a) Prove that any non-self-intersecting cyclic octagon is such that the sum of any four nonadjacent interior angles is 3π.

(b) An octagon is inscribed in a circle with vertices on any four diameters. Show that each alternate pair of exterior angles is complementary.

Packing problems

AMM E2524. **by T. H. Foregger**

Show that 41 $1 \times 2 \times 4$ bricks can be packed into a $7 \times 7 \times 7$ box. Is there a packing of 42 such bricks into this box?

AMM E2774.* **by James Propp**

Prove or disprove that given a convex two-dimensional figure S, six translates of S can fit inside a homothetic figure three times as large as S in linear dimensions.

CMB P276. **by H. S. M. Coxeter**

Find the radius of the smallest circle inside which discs of radius $1/n$ $(n = 1, 2, 3, \ldots)$ can all be packed.

OSSMB 75-15.

Circles of unit radius are packed, without overlapping of interior points, in a strip S of the plane whose parallel edges are a distance w apart. We say the circles form a k-cloud if every straight line that cuts across S makes contact with at least k circles. Prove that for a 2-cloud $w \geq 2 + \sqrt{3}$.

CRUX 135. **by Steven R. Conrad**

How many 3×5 rectangular pieces of cardboard can be cut from a 17×22 rectangular piece of cardboard so that the amount of waste is a minimum?

Paper folding: algorithms

AMM S4. **by Richard K. Guy**

In order to store a given length L of paper tape in an accessible way, I choose a length, λ, and an even integer, $2n$, so that $2n\lambda = L$. I then screenfold the tape with n "odd" folds in one sense at distances $f_1, f_3, \ldots, f_{2n-1}$ along the tape, and $n-1$ "even" folds, in the other sense, at distances $f_2, f_4, \ldots, f_{2n-2}$. The ends of the tape are $f_0 = 0$ and $f_{2n} = L = 2n\lambda$. I try to arrange that the quantities $f_{i+1} - f_i = \lambda_i$, $0 \leq i \leq 2n-1$, are each equal to λ, but in practice this rarely happens, so I then endeavor to improve the situation by lining up the ends and the even folds, $f_0, f_2, \ldots, f_{2n}$ and recreasing the odd folds at $f_1', f_3', \ldots, f_{2n-1}'$, so that hopefully better approximations, λ_i' to λ_i are produced, namely, $\lambda_i' = \lambda_{i+1}'$ $[= (\lambda_i + \lambda_{i+1})/2]$ for $i = 0, 2, \ldots, 2n-2$. I then line up the odd folds and recrease the even ones, giving $\lambda_{i-1}'' = \lambda_i''$ $\left[= \left(\lambda_{i-1}' + \lambda_i'\right)/2\right]$ for $i = 2, 4, \ldots, 2n-2$. I then repeat the process. Does it terminate or even converge?

Paper folding: cubes

JRM 628. **by Henry Larson**

A 9×12 sheet of paper is to be cut down into a pattern (consisting of a single piece) that can be folded into a cube. Find the largest cube that can be obtained, given that the pattern:

(a) consists of six squares;
(b) has arbitrary shape.

Paper folding: equilateral triangles

PARAB 399.

Show how to construct an equilateral triangle by folding a single (rectangular) sheet of paper. No rulers, compasses, or separate sheets for measuring are to be used.

SSM 3768. **by Charles W. Trigg**

The paper triangle ABC is equilateral with sides of length a. Vertex A is brought into contact with point D and BC, and the paper is flattened to form a crease EF, with E on AB and F on AC. If DF is perpendicular to BC, find

(a) the length of EF in terms of a; and
(b) the areas of triangles BED, DEF, and DFC in terms of a.

OSSMB 78-2.

A piece of paper in the shape of an equilateral triangle ABC is creased along a line XY, X on AB and Y on AC, so that A falls on some point D on BC. Show that the triangles XBD and DCY are similar. If $AB = 15$ and $BD = 3$, what is the length of the crease XY?

Paper folding: rectangles

SSM 3637. **by Charles W. Trigg**

A rectangular sheet of paper, $ABCD$, has the dimension $AB = CD = x$ and $BC = DA = y$. The point E is located on CD so that angle BEC is $60°$. The sheet is folded along BE so that C assumes a new position C'. The sheet is folded again so that a crease runs along EC' and meets DA in F. When a third fold along BF is made, AF falls along FE.

(a) Express y in terms of x.
(b) What are the lengths of the creases?
(c) What are the areas of the parts into which the creases divide the rectangle?
(d) Check your results by determining the sines of three angles using the computed dimensions.

Paper folding: regular pentagons

SSM 3661. **by Alan Wayne**
A piece of paper has the shape of a regular pentagon. The paper is folded over once and creased flat so that a vertex of the pentagon coincides with the midpoint of that side which is farthest from the vertex. Show that the length of the crease is one and a half times the length of a side of the pentagon.

Paper folding: regular polygons

CRUX 350. **by W. A. McWorter, Jr.**
What regular n-gons can be constructed by paper folding?

Paper folding: squares

CRUX 292. **by Charles W. Trigg**
Fold a square piece of paper to form four creases that determine angles with tangents of 1, 2, and 3.

Paper folding: strips

MSJ 464.
A strip of adding machine tape is folded making an angle A_0 of 80° with the bottom edge of the tape. Angles $A_1, A_2, A_3, \ldots$ are formed by successive folds of the edges of the tape to the creases previously obtained (and thereby halving the respective angles). Find the measure of A_{100}.

Parabolas

OSSMB G78.1-3.
Let P, Q, R be three points on a parabola such that their distances from the axis of the parabola are in geometric progression. Show that the tangents to the parabola at P and R meet on the line through Q perpendicular to the axis.

OSSMB G79.1-2.
A chord $y = mx + b$ intersects a parabola $y^2 = 4px$ at $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Find the coordinates of P, a point on the parabola, such that $\triangle PP_1P_2$ has maximum area.

NYSMTJ 94. **by H. O. Eberhart**
A nonaxial line passing through the focus of a parabola intersects it in two points, P and Q. Show that
(a) the tangent at P is perpendicular to the tangent at Q;
(b) these tangents intersect on the directrix.

CRUX 445. **by Jordi Dou**
Consider a family of parabolas escribed to a given triangle. To each parabola corresponds a focus F and a point S of intersection of the lines joining the vertices of the triangle to the points of contact with the opposite sides. Prove that all lines FS are concurrent.

Parallelograms

TYCMJ 153. **by K. R. S. Sastry**
Let $c \in (0, 1)$ be given and $A_1A_2A_3A_4$ be a parallelogram of one unit area with $E_i \in A_iA_{i+1}$ such that $A_iE_i/E_iA_{i+1} = c$, $(i = 1, 2, 3, 4; A_5 = A_1)$. Set $A_iE_{i+1} \cap A_{i+1}E_{i+2} = P_i$, $(i = 1, 2, 3, 4; A_5 = A_1, E_5 = E_1, E_6 = E_2)$. Determine the area of quadrilateral $P_1P_2P_3P_4$.

NYSMTJ 74. **by Norman Gore**
NYSMTJ OBG3. **by Norman Gore**
In parallelogram $ABCD$, L and M are interior points of sides $\overline{AD}$ and $\overline{BC}$ respectively. Let $P = \overline{BL} \cap \overline{AM}$ and $Q = \overline{MD} \cap \overline{CL}$. If the line determined by P and Q is parallel to line AD, show that it bisects $ABCD$.

CRUX 139. **by Dan Pedoe**
Let $ABCD$ be a parallelogram, and suppose a circle γ touches AB and BC and intersects AC in the points E and F. Show that there exists a circle δ which passes through E and F and touches AD and DC.

NYSMTJ OBG1. **by Norman Schaumberger**
Let $ABCD$ be a parallelogram. If a circle passes through A and cuts segments AB, AC, and AD at points P, Q, and R respectively, then prove that

$$AP \times AB + AR \times AD = AQ \times AC.$$

TYCMJ 117. **by Norman Schaumberger**
Let E be the intersection of the diagonals of a parallelogram $ABCD$, and let P and Q be points on a circle with center E. Prove that

$$PA^2 + PB^2 + PC^2 + PD^2 = QA^2 + QB^2 + QC^2 + QD^2.$$

CRUX 322. **by Harry Sitomer**
In parallelogram $ABCD$, $\angle A$ is acute and $AB = 5$. Point E is on AD with $AE = 4$ and $BE = 3$. A line through B, perpendicular to CD, intersects CD at F. If $BF = 5$, find EF.

NYSMTJ 43.
Given perpendicular rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, let $\overline{PQ}$ be any segment with an endpoint on each ray (other than A). Let X be the point of intersection of the bisectors of the exterior angles at P and Q of $\triangle APQ$. Introduce segments $\overline{XM}$ and $\overline{XN}$ perpendicular to rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, respectively. Prove that parallelogram $ANXM$ is a square.

SSM 3754. **by Fred A. Miller**
If θ is the angle between the diagonals of a parallelogram whose sides a and b are inclined at an angle α to each other, show that

$$\tan\theta = \frac{2ab\sin\alpha}{a^2 - b^2}\ .$$

Pentagons

CRUX 232. **by Viktors Linis**
Given are five points A, B, C, D, and E in the plane, together with the segments joining all pairs of distinct points. The areas of the five triangles BCD, EAB, ABC, CDE, and DEA being known, find the area of the pentagon $ABCDE$.

PME 383. **by Norman Schaumberger**
Find a pentagon such that the sum of the squares of its sides is equal to four times its area.

Perspective drawings

CRUX 406. **by W. A. McWorter Jr.**
The figure shows an unfinished perspective drawing of a railroad track with two ties drawn parallel to the line at infinity. Can the remaining ties be drawn, assuming that the actual track has equally spaced ties?

MM 980. **by Peter Ungar**
Show that in a perspective drawing of a straight railroad track which is at right angles to the image plane, the reciprocals of the images of the ties form an arithmetic progression.

Point spacing

CRUX 405. **by Viktors Linis**
A circle of radius 16 contains 650 points. Prove that there exists an annulus of inner radius 2 and outer radius 3 that contains at least 10 of the given points.

ISMJ J11.11.
Given a circle of radius 1, show that of any seven points on its perimeter at least two must be at a distance from each other of less than 1.

PUTNAM 1978/A.6.
Let n distinct points in the plane be given. Prove that fewer than $2n^{3/2}$ pairs of them are unit distance apart.

DELTA 6.1-3. **by Anthony Biagioli**
PARAB 324.
Given five points in a square with side a, show that two of them are within $a/\sqrt{2}$ of each other.

SIAM 78-13.* **by T. D. Rogers**
Given n points distributed uniformly in the unit circle, associate with each such point the region in the circle whose points are closer to it than the remaining $n-1$ a priori given points. If $A_1 \leq A_2 \leq \cdots \leq A_n$ is the ordered enumeration of the areas of these regions, what are the expected values of the A_i's?

JRM 554. **by Sidney Kravitz**
The town council of Erewhon proposes to relocate its 24 fire companies according to the following scheme: The square map of Erewhon is to be octasected by the two diagonals and the two segments connecting midpoints of opposite sides, and each of the eight interior half-segments divided into four equal segments by three points. Each fire company is then to be located at one of these 24 points, with the closest company responsible for handling a fire. Under this scheme, the areas of responsibility are to be as in a certain diagram. The firemen of Erewhon oppose this scheme and favor any plan that would ensure that the largest relative discrepancy in area between two regions be as small as possible. If, along each of the eight interior half-segments, the three fire companies may be spaced arbitrarily, what scheme comes closest to satisfying the firemen?

Points in plane: broken lines

MSJ 472.
Nine points, no two of which are the same distance apart, are given in a plane. Prove that if each point is connected to its nearest neighbor, then the line segments do not intersect one another except possibly at the endpoints.

PARAB 383.
Let n points be given in the plane. Show that the shortest broken line connecting the points does not cross itself.

Points in plane: circles

AMM E2746. **by George F. Shumm**
Let $A_1, \ldots, A_n$ be distinct noncollinear points in the plane. A circle with center P and radius r is called minimal if $A_kP \leq r$ for all k and equality holds for at least three values of k.

If $A_1, \ldots, A_n$ vary (n being fixed), what is the maximum number of minimal circles?

CRUX 165. **by Dan Eustice**
Prove that, for each choice of n points in the plane (at least two distinct), there exists a point on the unit circle such that the product of the distances from the point to the chosen points is greater than one.

Points in plane: distances

CRUX 233. **by Viktors Linis**
The three points (1), (2), (3) lie in this order on an axis, and the distances $[1,2] = a$ and $[2,3] = b$ are given. Points (4) and (5) lie on one side of the axis, and the distance $[4,5] = 2c > 0$ and the angles $(415) = v_1$, $(425) = v_2$, $(435) = v_3$ are also known. Determine the position of the points (1), (2), (3) relative to (4) and (5).

MSJ 482.
Find all possible arrangements of four points in the plane such that there are at most two different values for the set of distances between all possible pairs of points.

PME 406. **by P. Erdős**
Let there be given five distinct points in the plane. Suppose they determine only two distances. Is it true that they are the vertices of a regular pentagon?

Points in plane: parallel lines

ISMJ 14.19.
Given three points in the plane, in how many ways can one draw three equidistant parallel lines through them?

Points in plane: partitions

MM 957. **by Erwin Just**
Show that it is possible to partition the rational points of the plane into four sets, each of which is dense in the plane, and such that no straight line will contain a point from each of the four sets.

Can the partitioning also be into three sets?

JRM 557. **by David L. Silverman**
A set of points is called *Scottian* if, regardless of the way it is partitioned into two sets A and B, either A or B (or both) contains three points that are the vertices of a right triangle.

(a) Prove that the vertices of a square and the midpoints of the four sides constitute a Scottian set.

(b) Prove that the circumference of a circle is not Scottian, but the addition of the center of the circle makes it so.

(c) Prove that a triangle is Scottian if and only if it is not obtuse.

(d) Among all finite Scottian sets on a square lattice, what is the least number n of points possible and what shape must such an n-point set have? Is there an n-point Scottian set that cannot be embedded in a square lattice?

JRM 701. **by David L. Silverman**

A set of points is called *Scottian* if, regardless of how it is partitioned into two sets, at least one of the sets contains the vertices of a right triangle. Given a circle C and a point P in the plane of C, P is said to be *Scottian with respect to* C if the union of P and C is Scottian. What is the locus of points that are Scottian with respect to C?

Points in plane: perpendicular bisectors

NAvW 544. **by W. H. J. H. van Meeuwen, C. P. van Nieuwkasteele, and K. A. Post**

Prove the following statement: Let n be an integer, $n \geq 4$. Then n points can be chosen in the plane, such that their $\binom{n}{2}$ perpendicular bisectors dissect the plane into convex pieces among which an $(n-1)$-gon occurs.

Points in plane: rational distances

JRM 765. **by William C. Reil**

Given a set of noncollinear points in a plane, define a rational point as a point in the plane, but not in the set, that is a rational distance from each point in the set.

(a) Does any such set have an infinity of rational points?

(b) Does every such set have a rational point?

Points in plane: triangles

MSJ 494.

Prove that it is impossible to pick four points A, B, C, and D in the plane so that each of the interior angles of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$ is acute.

AMM E2531. **by V. F. Ivanoff**

Given points A, B, C, D, E, and F in the plane, let $[ABC]$ denote the directed area of triangle ABC, etc. Prove that

$$[AEF]\cdot[DBC]+[BEF]\cdot[DCA]+[CEF]\cdot[DAB] = [DEF]\cdot[ABC].$$

Polygons: 13-gons

CRUX 70. **by Viktors Linis**

Show that for any 13-gon there exists a straight line containing only one of its sides. Show also that for every $n > 13$ there exists an n-gon for which the above statement does not hold.

PARAB 347.

Is it possible to select four vertices of a regular 13-gon so that the four sides and two diagonals of the quadrilateral formed by the four chosen vertices have different lengths?

Polygons: 17-gons

CRUX PS3-1.

Does there exist a polygon of 17 sides such that some straight line intersects each of its sides in some point other than a vertex of the polygon? Note that the polygon need not be convex nor simple.

Polygons: convex polygons

AMM E2514. **by G. A. Tsintsifas**

Let P be a convex polygon, and let K be the polygon whose vertices are the midpoints of the sides of P. A polygon M is formed by dividing the sides of P (cyclically directed) in a fixed ratio $p{:}q$ where $p+q=1$. Show that

$$[M]=(p-q)^2[P]+4pq[K],$$

where $[X]$ denotes the area of polygon X.

CRUX 67. **by Viktors Linis**

Show that in any convex $2n$-gon there is a diagonal that is not parallel to any of its sides.

AMM E2641. **by Philip Straffin**

Given a convex polygon and a point p inside it, define $D(p)$ to be the sum of the perpendicular distances from p to the sides of the polygon (extended if necessary). Characterize those convex polygons for which $D(p)$ is independent of p.

MM 1018. **by H. Kestelman**

Let $P_1, P_2, \ldots, P_n$ be the vertices in order of a convex n-gon with θ_r, $0 < \theta_r < \pi$, as the angle at P_r. Rotations $R_1, R_2, \ldots, R_n$ are defined as follows: R_1 rotates $2\theta_1$ about P_1, R_2 rotates $2\theta_2$ about $R_1(P_2)$, R_3 rotates $2\theta_3$ about $R_2R_1(P_3)$, etc. Prove that $R_nR_{n-1}\cdots R_2R_1$ is the identity.

Polygons: equilateral polygons

NAvW 398. **by Hosia W. Labbers, Jr.**

Given an equilateral polygon $A_1A_2\ldots A_n$, $n \geq 3$, in the plane such that each of the $n-2$ angles $\angle A_{j-1}A_jA_{j+1}$, $1 < j < n$, is a rational multiple of π, prove that the angles $\angle A_{n-1}A_nA_1$ and $\angle A_nA_1A_2$ must also be rational multiples of π.

Polygons: interior point

MSJ 489.

Any interior point P of a given convex polygon having vertices $V_1, V_2, \ldots, V_n$ is called equitable if all the triangles $V_1PV_2, V_2PV_3, \ldots, V_nPV_1$ are of equal area. Prove that no such polygon can have more than one equitable point.

Polygons: visibility

PARAB 440.

Let Π be a polygon and let P be any point inside Π. If every line segment joining P to any other point inside or on Π lies completely in Π, we say that Π is visible from P. Prove that the set of all points from which Π is visible is a convex set.

AMM E2513. **by Neal Felsinger**

Let P be a simple (non-self-intersecting) planar polygon. If A is a point in the plane, and if E is an edge of P, then E is viewable from A if for every point x of E, the line segment joining A to x contains no point of P other than x.

(a) Let A and P be arbitrary. Must some edge of P be viewable from A? Examine the cases of A exterior to P and interior to P separately.

(b) Find sufficient conditions on A in order that some edge of P be viewable from A.

Geometry

Projective geometry

AMM 6267. **by A. E. Fekete**

We say that two collineations of the real projective space PR^n are of the same type if their invariant configurations are projectively equivalent (i.e., there is a real projective collineation mapping one configuration into the other). Find an explicit formula determining the number of all different nonidentity collineation types. For example, for $n = 1$ there are three types: hyperbolic (two fixed points), parabolic (one fixed point), and elliptic (no fixed point). Also, define collineation types for the complex projective space PC^n and find their number.

NAvW 547. **by O. Bottema**

Two coinciding three-dimensional projective spaces Σ and S have the homogeneous point coordinates X_i and x_i ($i = 1, 2, 3, 4$), respectively. A motion of S with respect to Σ is given by

$$X = (A + Bt)x,$$

where X and x are column vectors with the elements X_i and x_i, A and B are nonsingular 4×4 matrices, the eigenvalues of B are real and distinct, and the scalar t represents the time.

Obviously, the path of any moving point of S is a straight line. Determine the locus of the paths.

NAvW 512. **by O. Bottema**

In the projective plane, a quartic curve k with three cusps is given. The cusps D_i ($i = 1, 2, 3$) are taken as the vertices of the coordinate triangle, and the intersection of the three cuspidal tangents as the unit point.

Let P_1 be an arbitrary point on k. The fourth intersection of P_1D_1 and k is Q_1, that of Q_1D_2 and k is R_1, that of R_1D_3 and k is P_2. The construction is then repeated starting at P_2, etc., and the series $P_1, P_2, P_3, \ldots$, is obtained.

Determine $\lim_{n\to\infty} P_n$.

Quadrilaterals: angle bisectors

PME 346. **by R. S. Luthar**

The internal angle bisectors of a convex quadrilateral $ABCD$ enclose another quadrilateral $EFGH$. Let FE and GH meet in M and let GF and HE meet in N. If the internal bisectors of angles EMH and ENF meet in L, show that angle NLM is a right angle.

Quadrilaterals: area

CRUX 42. **by Viktors Linis**

Find the area of a quadrilateral as a function of its four sides, given that the sums of opposite angles are equal.

SSM 3789. **by Alan Wayne**

In the plane quadrilateral $ABCD$, angles A and B are complementary. Also $AB = 60$, $BC = 33$, $CD = 25$, and $DA = 16$. Find

(a) the area of $ABCD$ and

(b) the lengths of the diagonals AC and BD.

MSJ 442.

In convex quadrilateral $ABCD$, the diagonals intersect at E. If the areas of regions BEC, CED, DEA, and AEB are a, b, c, and d, respectively, prove that $ac = bd$.

MSJ 443.

In convex quadrilateral $ABCD$, the diagonals intersect at point E. If the areas of regions BEC, CED, and DEA are 6, 8, and 12, respectively, find the area of region AEB.

Quadrilaterals: circumscribed quadrilateral

CRUX 189. **by Kenneth S. Williams**

If a quadrilateral circumscribes an ellipse, prove that the line through the midpoints of its diagonals passes through the center of the ellipse.

CRUX 199. **by H. G. Dworschak**

If a quadrilateral is circumscribed about a circle, prove that its diagonals and the two chords joining the points of contact of opposite sides are all concurrent.

Quadrilaterals: determinants

MM 963. **by Hüseyin Demir**

Characterize convex quadrilaterals with sides a, b, c, and d such that

$$\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix} = 0.$$

SSM 3747. **by Alan Wayne**

The points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, and $P_4(x_4, y_4)$ are the vertices of a convex quadrilateral in the plane. What is the geometric significance of the following determinant?

$$\begin{vmatrix} x_1 & y_1 & 1 & 0 \\ x_2 & y_2 & 1 & 1 \\ x_3 & y_3 & 1 & 0 \\ x_4 & y_4 & 1 & 1 \end{vmatrix}$$

Quadrilaterals: diagonals

IMO 1976/1.

In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

Quadrilaterals: erected figures

PENT 308. **by John A. Winterink**
CRUX 37. **by Maurice Poirier**

On the sides of quadrilateral $ABCD$, isosceles right triangles ABP, BCQ, CDR, and DAS are constructed. Show that $PR = QS$ and $PR \perp QS$.

SPECT 11.9. **by A. J. Douglas**

Let $Z_1Z_2Z_3Z_4$ be a convex quadrilateral in the plane. Denote by W_1, W_2, W_3, W_4 the midpoints of the squares, drawn externally to the quadrilateral, with sides Z_1Z_2, Z_2Z_3, Z_3Z_4, Z_4Z_1 respectively. Let U_1, U_2, U_3, U_4 be the midpoints of the squares with sides W_1W_2, W_2W_3, W_3W_4, W_4W_1 respectively. Show that

(a) $W_1W_3 = W_2W_4$ and $W_1W_3 \perp W_2W_4$,

(b) U_1Z_2 and U_3Z_4 are perpendicular to Z_1Z_3, and

$$U_1Z_2 = U_3Z_4 = \frac{1}{2}Z_1Z_3.$$

Quadrilaterals: inscribed circles

PME 417. **by Clayton W. Dodge**

(a) Prove that the line joining the midpoints of the diagonals of a quadrilateral circumscribed about a circle passes through the center of the circle.

(b) Let the incircle of triangle ABC touch side BC at X. Prove that the line joining the midpoints of AX and BC passes through the incenter I of the triangle.

Quadrilaterals: maxima and minima

MSJ 485.
JRM 497. **by Sidney Kravitz**

Prove that among all quadrilaterals of given sides the one of maximum area is inscribable in a circle.

Quadrilaterals: sides

MM Q613. **by Sidney Penner**

Given quadrilateral $ABCD$ with sides a, b, c, and d, prove or disprove:

$$\text{If } a+b=c+d, \text{ then } a=c \text{ or } a=d.$$

Quadrilaterals: supplementary angles

NYSMTJ 52.

(a) Prove that if both pairs of opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle.

(b) Prove that if the sum of the lengths of one pair of opposite sides of a (convex) quadrilateral is equal to the sum of the lengths of the other pair of sides, then a circle can be inscribed in the quadrilateral.

Quadrilaterals: triangles

ISMJ 12.25.

Given any convex quadrilateral, consider all the triangles whose vertices lie on the quadrilateral. Show that the maximum area of such triangles can be achieved by a triangle with its vertices being vertices of the quadrilateral.

CANADA 1978/4.

The sides AD and BC of a convex quadrilateral $ABCD$ are extended to meet at E. Let H and G be the midpoints of BD and AC, respectively. Find the ratio of the area of the triangle EHG to that of the quadrilateral $ABCD$.

PENT 312. **by John A. Winterink**

Let L_1 and L_2 be the axes of a plane coordinate system that cut off line segments a_ib_i $(i=1,2,3,4)$ on the sides (extended if necessary) of a quadrilateral $ABCD$ in such a manner that each point a_i lies on L_1 and each point b_i lies on L_2. Let K denote the intersection of L_1 and L_2.

If similar triangles $a_ib_ic_i$ are drawn on each line segment a_ib_i such that each angle with its vertex at c_i is equal to the angle formed by L_1 and L_2, then show that the vertices c_i and the intersection K of the axes are collinear.

Rectangles

OMG 15.2.2.

If a rectangle is divided into four rectangular sections, prove that $A \cdot D = B \cdot C$ where A, B, C, and D are the areas of the sections, with area A and area D being diagonally adjacent.

SSM 3716. **by Alan Wayne**

Show that there cannot be two noncongruent rectangles having the same perimeter and the same area.

CRUX 244. **by Steven R. Conrad**

A rectangular strip of carpet 3 ft. wide is laid diagonally across the floor of a room 9 ft. by 12 ft. so that each of the four corners of the strip touches a wall. How long is the strip?

MM 960. **by Alan Wayne**

In an $a \times b$ rectangle, lines parallel to the sides divide the interior into ab square unit areas. Through the interior of how many of these unit squares will a diagonal of the rectangle pass?

Can the result be generalized to higher dimensions?

DELTA 6.2-1. **by R. C. Buck**

A man is standing in a rectangular field and is exactly 5 miles from one corner, 8 from another and 14 from a third.

(a) Can you tell how far he is from the remaining corner?

(b) If you know that the field is square, can you tell what its area is?

MM 966. **by Clayton W. Dodge**

A point P lies in the interior of a rectangle of sides a and b.

(a) Find a, b, and P so that all eight distances from P to the four vertices and the four sides are positive integers.

(b) Find an example of a square where seven of the distances are integers.

(c) Can all eight distances be integers for a square?

NYSMTJ 95. **by Samuel A. Greenspan**

The distance from a point in the interior of a rectangle to a given corner is 10 yards; to the opposite corner 11 yards; and to a third corner 5 yards. What is the distance from the point to the fourth corner?

PME 439. **by Richard I. Hess**

A bug starts at Monday noon in the upper-left corner (X) of a $p \times q$ rectangle, and crawls within the rectangle to the diagonally opposite corner (Y), arriving at 6 PM. Exhausted, he sleeps till noon Tuesday. At that time, he embarks for X, crawling along another path in the rectangle and arriving at X at 6 PM Tuesday. Prove that, at some time Tuesday, the bug was at a point no farther than p from where he was at the same time Monday.

PME 430. **by John M. Howell**

Given any rectangle, form a new rectangle by adding a square to the long side. Repeat. What is the limit of the long side to the short side?

Regular heptagons

OSSMB 77-16.

If A_0, $A_1, \ldots, A_6$ are the vertices of a regular 7-gon inscribed in the unit circle, show that

$$A_0A_1 \cdot A_0A_2 \cdot A_0A_3 \ldots A_0A_6 = 7.$$

Regular hexagons

SSM 3701. **by Fred A. Miller**

If, from any point on a circle, line segments are drawn to the vertices of an inscribed regular hexagon, prove that the sum of the longest two of these line segments equals the sum of the remaining four line segments.

Regular octagons

SSM 3653. **by Charles W. Trigg**

The diagonals of a regular octagon have three different lengths. Show that the area of a rectangle determined by a largest and a smallest diagonal is twice the area of a rectangle determined by an intermediate diagonal and a side of the octagon.

SSM 3656. **by Fred A. Miller**

Show that the diameter of a circle inscribed in a quadrant of a circle is equal to the side of a regular octagon circumscribed about the given circle.

Regular pentagons

ISMJ J11.7.

The diagonals AC and BD of the regular pentagon $ABCDE$ intersect at P. Show that $AP = PD = ED$.

FUNCT 3.2.5.

Let $ABCDE$ be a regular pentagon. The diagonals AD and EC meet at the point Q. Show that

$$\frac{AD}{AQ} = \frac{AQ}{QD},$$

and hence prove that the ratio AD/AQ is equal to the golden ratio $\left(1+\sqrt{5}\right)/2$.

CRUX PS2-1.

Prove that only one ellipse can be inscribed in a given regular pentagon.

DELTA 6.2-3. **by D. W. Crowe**

A drawing shows an incomplete "ring" of regular pentagons formed by placing each pentagon next to the other so that they have one side in common. A second drawing shows a "ring" of regular heptagons formed in the same way, except that the ring is completed by a square of side the same length as the side of one of the heptagons. Explain how each is incorrect.

FQ B-348. **by Sidney Kravitz**

Let $P_1, \ldots, P_5$ be the vertices of a regular pentagon and let Q_i be the intersection of segments $P_{i+1}P_{i+3}$ and $P_{i+2}P_{i+4}$ (subscripts taken modulo 5). Find the ratio of lengths Q_1Q_2/P_1P_2.

Regular polygons: cyclic polygons

OSSMB G75.3-1.

Two regular polygons are inscribed in a circle. The number of sides in one polygon is double the number in the other and an angle of one is to an angle of the other as 9:8. Prove that the areas are as $1 : \cos 18^\circ$.

Regular polygons: diagonals

ISMJ 12.13.

Suppose that all diagonals are drawn from some one vertex of a regular polygon. For which regular polygons are at least two of these diagonals perpendicular to each other?

PME 390. **by Robb Koether and David C. Kay**

Let the diagonals of a regular n-gon of unit side be drawn. Prove that the $n-2$ consecutive triangles thus formed which have their bases along one diagonal, their legs along two others or a side, and one vertex in common with a vertex of the polygon each have the property that the product of two sides equals the third.

Regular polygons: exterior point

SPECT 7.2.

Let $A_1A_2 \ldots A_n$ be a regular plane polygon with center O, and let P be a point in the plane outside the circumcircle of the polygon. Compare the geometric mean of the lengths A_rP $(1 \le r \le n)$ with the length OP in the following two cases:

(a) When OP passes through a vertex of the polygon.
(b) When OP bisects a side of the polygon.

Regular polygons: inscribed polygons

MM 1076. **by M. S. Klamkin**

Let B be an n-gon inscribed in a regular n-gon A. Show that the vertices of B divide each side of A in the same ratio and sense if and only if B is regular.

TYCMJ 146. **by M. S. Klamkin**

Prove that the smallest regular n-gon that can be inscribed in a given regular n-gon will have its vertices at the midpoints of the sides of the given n-gon.

Regular polygons: limits

JRM 394. **by Archimedes O'Toole**

From a fixed point P on the circumference of a circle, regular n-gons $(n = 3, 4, 5, 6, \ldots)$ are inscribed, all having one vertex at P. Prove or disprove: The limiting area common to all the n-gons as $n \to \infty$ is zero.

NAvW 410. **by J. van de Lune**

Let $P_1, P_2, \ldots, P_n$ be the vertices of a regular n-gon inscribed in the unit circle, and let a_n denote the average of the n^2 Euclidean distances $d\left(P_i, P_j\right)$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$. Prove that a_n is increasing and determine $\lim_{n\to\infty} a_n$.

SSM 3766. **by Herta T. Freitag**

(a) Consider the sequence of circles contained "inside" an equilateral triangle having side length a generated in the following manner. Begin with the inscribed circle. In one "corner" of the triangle inscribe a circle tangent to the first circle and to two sides of the triangle. Inscribe another circle tangent to the second circle and to the same two sides. Continue this process indefinitely. Find the sum of the radii of these circles.

(b) What is the corresponding result if a square having side length s is used instead of an equilateral triangle?

SSM 3772. **by Herta T. Freitag**

(a) Obtain a formula for the area of an equilateral triangle inscribed in a circle of radius R.

(b) Starting with the above triangle, inscribe a similar one in the incircle of the first one, and continue in this manner indefinitely. Obtain the total area (if it exists) of this set of triangles.

(c) Generalize the above for an arbitrary regular polygon with n sides.

Regular polygons: point on circumcircle

AMM E2646. **by William Wernick**

Let $A_1, \ldots, A_n$ be vertices of a regular n-gon inscribed in a circle with center O. Let B be a point on arc A_1A_n and $\theta = \angle A_nOB$. If a_k is the length of the chord BA_k, express

$$\sum_{k=1}^{n}(-1)^k a_k$$

as a function of θ.

Right triangles: angle measures

CRUX 18. **by Jacques Marion**

Show that in a right triangle with sides 3, 4 and 5, neither of the acute angles is a rational multiple of π.

Right triangles: circles

MATYC 119. **by Norman Shimmel**

A circle of radius R is inscribed in $\angle ABC$ of right triangle ABC (with right angle at C). The tangent to the circle parallel to AC and furthest from AC meets BC at D. If $AC = H$, $CD = L$, and $\angle ABC = \theta$, find θ in terms of L, H, and R.

Right triangles: erected figures

FUNCT 2.5.3.

A right triangle has area A and hypotenuse of length c. On each side of the triangle draw a square, exterior to the triangle. Imagine a tight rubber band placed around the figure. What area would it enclose?

Right triangles: incircle

PARAB 400.
ISMJ 10.17.
OMG 18.1.5.

Show that the diameter d of the incircle of a right triangle of legs a, b, and hypotenuse c satisfies

$$d = a + b - c.$$

OSSMB G75.3-2.

Find the radius of the greatest circle that can be inscribed in a right triangle whose perimeter is 100 inches. Find also each of the sides of the triangle when the radius is greatest.

Right triangles: mean proportionals

CRUX 218. **by Gilbert W. Kessler**

The altitude to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse. The median to the hypotenuse also has this property. Does any other segment from vertex to hypotenuse have the property?

Right triangles: perspectivities

PME 422. **by Jack Garfunkel**

Let perpendiculars be erected outwardly at A and B of a right triangle ABC ($C = 90°$), and at M, the midpoint of AB. Extend these perpendiculars to points P, Q, R such that

$$AP = BQ = MR = \frac{AB}{2}\ .$$

Show that triangle PQR is perspective with triangle ABC.

Right triangles: sequences

TYCMJ 61. **by Peter A. Lindstrom**

By the altitude of a right triangle, we mean the altitude which is not also a leg of that triangle. Construct the altitude of right triangle T_0. Call one of the subtriangles T and the other T_1. Construct the altitude of T_1 and call one of the subtriangles T_2. Continue the process so that, in general, T_n is one of the two subtriangles formed by constructing the altitude of T_{n-1}. It is known that there exist sequences $T_0, T_1, T_2, \ldots$, for which $\sum_{i=0}^{\infty}(\text{area } T_i)$ equals twice the area of T_0. Prove that the sum of the altitudes of the triangles in any one of these sequences equals the perimeter of T.

PME 461. **by David C. Kay**

(a) A right triangle with unit hypotenuse and legs r and s is used to form a sequence of similar right triangles $T_1, T_2, T_3, \ldots$ where the sides of T_1 are r times those of the given triangle, and for $n \geq 1$ the sides of T_{n+1} are s times those of T_n. Prove that the sequence T_n will tile the given triangle.

(b) What happens if the multipliers r and s are reversed?

(c) Given is a right triangle ABC with hypotenuse BC. A perpendicular is dropped from A onto BC, meeting BC at point P_1. Next, a perpendicular is dropped from P_1 onto AB, meeting AB at point P_2. This process is continued: perpendiculars are alternately dropped onto AB and BC to obtain a sequence of points $P_1, P_2, \ldots$. Show that the sum of the areas of $\triangle CAP_1$, $\triangle P_1P_2P_3$, $\triangle P_3P_4P_5, \ldots$ is equal to $(b^3c + bc^3)/(2b^2 + 4c^2)$.

Rolling

MENEMUI 1.2.1. **by R. J. E. Porkess**

A disc of radius R rolls without slipping around the inside of the circumference of a fixed circle whose radius is $2R$. Prove that the locus of a point at distance $R/2$ from the center of the disc is an ellipse of area $3\pi R^2/4$.

NYSMTJ 56.

Consider an object, such as a water glass in the shape of a frustum of a right circular cone, with base radii r and R, and slant height l. When such an object is placed on its side on a smooth, level surface, it can be rolled in a circle, returning to its starting point. Express the radius of this circle in terms of r, R, and l.

Semicircles

ISMJ 13.10.

Arc $ARPB$ is a semicircle. Prove that if R is above P then $AR + RB = AP + PB$.

CRUX 386. **by Francine Bankoff**

A square $PQRS$ is inscribed in a semicircle (O) with PQ falling along diameter AB. A right triangle ABC, equivalent to the square, is inscribed in the same semicircle with C lying on the arc RB. Show that the incenter I of triangle ABC lies at the intersection of SB and RQ, and that

$$\frac{RI}{IQ} = \frac{SI}{IB} = \frac{1+\sqrt{5}}{2}.$$

Simple closed curves

AMM 6129. **by E. H. Kronheimer**

Let S be a simple closed curve in the plane. Prove that unless S is a circle, it is always possible to find four points p, q, u, v on S and a point x inside S such that u and v belong to distinct components of $S \setminus \{p, q\}$, and x is nearer to both p and q than it is to either u or v.

PUTNAM 1977/B.4.

Let C be a continuous closed curve in the plane which does not cross itself and let Q be a point inside C. Show that there exist points P_1 and P_2 on C such that Q is the midpoint of the line segment P_1P_2.

MM 1006. **by G. A. Heuer**

A simple closed curve in the plane encloses a region R of area A. There is a point P in the interior of R such that every line through P intersects R in a line segment of length d. Find the greatest lower and least upper bounds for A. Are there curves where these bounds are attained?

Squares: 2 squares

CRUX 464. **by J. Chris Fisher and E. L. Koh**

(a) If the two squares $ABCD$ and $AB'C'D'$ have vertex A in common and are taken with the same orientation, then the centers of the squares together with the midpoints of BD' and $B'D$ are the vertices of a square.

(b) What is the analogous theorem for regular n-gons?

Squares: angles

CRUX 147. **by Steven R. Conrad**

In square $ABCD$, AC and BD meet at E. Point F is in CD and $\angle CAF = \angle FAD$. If AF meets ED at G and if $EG = 24$, find CF.

Squares: circles

OSSMB G77.1-3.

The square $ABCD$, with sides of length a, has circles of radius a drawn with centers A, B, C, D. Find the area of the central curvilinear quadrilateral.

Squares: circumscribed triangle

SSM 3652. **by Fred A. Miller**

Prove: The side of a square inscribed in a triangle is half the harmonic mean between the base and the altitude.

Squares: erected figures

IMO 1977/1.
PARAB 364.

Equilateral triangles ABK, BCL, CDM and DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN and AN are the twelve vertices of a regular dodecagon.

FUNCT 3.3.4. **by Lindsay Pope**

Given is a square with side s. Four quadrants of radius s are inscribed in the square, each having its center at one of the corners. Find the area of the intersection of the four quadrants.

MSJ 451. **by Saleh Rahman**

Let $ABCD$ be a square with $AB = 10$. Quadrants with centers at A and B, drawn interior to the square, intersect at E. Find the area of the region bounded by DC, arc DE, and arc EC.

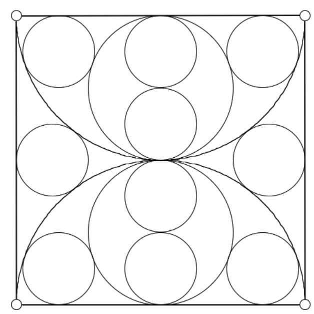
Squares: inscribed circles

CRUX 444. **by Dan Sokolowsky**

A circle is inscribed in a square $ABCD$. Point E is selected on BC so that the circle with diameter BE is tangent to the first circle. Show that $AB = 4BE$.

JRM 382. **by Leon Bankoff**

In the diagram shown, prove that the ten smaller circles are equal.



Squares: interior point

MATYC 130. **by Patrick J. Boyle**

Let P be a point in the square $ABCD$. If $PA = a$, $PB = a+b$, $PD = c$, and $a^2+b^2 = c^2$, prove $\angle APB = 90°$.

Squares: limits

OMG 15.3.3.

A circle is inscribed in a square of side 2. A square is inscribed in that circle. A circle is inscribed in that square, and so on ad infinitum. What is the sum of all the areas of the squares?

Squares: line segments

OSSMB 75-14.

A collection of line segments contained in a closed square of side 1 is said to be "opaque" if every straight line that crosses the square makes contact with at least one of the segments. Find an opaque set whose length is less than $1 + \sqrt{3}$.

Squares: lines

PARAB 413.

Let O be the center of the square $ABCD$ and let ℓ be a given line. The points O', A', B', C', and D' are the feet of the perpendiculars dropped from O, A, B, C, and D to the line ℓ. If $AA' \cdot CC' = BB' \cdot DD'$ and $AB = 2$, find OO'.

Squares: moats

OMG 15.1.1.

A student wishes to cross to a square island surrounded by a 4-meter wide moat. Can he do it with only two 3-meter long planks, and if so, how?

Stars

OMG 17.3.9.

Find the area of a star if

(a) the circumscribed circle has radius 10;

(b) the points of the star are the same distance apart;

(c) the star is formed by joining each point to the two opposite ones.

Symmetry

PARAB 374.

(a) A plane figure has one axis of symmetry and a point on that axis is a center of symmetry. Does the figure necessarily have a second axis of symmetry?

(b) A 3-dimensional figure has one plane of symmetry and a point in that plane is a center of symmetry. Does the figure necessarily have a second plane of symmetry?

Tesselations

ISMJ 14.2.

You are given an infinite supply of cardboard copies of a pentagon that has all sides one inch long and has two 90° angles that are not at opposite ends of the same side. Show how to cover the plane with these pentagons so that there are no overlaps and no uncovered spots.

CRUX 155. **by Steven R. Conrad and Ira Ewen**

A plane is tessellated by regular hexagons when the plane is the union of congruent regular hexagonal closed regions which have disjoint interiors. A lattice point of this tessellation is any vertex of any of the hexagons.

Prove that no four lattice points of a regular hexagonal tessellation of a plane can be the vertices of a square.

SSM 3677. **by Herta T. Freitag**

(a) Consider the following tessellation of equal-sized, regular hexagons of side a. After placing a tile, each following row is fitted so as to form a triangular array. Each time a row is completed, join the midpoints of the outer tiles of the tessellation to form a triangle. Obtain a formula for the area of these triangles in terms of the number of tiles used in the tessellation.

(b) Obtain corresponding formulas using (1) triangular tiles, (2) two different placements of square tiles.

JRM 388. **by Solomon W. Golomb**

Let $S_1, S_2, S_3, \ldots$ be a sequence of squares in the plane such that S_i has side length i. Can this sequence possibly tessellate the plane?

Tiling

PME 434. **by Sidney Penner**

Consider $(2n+1)^2$ hexagons arranged in a "diamond" pattern, the kth column from the left and also from the right consisting of k hexagons, $1 \le k \le 2n+1$. Show that if exactly one of the six hexagons adjacent to the center hexagon is deleted, then it is impossible to tile the remaining hexagons by pieces consisting of 3 mutually touching hexagons.

PARAB 315.

A large supply of small tiles is available for tiling the flat bottom of a large swimming pool. Each tile is in the shape of a regular polygon with edges all 1 cm long, and exactly 3 different shapes are used. The tiles are laid edge to edge in such a way that, although the vertices of 3 different tiles sometimes come together at the same point, no more than 3 vertices ever come together at the same point. Whenever 3 vertices do come together, the tiles at that point have different shapes. Prove that no tile used has an odd number of edges.

ISMJ 14.17.

Show that it is not possible to arrange ten equal squares in the plane so that no two overlap and so that one square touches each of the other nine squares.

PARAB 318.

Show how to place squares with sides of length $(1/m)$, where $m = 2, 3, 4, 5, \ldots$ (an infinite number of them) inside a square with side of length 1. None of the squares you use are allowed to overlap any other one.

Trapezoids

MSJ 470.

Trapezoid $APQB$ is inscribed in a semicircle and $AB = 4$ and $AP = BQ = 1$. Find the length of PQ.

SSM 3743. **by Steven R. Conrad**

Consider a trapezoid $ABCD$ having bases b and B with $b < B$. If each diagonal is divided into n equal parts, find the length of the line segment formed by connecting the ith division point on one diagonal to the ith division point on the other diagonal.

PME 409. **by Zazou Katz**

A point E is chosen on side CD of a trapezoid $ABCD$, $(AD \parallel BC)$, and is joined to A and B. A line through D parallel to BE intersects AB in F. Show that FC is parallel to AE.

Triangle inequalities: altitudes

NYSMTJ 92. **by Norman Schaumberger**

If h_a, h_b, and h_c are the lengths of the altitudes of a triangle, show that

$$h_a h_b + h_b h_c > h_a h_c.$$

Geometry

MM 936. **by Jack Garfunkel**

It is known that

$$h_a + h_b + h_c \leq \sqrt{3}s,$$

where the h's represent altitudes to sides a, b, and c and s represents the semiperimeter of triangle ABC. Prove or disprove the stronger inequality

$$t_a + t_b + m_c \leq \sqrt{3}s,$$

where the t's are the angle bisectors and m_c is the median to side c.

Triangle inequalities: angle bisectors and medians

PME 421. **by Murray S. Klamkin**

If $F(x, y, z)$ is a symmetric, increasing function of x, y, z, prove that for any triangle in which w_a, w_b, w_c are the internal angle bisectors and m_a, m_b, m_c the medians, we have

$$F(w_a, w_b, w_c) \leq F(m_a, m_b, m_c)$$

with equality if and only if the triangle is equilateral.

Triangle inequalities: angle bisectors extended

AMM S23. **by Jack Garfunkel and Leon Bankoff**

Prove that the sum of the distances from the incenter of a triangle ABC to the vertices does not exceed half of the sum of the internal angle bisectors, each extended to its intersection with the circumcircle of triangle ABC.

PME 374. **by Jack Garfunkel**

In a triangle ABC inscribed in a circle (O), angle bisectors AT_1, BT_2, CT_3 are drawn and extended to the circle with T_i lying on the circle. Perpendiculars T_1H_1, T_2H_2, T_3H_3 are drawn to sides AC, BA, CB respectively. Prove that

$$T_1H_1 + T_2H_2 + T_3H_3 \leq 3R,$$

where R is the radius of the circumcircle.

Triangle inequalities: angles

PME 394. **by Erwin Just and Bertram Kabak**

Prove that if A_1, A_2, and A_3 are the angles of a triangle, then

$$3\sum_{i=1}^{3} \sin^2 A_i - 2\sum_{i=1}^{3} \cos^3 A_i \leq 6.$$

Triangle inequalities: angles and radii

TYCMJ 85. **by Bertram Kabak**

(a) Let R, r, and P be the radius of the circumscribed circle, the radius of the inscribed circle, and the perimeter of a triangle, respectively. Prove that

$$54Rr \leq P^2.$$

(b) Let O be a point within triangle $A_1A_2A_3$ and let d_i be the distance from O to a_i, the side opposite angle A_i, $(i = 1, 2, 3)$. Prove that

$$\sum_{i=1}^{3} d_i \sin A_i = \prod_{i=1}^{3} \frac{a_i}{4R^2}\ .$$

Triangle inequalities: angles and sides

AMM E2649. **by A. Oppenheim**

Let a, b, c and α, β, γ be the sides and the corresponding opposite angles of a nonobtuse triangle. Show that

$$3(a + b + c) \leq \pi\left(\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}\right),$$

and

$$3(a^2 + b^2 + c^2) \geq \pi\left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}\right).$$

Triangle inequalities: centroids

AMM E2715. **by Jack Garfunkel**

Let G be the centroid of the triangle $A_1A_2A_3$ and let

$$\theta_i = \angle\left(\overrightarrow{A_iA_{i+1}}, \overrightarrow{A_iG}\right), \qquad i = 1, 2, 3.$$

Prove or disprove that $\sum \sin\theta_i \leq 3/2$.

Triangle inequalities: circumcenter and incenter

PME 442. **by Jack Garfunkel**

Show that the sum of the perpendiculars from the circumcenter of a triangle to its sides is not less than the sum of the perpendiculars drawn from the incenter to the sides of the triangle.

Triangle inequalities: circumradius

SIAM 77-9. **by I. J. Schoenberg**

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$, $x_1 < x_2 < x_3$, be points in the Cartesian (x, y)-plane and let R denote the radius of the circumcircle Γ of the triangle $P_1P_2P_3$ ($R = \infty$ if the triangle is degenerate). Show that

$$\frac{1}{R} < 2\left|\frac{y_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{y_2}{(x_2 - x_3)(x_2 - x_1)} + \frac{y_3}{(x_3 - x_1)(x_3 - x_2)}\right|$$

unless both sides vanish and that 2 is the best constant in the equation.

Triangle inequalities: Gergonne point

NAvW 478. **by W. J. Blundon and R. H. Eddy**

Let g_a, g_b, and g_c denote the cevians of a triangle ABC concurrent at the Gergonne point. Prove (in the usual notation) that

$$8Rr + 11r^2 \leq \sum g_a^2 \leq 4R^2 + 11r^2,$$

with equality if and only if the triangle is equilateral.

Triangle inequalities: half angles

NYSMTJ OBG7. **by Norman Schaumberger**

If A, B, and C are the angles of a triangle, then prove that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}\ .$$

Triangle inequalities: interior point

PME 410. **by Murray S. Klamkin**

If x, y, z are the distances of an interior point of a triangle ABC to the sides BC, CA, AB, show that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{2}{r},$$

where r is the inradius of the triangle.

MM 959. **by L. Carlitz**

Let P be a point in the interior of the triangle ABC and let r_1, r_2, and r_3 denote the distances from P to the sides of the triangle. Let R denote the circumradius of ABC. Show that

$$\sqrt{r}_1+\sqrt{r}_2+\sqrt{r}_3\leq 3\sqrt{R/2},$$

with equality if and only if ABC is equilateral and P is the center of ABC.

Triangle inequalities: medians and sides

MM Q638. **by Murray S. Klamkin**

Let a, b, and c denote the sides of an arbitrary triangle with respective medians m_a, m_b, and m_c. Determine all integral p and q so that

$$\left(\frac{\sqrt{3}}{2}\right)^p(a^pm_a^q+b^pm_b^q+c^pm_c^q)\geq \left(\frac{\sqrt{3}}{2}\right)^q(a^qm_a^p+b^qm_b^p+c^qm_c^p).$$

SIAM 79-19. **by M. S. Klamkin**

If a_1, a_2, a_3 and m_1, m_2, m_3 denote the sides and corresponding medians of a triangle, respectively, prove that

$$\left(a_1^2+a_2^2+a_3^2\right)(a_1m_1+a_2m_2+a_3m_3) \geq 4m_1m_2m_3\left(a_1+a_2+a_3\right).$$

Triangle inequalities: radii

NAvW 472. **by J. T. Groenman**

Let r, r_a, r_b, and r_c be the radii of the inscribed circles of a triangle ABC. Depending upon the fact of whether the triangle is acute, right, or obtuse, prove that one of the following statements holds:

$$\left(\frac{r_ar_br_c}{r}\right)^{\frac{1}{2}}>\frac{1}{2}\left(r+r_a+r_b+r_c\right),$$

(respectively = and <).

MM 1043. **by M. S. Klamkin**

If (a_i, b_i, c_i) are the sides, R_i the circumradii, r_i the inradii, and s_i the semiperimeters of two triangles $(i=1,2)$, show that

$$\sqrt{\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}}\geq 3\left\{\frac{1}{\sqrt{a_1a_2}}+\frac{1}{\sqrt{b_1b_2}}+\frac{1}{\sqrt{c_1c_2}}\right\}$$

with equality if and only if the two triangles are equilateral.

Also show that the analogous three triangle inequality

$$\sqrt{\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\frac{s_3}{r_3R_3}} \geq 9\left\{\frac{1}{\sqrt{a_1a_2a_3}}+\frac{1}{\sqrt{b_1b_2b_3}}+\frac{1}{\sqrt{c_1c_2c_3}}\right\}$$

is invalid.

Triangle inequalities: sides

SIAM 77-10. **by M. S. Klamkin**

Let P and P' denote two arbitrary points and let $A_1A_2A_3$ denote an arbitrary triangle of sides a_1, a_2, a_3. If $R_i = PA_i$ and $R_i' = P'A_i$, prove that

$$a_1R_1R_1'+a_2R_2R_2'+a_3R_3R_3'\geq a_1a_2a_3$$

and determine the conditions for equality. It is to be noted that when P' coincides with P, we obtain a known polar moment of inertia inequality.

TYCMJ 98. **by Norman Schaumberger**

Let a, b, and c be the lengths of the sides of a triangle with area K and perimeter P. Prove or disprove that

$$a^3+b^3+c^3\geq\frac{4\sqrt{3}}{3}KP$$

and

$$a^4+b^4+c^4\geq 16K^2.$$

TYCMJ 130. **by Aron Pinker**

Let a, b, and c be the sides of a triangle, P its perimeter, and K its area. Prove that:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{9}{P}$$
$$a^2+b^2+c^2\geq\frac{P^2}{3}$$
$$P^2\geq 12\sqrt{3K}$$
$$a^2+b^2+c^2\geq 4\sqrt{3}K$$
$$a^3+b^3+c^3\geq\frac{P^3}{9}.$$

Triangles: 2 triangles

AMM E2512. **by E. A. Herman**

Let T_1 and T_2 be two triangles with circumcircles C_1 and C_2, respectively. Show that if T_1 meets T_2, then some vertex of T_1 lies in (or on) C_2 or vice versa. Generalize.

NAvW 508. **by L. Kuipers**

In a plane, two congruent triangles ABC and $A'B'C'$ are in such a position that

$$AB\parallel\overrightarrow{B'A'},\quad BC\parallel\overrightarrow{C'B'},\quad\text{and}\quad CA\parallel\overrightarrow{A'C'}.$$

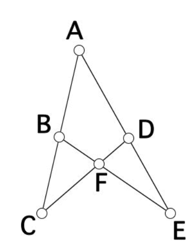
Now let A'' be the reflection of A' in the side BC, B'' that of B' in the side CA, and C'' that of C' in AB.

Prove that the triangles ABC and $A''B''C''$ are similar.

CRUX 171. **by Dan Sokolowsky**

Let P_1 and P_2 denote, respectively, the perimeters of triangles ABE and ACD. Without using circles, prove that

$$P_1=P_2\Longrightarrow AB+BF=AD+DF.$$



Triangles: 3 triangles

SSM 3660. **by Steven R. Conrad**
Triangles 1, 2, and 3 are coplanar. Every point of triangle 2 is interior to and 2 inches from triangle 1. Every point of triangle 3 is interior to and 2 inches from triangle 2. If the inch-lengths of the sides of triangle 2 are 13, 14, and 15, find the area of the region interior to triangle 1 but not also interior to triangle 3.

Triangles: 30 degree angle

OSSMB G78.3-6.
(a) If, in $\triangle ABC$, $b = a(\sqrt{3} - 1)$ and $\angle C = 30^\circ$, find $\angle A$ and $\angle B$.
(b) Given that $a = 2b$ in $\triangle ABC$, show that $\angle A > 2\angle B$.

PME 351. **by Jack Garfunkel**
Angle A and angle B are acute angles of a triangle ABC. If $\angle A = 30^\circ$ and h_a, the altitude issuing from A, is equal to m_b, the median issuing from B, find angles B and C.

Triangles: 60 degree angle

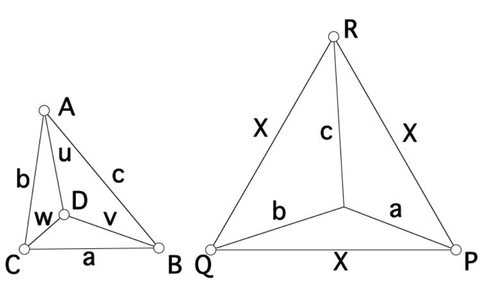
CRUX 148. **by Steven R. Conrad**
In $\triangle ABC$, $\angle C = 60^\circ$ and $\angle A$ is greater than $\angle B$. The bisector of $\angle C$ meets AB in E. If CE is a mean proportional between AE and EB, find $\angle B$.

OSSMB 79-16.
In triangle ABC, BE bisects angle ABC and angle AEB is 60°. Let F be a point on the side BC so that angle AFB is also 60°. Segment AF intersects BE at the point D. Prove that $DE = EC$.

AMM E2639. **by G. Tsintsifas**
Let ABC be a triangle with $\angle A = 40^\circ$ and $\angle B = 60^\circ$. Let D and E be points lying on the sides AC and AB, respectively, such that $\angle CBD = 40^\circ$ and $\angle BCE = 70^\circ$. Let F be the point where the lines BD and CE intersect. Show that the line AF is perpendicular to the line BC.

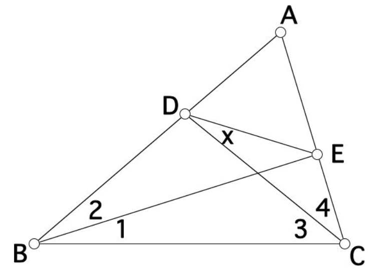
Triangles: 120 degree angle

CRUX 38. **by Léo Sauvé**
Consider the two triangles $\triangle ABC$ and $\triangle PQR$. In the triangle $\triangle ABC$, we have $\angle ADB = \angle BDC = \angle CDA = 120^\circ$. Prove that $X = u + v + w$.



Triangles: adventitious triangles

CRUX 255. **by Barry Hornstein**
In $\triangle ABC$, the measures of angles 1, 2, 3, 4 are given. Calculate angle x in terms of angles 1, 2, 3, 4.



ISMJ 12.32.
Let OPQ be an isosceles triangle with angles 20°, 80°, and 80°. The point B is chosen on side OQ so that $\angle OPD = 20^\circ$ and A is chosen on side OP so that $\angle OQA = 30^\circ$. Show that $\angle BAQ = 80^\circ$.

Triangles: altitudes

TYCMJ 74. **by Harley Flanders**
Let O be the intersection of the altitudes of acute triangle ABC. Choose B' on OB and C' on OC so that $AB'C$ and $AC'B$ are right angles. Prove that $AB' = AC'$.

CRUX 192. **by Ross Honsberger**
Let D, E, and F denote the feet of the altitudes of $\triangle ABC$, and let (X_1, X_2), (Y_1, Y_2), and (Z_1, Z_2) denote the feet of perpendiculars from D, E, and F, respectively, upon the other two sides of the triangle. Prove that the six points X_1, X_2, Y_1, Y_2, Z_1, and Z_2 lie on a circle.

NAvW 525. **by O. Bottema**
The altitudes of the triangle $A_1A_2A_3$ are A_1H_1, A_2H_2, and A_3H_3. The conic K is tangent to A_2A_3 at H_1, to A_3A_1 at H_2, and to A_1A_2 at H_3.
Show that the center of K coincides with the Lemoine point of the triangle.

TYCMJ 110. **by K. R. S. Sastry**
Let ABC be a triangle; AP, BQ, CR its altitudes; and AD, BE, CF the internal bisectors of the angles. Let BE and CF intersect AP in X_1 and X_2, respectively; CF and AD intersect BQ in Y_1 and Y_2, respectively; and AD and BE intersect CR in Z_1 and Z_2, respectively. Prove that $IX_1 \cdot IY_1 \cdot IZ_1 = IX_2 \cdot IY_2 \cdot IZ_2 = X_1X_2 \cdot Y_1Y_2 \cdot Z_1Z_2$, where I is the incenter of $\triangle ABC$.

CRUX 46. **by F. G. B. Maskell**
If p_1, p_2, and p_3 are the altitudes of a triangle and r is the radius of its inscribed circle, show that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}.$$

Triangles: angle bisectors

CRUX 365. **by Kesiraju Satyanarayana**
A scalene triangle ABC is such that the external bisectors of angles B and C (i.e., the segments intercepted by B. C and the opposite sides) are of equal length. Given the lengths of the sides b and c (with $b > c$), find the length of the third side, a, and show that its value is unique.

ISMJ 10.6.

In the triangle ABC, point P is on AB, the line CP bisects angle C, $m = CP$, $a = BC$, $b = AC$, $x = AP$, and $y = PB$. Show that $m^2 = ab - xy$.

ISMJ 14.18.

Let ABC be a triangle with $\angle A < \angle C < 90^\circ < \angle B$. Suppose the bisectors of the external angles at A and B, measured from the vertex to the opposite side (extended), are each equal to AB. Determine angle A.

ISMJ 11.2.

In triangle ABC, M is the midpoint of BC and the bisector of angle A cuts BC at X. The circle through A, X, and M cuts AB at P and AC at Q. Prove that $BP = CQ$.

CRUX 168. **by Jack Garfunkel**
MM Q646. **by Jack Garfunkel**

If a, b, c are the sides of a triangle ABC, t_a, t_b, t_c are the angle bisectors, and T_a, T_b, T_c are the angle bisectors extended until they are chords of the circle circumscribing the triangle ABC, prove that

$$abc = \sqrt{T_aT_bT_ct_at_bt_c}.$$

CRUX 309. **by Peter Shor**

Let ABC be a triangle with $a \geq b \geq c$ or $a \leq b \leq c$. Let D and E be the midpoints of AB and BC, and let the bisectors of angles BAE and BCD meet at R. Prove that

(a) $AR \perp CR$ if and only if $2b^2 = c^2 + a^2$;

(b) lf $2b^2 = c^2 + a^2$, then R lies on the median from B. Is the converse of (b) true?

OMG 18.3.4.

In $\triangle ABC$, show that the angle contained between the bisector of A and the perpendicular from A to BC is equal to the difference of angles B and C.

MM 998. **by Hüseyin Demir**

Characterize all triangles in which the triangle whose vertices are the feet of the internal angle bisectors is a right triangle.

Triangles: angle measures

AMM E2579. **by Benjamin Klein and Brian White**

Let $0 < \theta < \frac{1}{2}\pi$, and let p, q be arbitrary distinct points in the Euclidean plane E. Define $f_\theta(p,q)$ to be the unique point r in E such that triangle pqr is in the counterclockwise sense and $\angle rpq = \angle rqp = \theta$ radians. Show that $f_{\pi/3}(p,q)$ can be written as an expression involving only $f_{\pi/6}$, p, q, and parentheses.

PARAB 411.

Let D be a point on side AC of $\triangle ABC$. The angles ABD, DBC, and BCD are 20°, 20°, and 40°, respectively. Prove that $BC = BD + DA$.

Triangles: angle trisectors

JRM 706. **by Sidney Kravitz**

(a) Given an isosceles right triangle with unit legs, find the length of the sides of Morley's equilateral triangle.

(b) Solve the same problem for a general triangle.

OSSMB G78.2-5.

(a) If a, b, x, y are positive numbers such that

$$0^\circ < a + b < 180^\circ,$$
$$x + y = a + b, \text{ and}$$
$$\frac{\sin x}{\sin y} = \frac{\sin a}{\sin b},$$

show that $x = a$ and $y = b$.

(b) Show that $\sin 3\theta = 4\sin\theta\cdot\sin(60^\circ-\theta)\cdot\sin(60^\circ+\theta)$.

(c) Prove Morley's Theorem: The points of intersection of the adjacent trisectors of the angles of a triangle are the vertices of an equilateral triangle.

Triangles: area

USA 1977/2.

The triangles ABC and DEF have AD, BE and CF parallel. Show that

$$[AEF] + [DBF] + [DEC] + [DBC] + [AEC] + [ABF] = 3([ABC] + [DEF]),$$

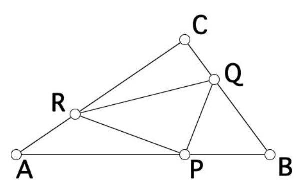
where $[XYZ]$ denotes the *signed* area of triangle XYZ. (Thus $[XYZ]$ is $+$area(XYZ) when the order X, Y, Z is counterclockwise and $-$area(XYZ) otherwise. For example, $[XYZ] = [YZX] = -[YXZ]$.)

CRUX 56. **by F. G. B. Maskell**

Find the area of a triangle in terms of its medians m_1, m_2, and m_3.

ISMJ 11.17.
ISMJ 12.15.

A triangle ABC has area F. Let P, Q, and R divide the sides AB, BC, and CA in the ratios 1:2. Let the triangle PQR have area f. Determine the ratio F/f.



TYCMJ 79. **by Martin Berman**

In triangle ABC, let D, E, and F be points on BC, CA, and AB, respectively, such that

$$AF/AB = BD/BC = CE/CA = r < 1/2.$$

Prove that the ratio of the area of the triangle determined by AD, BE, and CF to the area of triangle ABC is $4 - 3/(r^2 - r + 1)$.

Triangles: centroids

MSJ 438. **by Frank Eccles and Esmond McNutt**

Find necessary and sufficient conditions under which a line passing through the centroid of a triangle will divide the triangle into two regions of equal area.

TYCMJ 148. **by Martin Berman**

Form a triangle with line segments of uniform density and having lengths a, b, and c. Denote by g_1 the centroid of the three line segments and by g_2 the centroid of the triangular region bounded by the line segments. When do g_1 and g_2 coincide?

Geometry

MM 1028. **by Leon Gerber**

Let ABC be a triangle and P_1, P_2, and P_3 be arbitrary points in the plane of ABC. Let arbitrary lines perpendicular to AP_i, BP_i, and CP_i determine triangles $A_iB_iC_i$ for $i = 1, 2, 3$. Now, let A_0, B_0, and C_0, be the centroids of triangles $A_1A_2A_3$, $B_1B_2B_3$, and $C_1C_2C_3$, respectively. Show that the perpendiculars from A, B, and C on the sides of triangle $A_0B_0C_0$ concur.

CRUX 334. **by Philip D. Straffin**

Let A, B, and C be three fixed noncollinear points in the plane, and let X_0 be the centroid of $\triangle ABC$. Call a point P in the plane accessible from X_0 if there is a sequence of points X_0, $X_1, \ldots,$ $X_n = P$ such that X_{i+1} is closer than X_i to at least two of the points A, B, and C $(i = 0, 1, \ldots, n-1)$. Characterize the set of points in the plane which are accessible from X_0.

Triangles: Ceva's theorem

CRUX 414. **by Basil C. Rennie**

A few years ago a distinguished mathematician wrote a book saying that the theorems of Ceva and Menelaus were dual to each other. Another distinguished mathematician reviewing the book wrote that they were not dual. Explain why they were both right.

Triangles: cevians

CRUX 485. **by M. S. Klamkin**

Given three concurrent cevians of a triangle ABC intersecting at a point P, we construct three new points A', B', C' such that $AA' = kAP$, $BB' = kBP$, $CC' = kCP$, where $k > 0$, $k \neq 1$, and the segments are directed. Show that A, B, C, A', B', and C' lie on a conic if and only if $k = 2$.

CRUX 456. **by Orlando Ramos**

Let ABC be a triangle and P any point in the plane. Triangle MNO is determined by the feet of the perpendiculars from P to the sides, and triangle QRS is determined by the cevians through P and the circumcircle of triangle ABC. Prove that triangles MNO and QRS are similar.

Triangles: circles

OSSMB G78.2-3.

A triangle ABC is defined as follows: A has coordinates $(0, 0)$, C is on the positive x-axis, the slope of AB is $4/3$, the length of AB is 10, and the length of BC is $2\sqrt{17}$. Show that there are two values for C, say C_1 and C_2, and find the equation of the circle BC_1C_2.

SSM 3678. **by Fred A. Miller**

Prove or disprove that the circle determined by two vertices of a triangle and its incenter has its center on the circumcircle of the triangle.

OSSMB 79-8. **by Maurice Field**

Let ABC be a triangle and let D, E, F be points on the lines BC, AC, AB respectively; none of the points D, E, F are vertices of the triangle. Show that the circles AFE, BFD and CDE are concurrent. What interesting fact is obtained if, in addition, the points D, E, F are collinear?

CRUX 206. **by Dan Pedoe**

A circle intersects the sides BC, CA, and AB of a triangle ABC in the pairs of points X, X', Y, Y' and Z, Z' respectively. If the perpendiculars at X, Y and Z to the respective sides BC, CA and AB are concurrent at a point P, prove that the respective perpendiculars at X', Y' and Z' to the sides BC, CA and AB are concurrent at a point P'.

Triangles: circumcircles

MM 967. **by K. R. S. Sastry**

Let ABC be a triangle inscribed in a circle with the internal bisectors of the angles B and C meeting the circle again in the points B_1 and C_1, respectively.

(a) If $B = C$, prove $BB_1 = CC_1$.

(b) Characterize triangles ABC for which $BB_1 = CC_1$. Do these results hold if BB_1 and CC_1 are the external bisectors?

NAvW 425. **by O. Bottema**

Let a, b, and c be the sides and R the circumradius of an acute triangle. Show that

$$\rho = 0.344\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}$$

is an approximate value of R with a relative error $\frac{|\rho - R|}{R}$ that is less than 0.04.

AMM E2538. **by J. Garfunkel**

Let ABC be a triangle. If X is a point on side BC, let AX meet the circumcircle of ABC again at X'. Prove or disprove that if XX' has maximum length, then AX lies between the median and the internal angle bisector issuing from A.

SPECT 10.9. **by J. R. Alexander**

The following algorithm describes a geometrical procedure:

(1) take any triangle ABC;
(2) circumscribe a circle around ABC;
(3) draw tangents l, m, n at A, B, C;
(4) let $A = m \cap n$, $B = n \cap l$, $C = l \cap m$;
(5) go to (2).

Describe the angles of $\triangle ABC$ after reaching (4) for the nth time, and determine under what circumstances the angle at A takes its initial value again.

Now begin with a cyclic quadrilateral $ABCD$ instead of a triangle, and carry out the analogous construction. Show that if it is possible to pass beyond (2) for the second time, then

$$AB^2 + CD^2 = d^2,$$

where d is the diameter of the circle circumscribing $ABCD$.

Triangles: ellipses

CRUX 318. **by C. A. Davis**

Given any triangle ABC, thinking of it as in the complex plane, two points L and N may be defined as the stationary values of a cubic that vanishes at the vertices A, B, and C. Prove that L and N are the foci of the ellipse that touches the sides of the triangle at their midpoints, which is the inscribed ellipse of maximal area.

Triangles: equal angles

SSM 3668. **by Fred A. Miller**

In a triangle ABC a line has been drawn from vertex A to a given point in the opposite side BC. Find a point P on this line from which the two parts of BC subtend equal angles.

Triangles: equal areas

PENT 307. **by Fred A. Miller**

Let A, B, C denote the vertices of a triangle that lie on the sides DE, EF, and FD respectively of triangle DEF. Let $A'B'C'$ be a second triangle whose vertices lie on the sides of triangle DEF in such a way that A and A' are equidistant from the midpoint of DF, B and B' are equidistant from the midpoint of DE, and C and C' are equidistant from the midpoint of EF. Prove that triangles ABC and $A'B'C'$ have equal areas.

SSM 3707. **by Fred A. Miller**

Let RST be a triangle such that M, N, and L are the midpoints of its sides. If triangles ABC and DEF have vertices which lie on the sides of triangle RST at equal distances from M, N, and L, prove that these triangles have the same area.

MSJ 444.

For how many different positions of point P in the plane of triangle ABC will $\triangle PAB$, $\triangle PBC$, and $\triangle PAC$ all be the boundaries of regions that have equal areas?

Triangles: erected figures

ISMJ 10.4.

On the side AB of a given triangle ABC two equilateral triangles ABX and ABY are constructed. Prove that

$$(CX)^2 + (CY)^2 = (AB)^2 + (BC)^2 + (CA)^2.$$

PME 354. **by Arthur Bernhart and David C. Kay**

In a triangle ABC with angles less than $2\pi/3$, the Fermat point, defined as that point which minimizes the function $f(X) = AX + BX + CX$, may be determined as the point P of concurrence of lines AD, BE, and CF, where BCD, ACE, and ABF are equilateral triangles constructed externally on the sides of triangle ABC. If R, S, and T are the points where PD, PE, and PF meet the sides of triangle ABC, prove that PD, PE, and PF are twice the arithmetic means, and that PR, PS, and PT are half the harmonic means, of the pairs of distances (PB, PC), (PC, PA), and (PA, PB) respectively.

CRUX 363. **by Roland H. Eddy**

The following generalization of the Fermat point is known: If similar isosceles triangles BCA', CAB', ABC' are constructed externally to triangle ABC, then AA', BB', CC' are congruent.

Determine a situation in which AA', BB', CC' are concurrent if the constructed triangles are isosceles but not similar.

AMM E2802. **by M. Slater**

Given a triangle ABC (in the Euclidean plane), construct similar isosceles triangles ABC' and ACB' outwards on the respective bases AB and AC, and BCA'' inwards on the base BC (or ABC'' and ACB'' inwards and BCA' outwards). Show that $AB'A''C'$ (respectively, $AB''A'C''$) is a parallelogram.

PME 408. **by Clayton W. Dodge**

Squares are erected on the sides of a triangle, either all externally or all internally. A circle is centered at the center of each square with each radius a fixed multiple $k > 0$ of the side of that square. Find k so that the radical center of the three circles falls on the Euler line of the triangle, and find where it falls on the Euler line.

IMO 1975/3.
PARAB 379.

On the sides of an arbitrary triangle ABC, triangles ABR, BCP, CAQ are constructed externally with $\angle CBP = \angle CAQ = 45^\circ$, $\angle BCP = \angle ACQ = 30^\circ$, $\angle ABR = \angle BAR = 15^\circ$. Prove that $\angle QRP = 90^\circ$ and $QR = RP$.

Triangles: escribed circles

OSSMB G77.2-3.

Given $\triangle ABC$ with radius of incircle r and r_1, r_2, r_3 the radii of the escribed circles opposite angles A, B, C respectively, show that $ab = r_1r_2 + rr_3$.

PME 437. **by Zelda Katz**

Let N be the Nagel point of a triangle, which is the intersection of the lines from the vertices to the points of contact of the opposite escribed circles. In the triangle whose sides are $AB = 5$, $BC = 3$, and $CA = 4$, show that the areas of triangles ABN, CAN, and BCN are 1, 2, and 3 respectively.

Triangles: Euler line

CRUX 260. **by W. J. Blundon**

Given any triangle (other than equilateral), let P represent the projection of the incenter I on the Euler line $OGNH$ where O, G, N, H represent respectively the circumcenter, the centroid, the center of the nine-point circle and the orthocenter of the given triangle. Prove that P lies between G and H. In particular, prove that P coincides with N if and only if one angle of the given triangle has measure 60°.

Triangles: inscribed circles

SSM 3706. **by Irwin K. Feinstein**

In the coordinate plane, a line forms a Pythagorean triangle with the positive axes. A circle with radius r, r a positive integer, is inscribed in the triangle. The point (u, v) is the point of tangency of the line to the circle, where u and v are positive integers. What is the smallest value $u+v$ may assume?

Triangles: inscribed triangles

CRUX 372. **by Steven R. Conrad and Gilbert W. Kessler**

A triangle ABC has area 1. Point P is on side a, α units from B; point Q is on b, β units from C; and point R is on c, γ units from A. Prove that, if α/a, β/b, and γ/c are the zeros of a cubic polynomial f whose leading coefficient is unity, then the area of $\triangle PQR$ is given by $f(1) - f(0)$.

CRUX 210. **by Murray S. Klamkin**

Let P, O, and R denote points on the sides BC, CA, and AB, respectively, of a given triangle ABC. Determine all triangles ABC such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \quad (k \neq 0,\ 1/2,\ 1),$$

then PQR (in some order) is similar to ABC.

NAvW 401. **by O. Bottema**

Given the triangle $A = A_1A_2A_3$, determine the (real) triangle(s) $X = X_1X_2X_3$ such that

(1) X is inscribed in A with X_i on the side opposite A_i $(i = 1, 2, 3)$,

(2) X and A are similar,

(3) X and A are perspective.

Triangles: interior point

CRUX 270. **by Dan Sokolowsky**

A chord of a triangle is a segment with endpoints on the sides. Show that for every acute-angled triangle there is a unique point P through which pass three equal chords each of which is bisected by P.

PME 454. **by Marian Haste**

The point within a triangle whose combined distances to the vertices is a minimum is known as the Fermat-Torricelli point T. In a triangle ABC, if AT, BT, CT form a geometric progression with a common ratio of 2, find the angles of the triangle.

ISMJ 11.19.

Let ABC be a triangle and P a point inside or on this triangle.

(a) One of the three distances PA, PB, PC is least. Find the position(s) of P that makes this number as large as possible.

(b) One of the distances PA, PB, PC is largest. Find the position(s) of P that make this number least.

Triangles: isogonal conjugates

AMM E2793. **by E. D. Camier**

Let P and Q be two points isogonally conjugate with respect to a triangle ABC of which the circumcenter, orthocenter, and nine-point center are O, H, and N, respectively. If

$$\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ},$$

and U is the point symmetric to R with respect to N, show that the isogonal conjugate of U in the triangle ABC is the intersection V of the lines P_1Q and PQ_1, where P_1 and Q_1 are the inverses of P and Q in the circle ABC. (Assume that neither P nor Q is on the circle ABC.)

Triangles: isosceles triangles

OSSMB 79-5. **by H. Haruki**

An isosceles triangle ABC has an obtuse angle of 100° at A. The bisector of the base angle B meets AC at D. Show that $BD + AD = BC$.

CRUX 175. **by Andrejs Dunkels**

Given is an isosceles triangle ABC with $AB = AC$ and $\angle BAC = 20^\circ$. On AC a point D is marked off so that $AD = BC = b$. Find the measure of $\angle ABD$.

MSJ 456. **by Steven R. Conrad**

In an isosceles triangle with a 30° vertex angle, the length of the base is 12. Using only plane geometry, find the length of the altitude to the base.

MSJ 434. **by Gary Steiger**

Points D and E are outside isosceles triangle ABC such that CD and AE are the angle bisectors of base angles A and C. Segments CD and AE meet at H and points D, B, and E are collinear. If $DB = BE$, prove that $\angle BDH = \angle BEH$.

CRUX 134. **by Kenneth S. Williams**

Let ABC be an isosceles triangle with $\angle ABC = \angle ACB = 80^\circ$. Let P be the point on AB such that $\angle PCB = 70^\circ$. Let Q be the point on AC such that $\angle QBC = 60^\circ$. Find $\angle PQA$.

PARAB 344. **by G. Davis**

In $\triangle ABC$, $AB = AC$, D is on side AB and E is on side AC. Also, $\angle DAE = 20^\circ$, $\angle DCB = 60^\circ$, $\angle EBC = 50^\circ$, and $\angle CDE = x^\circ$. Find x without using trigonometric tables.

IMO 1978/4.

In triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB and AC at P and Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC.

CRUX 271. **by Shmuel Avital**

Find all possible triangles ABC which have the property that one can draw a line AD, outside the triangular region, on the same side of AC as AB, which meets CB (extended) in D so that triangles ABD and ACD will be isosceles.

MSJ 422. **by Ira Ewen**

In triangle ABC, $AB = BC$. There is a point P interior to the triangle for which $\angle APB = \angle CPB$. Line BP intersects AC at D. Prove that D is the midpoint of AC.

SSM 3733. **by Charles W. Trigg**

Suppose the median to the base of an isosceles triangle is equal to the base. Show that a leg, an altitude to the other leg, and one of the segments of that leg form a 3:4:5 triangle.

NYSMTJ 48. **by S. R. Conrad**

Establish a one-to-one correspondence between all isosceles triangles and all nonisosceles right triangles. Consider congruent triangles as the same triangle.

CRUX 376. **by V. G. Hobbes**

Isosceles triangles can be divided into two types: those with equal sides longer than the base and those with equal sides shorter than the base. Of all possible isosceles triangles what proportion are long-legged?

ISMJ J11.17.

Prove that in an isosceles triangle, the sum of the distances from any point on the base to the other two sides is a constant.

Triangles: line segments

NAvW 424. **by O. Bottema**

The endpoints B_1 and B_2 of a line segment with length 2ℓ move along the perimeters of the triangle $A_1A_2A_3$ with altitudes h_i; $h_1 \geq h_2 \geq h_3 > 2\ell$. A point B between B_1 and B_2 describes a path b. Prove that the area of the region inside $A_1A_2A_3$ and outside b is independent of h_i.

Triangles: lines

NAvW 482. **by O. Bottema and J. T. Groenman**

Let P and Q be two points in the plane of the triangle $A_1A_2A_3$. The line A_iP intersects the opposite side of the triangle at B_i. In the triangle $B_1B_2B_3$, the line B_iQ intersects the opposite side at C_i.

(a) Prove that the lines A_iC_i pass through one point R.

(b) Let Q be a fixed point and P a variable point; show that the relationship between P and R is a birational involutory correspondence.

Triangles: medians

CRUX 383. **by Daniel Sokolowsky**

Let m_a, m_b, and m_c be respectively the medians AD, BE, and CF of a triangle ABC with centroid G. Prove that

(a) if $m_a{:}m_b{:}m_c = a{:}b{:}c$; then $\triangle ABC$ is equilateral;

(b) if $m_b/m_c = c/b$, then either (i) $b = c$ or (ii) quadrilateral $AEGF$ is cyclic;

(c) if both (i) and (ii) hold in (b), then $\triangle ABC$ is equilateral.

CRUX 278. **by W. A. McWorter, Jr.**

If each of the medians of a triangle is extended beyond the sides of the triangle to 4/3 its length, show that the three new points formed and the vertices of the triangle all lie on an ellipse.

CRUX 144. **by Viktors Linis**

In a triangle ABC, the medians AM and BN intersect at G. If the radii of the inscribed circles in triangles ANG and BMG are equal, show that ABC is an isosceles triangle.

MSJ 458.

In a triangle, the lengths of the three medians are 9, 12, and 15. Find the length of the side to which the longest median is drawn.

ISMJ 12.14.

Let M be the midpoint of side BC of $\triangle ABC$. Show that, if $AM/BC = 3/2$, then the medians from B and C are perpendicular to each other.

Triangles: nine-point circle

CRUX 353. **by Orlando Ramos**

Prove that, if a triangle is self-polar with respect to a parabola, then its nine-point circle passes through the focus.

Triangles: orthocenter

OSSMB G78.2-4.

Given any triangle ABC, with orthocenter H, circumcenter O, and D on BC such that $OD \perp BC$, find the ratio OD/AH.

NAvW 494. **by J. T. Groenman**

The triangles $A_1B_1C_1$ and $A_2B_2C_2$ have the same circumcircle $O(R)$. The orthocenter of triangle $A_iB_iC_i$ is H_i $(i = 1, 2)$. Moreover:

A_1A_2 is parallel to B_1C_1,
B_1B_2 is parallel to C_1A_1, and
C_1C_2 is parallel to A_1B_1.

Prove that OL and H_1H_2 are parallel where L is the symmedian point of $\triangle A_1B_1C_1$.

Triangles: pedal triangles

NAvW 548. **by O. Bottema**

Do there exist triangles that coincide with one of their own pedal triangles?

Triangles: perpendiculars

CRUX 364. **by Sahib Ram Mandan**

In the Euclidean plane, if $x_1^i(x = a, b;\ i = 0, 1, 2)$ are the 2 triads of perpendiculars to a line p from two triads of points $X_i'(X = A, B)$ on p and (X) a pair of triangles with vertices X_i on x_1^i and sides x^i opposite X_i such that the three perpendiculars to b^i from A_i' concur at a point G, then it is true for every member of the 3-parameter family $f(B)$ of triangles like (B); and the 3 perpendiculars from B_i' to the sides a^i of any member of the 3-parameter family $f(A)$ of triangles like A concur at a point G' if and only if

$$\frac{A_0'A_1'}{A_1'A_2'} = \frac{B_0'B_1'}{B_1'B_2'}.$$

Triangles: ratios

CRUX 136. **by Steven R. Conrad**

In $\triangle ABC$, C' is on AB such that $AC'{:}C'B = 1{:}2$ and B' is on AC such that $AB'{:}B'C = 4{:}3$. Let P be the intersection of BB' and CC', and let A' be the intersection of BC and ray $\overrightarrow{AP}$. Find $AP{:}PA'$.

NYSMTJ 47. **by David Rosen**

We are given any triangle ABC and points P, between B and C, and Q, between A and C. Let AP meet BQ at X. Let CX intersect AB at R.

(a) If $AQ/QC = a/b$ and $BP/PC = c/d$, the following ratios are determined: AX/XP, BX/XQ, CX/XR, AR/RB. Find each.

(b) Are there any concurrent segments, other than the medians, which divide the three sides into equal ratios?

(c) If $\overline{AC}$ and $\overline{BC}$ are each divided into n congruent segments, and P and Q are the points of this division nearest C, prove that $AX/XP = BX/XQ = n$.

(d) With the same conditions as stated for part (c), prove that $\overline{CR}$ is the median to $\overline{AB}$ and that

$$\frac{CX}{XR} = \frac{2}{n-1}\ .$$

OMG 17.2.8.

In $\triangle ABC$, point D is on BC such that $BD{:}DC = 5{:}4$, and point E is on AC such that $AE{:}EC = 1{:}2$. If AD and BE intersect at point P, then $BP{:}PE = k{:}4$. Find k.

Triangles: relations among parts

OSSMB G78.1-5.

(a) Prove that the distance d between the circumcenter and the incenter of a triangle is given by the relation $d = R^2 - 2Rr$ where R and r are the circumradius and the inradius respectively.

(b) If the circumcenter of $\triangle ABC$ is on the inscribed circle, prove that

$$\cos A + \cos B + \cos C = \sqrt{2}.$$

CRUX 74. **by Viktors Linis**

Prove that if the sides a, b, and c of a triangle satisfy $a^2 + b^2 = kc^2$, then $k > \frac{1}{2}$.

Triangles: sides

CRUX 14. **by Viktors Linis**

If a, b, and c are the lengths of three segments that can form a triangle, show that the same holds true for

$$\frac{1}{a+c},\ \frac{1}{b+c},\ \frac{1}{a+b}.$$

Triangles: similar triangles

PENT 275. **by Kenneth M. Wilke**

One bright student observed that two similar triangles can be drawn which are not congruent even though two sides of one triangle are equal to two sides of the second triangle. How did he do it and what relationship is necessary for this to occur?

Triangles: special triangles

PARAB 409.

In a triangle ABC, $BC = 2AC$. Produce BA past A to D so that $AD = \frac{1}{3}AB$. Prove that $CD = 2AD$.

CRUX 102. **by Léo Sauvé**

Given a triangle ABC with $a = 4$, $b = 5$, and $c = 6$, show that $C = 2A$.

CRUX 213. **by W. J. Blundon**

(a) Prove that the sides of a triangle are in arithmetic progression if and only if

$$s^2 = 18Rr - 9r^2.$$

(b) Find the corresponding result for geometric progression.

CRUX 388. **by W. J. Blundon**

Prove that the line containing the circumcenter and the incenter of a triangle is parallel to a side of the triangle if and only if

$$s^2 = \frac{(2R-r)^2(R+r)}{R-r}.$$

JRM 626. **by Les Marvin**

(a) Prove that a triangle with side lengths of 4, 5, and 6 has a pair of angles one of which is twice the other.

(b) For what other integer triples (a, b, c), does a triangle with side lengths of a, b, and c have a pair of angles one of which is twice the other?

CRUX 229. **by Kenneth M. Wilke**

On an examination, one question asked for the largest angle of the triangle with sides 21, 41, and 50. A student obtained the correct answer as follows:

Let C denote the desired angle; then $\sin C = 50/41 = 1 + 9/41$. But $\sin 90^\circ = 1$ and $9/41 = \sin 12^\circ 40' 49''$. Thus

$$C = 90^\circ + 12^\circ 40' 49'' = 102^\circ 40' 49'',$$

which is correct. Find the triangle of least area having integral sides and possessing this property.

CRUX 313. **by Leon Bankoff**

In triangle ABC, we have $2b^2 = c^2 + a^2$. Prove that GK, the join of the centroid and the symmedian point, is parallel to the base b.

Triangles: squares

PME 361. **by Carl A. Argila**

Consider any triangle ABC such that the midpoint P of side BC is joined to the midpoint Q of side AC by the line segment PQ. Suppose R and S are the projections of Q and P respectively on AB, extended if necessary. What relationship must hold between the sides of the triangle if the figure $PQRS$ is a square?

Triangles: trisected sides

CRUX 320. **by Dan Sokolowsky**

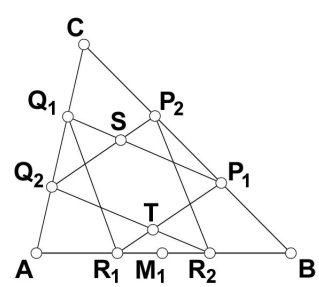
The sides of $\triangle ABC$ are trisected by the points P_1, P_2, Q_1, Q_2, R_1, R_2. Show that:

(a) $\triangle P_1Q_1R_1 \cong \triangle P_2Q_2R_2$;

(b) $[P_1Q_1R_1] = \frac{1}{3}[ABC]$, where the brackets denote area;

(c) the sides of $\triangle P_1Q_1R_1$ and $\triangle P_2Q_2R_2$ trisect one another;

(d) If M_1 is the midpoint of AB, then C, S, T, and M_1 are collinear.



CRUX 317. **by James Gary Propp**

In triangle ABC, let D and E be the trisection points of side BC with D between B and E, let F be the midpoint of side AC, and let G be the midpoint of side AB. Let H be the intersection of segments EG and DF. Find the ratio $EH{:}HG$ by means of mass points.

Algebras

AMM 6228. **by Ivan Vidav**

Let A be a C^*-algebra with unit 1, and let e and f be two projections of A such that $e+f$ is invertible in A. Show that $e \cap f = 2e(e+f)^{-1}f$. ($e \cap f$ is the supremum of the set of all projections $h \in A$ such that $h \leq e$ and $h \leq f$.)

AMM 6097. **by Glen E. Bredon**

Consider the polynomial

$$P(t) = 2^{-n}(1+t^{a_1})(1+t^{a_2})\cdots(1+t^{a_n}).$$

The first k derivatives of $P(t)$ evaluated at $t=1$, that is,

$$q_1 = P'(1),\ q_2 = P''(1), \ldots,\ q_k = P^{(k)}(1),$$

are symmetric functions of $a_1, a_2, \ldots, a_n$. Show that the polynomial algebra generated by these k symmetric functions coincides with that generated by S_1 and the S_{2j} for $2 \leq 2j \leq k$. Here S_i is the sum of ith powers, $S_i = a_1^i + a_2^i + \cdots + a_n^i$.

AMM 6068. **by Seth Warner**

Let A be an algebra over a commutative ring K, and let A_+ be the K-algebra $K \times A$ where addition and scalar multiplication are defined componentwise and multiplication by

$$(x,a)(y,b) = (xy, x \cdot b + y \cdot a + ab).$$

Let N and R be, respectively, the (Jacobson) radicals of K and A. It is standard that if $N = (0)$, $N \times R$ is the radical of A_+. What are the necessary and sufficient conditions for $N \times R$ to be the radical of A_+?

Binary operations

TYCMJ 43. **by Bernard C. Anderson**

Prove that there exists a noncommutative binary operation on the set of real numbers that is both right- and left-distributive over addition.

TYCMJ 81. **by Gino T. Fala**

Prove or disprove that any binary operation, $*$, on the rational numbers that is right- and left-distributive over addition is commutative.

AMM E2574. **by F. David Hammer**

Let $\mathbb{N}_0 = \{0,1,2,\ldots\}$ and let p be a prime. There is a binary operation $*$ on $\mathbb{N}_0$ satisfying $x * y \leq x + y$ for all $x, y \in \mathbb{N}_0$ such that $(\mathbb{N}_0, *)$ is an abelian group with every element (except 0) of order p: for example, write x and y to base p and add individual digits mod p. Prove or disprove that this gives the only such operation.

AMM 6238. **by F. David Hammer**

To see if a binary operation on a set with n elements is associative, one might think it necessary to verify directly n^3 instances of the associative law. Often, however, for instance if the operation is commutative and has an identity, considerably fewer need be verified. Is there a set of n elements and an operation on them for which all n^3 verifications are necessary?

PUTNAM 1978/A.4.

A bypass operation on a set S is a mapping from $S \times S$ to S with the property

$$B(B(w,x), B(y,z)) = B(w,z) \quad \text{for all } w,x,y,z \text{ in } S.$$

(a) Prove that $B(a,b) = c$ implies $B(c,c) = c$ when B is a bypass.

(b) Prove that $B(a,b) = c$ implies $B(a,x) = B(c,x)$ for all x in S when B is a bypass.

(c) Construct a table for a bypass operation B on a finite set S with the following three properties:

(1) $B(x,x) = x$ for all x in S.
(2) There exist d and e in S with $B(d,e) = d \neq e$.
(3) There exist f and g in S with $B(f,g) \neq f$.

NAvW 477. **by M. N. van Ulvenhout**

Define an operation $\cup_*$ (called "uglification") on the nonnegative integers by the inductive rule:

(1) $\cup_*$ is distributive over Nim-addition $\oplus$ (Nim-addition of integers written in binary is vector addition over GF(2) — i.e., 'add without carry' — or 'exclusive or').

(2) $2^m \cup_* 2^n$ is the smallest number different from all numbers

$$x \cup_* 2^n \qquad (x < 2^m),$$
$$2^m \cup_* y \qquad (y < 2^n).$$

Determine the numbers x such that $x \cup_* y = 0$ implies $y = 0$.

ISMJ 13.17.

Given two real numbers x and y, they can be combined by the new operation $\diamond$ so as to give the real number $x \diamond y$. Assume the following properties of $\diamond$:

(1) $(x+y)(x \diamond y) = x^2 \diamond y^2$ for all x and y.
(2) $(x \diamond y) = (x+z) \diamond (y+z)$ for all x, y, and z.
(3) $1 \diamond 0 = 1$.

Use these properties to show that $x \diamond y = x - y$ for all x and y.

PARAB 351.

A product $x \circ y$ is defined for all pairs of real numbers x, y so that the following hold for any x, y, z:

(1) $x \circ y = y \circ x$.
(2) $(x \circ y)z = xz \circ yz$.
(3) $(x \circ y) + z = (x+z) \circ (y+z)$.

What is the value of $99 \circ 100$?

Category theory

AMM 6169. **by Joseph Rotman**

Prove that the category of all Lie algebras over a field K has no injective objects other than 0.

Fields: complex numbers

CMB P252. **by D. Ž. Djoković**

Let F be a subfield of $\mathbb{C}$ such that $\mathbb{C}$ is a quadratic extension of F, i.e., $(\mathbb{C} : F) = 2$. It is well known that this implies that F is a real closed field and hence $i \notin F$ ($i =$ the imaginary unit). Is it true or not that F must be isomorphic to $\mathbb{R}$?

Fields: extension fields

AMM 6043. **by Brian Peterson**

Let P be a nonempty proper subset of the primes. Consider algebraic extensions F of the rationals $\mathbb{Q}$ with the property:

(*) Every x in F has degree over $\mathbb{Q}$ divisible only by primes in P.

A Zorn's lemma argument shows that there exist maximal extensions satisfying (*). Is such a maximal extension unique up to isomorphism?

Fields: finite fields

AMM E2540. **by Richard Stanley**

Let F be a finite field of order q, let n be a divisor of $q-1$, and let α be a nonzero element of F. Evaluate

$$S(n,q;\alpha) = \sum \left(t^n - \alpha\right)^{-1},$$

the sum being over all $t \in F$ with $t^n \neq \alpha$.

NAvW 435. **by M. van Rijk**

Let F_p be the prime field with p elements, and let ξ and η be algebraically independent over F_p.

Let

$$P = F_p(\xi, \eta),$$

and let E be the subfield

$$F_p\left(\xi^p - \xi, \eta^p - \xi\right)$$

of P. Determine the field of all elements of P that are purely inseparable over E and determine $\mathrm{Aut}(P|E)$, the group of all automorphisms of P that fix E.

AMM 6201. **by Daniel D. Anderson**

Let $\mathrm{GF}(p^n)$ be the finite field of order p^n. For which positive integers k is every element of $\mathrm{GF}(p^n)$ a sum of kth powers?

Fields: number fields

NAvW 486. **by G. J. Rieger**

Let K be a quadratic number field. Let N_K denote the norm of K and $\{\rho, \sigma\}$ be a complete basis of K. Given

$$\alpha = a\rho + b\sigma, \qquad \beta = c\rho + d\sigma$$

where a, b, c, and d are real numbers, show that

$$\gcd(\alpha, \beta) = 1 \Longleftrightarrow \gcd\left(N_K(\alpha), ad - bc, N_K(\beta)\right) = 1,$$

where $|ad - bc|$ is basis-independent.

Fields: perfect fields

AMM 6177. **by Adrian R. Wadsworth**

Let K be a perfect field of prime characteristic. Prove that if R is a Noetherian integral domain with quotient field K, then $R = K$.

Fields: polynomials

AMM 6066. **by C. W. Anderson**

For $n = 3$ and $x \in (0,1)$ rational, show that

$$f_n(x) = (1 - x^n)^{1/n}$$

is algebraic of degree n.

AMM 6101. **by Michael Slater**

Suppose F is an ordered field in which Rolle's theorem holds for polynomials. Show that any sum of squares in F is a square in F.

CMB P253. **by D. Ž. Djoković**

Let F be a field of characteristic zero and let

$$f(X) = \prod_{i=1}^{r} f_i(X)^{m_i}, \quad g(X) = \prod_{i=1}^{s} g_j(X)^{n_j}$$

be prime factorizations of two polynomials $f(X)$ and $g(X)$ in one variable X over F. Further, suppose $E_i = F(\alpha_i)$ and $K_j = F(\beta_j)$ $(1 \le i \le r,\ 1 \le j \le s)$ are simple extensions of F considered as F-algebras, where α_i is a root of $f_i(X)$ and β_j is a root of $g_j(X)$. Show that the following are equivalent:

(1) There exists a polynomial $h(X)$ such that $f(h(X))$ is divisible by $g(X)$.

(2) For each j there exists an i such that E_i is isomorphic to an F-subalgebra of K_j.

How should (2) be modified if F has prime characteristic?

PUTNAM 1979/B.3.

Let F be a finite field having an odd number m of elements. Let $p(x)$ be an irreducible polynomial over F of the form

$$x^2 + bx + c, \quad b, c \in F.$$

For how many elements k in F is $p(x) + k$ irreducible over F?

AMM 6046. **by Stephen McAdam**

Let f and g be two nonconstant monic irreducible polynomials over the field K. Let u and v be roots of f and g, respectively, in some extension field of K. Suppose that over $K[v]$, the irreducible decomposition of f is $f = f_1^{e_1}, \ldots, f_n^{e_n}$ while over $K[u]$, g decomposes into $g = g_1^{d_1}, \ldots, g_m^{d_m}$. Then $n = m$ and, when appropriately ordered, $e_i = d_i$ and

$$\frac{\deg g_i}{\deg f_i} = \frac{\deg g}{\deg f}\ .$$

AMM E2578. **by Carl Pomerance**

Prove that $x^4 + 1$ is reducible over every field of prime characteristic. Do the same for $x^4 - x^2 + 1$.

Fields: rational functions

AMM 6082. **by Thomas C. Craven**

Let $K(t)$ be the rational function field in one variable over a field K of arbitrary characteristic. Does the equation

$$x^n - y^2 = 1$$

have a nonconstant solution in $K(t)$ when $n > 2$?

Fields: subfield chains

AMM 6268. **by Gene Smith and Hugh M. Edgar**

Assume that the algebraic number field K possesses at least one proper intermediate field E, i.e, $Q \subset E \subset K$. Prove or disprove the following: K must have a strictly increasing chain

$$Q = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n-1} \subset K_n = K,$$

$n \ge 2$, of subfields such that K_i has a relative integral basis over K_{i-1} for $1 \le i \le n$.

Fields: subfields

AMM 6119.* **by M. J. Pelling**
AMM 6216.* **by M. J. Pelling**

Are there any algebraic number fields A with the property that $A = A_1 + A_2$ (qua abelian groups), where A_1, A_2 are proper subfields of A?

Fields: vector spaces

NAvW 497. **by J. H. van Lint**

Let p be a prime, $F = GF(p^{2t})$, and K be the subfield $GF(p^t)$. The field F can be interpreted as a $2t$-dimensional vector space over $GF(p)$. Let V be a $(2t-1)$-dimensional linear subspace of this space. Show that exactly one coset of K^* in F^* is completely contained in V.

Galois theory

AMM E2650. **by M. J. Pelling**

Find the Galois group of the equation $x^9 + x^3 + 1 = 0$ over the rationals.

Groupoids

AMM 6150. **by Albert A. Mullin**

Let G be any groupoid. Call $e \in G$ a near identity of G if e is idempotent and $ex = xe = x$ fails for at most one $x \in G$. It is well known that G can have at most one identity.

(a) Show that if G is a semigroup, then it can have at most two near-identity elements. Every group has precisely one near-identity element.

(b) Give an example of an uncountably infinite semigroup with precisely two near identities that contains a countably infinite semigroup with precisely two near identities.

Groups: abelian groups

MM Q612. **by Kenneth Taylor**

Let $(G, \cdot)$ be a group with the following special cancellation property:

$$x \cdot a \cdot y = b \cdot a \cdot c \text{ implies } x \cdot y = b \cdot c$$

for all x, y, b, c, and a in G. Prove that G is abelian.

AMM 6011. **by M. Slater**

The group $\mathbb{Z}$ of integers has the following property X: For any n, suppose that A is a list of $(2n+1)$ terms in $\mathbb{Z}$, such that on removal of any one term, the remainder can be divided into two batches of n terms having equal sums. Then all the terms of A are equal.

Determine exactly what abelian groups G have property X.

Groups: alternating groups

CMB P266. **by D. Ž. Djoković and J. Malzan**

Is there a subgroup G of A_n such that its normalizer in S_n actually lies in A_n?

Groups: associativity

AMM E2659. **by Arthur L. Holshouser**

The sequence a, b, c, d can be parenthesized in five ways. Equating these two at a time, we obtain the following "identities":

$$(1)\ (ab)(cd) = a\,(b(cd))\,, \qquad (2)\ (ab)(cd) = a\,((bc)d)\,,$$

$$(3)\ (ab)(cd) = ((ab)c)\,d, \qquad (4)\ (ab)(cd) = (a(bc))\,d,$$

$$(5)\ a\,(b(cd)) = a\,((bc)d)\,, \qquad (6)\ a\,(b(cd)) = ((ab)c)\,d,$$

$$(7)\ a\,((bc)d) = (a(bc))\,d, \qquad (8)\ ((ab)c)\,d = (a(bc))\,d,$$

$$(9)\ a\,(b(cd)) = (a(bc))\,d, \qquad (10)\ a\,((bc)d) = ((ab)c)\,d.$$

Which of these identities implies that a quasigroup satisfying it is necessarily a group?

Groups: finite groups

NAvW 540. **by N. Hekster and R. Schoof**

Let $n \in \mathbb{N}$. Prove that there is exactly one group of order n if and only if $\gcd(\phi(n), n) = 1$.

JRM 479. **by Garland Hopkins**

Write a program capable, for a given value of n, of generating and listing all groups of order n, weeding out isomorphic repeats. For what composite values of n ($1 \le n \le 100$) is there only one group, viz., the cyclic group?

AMM 6202. **by A. A. Jagers**

Let S be a set of generators of a finite group G. For $g \in G$, let $m(g)$ be the least number of terms in a representation of g as a product of elements of S. Let $n_1, n_2, \ldots, n_k$ be the degrees of the irreducible characters of G. Prove that

$$m(g) \le n_1 + n_2 + \cdots + n_k - 1.$$

AMM E2592. **by Melvin Hausner**

Let G be a finite group of even order $n = 2m$. Let H be the set of all x in G with $x^m = 1$. Prove

(a) H is a subgroup of G, and

(b) either $H = G$ or the index $[G : H]$ is 2.

CRUX 57. **by Jacques Marion**

Let G be a group of order pn where p is prime and $p \ge n$. Show that if H is a subgroup of order p then H is normal in G.

NAvW 555. **by H. W. Lenstra, Jr. and R. W. van der Waall**

Which finite groups G have the property that for all $a, b \in G$, with $\gcd(\text{order}(a), \text{order}(b)) = 1$, we have $\text{order}(ab) = \text{order}(a) \cdot \text{order}(b)$?

AMM 6176. **by Morris Newman and Daniel Shanks**

Prove that for the most common type of simple group, which is designated $\mathrm{PSL}_2(p^n)$, its order N is never a perfect square. Find at least one simple group that does have square order.

NAvW 501. **by J. C. Bioch**

Let G be a finite group. It is well known that the intersection of the commutator subgroup G' and the center $Z(G)$ of G is contained in the Frattini subgroup $\Phi(G)$ of G. Prove the following extension of this result:

$$G' \cap Z_\infty(G) \le \Phi(G),$$

where $Z_\infty(G)$ is the hypercenter of G.

NAvW 448. **by J. C. Bioch and R. W. van der Waall**

Let G be a finite group. Let M be a subgroup of G such that $\gcd(|M|, t-1) = 1$, where t is the index of M in G. If t is prime, then prove that M is a normal subgroup of G.

AMM 6026. **by Fred Commoner**

Prove the following theorem: Let p be an odd prime. If G is a finite nonabelian group such that p is less than or equal to the least prime dividing $|G|$, then no automorphism of G can send more than $|G|/p$ elements of G to their inverses. There is a nonabelian group G of order p^3 and an automorphism of G sending exactly $|G|/p$ elements of G to their inverses.

AMM 6059. **by S. Baskaran**

A group G is called metacyclic if the derived group G' and the factor group G/G' are both cyclic. Prove that if G is a finite metacyclic group and p is the smallest prime dividing the order of G, then a Sylow p-subgroup of G is cyclic.

Groups: group presentations

PUTNAM 1976/B.2.

Suppose that G is a group generated by elements A and B. Also, suppose that $A^4 = B^7 = ABA^{-1}B = 1$, $A^2 \neq 1$, $B \neq 1$.

(a) How many elements of G are of the form C^2 with C in G?

(b) Write each such square as a word in A and B.

CMB P259. **by Jerome B. Minkus**

Let G_n denote the group generated by $a_1, a_2, \ldots, a_n$ subject to the relations

$$a_1a_2^4a_3 = a_2a_3^4a_4 = \cdots = a_{n-2}a_{n-1}^4a_n = a_{n-1}a_n^4a_1 = a_na_1^4a_2 = 1.$$

Show that G_n is infinite for all $n \ge 6$.

AMM 6099. **by Jerome Minkus**

For $n \ge 3$, let G_n denote the group generated by the elements $a_1, a_2, \ldots, a_n$ subject to the relations

$$a_1a_2^{-1}a_3 = a_2a_3^{-1}a_4 = \cdots = a_{n-2}a_{n-1}^{-1}a_n = a_{n-1}a_n^{-1}a_1 = a_na_1^{-1}a_2 = 1.$$

Show that

(a) G_5 is isomorphic to the binary dodecahedral group

$$\{a, u \mid a^5 = u^3 = (au)^2\},$$

(b) G_n is nonabelian for all $n \ge 3$.

NAvW 502. **by J. C. Bioch**

Let G be a finite supersoluble non-nilpotent group. If every proper factor group of G is nilpotent, then prove that G is metacyclic with presentation:

$$G = \langle a, b \mid a^p = 1, \quad bab^{-1} = a^j, \quad b_n = 1;$$

$$p \text{ prime}, \quad 1 < j < p\rangle,$$

where $n \,|(p-1)$ and $j^n \equiv 1 \pmod p$.

Groups: matrices

AMM E2545. **by Ron Evans**

Let V be an invertible $n \times n$ matrix with rational entries, and let G denote the group of all $n \times n$ matrices with integral entries and determinant 1. Prove that if H and VHV^{-1} are subgroups of G of finite index p and q, respectively, then $p = q$.

Groups: permutation groups

AMM 6049. **by D. E. Knuth**

What group is generated by the two cyclic permutations $(1, 2, \ldots, m)$ and $(1, 2, \ldots, n)$ when $1 < m < n$?

AMM E2708. **by Edward T. H. Wang**

Find all n for which the symmetric group S_n has the following property: If $\alpha, \beta \in S_n$ are n-cycles, then either $\langle\alpha\rangle = \langle\beta\rangle$ or $\langle\alpha\rangle \cap \langle\beta\rangle = \{1\}$.

CRUX 66. **by John Thomas**

What is the largest non-trivial subgroup of the group of permutations on n elements?

Groups: subgroups

AMM 6204.* **by F. David Hammer**

(a) If all proper subgroups of an infinite abelian group are free (as abelian groups), then show that the group is free.

(b) Find a weaker hypothesis for (a).

(c) Delete abelian in (a).

AMM 6205. **by Alan McConnell and Louis Shapiro**

Let G be a group with no nontrivial elements of finite order, and let H be a cyclic subgroup of finite index in G. Show that G is itself cyclic.

AMM 6221. **by F. David Hammer**

Recently, Shelah found a group of cardinality $\aleph_1$ with no proper subgroups of that cardinality. Prove that this cannot happen with abelian groups. In fact, every uncountable abelian group has a proper subgroup of the same cardinality.

MM 935. **by Qazi Zameeruddin**

It is known that the additive group Q of the rational numbers has no maximal subgroup. Is this statement true for the multiplicative group Q^* of nonzero rational numbers? If the answer is no, then characterize all maximal subgroups of Q^*.

NAvW 506. **by R. Jeurissen**

Prove or disprove the following statement. If G is a group with subgroups H and K, and if there are elements h and k in G such that $h^{-1}Hh \subseteq K$ and $k^{-1}Kk \subseteq H$, then H and K are conjugate in G.

PUTNAM 1975/B.1.

In the additive group of ordered pairs of integers (m, n) [with addition defined componentwise: $(m, n) + (m', n') = (m + m', n + n')$] consider the subgroup H generated by the three elements

$$(3, 8), \quad (4, -1), \quad (5, 4).$$

Then H has another set of generators of the form

$$(1, b), \quad (0, a)$$

for some integers a and b with $a > 0$. Find a.

PUTNAM 1977/B.6.

Let H be a subgroup with h elements in a group G. Suppose that G has an element a such that for all x in H, $(xa)^3 = 1$, the identity. In G, let P be the subset of all products $x_1ax_2a\cdots x_na$, with n a positive integer and the x_i in H.

(a) Show that P is a finite set.

(b) Show that, in fact, P has no more than $3h^2$ elements.

Groups: torsion groups

AMM 6052. **by J. R. Gard**

If G is a torsion group such that there exists an element $x \in G$ with the property that x and y generate G whenever $y \in G$ is not a power of x, is G finite? What other properties does G have?

Groups: transformations

AMM E2542. **by Ron Evans**

Let G be the group generated by the transformations T and S on the extended complex plane, where $zT = -1/z$ and $zS = z + 2i$. Suppose that z_0 is fixed by some non-identity transformation in G. Prove that z_0 must lie on the extended imaginary axis.

AMM 6102. **by Barbara Osofsky**

Let A and B be nontrivial rotations of $\mathbb{R}^3$ about l_1 and l_2, respectively, which are axes through $(0, 0, 0)$ such that

$$A^2 = B^3 = I$$

where I is the identity transformation. Hausdorff has shown that if $\cos 2\theta$ is transcendental, where θ is the angle between l_1 and l_2, then all relations between A and B are generated by $A^2 = I$ and $B^3 = I$. Show that the same is true for $\theta = \pi/4$.

AMM 6276. **by R. K. Oliver**

Let g and h be two screw motions of Euclidean three-space with positive angles less than $\pi/3$ and nonparallel axes. Show that the group generated by g and h is not discrete.

MM 1086. **by Barbara Turner**

Consider the following transformations on 4×4 matrices. Let R move the top row to the bottom and the other rows cyclically up; let D be the reflection across the main diagonal; let S be the interchange of the 1st and 2nd rows followed by the interchange of the 1st and 2nd columns. What is the order of the group generated by R, D, and S?

Lattices

AMM 6032. **by D. J. Johnson**

Suppose L and M are distributive lattices. Let $[\mathcal{G}, \leq]$ be the partially ordered set of lattice morphisms from L to M, ordered according to the following rule: $f \leq g$ if and only if for all x in L, $f(x) \leq g(x)$ in M. Is $[\mathcal{G}, \leq]$ necessarily a lattice?

AMM E2700. **by Richard Stanley**

Let L be a finite lattice with minimum element 0 and maximum element 1. Suppose that for all $x \neq 0$ in L, the interval $[0, x]$ contains an even number of elements. Show that L is complemented, i.e., for all x in L there is a y in L such that $x \wedge y = 0$ and $x \vee y = 1$.

NAvW 541. **by C. B. Huijsmans and B. de Pagter**

Prove that, in an Archimedean Riesz space L, the following are equivalent:

(1) L is of finite dimension.
(2) Every ideal in L is principal.
(3) Every prime ideal in L is principal.

Loops

MATYC 109. **by Dean Jordan**

(a) What is the fewest number of elements a set may contain and be a loop without also being a group?

(b) What is the fewest number of elements a set may contain and be a commutative loop without also being a group?

Quaternions

NAvW 431. **by L. Kuipers**

Let p be an odd prime. Let J be the set of (Hurwitz) integral quaternions, and let Λ be a complete system of residues of J (mod p). Let $N(a)$ be the norm of a ($a \in J$). Let $s \in \Lambda$, $s \not\equiv 0$ (mod p), and let

$$t \in \{1, 2, \ldots, p-1\}.$$

Determine the number of solutions of the system of quaternion congruences (in x and f):

$$N(f) \equiv t \pmod{p}, \qquad xf \equiv s \pmod{p}.$$

Rings: Boolean rings

AMM E2536. **by Jacob Brandler**

If $x^6 = x$ for every element x in the ring R, prove that R is a Boolean ring. Generalize.

MM 1052. **by F. David Hammer**

Show that Boolean rings (idempotent commutative rings with identity) are isomorphic if their multiplicative semigroups are isomorphic.

Rings: characteristic

MM 1019. **by Daniel Mark Rosenblum**

Let R be a ring for which there is an integer n, $n > 1$, such that $x^n = x$ for each element x of R. Prove that the characteristic of R is a (square-free) product of distinct primes p such that $(p-1)\,|\,(n-1)$.

Rings: commutative rings

TYCMJ 40. **by Steven R. Conrad**

Assume that R is a ring in which, for each $x \in R$, $x^2 - x$ is contained in the center of R. Prove that R is commutative.

Rings: finite rings

AMM 6284. **by William P. Wardlaw**

Let R be a finite ring with more than one element and with no nonzero nilpotent element. Show that R is a direct sum of fields.

MM 991. **by F. S. Cater**

Let a and b be elements of a finite ring such that $ab^2 = b$. Prove that $bab = b$.

TYCMJ 65. **by Kenneth V. Turner, Jr.**

Let x and y be respectively left and right divisors of zero in a finite ring with $xy \neq 0$. Prove that xy is both a left and a right divisor of zero.

Rings: ideals

AMM 6152. **by R. Raphael**

In some rings one has unique factorization for ideals. Show that the following limited form of factorization holds in all rings: If I_j, $j = 1, \ldots, n$, are distinct nonzero ideals in a ring R, and if a_j and b_j are positive integers with $a_j < b_j$ for each j, then

$$\prod_{j=1}^{n} I_j^{a_j} = \prod_{j=1}^{n} I_j^{b_j} \qquad \text{implies} \qquad \prod_{j=1}^{n} I_j^{a_j} = \prod_{j=1}^{n} I_j^{c_j},$$

where c_j, $j = 1, \ldots, n$, are any integers satisfying

$$a_j \leq c_j \leq b_j.$$

In particular, $\prod I_j^{a_j} = \prod I_j^{a_j+1}$. Show by an example that this is best possible, that is, show that one can have the products equal when the exponents are not.

DELTA 5.1-3. **by Robert C. Davis, Jr.**

Let A be the ring of all polynomials $f(x)$ with rational coefficients such that $f(1)$ is an integer. Let

$$I = \{f(x) \in A \,|\, f(1) = 0\}.$$

Show that I is an ideal of A that is not finitely generated.

AMM 6180. **by L. C. Larson**

Let A and B be ideals of a commutative ring R with unity. Show that $\{x \in R \,|\, xB \subseteq xA\}$ is an ideal if R is either an integral domain or a principal ideal ring, but that in general it need not be.

Rings: integral domains

AMM 6170. **by Paul W. Haggard**

Let D be an integral domain with prime characteristic p, and let x and y be indeterminates. In $D[x, y]$, consider expansions of $(x + y)^n$ for nonnegative integers n.

(a) If p_1 is an odd prime, prove that the expansion of

$$(x + y)^{p_1}$$

has an even number, N, of terms.

(b) When and how can n be obtained such that the expansion of $(x+y)^n$ will have a given number, N, of terms?

AMM 6069. **by A. R. Charnow**

Let R be an integral domain, G a torsion-free group, and $R[G]$ the group ring of G over R. Let $x = r_1g_1 + r_2g_2$, $r_i \in R$, $r_i \neq 0$, $g_i \in G$, $g_1 \neq g_2$. Prove that x is neither a zero divisor nor a unit in $R[G]$.

AMM 6116. **by S. H. Cox, Jr.**

Let A be an integral domain satisfying the following condition: For every nonzero ideal I of A, there is an epimorphism $A \to A'$ of rings such that I and A' are isomorphic A-modules. For example, a principal ideal domain satisfies the condition with $A \to A'$ the identity $A = A'$. Show that each domain satisfying the condition is a principal ideal domain.

AMM 6264. **by William C. Waterhouse**

A mathematician once assumed that when he had two elements with no common factor, he could write 1 as a linear combination of them. Show that for a Noetherian integral domain, this assumption implies unique factorization.

Rings: matrices

AMM E2742. **by P. M. Gibson**

In a ring with identity, find two matrices such that only the scalar matrices commute with both.

AMM E2528. **by L. W. Shapiro**

Let R denote the ring of $n \times n$ real matrices with the property that every element not in the first row or on the main diagonal is 0. How many two-sided ideals does R have?

AMM E2676. **by Robert Gilmer**

Let R be a ring (not necessarily with identity). We denote by R_n the ring of $n \times n$ matrices over R. Show that the following are equivalent:

(1) Every ideal of R_n is of the form I_n, where I is an ideal of R.

(2) $I = IR = RI$ holds for every ideal I of R.

Rings: nonassociative rings

AMM 6263. **by David Pokrass**

In a simple nonassociative ring R, let

$$(a, b, c) = (ab)c - a(bc),$$
$$[a, b] = ab - ba,$$
$$a \circ b = ab + ba.$$

If R satisfies the identity $w \circ (x, y, z) = 0$ and has no elements of additive order 2, show that R is either associative or anticommutative, i.e., R satisfies either $(x, y, z) = 0$ or $x \circ y = 0$ identically.

Rings: number of idempotents

AMM 6183. **by Albert A. Mullin**

Let R be a ring with a finite number n of multiplicative idempotents.

(a) If R is commutative, show that n is a power of 2.

(b) If R has a unit, show that n is even but need not be a power of 2.

(c) Is there an R for which n is an odd prime?

Rings: polynomials

AMM 6259. **by William D. Blair and James E. Kettner**

Let R be a commutative ring with unity and $R\left[x, x^{-1}\right]$ be the ring of Laurent polynomials

$$f(x) = \sum_{i=-m}^{n} a_i x^i$$

over R. Find necessary and sufficient conditions on the coefficients a_i of $f(x)$ for $f(x)$ to be invertible.

Rings: power series

AMM 6039. **by Robert Gilmer**

Let R be an associative ring, and let $\{X_i\}_1^n$ be a finite set of commuting indeterminates over R. Prove that each central idempotent of the power series ring $R\left[[X_1, \ldots, X_n]\right]$ is in R.

Rings: regular rings

CMB P258. **by R. Raphael**

A ring is regular if for each x there is a y such that $x = xyx$. Prove that for regular rings the following are equivalent: (1) The ideals are totally ordered by inclusion. (2) The prime ideals are totally ordered by inclusion. (3) All the ideals are prime.

Rings: subrings

AMM 6134. **by Barbara Osofsky**

Let R be a ring, not necessarily with identity, and let R^n be the subring generated by n-fold products of elements of R. Prove that if R has the descending chain condition on right ideals, then so does R^n. Does this result hold if "descending chain condition" is replaced by "ascending chain condition"?

Affine spaces

AMM E2779.* **by H. Schwerdtfeger**

(a) Let $A = (a^{(1)}\ a^{(2)} \cdots\ a^{(n)})$ be a nonsingular matrix over a field F, whose columns $a^{(j)}$ represent points in the n-dimensional affine space S_n. Let π be the hyperplane passing through the points $a^{(1)}, \dots, a^{(n)}$. Let $b \in S_n$, $b \neq 0$, and B be the matrix $(b\, b \cdots b)$. Show that the determinant $|A - B| = 0$ if and only if $b \in \pi$.

(b) Generalize (a) to a more general matrix of rank 1, namely $B = (\gamma_1 b\ \gamma_2 b\ \cdots\ \gamma_n b)$, $\gamma_1\gamma_2\cdots\gamma_n \neq 0$, $\gamma_j \in F$.

(c) If A is singular and Σ is the subspace of S_n generated by the columns of A, show that there is no b in Σ such that $|A - B| \neq 0$, with $B = (b\, b \cdots\ b)$.

Determinants: block matrices

AMM E2556. **by Leon Gerber**

Let $A = (\mathbf{a_1}|\cdots|\mathbf{a_n})$ and $B = (\mathbf{b_1}|\cdots|\mathbf{b_n})$ be two $2n \times n$ real matrices, partitioned into columns. Assume that $n \geq 3$ and that the rank of A does not exceed $n - 3$. Let $r_1, \dots, r_n$, $s_1, \dots, s_n$ be arbitrary positive numbers. For $i, j = 1, 2, \dots, n$, define

$$t_{ij} = \frac{|\mathbf{a_i} - \mathbf{b_j}|^2 - r_i^2 - s_j^2}{2r_i s_j}\ .$$

Show that $\det(t_{ij}) = 0$.

AMM 6057. **by Anon**

Let A, B, C, and D be $n \times n$ matrices such that $CD^{\mathrm{T}} = DC^{\mathrm{T}}$. Prove that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD^{\mathrm{T}} - BC^{\mathrm{T}}|.$$

Determinants: complex numbers

AMM E2525. **by D. Ž. Djoković**

Let A be a complex $n \times n$ matrix, let $\overline{A}$ be its complex conjugate, and let I be the $n \times n$ identity matrix. Prove that $\det(I + A\overline{A})$ is real and nonnegative.

AMM 6258.* **by John S. Lew**

Let $X = (x_{jk})$ be an $m \times n$ matrix, where $1 < m < n$ and the x_{jk} are algebraically independent indeterminates over the field C of complex numbers. Let X' be the transpose of X. Prove that $\det(XX')$ is an irreducible polynomial over C.

Determinants: evaluations

AMM E2559. **by Hugh L. Montgomery**

Determine whether the following matrix is singular or nonsingular:

$$\begin{pmatrix} 51237 & 79922 & 55538 & 39177 \\ 46152 & 16596 & 37189 & 82561 \\ 71489 & 23165 & 26563 & 61372 \\ 44350 & 42391 & 91185 & 64809 \end{pmatrix}.$$

AMM E2552. **by Philip Castevens**

Let A be an $n \times n$ real matrix with zeros on the main diagonal and ± 1 off the diagonal. Show that A is nonsingular if n is even, but that A may be singular if n is odd.

AMM E2586. **by Walter Egerland**

Evaluate $\det A$, where $A = (a_{ij})$ is the $(n+1) \times (n+1)$ matrix defined by

$$\begin{aligned} a_{ij} &= 0 && \text{if } i - j \neq 0, 2, -2, \\ a_{ii} &= \lambda_i + \lambda_{i-1} && \text{where } \lambda_0 = \lambda_{n+1} = 0, \\ a_{i+2,i} &= 1, && \text{and} \\ a_{i,i+2} &= \lambda_i \lambda_{i+1}. \end{aligned}$$

The scalars λ_i may belong to any commutative ring.

Determinants: identities

AMM E2703. **by David Jackson**

Let J be the $n \times n$ matrix whose entries are all 1's and write $J = L + U$, where L (resp. U) is a lower (resp. upper) triangular matrix and the diagonal entries of L are zeros. Let $X = \operatorname{diag}(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are variables. Prove that

$$\det\Big(I - (XU)^{k-1} XL\Big) = \sum_{s \geq 0} (-1)^s a_{sk}, \quad k = 1, 2, 3, \dots,$$

where the a_j are defined by

$$\prod_{i=1}^{n} \frac{1 - (tx_i)^k}{1 - tx_i} = \sum_{j \geq 0} a_j t^j,$$

where t is a new variable.

Determinants: recurrences

FQ B-411. **by Bart Rice**

A tridiagonal $n \times n$ matrix $A_n = (a_{ij})$ is of the form

$$a_{ij} = \begin{cases} 2a, & (a \text{ real}) \text{ for } j = i, \\ 1, & \text{for } j = i \pm 1, \\ 0, & \text{otherwise}. \end{cases}$$

Let $d_n = \det A_n$.

(a) Show that (d_n) satisfies a second-order homogeneous linear recursion.

(b) Find closed-form and asymptotic expressions for d_n.

(c) Derive the combinatorial identity

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} (-x)^k = (x+1)^{(n-1)/2}\ \frac{\sin rn}{\sin r}$$

for $x > 0$, $r = \tan^{-1} \sqrt{x}$.

Determinants: symmetric matrices

SIAM 79-3. **by A. E. Barkauskas and D. W. Bange**

Find either a closed form solution or a simple recurrence to evaluate the $n \times n$ determinant $\left|a_{ij}\right|$ where $a_{ij} = a_{ji}$, $a_{ii} = c+1$ (c an integer > 1), $a_{12} = 1$, $a_{i,ci+k} = 1$ for $k = 1$ to c and $ci + k \leq n$; all other $a_{ij} = 0$.

Eigenvalues

SIAM 76-20. **by L. B. Bushard**

Find estimates, as functions of n, on the largest and smallest eigenvalues of the $n \times n$ matrix

$$A_n = (a_{ij}) : a_{ij} = \frac{1}{1+|i-j|} ,$$

$i, j = 1, \ldots, n$.

SIAM 75-15. **by E. Wasserstrom**

Let

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix},$$

$$T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

where d_1, d_2, and d_3 are positive and $d_3 \le d_1$. Show that if $d_3 < d_1/3$, then there are two other positive diagonal matrices D_1 and D_2 such that D, D_1, and D_2 are distinct but DT, D_1T, and D_2T have the same eigenvalues. Show also that if $d_3 > d_1/3$ and D_1 is a positive diagonal matrix distinct from D, then DT and D_1T must have different sets of eigenvalues.

CMB P251. **by D. Ž. Djoković**

Find the eigenvalues and the eigenvectors of the two-diagonal matrix $A = (a_{ij})$, where $a_{ij} = 0$ if $|i-j| \ne 1$ and $a_{i,i+1} = a_{n+2-i,n+1-i} = i$ $(1 \le i \le n)$.

SIAM 79-2. **by G. Efroymson, A. Steger, and S. Steinberg**

Let M_n denote the $n \times n$ matrix whose (j, k) entry $M_n(j, k)$ is given by

$$\frac{\omega^{(j-1)(k-1)}}{\sqrt{n}} , \qquad 1 \le j, k \le n$$

where $\omega = e^{2\pi i/n}$. Determine all of the eigenvalues of M_n.

AMM 6168. **by Edmond Dale Dixon**

Let A be a diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots$ such that $|\lambda_1| > |\lambda_2| \ge \cdots$, and let X be any vector not in the subspace spanned by the eigenvectors associated with $\lambda_2, \lambda_3, \ldots$. Let E_i be the vector with 1 in the ith position and zeros elsewhere. Then show that it is not necessarily true that $E_i \cdot A^{n+1}X/E_i \cdot A^nX \to \lambda_i$, for each i, where the denominators are nonzero.

MM Q624. **by I. J. Good**

Think of a square matrix as placed on a checkerboard, so that the leading diagonal consists entirely of white squares. Then if the signs of all the entries on black squares are changed, prove that the eigenvalues are unchanged.

Lattices

AMM 6172.* **by Doug Hensley**

Give an example, if possible, of two planar lattices of unit determinant that do not possess a common bounded measurable fundamental domain. Do any two distinct lattices possess a common fundamental domain?

Linear transformations

AMM 6236. **by Antal E. Fekete**

We say that two endomorphisms of the complex vector space $\mathbb{C}^n$ are of the same type if there is a bijection between their respective sets of eigenvalues that maps the Jordan normal form of one endomorphism into that of the other. Find a formula determining the number of different endomorphism types of $\mathbb{C}^n$. Define what is meant by an endomorphism type of the real vector space $\mathbb{R}^n$ and determine their number.

AMM 6051.* **by Jochem Zowe**

Let X be a real vector space, Y an ordered vector space, and p a sublinear map of X into Y, i.e., $p(\lambda x) = \lambda p(x)$ and $p(x + x') \le p(x) + p(x')$ for all $x, x' \in X$ and all real nonnegative λ. Does there always exist a linear map T of X into Y such that $Tx \le p(x)$ for all $x \in X$?

AMM S22. **by Edward T. H. Wang and Roy Westwick**

Let V and W be two vector spaces over the same field. Suppose f and g are two linear transformations $V \to W$ such that for every $x \in V$, $g(x)$ is a scalar multiple (depending on x) of $f(x)$. Prove that g is a scalar multiple of f.

AMM E2712. **by A. Wilansky**

Let A be a linear map from real bounded sequences to the real numbers, such that for each sequence x some subsequence of x converges to $A(x)$. Must $A(xy) = A(x)A(y)$?

Matrices: 0-1 matrices

AMM E2662. **by Edward T. H. Wang**

For an $n \times n$ $(0, 1)$-matrix A, let A' denote the complementary matrix, i.e., $A' = J - A$, where J is the matrix with all entries equal to 1. Define $\sigma_n = \max \Sigma(AA')$, where $\Sigma(X)$ denotes the sum of all entries of a matrix X and the maximum is taken over all $n \times n$ $(0, 1)$-matrices A.

Show that

$$\sigma_n \ge \frac{n^3 - n}{3}.$$

Does the equality hold for all n?

AMM E2678. **by Edward T. H. Wang**

Find the maximum number of 1's in an $n \times n$ $(0, 1)$-matrix whose square is again a $(0, 1)$-matrix.

MM 1065. **by H. Kestelman**

Let A be an $(n+1) \times (n+1)$ matrix; its $(1, 1)$-th element is 0 and all others are 1. Find a formula for the elements of A^k when $k \ge 2$.

FQ H-281. **by V. E. Hoggatt, Jr.**

(a) Consider the matrix equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^n = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & G_n \\ H_n & I_n & J_n \end{pmatrix}, \quad n \ge 1.$$

Identify A_n, B_n, C_n, …, J_n.

(b) Consider the matrix equation

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} A'_n & B'_n & C'_n \\ D'_n & E'_n & G'_n \\ H'_n & I'_n & J'_n \end{pmatrix}, \quad n \ge 1.$$

Identify A'_n, B'_n, C'_n, …, J'_n.

Matrices: adjoints

AMM 6222. **by Emilie V. Haynsworth**

Let A be an $n \times n$ matrix over the complex field. Let Adj$A$ denote the standard adjoint matrix for A, that is, Adj$A = (C_{ji})$, where C_{ij} is the cofactor of a_{ij} in A. Prove that if $A + \text{Adj}A = kI$, then

(i) A has at most two distinct eigenvalues, λ_1 and λ_2;

(ii) the Jordan form, J, for A has blocks no larger than 2×2, and if $\lambda_1 \neq \lambda_2$, A is diagonalizable;

(iii) if $\lambda_1\lambda_2 \neq 0$, and λ_1 has multiplicity m, then

$$\lambda_1^{m-1}\lambda_2^{n-m-1} = 1;$$

(iv) if $\lambda_1 = 0$, $A \neq 0$, $n > 2$, then λ_1 is a simple root and $\lambda_2^{n-2} = 1$;

(v) if $S = A + J - kI$, then S^2 commutes with both A and J and if S is nonsingular, $S^{-1}AS = J$;

(vi) if A is nonnegative and λ_1 and λ_2 are both positive, then A^{-1} is an M-matrix.

Conversely, if properties (i), (ii) and (iii) hold, then $A + \text{Adj}A = kI$.

Matrices: block matrices

AMM E2762. **by Peter Hoffman**

Let $A_1, \ldots, A_n$ be $k \times k$ matrices over a field F, such that $A = A_1 + \cdots + A_n$ is invertible. Show that the block matrix $B =$

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_{n-1} & A_n & 0 & \ldots & 0 \\ 0 & A_1 & A_2 & \ldots & A_{n-1} & A_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & A_1 & A_2 & \ldots & A_{n-1} & A_n \end{pmatrix}$$

has full rank, i.e., rank $(B) = mk$, where m is the number of block rows.

Matrices: characteristic polynomial

AMM E2635. **by Kirby C. Smith**

Let F be a field of characteristic $p \neq 0$. Let $A = CD$, where C is a cyclic $p \times p$ matrix over F and D is the diagonal matrix with diagonal entries $0, 1, 2, \ldots, p-1$. Compute the characteristic polynomial of A. Generalize.

AMM E2711. **by Frank Uhlig**

Let A and B be $m \times m$ matrices over a field. If the characteristic polynomial of A is irreducible, show that rank $(AB - BA) \neq 1$.

MATYC 91. **by Richard Gibbs**

Let A be a nonsingular matrix with characteristic polynomial

$$|xI - A| = x^n + d_1x^{n-1} + \cdots + d_{n-1}x + d_n.$$

What is the trace of A^{-1}?

Matrices: Hermitian matrices

SIAM 76-8. **by W. Anderson, Jr. and G. Trapp**

Let A and B be Hermitian positive definite matrices. Write $A \geq B$ if $A - B$ is Hermitian positive definite. Show that

$$A^{-1} + B^{-1} \geq 4(A + B)^{-1}.$$

AMM 6072. **by Wayne Lawton**

Let $a_1, \ldots, a_n$ be n distinct complex numbers such that $0 < |a_k| < 1$ for $1 \leq k \leq n$. Let $B = (b_{ij})$ be the $n \times n$ Hermitian matrix defined by

$$b_{ij} = \frac{a_i\overline{a}_j}{(1 - a_i\overline{a}_j)}$$

for $1 \leq i, j \leq n$. Prove that B is positive definite and that the following equality is valid:

$$\max_{x_l \in \mathbb{C}} \left\{ |x_1 + \cdots + x_n|^2 : \sum_{1 \leq i,j \leq n} b_{ij}x_i\overline{x}_j = 1 \right\} = \prod_{k=1}^{n} |a_k|^{-2} - 1.$$

AMM 6061. **by Hung C. Li**

For any $n \times n$ positive semidefinite Hermitian matrix H, the set

$$S = \{A \,|\, \text{tr}(AA^*)H \leq \lambda\}$$

is convex in A, where A is $n \times m$, X^* is the complex conjugate and transpose of X, and tr X is the trace of X.

MM Q644. **by John Z. Hearon**

Let A be a nonzero matrix of rank one so that $A = ab^*$ where a and b denote column vectors and * denotes conjugate transpose. Show that A is Hermitian if and only if a is a scalar multiple of b. Given that A is Hermitian, show that A is positive semidefinite if and only if the inner product $a * b$ is positive.

Matrices: identity matrix

MM 951. **by G. A. Heuer**

Let A be a square matrix, some scalar multiple of which differs from the identity matrix by a matrix of rank one. Give a simple necessary and sufficient condition that A be nonsingular, and find A^{-1} in this case.

TYCMJ 139. **by Gregory P. Wene**

Find all positive integers n such that if M is an $n \times n$ matrix and I is the $n \times n$ identity matrix over the real numbers, then one of the following is true:

(1) M is invertible,

(2) $M - I$ is invertible,

(3) M is idempotent.

Matrices: maxima and minima

AMM E2555. **by T. W. Cusick**

Let $A = (\mathbf{a_1}|\mathbf{a_2})$ be a nonsingular 2×2 matrix partitioned into columns. Show that

$$\min_A \max_{\mathbf{x}} \frac{(\mathbf{a_1} \cdot \mathbf{x})(\mathbf{a_2} \cdot \mathbf{x})}{\det A} = \frac{1}{2},$$

where the max is over all $\mathbf{x}$ in the box $|x_i| \leq 1$, and the min is over all such matrices A.

Establish a corresponding result for higher dimensions.

Matrices: Moore-Penrose inverse

SIAM 76-15. by A. Berman and M. Neumann

A square matrix is monotone if it is nonsingular and if its inverse is nonnegative. A rectangular matrix is semimonotone if its Moore-Penrose inverse is nonnegative. Let A be a semimonotone matrix of rank r. Prove, or give a counterexample, that A possesses an $r \times r$ monotone submatrix.

Matrices: norms

AMM 6125. by Simeon Reich

For a given $n \times n$ matrix A of rank r and an integer k, $1 \le k \le r$, a best rank k approximation of A is a matrix $A_{(k)}$ satisfying

$$\begin{aligned}&\|A - A_{(k)}\| \\ &\quad = \inf\{\|A - X\| : X \text{ is an } n \times n \text{ matrix of rank } k\},\end{aligned}$$

where $\|A\| = (\text{tr } A^*A)^{1/2}$.

Show that if A is normal, then $A^j_{(k)}$ is a best rank k approximation of A^j for all $j \ge 1$, but that this is no longer true for arbitrary A.

AMM 6249. by H. Kestelman

The norm $\|A\|$ of a real 2×2 matrix A is by definition the maximum of $\|A\hat{x}\|$ when $\|\hat{x}\| = 1$. If $\|x\|$ is the Euclidean norm $\left(x^T x\right)^{1/2}$, then $\|A\| \le \||A|\|$, where $|A|$ is the matrix whose elements are the absolute magnitudes of those of A. Find necessary and sufficient conditions on an invertible 2×2 matrix N in order that $\|A\| \le \||A|\|$ for all A when $\|x\|$ is defined as the Euclidean norm of Nx.

Matrices: orthogonal matrices

MM 1035. by H. Kestelman

Let A be a real $n \times n$ matrix. Do there exist orthogonal matrices B such that $A + B$ is real orthogonal?

Matrices: permutations

AMM 6171. by R. W. K. Odoni and J. B. Wilker

Let F be a field, and let n and d be positive integers, each ≥ 2. Let σ be any permutation of $\{1, 2, \ldots, n\}$, and let σ_0 be the n-cycle $j \to j + 1 \pmod{n}$. Prove that σ is a power of σ_0 if and only if for every sequence of n $d \times d$ matrices over F, $\text{tr}\left(\prod_{j=1}^n M_j\right) = \text{tr}\left(\prod_{j=1}^n M_{\sigma(j)}\right)$.

AMM E2516. by Morris Newman and Charles Johnson

Two matrices A and B are permutation-equivalent if B can be obtained from A by first permuting the rows of A and then permuting the columns of the resulting matrix.

Call an $n \times n$ matrix of 0's and 1's a $k-k$ matrix if there are precisely k 1's in each row and each column. Show that if $n \le 5$, then every $k - k$ matrix is permutation-equivalent to its transpose, but that this is no longer true if $n \ge 6$.

Matrices: polynomials

AMM E2597. by R. W. Farebrother

Let j and n be integers such that $0 \le j \le n$, and let

$$(1 - x)^j (1 + x)^{n-j} = \sum_{i=0}^{n} c_{ij}(n) x^i.$$

If $C(n)$ is the matrix $(c_{ij}(n))$, where $i, j = 0, 1, \ldots, n$, show that

$$\begin{aligned} C(n)^2 &= 2^n I, \\ \det C(n) &= (-2)^{n(n+1)/2}, \\ \text{tr } C(n) &= \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 2^{n/2}, & \text{if } n \text{ is even}. \end{cases} \end{aligned}$$

AMM 6006. by Frank Uhlig

Let A_i be a finite family of complex square matrices that have no eigenvalues in common. Let p_i be a family of real polynomials and define $B_i = p_i(A_i)$ for each i. If each A_i is similar to a real matrix, prove that there is a real polynomial p such that $p(A_i) = B_i$ for every i.

Matrices: positive definite matrices

AMM 6095. by Anon

Let P, Q, and B be $m \times m$, $n \times n$, and $n \times m$ complex matrices with P and Q positive definite. Show that $P - B^*Q^{-1}B$ is positive definite if and only if $Q - BP^{-1}B^*$ is positive definite.

Matrices: power series

AMM E2734. by Melvin Hausner

Let $A = (a_{ij})$ be a real square matrix such that $a_{ij} > 0$ for $i \ne j$. Show that all entries of e^A are positive.

NAvW 517. by M. L. J. Hautus

Let A and B be $n \times n$ matrices. If e^{tA} is bounded for $t \ge 0$, show that e^{tA+B} is also bounded for $t \ge 0$.

Matrices: powers

MM 1017. by Stanley Friedlander

(a) Given an $n \times n$ matrix A over the rationals, show that $A^p = I$ for a prime $p > n + 1$ implies $A = I$.

(b) For each k, $1 < k \le n + 1$, show that there exists an $n \times n$ non-identity matrix over the rationals such that $A^k = I$.

NYSMTJ 61. by Samuel A. Greenspan and Sidney Penner

Let A be a 2×2 matrix over the reals, and let n be a positive integer. Is there an $n > 1$ such that $A^n = I$ implies $A = I$?

Matrices: products

AMM 6251. by William P. Wardlaw

Let m and n be positive integers. What pairs of matrices C and D, over any field K, have the property that if A is an $m \times n$ matrix over K and B is an $n \times m$ matrix over K such that $AB = C$, then $BA = D$?

Matrices: similar matrices

MM 1058. **by H. Kestelman**

Is it true that a square matrix that is not a scalar multiple of the identity is always similar to a matrix with all nonzero elements?

Matrices: spectral radius

SIAM 76-9. **by S. Venit**

Let

$$P = \begin{bmatrix} B & C \\ I & 0 \end{bmatrix},$$

be a real, square matrix of order $2n$, partitioned into four $n \times n$ blocks. Assume that I and 0 are the identity and null matrices (of order n), respectively, and that the only nonzero elements of B and C are given by

$$b_{ij} = \frac{2r_j}{1+2r_j}$$

when $|i-j| = 1$, and

$$c_{ij} = \frac{1-2r_j}{1+2r_j}$$

when $i = j$ $(i, j = 1, 2, \ldots, n)$, where the r_j are arbitrary positive numbers.

Show either that the spectral radius of P is less than 1 for all positive integers n, or find a counterexample.

AMM 6209. **by Marcel F. Neuts**

Let A be a primitive nonnegative matrix of order m, and let B be a finite real matrix of order m. Denote the spectral radius of A by ρ. Show that

$$\lim_{n\to\infty} \frac{1}{n} \rho^{-(n-1)} \sum_{\nu=0}^{n-1} A^{\nu} B A^{n-1-\nu}$$

exists and identify the limit.

AMM S13. **by H. Kestelman**

A nonnegative real matrix A with spectral radius 1 has the property that for some pair (p, q), the p, q element of A^j tends to 0 as $j \to \infty$. Show that for some pair (r, s), the r, s element of A^j is 0 for all positive integers j.

SIAM 75-7. **by D. A. Voss**

The $n \times n$ matrix $D_n = \left[d_{ij}\right]$ satisfies

$$d_{ij} = \begin{cases} \frac{j(i-n)}{n^3}, & j < i, \\ \frac{(6i^2-6in+n)}{6n^3}, & j = i < n, \\ \frac{i(j-n)}{n^3}, & j > i, \\ 0, & j = i = n. \end{cases}$$

Prove or disprove that the spectral radius $\rho(D_n)$ of D_n satisfies

$$\rho(D_n) < \frac{1}{\pi^2}$$

and

$$\lim_{n\to\infty} \rho(D_n) = \frac{1}{\pi^2}\ .$$

SIAM 78-12. **by P. J. Schweitzer**

Investigate the spectral properties of the $N \times N$ matrix

$$Q_{ij} = P_i \delta_{ij} - P_i P_j, \qquad i, j = 1, 2, \ldots, N,$$

where

$$P_i \geq 0, \qquad \sum_{i=1}^{N} P_i = 1.$$

SIAM 77-14.* **by G. K. Kristiansen**

Let $P = \{p_{rs}\}$ be a symmetric matrix having

(1) $p_{rs} = 0$ for $|r-s| > 1$ and $p_{rs} > 0$ otherwise,

(2) spectral radius 1, and

(3) $p_{s-1,s} + p_{s+1,s} \leq 1$ for all s.

Denote by e^T the $1 \times n$ matrix with all entries 1, and let

$$I = \{\delta_{rs}\}$$

be the $n \times n$ unit matrix. Let c be a nonnegative $n \times 1$ matrix with $e^T c = 1$. Prove or disprove that the matrix

$$F = \left(I - ce^T\right) P$$

has spectral radius at most equal to 1. If a counterexample is found, try to minimize the order n.

Matrices: stochastic matrices

AMM E2652. **by Jeffrey L. Rackusin**

Let $A = (a_{ij})$ be a row-stochastic $n \times n$ matrix. Show that

$$\sum_{\sigma \in S_n} \prod_{i=1}^{n} \left(\frac{a_{i,\sigma(i)}}{\sum_{j=i}^{n} a_{i,\sigma(j)}} \right) = 1,$$

where S_n is the symmetric group.

SIAM 75-13.* **by M. Golberg**

Let $\mathbf{P}$ denote an $n \times n$ primitive stochastic matrix and let $\mathbf{R}$ denote a diagonal matrix with diagonal $(r_1, r_2, \ldots, r_n)$, where $0 \leq r_i \leq 1$. Determine

$$\lim_{N\to\infty} \frac{1}{N} \left\{ \sum_{k=1}^{N} \frac{(\mathbf{P}+\mathbf{R})^k}{\left(1 + \sum_{i=1}^{n} \frac{r_i}{n}\right)^k} \right\}.$$

Matrices: symmetric matrices

MM 995. **by Edward T. H. Wang**

Call an $n \times n$ matrix $(n \geq 2)$ R-symmetric if the interchange of any two distinct rows yields a symmetric matrix. Find a characterization of all R-symmetric matrices.

Matrices: unitary matrices

AMM E2741. **by H. S. Witsenhausen**

Given a complex square matrix A, show that there exists a unitary matrix U such that U^*AU has all diagonal entries equal. If A is real, U can be taken real orthogonal.

Linear Algebra

Matrix equations

AMM 6162. **by Ray Latham**

If $A = (a_{ij})$ is the $n \times n$ matrix defined by

$$a_{ij} = \frac{1}{1 - 4(i - j)^2} ,$$

and $\mathbf{x} = (x_i)$ is the unique vector such that $A\mathbf{x} = \mathbf{e}$ (the all 1's vector), show that

$$\sum_{i=1}^{n} x_i = \binom{n+1}{2}.$$

MM 1040. **by H. Kestelman**

If A is an $m \times n$ matrix that is not invertible, show that there are infinitely many $n \times m$ matrices X satisfying $AXA = A$.

CRUX 208. **by Kenneth S. Williams**

Let a and b be real numbers such that $a \geq b \geq 0$. Determine a matrix X such that

$$X^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

FQ H-252. **by V. E. Hoggatt, Jr.**

Let $A_{n\times n}$ be an $n \times n$ lower semi-matrix and $B_{n\times n}$, $C_{n\times n}$ be matrices such that $A_{n\times n}B_{n\times n} = C_{n\times n}$. Let $A_{k\times k}$, $B_{k\times k}$, $C_{k\times k}$ be the $k \times k$ upper left submatrices of $A_{n\times n}$, $B_{n\times n}$, and $C_{n\times n}$. Show that $A_{k\times k}B_{k\times k} = C_{k\times k}$ for $k = 1, 2, \ldots, n$.

Matrix sequences

MM 1038. **by Douglas Lewan**

Define the following sequence of square matrices:

$$M(1) = [1], \quad M(2) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix},$$

$$M(3) = \begin{bmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \end{bmatrix}, \ldots .$$

Find the sum of the elements on the main diagonal of $M(n)$.

NAvW 418. **by M. L. J. Hautus**

For which $n \times n$ matrices B does there exist an $n \times n$ matrix A and a sequence of real numbers u_k, such that

$$u_k A^k \to B \qquad (k \to \infty)?$$

Normed spaces

AMM 6017. **by Albert Wilansky**

In a popular text it is proposed to find a strictly smaller norm for any normed space E by first constructing a strictly larger norm on E'. Show that this construction must fail. The new norm must be equivalent to the old. Give a correct construction.

Vector spaces

ISMJ 13.9.

Let B be a Hamel basis for $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$. Given any function $\lambda: B \to \mathbb{R}$, define the function $f_\lambda: \mathbb{R} \to \mathbb{R}$ by setting $f_\lambda(x) = r_1\lambda(b_1) + \cdots + r_n\lambda(b_n)$ where $x = r_1b_1 + \cdots + r_nb_n$ ($r_i \in \mathbb{Q}$, $b_i \in B$, $i = 1, \ldots, n$). Show that f_λ is additive.

MM 984. **by Peter Ørno**

Let $(b_1, b_2, \ldots, b_n)$ be a nonzero element of $\mathbb{R}^n$. For which n, $2 \leq n \leq 8$, is it true that one can choose an orthogonal basis for $\mathbb{R}^n$ from the collection

$$\{(\pm b_{\pi(1)}, \pm b_{\pi(2)}, \ldots, \pm b_{\pi(n)}) \mid \pi \in P_n\},$$

where P_n is the set of all permutations of $(1, 2, \ldots, n)$?

AMM E2785. **by Stephen M. Gagola, Jr.**

A flat X in a vector space V over a field F is defined to be a coset of a maximal subspace of V. Assume that F is finite with q elements. If V has dimension n and $V \setminus \{0\}$ is the union of m flats, prove that $m \geq n(q - 1)$.

AMM 6215. **by Ki Hang Kim and Fred Roush**

Heawood's system for the four-color theorem for a map with n faces amounts to a linear system of rank $n - 2$ in a $(2n - 4)$-dimensional vector space over GF(3). Prove that for a random rank n system in a $2n$-dimensional vector space over GF(3), the probability that there is at least one solution vector with no zero component tends to 1 as $n \to \infty$.

Abundant numbers

AMM 6138. **by Harry D. Ruderman**

Let $p_1, p_2, \ldots$ be consecutive primes with $p_1 = 2$.

(a) Show that for every n, there is a k for which

$$\prod_{i=n}^{n+k} p_i$$

is an abundant number.

(b) Find an upper bound for k in terms of n.

Algorithms

PENT 286. **by Kenneth M. Wilke**

In Alcatraz Prison an eccentric jailer decided to effect a "selective release" of the prisoners. The cells are numbered consecutively beginning with the number 1. First he unlocked all the cells. Then after returning to the place of beginning, he turned the key in the lock of every second cell. Next he repeated the process by returning to the place of beginning and turning the key in the lock of every third cell. The jailer repeats this process and on the ith trip he turns the key in every ith cell after returning to the place of beginning at cell number 1. Assuming that Alcatraz has 200 cells and that no prisoner escapes during the process, how many prisoners are released and what cells do they occupy?

JRM 739. **by Frank Rubin**

Write an efficient algorithm to compute the geometric mean of a list of N positive real numbers. To be efficient, your algorithm must not use more than a fixed number, independent of N, of higher functions (roots, exponentials, logarithms, etc.). The total number of operations must be at most proportional to N.

If you have access to a computer, test your program by finding the geometric means of the following two lists:

(a) $1, 2, 3, \ldots, 5000$.

(b) $2^1, 3^{-2}, 2^3, 3^{-4}, 2^5, \ldots, 2^{99}, 3^{-100}$.

OSSMB G78.1-2.

A given natural number N is a perfect square whose square root contains $2n+1$ digits. Show that when the $n+1$ high-order digits have been obtained by the usual method, the remaining n digits may be found by simple division.

Approximations

CRUX 202. **by Daniel Rokhsar**

Prove that any real number can be approximated within any $\varepsilon > 0$ as the difference of the square roots of two natural numbers.

PME 375. **by Richard S. Field**

Approximate the value of 2^{10000} without using pencil and paper.

MM Q617. **by Norman Schaumberger and Erwin Just**

If a and b are positive real numbers, show that for any positive integers m and n there is always a rational number of the form x^m/y^n between a and b with x and y integers.

Arithmetic operations

MATYC 113. **by Mark Butler**

What is the largest possible number that can be "carried" from one column to the next when adding n whole numbers?

SSM 3670. **by Herta T. Freitag**

Is there an infinitude of triples of nonzero real numbers for which addition distributes over multiplication and multiplication over addition?

ISMJ 11.8.

By inserting parentheses in

$$1 \div 2 \div 3 \div 4 \div 5 \div 6 \div 7 \div 8 \div 9$$

the value of the expression can be made 7/10. How? What are the largest and the smallest values that can be obtained by insertion of parentheses?

Arithmetic progressions: coprime integers

AMM E2684. **by Charles A. Nicol**

Let A_n be the set of positive integers that are less than n and relatively prime to n. For which n is A_n an arithmetic progression?

Arithmetic progressions: geometric progressions

ISMJ 12.18.

Suppose an arithmetic progression and a geometric progression have positive terms and the first two terms are the same in the two progressions. Show that any other term of the arithmetic progression does not exceed the corresponding term of the geometric progression.

Arithmetic progressions: maxima and minima

TYCMJ 141. **by Thomas E. Elsner**

Call (G_i), $i = 1, 2, \ldots$, a "nonarithmetic sequence" if it is an increasing sequence of positive integers with no three terms in arithmetic progression. Let (H_i), $i = 1, 2, \ldots$, be called a "minimal nonarithmetic sequence" if, for each i, H_i does not exceed the ith term of each "nonarithmetic sequence". Prove or disprove that a "minimal nonarithmetic sequence" exists.

Arithmetic progressions: primes

JRM 712. **by Friend H. Kierstead, Jr.**

The longest known arithmetic progression of primes has 16 terms. Without knowing the common difference of such a progression, is it possible to infer what some of its factors must be?

SSM 3697. **by Charles W. Trigg**

In the decimal system, find a four-term arithmetic progression of three-digit prime numbers in which the fourth term is the reverse of the first term.

SSM 3776. **by Charles W. Trigg**

Find three three-digit prime numbers in arithmetic progression which contain no duplicated digits.

Number Theory

AMM E2561. **by J. M. Simon**

Let (p_1, p_2, p_3) be a prime triplet spaced by the common interval d. Show that if d is not a multiple of 6, then $p_1 = 3$ and necessarily the triplet is unique. Discuss the situation if d is a multiple of 6.

JRM 627. **by Henry Larson**

What is the longest arithmetic progression of primes (negative primes permitted) in which no member has more than two digits?

Arithmetic progressions: ratios

CRUX 114. **by Léo Sauvé**

An arithmetic progression has the following property: for any even number of terms, the ratio of the sum of the first half of the terms to the sum of the second half is always equal to a constant k.

Show that k is uniquely determined by this property, and find all arithmetic progressions having this property.

Arithmetic progressions: roots

AMM E2628. **by Richard J. Hall**

Let a, b, and c be distinct positive integers, at least two of which are prime. Show that $a^{1/n}$, $b^{1/n}$, and $c^{1/n}$ cannot be terms of an arithmetic progression.

Arithmetic progressions: subsequences

JRM 377. **by David L. Silverman**

Let S be an increasing sequence of positive integers that contains arithmetic subsequences of arbitrary length; that is, for every positive integer n, there is an arithmetic subsequence of S of length n.

Prove or disprove: S must contain an infinite arithmetic subsequence.

AMM E2522. **by Joel Spencer**

An infinite subset

$$S = \{s_1, s_2, \ldots\}$$

of $\mathbb{N}$ $(s_1 < s_2 < \cdots)$ has bounded gaps if $(s_{n+1} - s_n)$ is bounded. Show that if S has bounded gaps, then it contains arbitrarily long arithmetic progressions.

Arithmetic progressions: sum of terms

OSSMB G75.2-1.

Two arithmetic progressions P and P' are such that the sum of n terms of P is $2n^2 - 5n$, and the sum of n terms of P' is $\frac{n}{2}(7n-3)$ and there are a number of terms common to both. Find the first five ordered pairs (k, r) such that t_k of P is equal to t_r of P'.

Arrays

TYCMJ 147. **by Charles W. Trigg**

In the square array

$$\begin{matrix} 5 & 8 & 7 \\ 3 & 6 & 9 \\ 1 & 4 & 2 \end{matrix}$$

all but one of the twelve adjacent digit pairs, taken horizontally and vertically, have prime absolute differences. Show that there is no rearrangement of the digits in which all of the differences are (a) different, (b) the same, (c) composite, or (d) prime.

CRUX 345. **by Charles W. Trigg**

It has been shown that when the nine nonzero digits are distributed in a square array so that no column, row or unbroken diagonal has its digits in order of magnitude, the central digit must always be odd.

(a) Can such a distribution be made for every odd central digit?

(b) Do any such distributions exist in which odd and even digits alternate around the perimeter of the array?

OSSMB 76-16.

The integers 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged in a 3×3 array in such a way that no three numbers in a line (row, column, or diagonal) occur in order of magnitude (increasing or decreasing). Prove that, in every such arrangement, the number in the center must be an odd number.

AMM E2732. **by Peter Sjögren**

It is easy to see that one can label the squares of an $n \times n$ chessboard by integers from 1 to n^2 so that the difference between labels of neighboring squares does not exceed n. Is this best possible? (Two squares are neighbors if they share a common side.)

SIAM 79-4.* **by K. L. McAvaney**

For positive integer n, maximize the number of $n \times n$ matrices each containing all of $1, 2, \ldots, n^2$ such that any two entries appear simultaneously in at most one row of all the matrices.

PARAB 329.

Consider the array of natural numbers similar to Pascal's triangle. If we denote the nth row of the triangle by

$$a_{n,1}, a_{n,2}, a_{n,3}, \ldots, a_{n,n-1}, a_{n,n},$$

then the law of formation is given by

$$a_{n,1} = a_{n,n} = 1$$

and for $2 \le i \le n-1$,

$$a_{n,i} = (n-i+1)a_{n-1,i-1} + i a_{n-1,i}.$$

Find a simple formula involving n, for the sum S_n of the nth row, $S_n = a_{n,1} + a_{n,2} + a_{n,3} + \cdots + a_{n,n}$.

$$\begin{matrix} & & & & 1 \\ & & & 1 & 1 \\ & & 1 & 4 & 1 \\ & 1 & 11 & 11 & 1 \\ 1 & 26 & 66 & 26 & 1 \end{matrix}$$

FQ H-254.* **by R. Whitney**

Find a formula for the row-sums of the Fibonacci-Pascal type array below.

$$\begin{matrix} & & & & F_1 & & & & \\ & & & F_1 & & F_1 & & & \\ & & F_1 & & F_2 & & F_1 & & \\ & F_1 & & F_3 & & F_3 & & F_1 & \\ F_1 & & F_4 & & F_6 & & F_4 & & F_1 \\ & & & & \cdots & & & & \end{matrix}$$

AMM E2534. **by C. H. Kimberling**

Consider the array of numbers $a(j,k)$ defined for

$$j,k = 0,1,\ldots$$

as follows: $a(j,0) = 1$ for $j = 0,1,\ldots$; $a(0,k) = 2$ for $k = 1,2,\ldots$; $a(j,k) = a(j,k-1) + a(j-1,k)$ for $j,k \geq 1$. Prove the following:

(a) If p is prime, then $p \,|\, a(j, p-j+1)$ for $j = 2,3,\ldots,p-1$.

(b) If $j+2k$ is prime, then $j+2k$ divides $a(j,k)$.

(c) If $a(j,k)$ is prime, then $a(j,k)$ divides $a(mj,mk)$ for $m = 1,2,\ldots$.

JRM 740. **by Frank Rubin**

A multiplex cipher uses a number of randomly-chosen shuffled alphabets, usually 25-30. Encipherment consists of picking, for each plaintext letter, a letter in the corresponding alphabet a fixed distance away. For instance, the figure shown shows four five-letter alphabets. Suppose we decide to use the second letter below each plaintext letter; then the word BEAD would be enciphered as DBBC.

Breaking the cipher depends upon the fact that in a randomly-chosen set of alphabets, the set of all letters at a given distance from a given letter does not contain all the other 25 letters. In our example the set one down from A does not contain C; the set two down does not contain D; and the set three down does not contain E.

Is it possible to make the cipher secure by providing a set of 25 alphabets in which, for each of the 26 letters at each distance from 1 to 25, all other letters occur? If not, what is the minimum number of alphabets required?

A	B	D	E
B	D	C	C
C	A	A	B
D	E	E	A
E	C	B	D

Base systems: cubes

CRUX 157. **by Steven R. Conrad**

In base fifty, the integer x is represented by $\mathtt{CC}$ and x^3 is represented by $\mathtt{ABBA}$. If $\mathtt{C} > 0$, express all possible values of $\mathtt{B}$ in base ten.

Base systems: digit permutations

JRM C1. **by David L. Silverman**

Find the smallest positive integer N such that in base N there are digits A, B, and C ($0 < A < B < C < N$) with the property that all six base-N permutations, ABC, ACB, BAC, BCA, CAB, and CBA are primes. Generalize by investigating the 24 permutations of the digits A, B, C, and D for primality in base N.

SSM 3580. **by Charles W. Trigg**

In the four-digit integer $abcd$ in base seven, $\sqrt{ab} = cd$. A permutation of the digits in this integer represents its equivalent in base ten. Find the integer.

MM 1045. **by J. L. Murphy**

Define N to be an absolute perfect square, relative to a given base, if every permutation of the digits of N is a perfect square in that base. In base ten, 1, 4, and 9 are obviously absolute perfect squares. Show that these are the only ones.

Base systems: digit reversals

JRM 657. **by John Michael Schram**

Consider the equality $\mathtt{xy}_b = \mathtt{yx}_c, \mathtt{y} < \mathtt{x}$, where $\mathtt{x}$ and $\mathtt{y}$ denote digits in both bases b and c (e.g., $21_4 = 12_7$).

Characterize the values of c that never occur in such an equality.

JRM 760. **by Klaus Lunstroth**

Find all 2-digit numbers in all bases such that reversing the order of the digits multiplies the number by 2.

MSJ 417. **by Charles W. Trigg**

When the order of the digits of a 4-digit number in the decimal system is reversed, its equivalent in base 7 is formed. Furthermore, the square roots of the number in the two base systems contain the same digits. Identify the integer.

OSSMB G77.2-2.

(a) Determine the three digit integer in base 7 whose digits are reversed when expressed in base 9.

(b) Find all three digit integers (base 10) that are n times the sum of their digits when $n = 17$. Prove that there is no such integer for $n = 9$.

SSM 3595. **by R. F. Wardrop**

(a) Find three different-digit numbers $\mathtt{xyz}$ such that for each $\mathtt{xyz}$, $\mathtt{xyz}_9 = \mathtt{zyx}_b$ for $b < 9$.

(b) Are there any numbers in base eight such that $\mathtt{xyz}_8 = \mathtt{zyx}_b$, for $b < 8$?

(c) How about base seven, six, five, four, and three?

SSM 3600. **by Alan Wayne**

Find a three-digit numeral in the base sixteen system of numeration that has the same digits as a decimal numeral, but in reverse order, and that represents the same positive integer.

SSM 3631. **by Charles W. Trigg**

Find a three-digit integer in base five that has the order of its digits reversed when multiplied by 2. Generalize.

SSM 3679. **by R. F. Wardrop**

Find all three-digit numbers xyz (base ten) such that the following holds: xyz (base ten) $= zyx$ (base b) where $2 \leq b \leq 9$ and x, y, and z are distinct.

SSM 3614. **by Charles W. Trigg**

In the equation, $N - N' = M$, N is an integer, its reverse is $N' < N$, and M is a permutation of the digits on N. For example, in the decimal system $954 = 459 = 495$. In the scale of notation with base four, find a four-digit integer that is both an M and an N.

PME 348. **by Bob Prielipp and N. J. Kuenzi**

When the digits of the positive integer N are written in reverse order, the positive integer N' is obtained. Let $N + N' = S$. Then S is called the sum after one reversal addition.

Prove that there are infinitely many triangular numbers which have a palindromic sum after one reversal addition in the base b, where b is an arbitrary positive integer greater than or equal to 2.

Base systems: divisibility

SSM 3590. **by Herta T. Freitag**

In the base-ten system of numeration, divisibility of a number $N = a_n a_{n-1} \ldots a_2 a_1 a_0$ by 2 is tested by seeing if a_0 is divisible by 2; for divisibility by 4, one checks $a_1 a_0$; and if $a_2 a_1 a_0$ is divisible by 8, so is N.

Generalize these criteria for base system b and divisibility by a number m^t where m and t are positive integers and m is greater than 1.

Base systems: factorials

JRM 598. **by Sherry Nolan**

Given: $a_n a_{n-1} \ldots a_2 a_1$ is the factorian representation of $a_n \cdot n! + a_{n-1} \cdot (n-1)! + \ldots + a_2 \cdot 2! + a_1 \cdot 1!$. Uniqueness of representation is assured by requiring that only the digits $0, 1, 2, \ldots, n$ are allowed in the nth position (from the right).

(a) 1(factorian)=1(decimal). For what other positive integer value does such an equality hold?

(b) Four is the first integer more efficiently represented in factorian (20) than in binary (100). For bases 3 through 12 determine at what integer value the factorian system becomes more efficient than each of these systems.

Base systems: limits

NAvW 507. **by L. Kuipers**

Let $g > 1$ be a fixed positive integer. Let the positive integer x be represented with respect to base g:

$$x = a_1 g^{n_1} + a_2 g^{n_2} + \cdots + a_t g^{n_t},$$

where

$$n_1 > n_2 > \cdots > n_t \geq 0,$$

$$0 \leq a_i \leq g - 1 \qquad (i = 1, 2, \ldots, t).$$

Let

$$\beta(x) = \sum_{i=1}^{t} a_i^2$$

and let

$$B(x) = \sum_{y \leq x} \beta(y).$$

Prove that

$$B(x) = \frac{(g-1)(2g-1)}{6} \frac{x \log x}{\log g} + O(x), \qquad x \to \infty.$$

Base systems: maxima and minima

CANADA 1977/3.
OMG 16.2.3.

Let N be an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

Base systems: modular arithmetic

SSM 3765. **by Alan Wayne**

Prove that in any system of numeration with base b (where b is an integer greater than or equal to 2), if each digit in turn is multiplied by $b-1$ and divided by b, then the resulting set of nonnegative integer remainders is the set of all digits.

Base systems: number of digits

NYSMTJ 76. **by Charles D. Smith**

How many digits does 9^{9^9} have when written
(a) in base nine?
(b) in the decimal system?

Base systems: palindromes

SSM 3712. **by Charles W. Trigg**

Find two palindromic squares in base 8 each of which contains every one of the seven nonzero octal digits.

Base systems: pandigital numbers

SSM 3626. **by Alan Wayne**

Find an integer N, in base 8, which is a multiple of the cube of three and whose square has the eight octal digits once each.

JRM 649. **by Harry Nelson**

List all primes in all bases which are composed of exactly one of each of the digits in that base.

Base systems: polygonal numbers

PME 415. **by Charles W. Trigg**

A hexagonal number has the form $2n^2 - n$. In base 9, show that the hexagonal number corresponding to an n that ends in 7 terminates in 11.

Base systems: powers

TYCMJ 138. **by Warren Page**

Given any natural number n, do there exist numbers B and N in base 10 such that N is a perfect nth power in every base greater than B?

Base systems: products

JRM 440. **by Edmund Charles**

"That integer you came up with was three times what it should have been," said the Data Reduction Specialist.

"You knew it was written in octal, didn't you?" replied the Programmer.

"Oh, I thought it was in duodecimal," said the D.R.S.

What was the integer?

Base systems: repeating fractions

MM 973. **by Robert Cranga**

Let N be an odd integer. If the period of N^{-1} is P in base b, and if $N^2 \nmid \left(b^P - 1\right)$, then prove that the period of N^{-n} in base b is PN^{n-1}.

MM Q627. **by Michael Golomb**

It is a curious fact that $80/81 = .9876543210\ldots$ is accurate to ten decimal places. Show that if $b \geq 4$ is an integer, then in the base b,

$$(\overline{b-2}\ 0)/(\overline{b-2}\ 1) = .b-1\ b-2 \ldots 210\ldots$$

is accurate to b b-places with error less than $(b^{-b})/2$.

Base systems: square roots

SPECT 9.4. **by C. J. Knight**
What is the representation in base 7 of the square root of the number whose representation in base 7 is 14,641?

Base systems: squares

SSM 3594. **by Charles W. Trigg**
In the system of notation with base eleven, find six-digit numerals of the form *abcabc* that are squares.

CRUX 197. **by Charles W. Trigg**
In the octonary system, find a square number that has the form `aaabaaa`.

TYCMJ 90. **by Charles W. Trigg**
A plateau number has the form `abb...bba` with `a` < `b`. In base 8, $(33)^2 = 1331$ is a plateau square. In this same system, find another plateau square.

OMG 15.3.5.
In base 8, what digits can odd squares end in?

SSM 3610. **by Charles W. Trigg**
In the decimal system, there are no six-digit numerals with the form *abcabc* which are squares. Find the system of numeration with the smallest base wherein such a square exists, with a, b, and c being distinct nonzero digits.

JRM 677. **by David L. Silverman**
Find all integers n in all bases such that the sum of the digits of n^2 is n.

Base systems: sum of digits

NYSMTJ 88. **by Alan Wayne**
Show that, in every numeration base b, there is a unique three-digit integer that is $(b+1)$ times the sum of its digits.

SSM 3689. **by R. F. Wardrop**
Find all three-digit numbers `abc` such that

$$\mathtt{a}+\mathtt{b}+\mathtt{c}=\mathtt{abc}_B$$

where B is a positive integer, $2 \le B \le 12$.

Base systems: triangular numbers

OSSMB 76-12.
Prove that every odd square in base 8 ends in 1, and if this 1 be cut off, the remaining part is always a triangular number.

Binomial coefficients: arithmetic progressions

PARAB 414.
Find all positive integers n and k such that the three binomial coefficients $\binom{n}{k}$, $\binom{n}{k+1}$, and $\binom{n}{k+2}$ are in arithmetic progression.

SPECT 8.8.
Is it possible for three consecutive binomial coefficients to be
(a) in arithmetic progression,
(b) in geometric progression?

Binomial coefficients: congruences

MM Q650. **by Edward T. H. Wang**
Prove that for any positive integer n, $\binom{n}{k} \equiv 0 \pmod{n}$ if $\gcd(n,k)=1$ and $k=1,2,\ldots,n-1$.

PUTNAM 1977/A.5.
Prove that

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$$

for all integers p, a and b with p a prime, $p>0$, and $a \ge b \ge 0$.

Binomial coefficients: divisibility

FQ B-310. **by Daniel Finkel**
Find some positive integers n and r such that the binomial coefficient $\binom{n}{r}$ is divisible by $n+1$.

NAvW 396. **by P. Erdős**
Let $n>6$. Show that for some i, with $1<i\le \frac{n}{2}$, the binomial coefficient $\binom{n}{i}$ is divisible by n.

NAvW 397. **by P. Erdős**
Show that, for every positive integer k, there is an n_k such that for every $n>n_k$ there is an integer ℓ, with $k<\ell\le\frac{n}{2}$, for which $\binom{n}{\ell}$ is divisible by $\binom{n}{k}$.

PARAB 355.
If p is a prime, prove that $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by p.

Binomial coefficients: finite sums

FQ B-338. **by George Berzsenyi**
Let k and n be positive integers. Let $p=4k+1$, and let h be the largest integer with $2h+1\le n$. Show that

$$\sum_{j=0}^{h} p^j \binom{n}{2j+1}$$

is an integral multiple of 2^{n-1}.

FQ H-264. **by L. Carlitz**
Show that

$$\sum_{i=0}^{m-r}\binom{s+i}{i}\binom{m+n-s-i+1}{n-s} = \sum_{i=0}^{n-s}\binom{r+i}{i}\binom{m+n-r-i+1}{m-r}.$$

FQ H-276. **by V. E. Hoggatt, Jr.**
Show that the sequence of Bell numbers, $\{B_i\}_{i=0}^{\infty}$, is invariant under repeated differencing.

$$B_0=1, \quad B_{n+1}=\sum_{k=0}^{n}\binom{n}{k}B_k \quad (n\ge 0).$$

FQ H-269. **by George Berzsenyi**

The sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, defined by

$$a_n = \sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n-2k}{k}, \quad (n \geq 1),$$

and

$$b_{2n} = \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n-k \\ 2k \end{bmatrix},$$

$$b_{2n+1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{bmatrix} n-k \\ 2k+1 \end{bmatrix}, \quad (n \geq 0)$$

are obtained as diagonal sums from Pascal's triangle and from a similar triangular array of numbers formed by the coefficients of powers of x in the expansion of $(x^2+x+1)^n$, respectively. (More precisely, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is the coefficient of x^k in $(x^2+x+1)^n$.) Verify that $a_n = b_{n-1} + b_n$ for each $n = 1, 2, \ldots$.

SIAM 75-10. **by G. E. Andrews**

It is known that if

$$H(m,n) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{i+j}{i} \binom{m-i+j}{j} \cdot \binom{i+n-j}{i} \binom{m+n-i-j}{n-j},$$

then

$$H(m,n) - H(m-1,n) - H(m,n-1) = \binom{m+n}{n}^2.$$

Prove also that

$$(2m+1)H(m,n) = (n+m+1)\left\{2H(m-1,n) + \binom{m+n}{n}^2\right\}.$$

Binomial coefficients: generating functions

FQ B-390. **by V. E. Hoggatt, Jr.**

Find, as a rational function of x, the generating function

$$G_k(x) = \binom{k}{k} + \binom{k+1}{k}x + \binom{k+2}{k}x^2 + \cdots + \binom{k+n}{k}x^n + \cdots, \qquad |x| < 1.$$

Binomial coefficients: maxima and minima

AMM E2640. **by James E. Desmond and William R. Hastings**

Prove or disprove: The largest power of 2 that divides

$$\binom{2^{n+1}}{2^n} - \binom{2^n}{2^{n-1}}, \qquad n > 1$$

is 2^{3n}.

Binomial coefficients: number representations

FQ H-261. **by A. J. W. Hilton**

Show that, if $k \geq 2$, $n = r+s$, where $r \geq 1$, $s \geq 1$, and if the k-binomial representations of n, r, and s are

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t}$$

$$r = \binom{b_k}{k} + \binom{b_{k-1}}{k-1} + \cdots + \binom{b_u}{u}$$

$$s = \binom{c_k}{k} + \binom{c_{k-1}}{k-1} + \cdots + \binom{c_v}{v}$$

then

$$\binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1} \leq \binom{b_k}{k-1} + \binom{b_{k-1}}{k-2} + \cdots + \binom{b_u}{u-1} + \binom{c_k}{k-1} + \binom{c_{k-1}}{k-2} + \cdots + \binom{c_v}{v-1}.$$

Binomial coefficients: odd and even

CRUX 90. **by Léo Sauvé**

(a) Determine, as a function of the positive integer n, the number of odd binomial coefficients in the expansion of $(a+b)^n$.

(b) Do the same for the number of odd multinomial coefficients in the expansion of $(a_1 + a_2 + \cdots + a_r)^n$.

Binomial coefficients: primes

PME 369. **by P. Erdős**

Determine all solutions of

$$\binom{n}{k} = \prod_{p \leq n} p.$$

Collatz problem

SSM 3608. **by R. F. Wardrop**

For any number N_0, if N_0 is even, divide by 2; if N_0 is odd, triple and add 1; thus obtaining N_1. Continuing in this manner either dividing by 2 if even or tripling and adding 1 if odd, eventually the integer 1 is reached. In this manner every number N_i can be changed to 1 through a series of operations.

The successive numbers 12 and 13 each take nine operations to get to 1.

Some possible questions that can be asked are:

(a) Is it ever true that three successive numbers require the same number of operations to get to 1?

(b) Is it ever true that four successive numbers require the same number of operations to get to 1?

(c) Find seventeen consecutive numbers less than $10,000$ such that each one takes the same number of operations to get to 1.

(d) Can you find x consecutive numbers such that each one takes the same number of operations to get to 1, where $x = 18, 19, \ldots$?

(e) What is the average number of operations for numbers $1-100$, $100-260$, etc?

Number Theory

CRUX 133.* **by Kenneth S. Williams**
FUNCT 2.1.4.

Let f be the operation that takes a positive integer n to $\frac{1}{2}n$ (if n even) and to $3n+1$ (if n odd). Prove or disprove that any positive integer can be reduced to 1 by successively applying f to it.

Composed operations

JRM 737. **by Frank Rubin**

Every positive integer can be generated by successive applications of the functions *factorial, square root*, and *floor* in some appropriate order to the starting integer $N = 3$. For example, $\left\lfloor\sqrt{(3!)!}\right\rfloor = 26$. Let M be the largest intermediate value to which the factorial function is applied. In the example, $M = 6$.

(a) For $N = 3$, what is the largest value of M which *must* occur to achieve the generation of all integers from 1 to 10?

(b) Is there any starting value for N that permits a smaller maximum value of M in generating all integers from 1 to 10?

Composite numbers

AMM E2800. **by B. de la Rosa**

Show that an odd positive integer c is composite if and only if there exists a positive integer $a \le (c-3)/3$ such that $(2a-1)^2 + 8c$ is a square.

NYSMTJ 93. **by Erwin Just and Sidney Penner**

For each positive integer n, show that

$$1 + 9 + 9^2 + \cdots + 9^n$$

is composite.

OSSMB 75-11.

Find 40 consecutive values of x for which $x^2 + x + 41$ yields only composite numbers.

ISMJ 14.20.

Show that the sequence $an^2 + bn + c$, $n = 1, 2, 3, \ldots$, where a, b, and c are positive integers with no common factors, contains infinitely many composite numbers.

MSJ 481.

Prove that there are infinitely many values of n for which the expression $n^2 - 39n + 421$ yields a composite number.

AMM E2679. **by Solomon W. Golomb**

If a positive integer m has a prime factor greater than 3, show that $4^m - 2^m + 1$ is composite.

Continued fractions: convergents

ISMJ 13.1.

Show that the convergents of a continued fraction with positive integral coefficients are all in lowest terms.

ISMJ 13.2.

Consider the continued fraction:

$$1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{\ddots}}}$$

Find formulas for the numerators and denominators of the convergents of continued fractions of this type.

FQ H-308. **by Paul Bruckman**

Let

$$[a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} = \frac{p_n(a_1, a_2, \ldots, a_n)}{q_n(a_1, a_2, \ldots, a_n)}$$

denote the nth convergent of the infinite simple continued fraction $[a_1, a_2, \ldots]$, $n = 1, 2, \ldots$. Also, define $p_0 = 1$ and $q_0 = 0$. Further, define

$$\begin{aligned} W_{n,k} &= p_n(a_1, a_2, \ldots, a_n)q_k(a_1, a_2, \ldots, a_k) \\ &\quad - p_k(a_1, a_2, \ldots, a_k)q_n(a_1, a_2, \ldots, a_n) \\ &= p_n q_k - p_k q_n, \quad 0 \le k \le n\,. \end{aligned}$$

Find a general formula for $W_{n,k}$.

Continued fractions: evaluations

CRUX 163. **by Charles Stimler**

Evaluate:

$$\cfrac{2}{1 + \cfrac{3}{2 + \cfrac{4}{3 + \cfrac{5}{4 + \cfrac{6}{\ddots}}}}}.$$

Continued fractions: identities

SSM 3732. **by Herta T. Freitag**

Show that

(a)

$$\frac{2}{3+}\,\frac{1}{4+}\,\frac{1}{4+}\,\frac{1}{4+}\cdots = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots$$

and

(b)

$$\left(1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots\right)\cdot\left(1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots\right)$$

$$= 2 + \frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\cdots\,.$$

Continued fractions: periodic continued fractions

CRUX 349. **by R. Robinson Rowe**

Solve in positive integers a and b the continued fraction equation

$$\cfrac{2}{a+\cfrac{1}{a+\cfrac{1}{\ddots}}} - \cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{\ddots}}} = 1.$$

PME 392. **by R. Robinson Rowe**

Solve in distinct positive integers:

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{\ddots}}}}} - \cfrac{3}{c+\cfrac{1}{d+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{\ddots}}}}} = \frac{1}{2}.$$

Continued fractions: pi

ISMJ 13.3.

Prove or disprove:

$$\frac{\pi}{4} = 1 + \cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{\ddots}}}$$

Continued fractions: radicals

CRUX 227. **by W. J. Blundon**

It is known that

$$\sqrt{a^2+1} = \langle a, 2a\rangle = a + \cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{\ddots}}}$$

for all positive integers a. Solve completely in positive integers each of the equations

$$\sqrt{a^2+y} = \langle a, x, 2a\rangle \quad\text{and}\quad \sqrt{a^2+y} = \langle a, x, x, 2a\rangle,$$

where in both cases $x \neq 2a$.

ISMJ 13.4.

Find the simple continued fraction expansion for $\sqrt{3}$.

Decimal representations

MSJ 498.

Characterize the block of digits that repeat endlessly in the decimal expansion for fractions of the form $n/(10^m+1)$, where n and m are positive integers.

PARAB 271.

Prove that if the sum of the fractions

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$$

(where n is a positive integer) is put in decimal form, it forms a nonterminating decimal which is periodic after several terms.

MATYC 87. **by James M. Thelen**

Let n be a positive integer, with $\gcd(n, 10) = 1$. Then $1/n$ has an infinite repeating decimal representation. Show that the repeating cycle begins immediately after the decimal point.

CANADA 1975/4.

Find a positive number which is such that its decimal part, its integral part, and the number itself are three terms in geometric progression.

Determinants: 0-1 matrices

AMM E2588. **by Stephen B. Maurer**

Let A_n be the matrix of order $(2^n-1)\times n$, where the kth row is the binary expression for k. Let $M_n = A_n A_n^t \pmod 2$. If M_n is regarded as a matrix over the integers, what is its determinant?

Determinants: binomial coefficients

AMM E2729. **by John Goth**

Evaluate $\det(A)$ where $A = (a_{ij})$ is the $n\times n$ matrix given by

$$a_{ij} = \binom{im+j-1}{j}, \quad i, j = 1, \ldots, n,$$

m being a fixed positive integer.

MENEMUI 1.3.1. **by S. L. Lee**

Evaluate

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \binom{2}{1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \binom{3}{1} & \binom{3}{2} & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \binom{n-1}{1} & \binom{n-1}{2} & \binom{n-1}{3} & \binom{n-1}{4} & \cdots & \binom{n-1}{n-2} & 1 \\ 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \binom{n}{4} & \cdots & \binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}$$

AMM E2709. **by R. M. Norton**

Let $A = (a_{ij})$, $0 \le i, j \le n$, be a Hankel matrix defined by

$$a_{ij} = \begin{cases} 0, & \text{if } i+j \text{ is odd,} \\ \binom{i+j}{\frac{i+j}{2}}, & \text{if } i+j \text{ is even.} \end{cases}$$

Compute $\det A$.

SIAM 78-15. **by R. Shantaram**

Define $m[2n] = \binom{2n}{n}$, $n = 0, 1, 2, \ldots$. Let $T(n)$ be the $n\times n$ matrix whose (i,j) element is $m\,[2(i+j-1)]$, $i, j = 1, 2, \ldots, n$ and $S(n)$ be the $(n+1)\times(n+1)$ matrix whose (i,j) element is $m\,[2(i+j)]$, $i, j = 0, 1, \ldots, n$. Prove that

$$\det T(n) = \det S(n) = 2^n.$$

Number Theory

Determinants: congruences

CRUX 324. **by Gali Salvatore**

In the determinant

$$\begin{vmatrix} 6 & a & 6 & b \\ c & 8 & d & 2 \\ 1 & e & 5 & f \\ g & 1 & h & 1 \end{vmatrix}$$

replace the letters $a, b, \ldots, h$ by eight different digits so as to make the value of the determinant a multiple of the prime 757.

AMM E2683. **by Ira Gessel**

Let A be the cyclic matrix with $(a_0, a_1, \ldots, a_{p-1})$ as first row, p a prime. If the a_i are integers, show that

$$\det A \equiv a_0 + a_1 + \cdots + a_{p-1} \pmod{p}.$$

Determinants: counting problems

AMM 6086. **by Raymond M. Redheffer**

Let d_{ij} be the number of divisors common to i and j. Prove that the determinant $|d_{ij}|$ for $2 \le i, j \le n$ equals the number of square-free integers from 1 to n.

Determinants: factorials

AMM E2747. **by H. L. Krall and Emil Grosswald**

(a) Compute the determinant of the matrix $A = (a_{ij})$, where $0 \le i, j \le n-1$ and $a_{ij} = 1/(i+j+1)!$.

(b) Compute the determinant of the matrix $B = (b_{ij})$, where

$$b_{ij} = (-1)^{i+j+1} 2^{i+j+1} / (i+j+1)!$$

for $i, j \in \{1, 2, \ldots, n\}$.

Determinants: identities

SIAM 78-14. **by D. Slepian**

Denote by $R(n, N)$ the determinant of the $(n+1) \times (n+1)$ matrix that has

$$\sum_{l=0}^{N-1} l^{i+j}$$

for the element in its ith row and jth column, $i, j = 0, 1, \ldots, n$. Here $0^0 \equiv 1$. Show that

$$R(n, N) = \sum_{\substack{l_0 < l_1 < \cdots < l_n \\ 0}}^{N-1} \prod_{\substack{i<j \\ 0}}^{n} \left(l_i - l_j\right)^2$$

$$= N^{n+1} \prod_{j=1}^{n} \frac{(j!)^4 \left(N^2 - j^2\right)^{n+1-j}}{(2j)!(2j+1)!} .$$

Determinants: solution of equations

TYCMJ 150. **by Aron Pinker**

Let A_1, A_2, A_3, and A_4 be nonzero integers and α a positive integer that is not a perfect square. Is it possible that

$$\begin{vmatrix} A_1 & \alpha A_4 & \alpha A_3 & \alpha A_2 \\ A_2 & A_1 & \alpha A_4 & \alpha A_3 \\ A_3 & A_2 & A_1 & \alpha A_4 \\ A_4 & A_3 & A_2 & A_1 \end{vmatrix} = 0?$$

NAvW 533. **by R. J. Stroeker**

For $n \in \mathbb{N}$ $(n \ge 3)$ and $x \in \mathbb{R}$, $y \in \mathbb{R}$, we define

$$F_n(x,y) = \begin{vmatrix} y & 1 & 0 & 0 & . & . & . \\ 1 & x & 1 & 0 & . & . & . \\ 0 & 1 & x & 1 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & 1 & x & 1 \\ . & . & . & 0 & 0 & 1 & y \end{vmatrix}_{n \times n} .$$

Find all $(x, y) \in \mathbb{N}^2$ such that $F_n(x, y) = 1$.

Difference equations

FQ B-389. **by Gregory Wulczyn**

Find the complete solution, with two arbitrary constants, of the difference equation

$$(n^2 + 3n + 3)U_{n+2} - 2(n^2 + n + 1)U_{n+1} + (n^2 - n + 1)U_n = 0.$$

FQ H-274. **by George Berzsenyi**

It has been shown that if

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

then

$$Q^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1} - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix}.$$

Generalize the matrix Q to solutions of the difference equation

$$U_n = rU_{n-1} + sU_{n-2},$$

where r and s are arbitrary real numbers, $U_0 = 0$ and $U_1 = 1$.

JRM 705. **by Friend H. Kierstead, Jr.**

Let $f(n) = pf(n-1) + 1$, with $f(0) = 1$, where p is independent of n. Find an explicit expression for $f(n)$ in terms of n and p.

FQ B-370. **by Gregory Wulczyn**
FQ B-383. **by Gregory Wulczyn**

Solve the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = F_n.$$

Digit problems: arithmetic progressions

CRUX 378. **by Allan Wm. Johnson Jr.**

(a) Find four positive decimal integers in arithmetic progression, each having the property that if any digit is changed to any other digit, the resulting number is always composite.

(b)* Can the four integers be consecutive?

Digit problems: base systems

OSSMB 75-6. **by Michael Webster**

Prove that

$$\frac{x_1x_2x_3\ldots x_n - \sum_{k=1}^{n} x_k}{\sum_{k=0}^{n-2}(x_1+x_2+x_3+\cdots+x_{n-k-1})R^k} = R-1$$

where $x_1x_2x_3\ldots x_n$ is an n-digit numeral, base R, $n \geq 2$.

Digit problems: cancellation

ISMJ 14.11.

A mathematics student complained that he was not given credit for a correct answer when he cancelled the 6's in the fraction 16/64. Find all the fractions less than 1 in value with 2-digit numerators and denominators for which this kind of cancellation works. How about 3-digit numerators and denominators?

PME 365. **by Clayton W. Dodge**

Find all fractions `abc`/`cde` such that cancelling the digit `c` yields an equivalent fraction.

Digit problems: consecutive digits

FQ B-364. **by George Berzsenyi**

Find and prove a formula for the number of positive integers less than 2^n whose base 2 representations contain no consecutive 0's.

CRUX 267. **by John Veness**

Some products, like $56 = 7 \cdot 8$ and $17820 = 36 \cdot 495$, exhibit consecutive digits without repetition. Find all such products $c = a \cdot b$ which exhibit without repetition four, five,..., ten consecutive digits.

Digit problems: counting problems

FUNCT 2.4.2.

There are 700 hymns in a church hymnal. It is required to print a set of cards, each with one digit on it, so that the numbers of any four hymns can be displayed on a notice board. How many cards are required? (Give two answers, one assuming that an inverted 6 can be used as a 9, the other without that option.)

SSM 3665. **by Alan Wayne**

In the sets of decimal integers $S_1 = \{1, 2, \ldots, 10\}$, $S_2 = \{1, 2, \ldots, 100\}$, and $S_3 = \{1, 2, \ldots, 1000\}$, the number of zero digits in S_2 is the same as the number of digits in S_1; and the number of zero digits in S_3, is the same as the number of digits in S_2. Show that this equality relation holds in general.

Digit problems: cubes

CRUX 385. **by Charles W. Trigg**

In the decimal system, there is a 12-digit cube with a digit sum of 37. Each of the four successive triads into which it can be sectioned is a power of 3. Find the cube and show it to be unique.

Digit problems: cyclic shift

FUNCT 1.1.4.

The left-hand digit of a natural number is removed and replaced at the right-hand end, and this results in increasing the original number by fifty percent. Find such a natural number.

FUNCT 1.2.5.

(a) The right-hand digit of a natural number is to be removed and replaced at the left-hand end, so increasing the original number by 50%. Find such natural numbers.

(b) Repeat (a) with 50% replaced by 75%.

ISMJ J11.5.

The last two digits of a six digit number are 4 and 2 respectively. When these two digits are shifted to be the first two digits the new six digit number is exactly half of the original. Find the original number.

NYSMTJ 63. **by Haralyn Kuckes**

Find the smallest positive integer such that, when the first digit is transposed to the end, the resultant number is 3/2 times the original.

OSSMB 79-2.

Find the smallest positive integer whose value is tripled if the left-hand digit is transferred to the right-hand end.

Digit problems: digit reversals

ISMJ 13.21.

(a) If $10 \leq n \leq 99$, show that the number obtained from n by reversing its digits is given by the formula $10n - 99\lfloor n/10 \rfloor$.

(b) For $1 \leq n \leq 9999$, write a formula for the sum of the digits of n. This formula may involve the usual arithmetic operations and also the floor function.

PARAB 369.

Find a 5-digit number which, when divided by 4, yields another 5-digit number using the same five digits but in the opposite order.

SSM 3575. **by Bob Prielipp and N. J. Kuenzi**
SSM 3591. **by Bob Prielipp and N. J. Kuenzi**

When the digits of the positive integer N are written in reverse order, the positive integer N' is obtained. Let $N + N' = S$. Then S is called the sum after one reversal addition. The kth pentagonal number is given by

$$P_k = k(3k-1)/2, \quad k = 1, 2, 3, \ldots .$$

Prove that there are infinitely many pentagonal numbers that have a palindromic sum after one reversal addition.

MSJ 435. **by Peter A. Lindstrom**

Let abcd be a four-digit numeral, written in base 10, whose digits a, b, c, and d are such that a > b > c > d. Reverse the order of the digits to form another four-digit base ten number, dcba. Show that the sum of the digits of the differences of these two numbers is 18.

Digit problems: digital roots

CRUX 203. **by Charles W. Trigg**

Prove or disprove: The digital root of every even perfect number greater than 6 is 1.

SSM 3674. **by Richard L. Francis**

Let S denote the set of positive integers divisible by 7 and having a digital root of 7. Show that S contains infinitely many squares not ending in zero.

SSM 3779. **by Richard L. Francis**

Is it true that the digital root of a prime of the form $x^3 - y^3$ is 1 or 7?

Digit problems: distinct digits

CRUX 486. **by Gilbert W. Kessler**

(a) Find all natural numbers N whose decimal representation

$$N = \mathtt{abcdefghi}$$

consists of nine distinct nonzero digits such that

$$2|(a-b),\quad 3|(a-b+c),\quad 4|(a-b+c-d),\quad \ldots,$$

$$9|(a-b+c-d+e-f+g-h+i).$$

(b) Do the same for natural numbers $N = \mathtt{abcdefghij}$ consisting of ten distinct digits (leading zeros excluded) such that

$$2|(a-b),\quad 3|(a-b+c),\quad \ldots,$$

$$10|(a-b+c-d+e-f+g-h+i-j).$$

OSSMB 79-13.

The digits in the set $\{0, 1, 2, \ldots, 9\}$ can be uniquely arranged so that, starting from the left, the number formed by the first k digits is divisible by k for $k = 1, 2, \ldots, 10$. Find this arrangement.

JRM 671. **by Frank Rubin**

A used car has a standard 6-digit odometer and a 4-digit trip odometer. Assuming that the new purchaser will never reset the trip odometer, how can one determine from the present settings at what mileage (if ever) the ten digits will first be all distinct?

OSSMB 77-11.

The digits $0, 1, \ldots, 7$ can be arranged to form integers whose sum is 100. Is it possible to form such an arrangement using the digits $0, 1, 2, \ldots, 9$? Note that each digit must be used once and only once.

OSSMB G77.1-1.

Find the sum of all 3-digit numbers that can be formed from the digits 2, 3, 4, 7, 8, 9 where each number consists of 3 distinct digits.

Digit problems: divisibility

ISMJ J11.8.

Any three digit number abc is divisible by 7 if and only if $2a + 3b + c$ is divisible by 7. Why is this so? Can you generalize this to a rule for four or more digits?

JRM 596. **by Dan Wm. Burns**

Let n be any nonnegative integer. Prove that the number formed by placing 2^n and 2^{n+1} side by side in either order is a multiple of 3.

Digit problems: division

OMG 14.1.1.

An eight-digit number is divided by a three-digit number. The quotient is a five-digit number beginning with the digit 8, and the remainder is 0. Reconstruct the division.

Digit problems: factorials

MM 1075. **by Philip M. Dunson**

Counting from the right end, what is the 2500th digit of 10000!?

SSM 3717. **by Merrill Barnebey**

For what values of n, $n > 1$, does the expanded form of $n!$ have exactly n digits?

PARAB 432.

Find all three-digit numbers that are equal to the sum of the factorials of their digits.

Digit problems: fractions

AMM E2511. **by Morris Olitsky**

We observe that

$$1/3 = 0.333333\cdots = 3\left[(0.1) + (0.1)^2 + (0.1)^3 + \cdots\right]$$

and that

$$1/7 = 0.142857\cdots = 7\left[(0.02) + (0.02)^2 + (0.02)^3 + \cdots\right].$$

Are there any other positive integers x for which

$$\frac{1}{x} = x\left[\sum_{j=1}^{\infty}\left(m10^{-n}\right)^j\right]$$

for suitable integers m and n?

CRUX 131. **by André Bourbeau**

Let $p \geq 7$ be a prime number. If $p^{-1} = 0.\dot{a}_1 a_2 \ldots \dot{a}_k$, show that the integer

$$N = \overline{a_1 a_2 \ldots a_k}$$

is divisible by 9.

PME 366. **by Richard Field**

Let $Q = \lfloor 10^n/p \rfloor$, where p is a prime greater than 5, and n is the cycle length of the repeating decimal $1/p$. Can Q be a prime?

Digit problems: juxtapositions

JRM 380. **by J. A. H. Hunter**

Find a seven-digit number ABCDEFG with the property that half the square of ABCD plus twice the square of EFG is equal to ABCDEFG. The digits A through G are not necessarily distinct.

SSM 3751. **by Herta T. Freitag**

Consider an n-digit base ten number, $n > 1$. Write the same number next to it so as to obtain a $2n$-digit number. Such a number will be called a "$2n$-number."

(a) For what values of n, if any, will the set of $2n$-numbers be such that no pair of them is relatively prime?

(b) For what values of n, if any, will the set of $2n$-numbers contain prime numbers?

(c) What about the same questions if numbers are considered in a numeration system with a different base?

MATYC 101. **by Lawrence Sher**

Take a 3-digit number, base 10. Repeat the digits to form a 6-digit number. Which primes, smaller than 15, divide evenly into the 6-digit number? Generalize to division by any prime of a $2N$-digit number constructed as above.

CRUX 457. **by Allan Wm. Johnson Jr.**

Here are examples of two n-digit squares whose juxtaposition forms a $2n$-digit square:

$$\begin{array}{ccccccc} 4 & \text{and} & 9 & \text{form} & 49 & = & 7^2, \\ 16 & \text{and} & 81 & \text{form} & 1681 & = & 41^2, \\ 225 & \text{and} & 625 & \text{form} & 225625 & = & 475^2. \end{array}$$

Is there at least one such juxtaposition for each $n = 4, 5, 6, \ldots$?

Digit problems: leading digits

JRM 786. **by Daniel P. Shine**

A multiplication table showing the products of all two-digit numbers contains 8,100 entries. The distribution of the first digit of these numbers is as follows:

1	1954
2	1481
3	1181
4	952
5	767
6	618
7	494
8	372
9	281

These results may be approximated analytically by considering real numbers distributed uniformly in the interval $[10, 100]$. How good is the approximation?

NAvW 455. **by J. van de Lune**
FUNCT 3.3.5.

For any natural number n, written in the scale of ten, let $f(n)$ be the first digit of n. For $1 \le k \le 9$, determine the frequency of the digit k in the sequence $(f(2^n))_{n\in\mathbb{N}}$.

CRUX 266.* **by Daniel Rokhsar**

Let d_n be the first digit in the decimal representation of $n!$, so that

$$d_0 = 1,\ d_1 = 1,\ d_2 = 2,\ d_3 = 6,\ d_4 = 2, \ldots.$$

Find expressions for d_n and $\sum_{i=0}^{n} d_i$.

Digit problems: matrices

JRM 768. **by Peter MacDonald**

Using each of the digits 0 through 9 at least once, fill in a 4×4 matrix such that (a) in each row the digit in the first column times the digit in the fourth column equals the two-digit number formed from the digits in columns two and three; and (b) in each column the digit in the first row times the digit in the fourth row equals the two-digit number formed from the digits in rows two and three.

Arrange your solutions so that the smallest corner digit is at the upper left and the next smallest is at the upper right.

Digit problems: maxima and minima

SPECT 11.8.
PARAB 346.

Arrange the digits 0 to 9 to form five 2-digit numbers in such a way that the product of these five numbers is maximal.

ISMJ 14.14.

Form one- and two-digit numbers from the digits from 0 to 9. Use each digit once in doing so. Add the numbers. What is the largest and the smallest sum that can be obtained this way?

Digit problems: missing digits

OMG 18.3.9.

An old invoice showed that 72 turkeys had been purchased for $\$*67.9*$. The first and last digits were illegible. What were they?

PARAB 431.

Adjoin to the digits 632 three more digits so that the resulting six-digit number is divisible by each of 7, 8, and 9.

PENT 292. **by Léo Sauvé**

The number $9{,}x29{,}50y{,}zt7$ is known to be divisible by 73 and 137. Determine the digits x, y, z, t and thereby identify the number.

SSM 3741. **by Charles W. Trigg**

The product of three consecutive odd integers is given as $39x, xxx, xx7$ (where each x represents a digit, and not necessarily always the same digit). Find the integers and supply the missing digits in the product.

Digit problems: multiples

ISMJ 13.25.

Show that for any integer n, there is an integer q such that the digits of nq (in the decimal notation) are all either 0 or 1.

PARAB 428.

Let n be an integer whose last digit is 7. Show that some multiple of n has no digit equal to zero.

Number Theory

Digit problems: number of digits

FUNCT 1.3.4.
A large textbook has every page numbered. The printer used 1,890 digits to number the pages. How many pages were there?

JRM 604.
An n-digit number in base n is called an "inventory number" if it tallies its digits accurately in increasing order of digit. For example $3,211,000$ is a tally number in base 7, listing 3 zeros, 2 ones, 1 two, 1 three, 0 fours, 0 fives, and 0 sixes.

Prove that base n has an inventory number if and only if n is not a factor of 6. What bases have more than one inventory number?

PARAB 396.
Find a 10-digit number whose first digit tells the number of zeros that appear in it, whose second digit tells the number of ones, and so on (thus the tenth digit tells the number of nines in the number). Is there another such number?

Digit problems: operations

CRUX 21. **by H. G. Dworschak**
What single standard mathematical symbol can be used with the digits 2 and 3 to make a number greater than 2 but less than 3?

CRUX 285. **by Robert S. Johnson**
Using only the four digits 1, 7, 8, 9 (each exactly once) and four standard mathematical symbols (each at least once), construct an expression whose value is 109.

Digit problems: pandigital numbers

SSM 3681. **by Joe Dan Austin**
Find natural numbers $n(2)$, $n(3), \ldots, n(11)$ such that each $n(i)$:

(a) uses each of the digits $0, 1, 2, \ldots, 9$ exactly once;
(b) is divisible by i; and
(c) is the largest number satisfying (a) and (b).

CRUX PS5-2.
It has been stated that the number

$$526315789473684210$$

is a persistent number, that is, if multiplied by any positive integer the resulting number always contains the ten digits $0, 1, 2, \ldots, 9$ in some order with possible repetitions.

(a) Prove or disprove the above statement.
(b) Are there any persistent numbers smaller than the above number?

Digit problems: permutations

MSJ 455. **by Mike Conwill**
For how many of the 720 permutations of the digits 1, 2, 5, 6, 7, and 9 is the result a number divisible by 6?

MSJ 465.
Let N and n be positive integers, and suppose that the base 10 representation of N consists of the following digits: n 3's, one 4 and one 6. Prove that there exists a permutation of the digits of N so that the resulting number is divisible by 7.

PARAB 404.
ISMJ 10.8.
ISMJ 10.13.
The number 1234567 is not divisible by 11, but 3746512 is. How many different multiples of 11 can be obtained by appropriately ordering these digits?

MM 1016. **by Michael W. Ecker**
For n a positive integer, describe all n-digit numbers x with the property that there exists a permutation y of the digits of x such that $x + y = 10^n$.

Digit problems: powers

CRUX 164. **by Steven R. Conrad**
In the 5-digit decimal number ABCDE (with $\mathtt{A} \neq 0$), different letters do not necessarily represent different digits. If this number is the fourth power of an integer, and if $\mathtt{A} + \mathtt{C} + \mathtt{E} = \mathtt{B} + \mathtt{D}$, find the digit C.

Digit problems: primes

PENT 296. **by Charles W. Trigg**
Using three consecutive digits repeated, form an arithmetic progression of three-digit primes in the decimal system.

JRM 531. **by David L. Silverman**
Can the numbers 0 through 9 be arranged in a bracelet in such a way that every pair of adjacent links forms a two-digit prime or the reversal of one? (The pair 0p will be considered a two-digit prime if p is a prime digit.)

JRM 570. **by Alvin Owen**
The smallest positive integer that is not a factor of any number that has no repeated digits is 100. What is the smallest prime that is not a factor of any number that has no repeated digits?

MM 1029. **by Murray S. Klamkin**
Does there exist any prime number such that if any digit (in base 10) is changed to any other digit, the resulting number is always composite?

OSSMB 79-6.
In the multiplication of a three-digit number by a two-digit number which yields a five digit number, all the digits are prime, including those that appear in the standard multiplication algorithm. Find the digits.

MM 953. **by Allan W. Johnson, Jr.**
An absolute prime is a prime number all of whose decimal digit permutations are also prime numbers. Show that no absolute prime number exists that contains three of the four digits 1, 3, 7, and 9.

Are there any absolute primes of more than three digits that contain two of the digits 1, 3, 7, and 9?

SSM 3753. **by Bob Prielipp**
If q is a prime number and

$$1/q = .a_1a_2 \ldots a_ta_{t+1}a_{t+2} \ldots a_{2t}$$

prove that

$$a_{t+1}a_{t+2} \ldots a_{2t} = (q-1)(a_1a_2 \ldots a_t) + (q-2).$$

JRM 555. **by Henry Larson**

(a) There are two different ways of expressing the prime 809 as the sum of smaller primes with no digit used more than once, on either side of the equation. One of them is: $809 = 761 + 43 + 5$. Find the other.

(b) Can a similar all-prime equation be written using all ten digits only once?

(c) If primes are formed using each of the digits 1 through 9, the smallest obtainable sum comes from $89 + 61 + 43 + 7 + 5 + 2$. What is the smallest possible sum of primes using all ten digits?

(d) What is the smallest possible product obtainable using primes made up of the digits 1 through 9? 0 through 9?

MSJ 420. **by John Murphy**

An "upside down" prime is a positive prime number that remains a positive prime number when the paper on which it is written is turned upside down. Find all upside down primes smaller than 1,000.

Digit problems: products

ISMJ 10.16.

Prove that if an integer n exceeds 10, the product of its digits is less than n.

PME 444. **by Peter A. Lindstrom**

In terms of n, which is the first nonzero digit of

$$\prod_{i=1}^{n/2}(i)(n-i+1)$$

for even $n \geq 6$?

Digit problems: squares

CRUX 470. **by Allan Wm. Johnson Jr.**

Construct an integral square of eleven decimal digits such that, if each digit is increased by unity, the resulting integer is a square.

CRUX 95. **by Walter Bluger**

Said a math teacher, full of sweet wine:
"Your house number's the exact square of mine."
— "You are tight and see double
Each digit. That's your trouble,"
These 2-digit numbers you must divine.

MM Q642. **by Steven R. Conrad**
OSSMB 75-13.

Prove that the numbers 49, 4489, 444889, …, obtained by inserting 48 into the middle of the preceding numbers, are all perfect squares.

PARAB 352.

How many square numbers are there whose digits, when written in base 10 notation, contain three hundred 1's and some number of 0's?

AMM E2786. **by Walter Stromquist**

The consecutive integers 31 and 32 have these properties: The larger one is twice a square, and the sum of the digits in both numbers is a square.

(a) How many pairs of consecutive integers have the same properties?

(b) Would there exist such a pair if we used base 3 instead of decimal notation?

(c) Does such a pair exist in any odd base other than 3?

SSM 3685. **by Douglas E. Scott**

The number 81 has the following interesting property: "Bisect" the number, obtaining 8 and 1. Add 8 and 1 and square the result. The answer is the original number 81. Find two 4-digit and two 6-digit numbers with this property. Can you generalize the results?

CRUX 443.* **by Allan Wm. Johnson Jr.**

(a) Here are seven consecutive squares for each of which its decimal digits sum to a square:

81, 100, 121, 144, 169, 196, 225.

Find another set of seven consecutive squares with the same property.

(b)* Does there exist a set of more than seven consecutive squares with the same property?

ISMJ J11.16.

Find all fractions a/b such that
(i) $a/b = 2/7$,
(ii) $a + b$ is a two digit number, and
(iii) $a + b$ is a perfect square.

SSM 3705. **by Alan Wayne**

Show that for each positive integer n there is an n-digit positive integer N such that N^2 starts with precisely n ones.

CRUX 65. **by Viktors Linis**

Find all natural numbers whose square (in base 10) is represented by odd digits only.

NYSMTJ OBG9. **by Alan Wayne**

In the decimal system, find a six-digit, positive integer whose square ends at the right in eleven times the integer.

PME 457. **by R. Robinson Rowe**

Defining the last n digits of a square as its n-tail, what is the longest n-tail consisting of some part of the cardinal sequence $0, 1, 2, 3, \ldots, 9$? What is the smallest square with that n-tail?

Digit problems: sum of cubes

CRUX 407. **by Allan Wm. Johnson Jr.**

There are decimal integers whose representation in some number base $B = 2, 3, 4, \ldots$ consists of three nonzero digits whose cubes sum to the integer. For example,

$$43_{10} = 223_4 = 2^3 + 2^3 + 3^3,$$
$$134_{10} = 251_7 = 2^3 + 5^3 + 1^3,$$
$$433_{10} = 661_8 = 6^3 + 6^3 + 1^3.$$

Prove that infinitely many such integers exist.

Digit problems: sum of digits

CRUX 426. **by Charles W. Trigg**

There are two positive integers less than 10^{10} for each of which

(i) its digits are all alike;

(ii) its square has a digit sum of 37.

Find them and show that there are no others.

OSSMB 76-1.

What is the sum of all the digits occurring in the numbers from one to a billion?

OSSMB 77-1.

The positive integers x and y add up to z. The sum of the digits in x is 43 (in the usual base 10 representation) and the sum of the digits in y is 68. If in performing the addition of x and y there are exactly five "carries", what is the sum of the digits in z?

IMO 1975/4.
OSSMB 76-2.
PARAB 380.

When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

AMM 6077. **by H. L. Montgomery**

Let $s(n)$ denote the sum of the base 10 digits of $(1974)^n$. Show that $s(n) \to \infty$ as $n \to \infty$.

CRUX 430. **by Allan Wm. Johnson, Jr.**

(a) For $n = 1$, 8^{-n} equals a decimal fraction whose digits sum to 8. Prove that 8^{-n} for $n = 2, 3, 4, \ldots$ never again equals a decimal fraction whose digits sum to 8.

(b) The cube of 8 has decimal digits that sum to 8. For $n = 4, 5, 6, \ldots$, is there another 8^n whose decimal digits sum to 8?

CRUX 228. **by Charles W. Trigg**

(a) Find four consecutive primes having digit sums that, in some order, are consecutive primes.

(b) Find five consecutive primes having digit sums that are distinct primes.

SSM 3573. **by Charles W. Trigg**

Where $\sum d_i$ is the sum of the digits of N_i, the reiterated operation $N_i + \sum d_i = N_{i+1}$ produces an infinite sequence. Find such a sequence in the decimal system wherein three consecutive terms are palindromes.

SSM 3686. **by Charles W. Trigg**

Find a prime number such that

(a) the sum of the digits of its square is a square, and

(b) the square of the prime number is also a sum of five consecutive prime numbers.

CRUX 34. **by H. G. Dworschak**

Once a bright young lady called Lillian
Summed the numbers from one to a billion.
But it gave her the fidgets
To add up the digits;
If you can help her, she'll thank you a million.

Digit problems: sum of powers

NYSMTJ 58. **by Gary Wernsing**

An Armstrong number is an n-digit number equal to the sum of the nth powers of its digits.

Prove that there are a finite number of Armstrong numbers.

Digit problems: sum of squares

PENT 304. **by Charles W. Trigg**

Does any three-digit number, N, equal 11 times the sum of the squares of its digits?

Digit problems: terminal digits

JRM 764. **by John Brinn and Romae Cormier**

(a) Characterize the 2-digit numbers that do not occur as the last two digits of a cube.

(b) Characterize the n-digit numbers that do not occur as the last n digits of a cube.

PENT 289. **by Charles Trigg**

Each of the three consecutive integers 4, 5, and 6 terminates its own cube. Find four pairs of larger, consecutive integers in which each integer terminates its own cube.

FUNCT 2.4.3.

Find the number of 0's at the end of the number 1000!.

SSM 3597. **by Herta T. Freitag**

Given that a and b are positive integers such that b divides a, let U_b be the units digit of b.

(a) If $U_b = 0$, no prediction can be made about U_q, the units digit of the quotient q; however if $U_b = 5$, U_q and U_b must be of the same parity. Prove this.

(b) State and prove a relationship that predicts U_q for all other cases.

SSM 3612. **by Bob Prielipp**

Verify the following rule for multiplying two natural numbers, each of which has 5 as its units digit:

$$a5 \times b5 = \left\{ (a \times b) + \left\lfloor \frac{a+b}{2} \right\rfloor \right\} c$$

where $c = 25$ if $a + b$ is even and $c = 75$ if $a + b$ is odd.

AMM E2776. **by Alan Wayne**

(a) In the decimal system, find all twelve-digit positive integers n such that n^{102} ends at the right in the digits of n.

(b) Is there a corresponding solution to the problem in numeration systems other than base ten?

CRUX 149. **by Kenneth S. Williams**

Find the last two digits of 3^{1000}.

CRUX 253. **by David Fischer**

Let $x \uparrow y$ denote x^y. What are the last two digits of

$$2 \uparrow (3 \uparrow (4 \uparrow 5))?$$

JRM 741. **by Frank Rubin**

(a) Find the eight least significant digits of 7^{9999}.

(b) Find the ten least significant digits of 3^{999999}.

IMO 1978/1.

Given natural numbers m and n with $1 \leq m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m + n$ has its least value.

OMG 17.2.3.

If 8888^{8888} is multiplied out, what is the units digit in the final product?

ISMJ 14.1.

Consider the last (units) digits of the numbers $1^1 = 1$, $2^2 = 4$, $3^3 = 27$, $4^4 = 256$, $5^5 = 3125, \ldots$. Show that the sequence of last digits is periodic with period 20.

PARAB 268.

Prove that $11^{10} - 1$ is divisible by 100.

PARAB 358.
ISMJ 10.5.
ISMJ 14.12.

What are the last two digits of $2^{2^{73}}$?

MSJ 496.

Let $a_1 = 1$ and for $n > 1$ define $a_n = n^{a_{n-1}}$. What are the last two digits of a_9?

MSJ 490.

Consider the sequence (a_n):

$$6, 76, 376, 9376, 09376, 109376, \ldots$$

and note that for each $n = 1, 2, \ldots$, the product of two numbers terminating in a_n is again a number terminating in a_n. Find a_7.

CANADA 1978/1.

Let n be an integer. If the tens digit of n^2 is 7, what is the units digit of n^2?

ISMJ 14.15.

What integer has a square ending in the longest string of nonzero digits (in base 10)?

CRUX 55. **by Viktors Linis**

What is the last digit of $1 + 2 + \cdots + n$ if the last digit of $1^3 + 2^3 + \cdots + n^3$ is 1?

Digit problems: triangular numbers

CRUX 274. **by Charles W. Trigg**

Find triangular numbers of the form `abcdef` such that

$$\texttt{abc} = 2\texttt{def}.$$

Diophantine equations: degree 2

AMM E2624. **by Robert M. Hashway**

Solve the Diophantine equation

$$a + b \cdot 10^k = (a + b)^2,$$

where $0 < a, b < 10^k$ and $k \leq 5$.

CANADA 1977/1.
OMG 16.2.1.

If $f(x) = x^2 + x$, prove that the equation $4f(a) = f(b)$ has no solutions in positive integers a and b.

FQ B-387. **by George Berzsenyi**

Prove that there are infinitely many ordered triples of positive integers (x, y, z) such that

$$3x^2 - y^2 - z^2 = 1.$$

IMO 1977/5.
PARAB 367.

Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

JRM 81a. **by D. Silverman**

The Diophantine equation $\sum_{k=1}^{N} x_k^2 = \prod_{k=1}^{N} x_k$ ($x_k \neq 0$, for all k), has known solutions for $n = 1, 3, 4, 5, 7$, and most larger values of n. No solutions are possible for $n = 2$, since $x_1^2 + x_2^2 > x_1 \cdot x_2$ for all such x_1 and x_2.

Find a solution for $n = 6$ or show that no such solutions are possible.

MSJ 446.

Solve the following Diophantine equation in positive integers x and y:

$$x^2 + x + 29 = y^2.$$

NYSMTJ 83. **by Steven R. Conrad**

Find all ordered pairs of integers (x, y) such that

$$4x + 3y - 8 = xy.$$

CRUX 153. **by Bernard Vanbrugghe**

Show that the only positive integers that satisfy the equation $a \cdot b = a + b$ are $a = b = 2$.

MSJ 449. **by Steven R. Conrad**

Solve the Diophantine equation

$$x^2 + 4x + y^2 = 9.$$

FQ B-391. **by M. Wachtel**

Some of the solutions of $5x^2 + 1 = y^2$ in positive integers x and y are $(x, y) = (4, 9), (72, 161), (1292, 2889), (23184, 51841)$, and $(416020, 930249)$. Find a recurrence formula for the x_n and y_n of a sequence of solutions (x_n, y_n) and find

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n}$$

in terms of $\alpha = (1 + \sqrt{5})/2$.

FQ B-410. **by M. Wachtel**

Some of the solutions of $5(x^2 + x) + 2 = y^2 + y$ in positive integers x and y are:

$$(x, y) = (0, 1),\ (1, 3),\ (10, 23),\ (27, 61).$$

Find a recurrence formula for the x_n and y_n of a sequence of solutions (x_n, y_n). Also find $\lim(x_{n+1}/x_n)$ and $\lim(x_{n+2}/x_n)$ as $n \to \infty$ in terms of $\alpha = (1 + \sqrt{5})/2$.

Diophantine equations: degree 3

CANADA 1978/2.

Find all pairs a, b of positive integers satisfying the equation $2a^2 = 3b^3$.

CRUX 101. **by Léo Sauvé**

Show that the cube of any rational number is equal to the difference of the squares of two rational numbers.

CRUX PS6-1.

Solve the Diophantine equation

$$x^3 + y^3 + z^3 = (x + y + z)^3.$$

MM Q611. **by Erwin Just**

Prove that the only solutions to the Diophantine equation, $x^3 - 2 = 6y^2$, are $x = 2,\ y = \pm 1$.

PUTNAM 1978/B.4.

Prove that for every real number N, the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$

has a solution for which x_1, x_2, x_3 and x_4 are all integers larger than N.

MSJ 453. **by Daniel Flegler**

Solve the Diophantine equation

$$x^3 + y^3 - 3x^2 + 6y^2 + 3x + 12y + 6 = 0.$$

MSJ 471.

Prove that there are no integers a and b for which the roots of

$$x^3 + ax^2 + 7x + b = 0$$

form an arithmetic progression.

SSM 3599. **by Robert Fink and Bob Prielipp**

Find ten, positive integer solutions of the equation

$$x^3 + y^3 + z^3 = u^3$$

where $(x, y, z, u) = 1$ and $x \le y \le z \le u \le 250$.

TYCMJ 80. **by Steve Kahn**

Prove that if $\sum_{i=1}^{6} x_i^3 = x_7^3$ has a solution in integers, then

$$\prod_{i=1}^{7} x_i \equiv 0 \pmod 3.$$

MATYC 71. **by Steve Kahn**

Find all primes p for which the equation $x^3 + y^3 = p$ has a solution in positive integers.

SSM 3680. **by Charles W. Trigg**

Are there any prime values of $p < 2200$ for which the equation $x^3 - y^3 = p$ has a solution in positive integers x and y?

TYCMJ 58. **by J. Orten Gadd**

Prove or disprove that the only values of the prime, p, and the integer, k, for which the zeros of $x^3 + kx + (p - k - 1)$ are integers are $p = 5$ and $k = -12$.

Diophantine equations: degree 4

CMB P255. **by M. D. Nutt**

Show that for integral A, the diophantine equation $A^2x^3(x + 2) + 1 = y^2$ can have only a finite number of solutions.

CRUX 217. **by David R. Stone**

Solve the Diophantine equation

$$n^2(n - 1)^2 = 4(m^2 - 1).$$

CRUX 496. **by E. J. Barbeau**

Solve the Diophantine equation

$$(x + 1)^k - x^k - (x - 1)^k = (y + 1)^k - y^k - (y - 1)^k$$

for $k = 2, 3, 4$ and $x \ne y$.

FQ B-360. **by T. O'Callahan**

Show that for all integers a, b, c, d, e, f, g, and h, there exist integers w, x, y, and z such that

$$(a^2 + 2b^2 + 3c^2 + 6d^2)(e^2 + 2f^2 + 3g^2 + 6h^2) = (w^2 + 2x^2 + 3y^2 + 6z^2).$$

FUNCT 1.4.2.

Show that the only integral values of n making

$$n^4 + n^3 + n^2 + n + 1$$

a perfect square are $n = 3$, $n = 0$, and $n = -1$.

USA 1976/3.
MSJ 441.

Solve the Diophantine equation

$$a^2 + b^2 + c^2 = a^2b^2.$$

USA 1979/1.

Find all nonnegative solutions (apart from permutations) of the Diophantine equation:

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1,599.$$

Diophantine equations: degree 5

PME 440. **by Charles W. Trigg**

Are there any prime values of $p < 10^5$ for which the equation $x^5 - y^5 = p$ has a solution in positive integers? How about $x^5 + y^5 = p$?

Diophantine equations: degree 6

MM Q647. **by Robert Scherrer**

Find all integer solutions, (a, c), of

$$a^4 + 6a^3 + 11a^2 + ba + 1 = q\frac{(a^2 - 1)(c^2 - 1)}{a^2 + c^2},$$

where q is the product of arbitrary, nonnegative powers of alternate primes, i.e.,

$$q = 2^{b_1} \cdot 5^{b_2} \cdot 11^{b_3} \cdots p^{b_n},\ b_i \ge 0.$$

Diophantine equations: degree n

AMM E2532. **by Erwin Just**
Solve the following Diophantine equations:
(a) $x^m(x^2+y)=y^{m+1}$,
(b) $x^m(x^2+y^2)=y^{m+1}$.

AMM E2621. **by Barry Powell**
Prove that $x^n+1=y^{n+1}$ has no solutions in positive integers x, y, and n, $n\geq 2$, with $\gcd(x,n+1)=1$.

AMM E2642. **by Antonio Rocha**
Let x, y, and z be integers such that

$$x^2+y^2=z^{2m}, \qquad \gcd(x,y)=1,$$

where m is a positive integer. If $4m-1=p$ is a prime, show that $p\,|\,xy$.

CRUX 99. **by H. G. Dworschak**
If a, b, and n are positive integers, prove that there exist positive integers x and y such that

$$(a^2+b^2)^n=x^2+y^2.$$

If $a=3$, $b=4$, and $n=7$, find at least one pair (x,y) of positive integers that satisfies this equation.

Diophantine equations: exponentials

AMM E2749. **by Leo J. Alex**
(a) Show that neither of the equations

$$3^a+1=5^b+7^c,$$

$$5^a+1=3^b+7^c$$

has a solution in integers a, b, and c other than $a=b=c=0$.
(b) Show that the only solutions to the equation

$$7^a+1=3^b+5^c$$

in integers a, b, and c are $(a,b,c)=(0,0,0)$ or $(1,1,1)$.

AMM E2750. **by A. P. Hillman**
Find all solutions in integers a, b, and c of the equation

$$9+5^a=3^b+7^c.$$

CRUX 188. **by Daniel Rokhsar**
ISMJ 12.30.
Show that the only positive integer solution of the equation $a^b=b^a$, $a<b$, is $a=2$, $b=4$.

ISMJ 14.16.
Show that neither 2^n-1 nor 2^n+1 is a cube if n is a positive integer larger than 1.

JRM 496. **by Steven Kahan**
Solve for integer m: $(m^2-7)^{m+1}=(m+1)^{m^2-7}$.

MATYC 136. **by John Annulis**
Prove: If n is a positive integer and $n^{1/(n-1)}$ is an integer, then $n=2$.

MATYC 75. **by James Chilaka**
Find all positive integers n for which there exist positive integers x and y ($x\neq 1$, $y\neq 1$) with $2^x-2=n^y-n$.

MM 1012. **by Gerald E. Gannon and Harris S. Shultz**
Find all solutions (x,y) of $x^y=y^{x-y}$, where x and y are positive integers.

NAvW 467. **by R. J. Stroeker**
Show that the only solution in positive integers of the equation

$$x^y-y^x=x+y$$

is $x=2$ and $y=5$.

NAvW 500. **by R. J. Stroeker and R. Tijdeman**
Solve the following Diophantine equation in nonnegative integers x, y, z, and w:

$$3^x+3^y=5^z+5^w.$$

NYSMTJ 66. **by Steven R. Conrad**
(a) Find the sum of the series

$$\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)+\cdots+\left(\frac{1}{100}+\frac{2}{100}+\frac{3}{100}+\cdots+\frac{99}{100}\right).$$

(b) Find all ordered pairs of integers (x,y) such that $x^{x+y}=y^4$ and $y^{x+y}=x$.

PME 432. **by Erwin Just**
Does there exist an integer m for which the equation

$$\sum_{i=0}^{m}3^{ix}=7^y$$

has solutions in positive integers?

TYCMJ 127. **by Sidney Penner**
Find all rational solutions of $y^x=xy$.

CRUX 219. **by R. Robinson Rowe**
Find the least integer N which satisfies

$$N=a^{a+2b}=b^{b+2a}, \quad a\neq b.$$

CRUX 230. **by R. Robinson Rowe**
Find the least integer N that satisfies

$$N=a^{ma+nb}=b^{mb+na}$$

with m and n positive and $1<a<b$.

PARAB 436.
Find all solutions in nonnegative integers x, y of the equation $3\cdot 2^x+1=y^2$.

TYCMJ 95. **by R. S. Luthar**
Solve the equation $2^x+1=y^2$ in positive integers.

NAvW 421. **by O. P. Lossers**
Let p be a prime. Consider the Diophantine equation

$$2^n-3p=x^2.$$

If this equation has two solutions, then determine p.

PME 423. **by Richard S. Field**

Find all solutions in positive integers of the equation $a^d - b^d = c^c$, where d is a prime number.

PUTNAM 1976/A.3.

Find all integral solutions of the equation

$$|p^r - q^s| = 1,$$

where p and q are prime numbers and r and s are positive integers larger than unity. Prove that there are no other solutions.

Diophantine equations: factorials

CRUX 434.* **by Harold N. Shapiro**

(a) It is not hard to show by Bertrand's Postulate that all the solutions in positive integers x, y, m, n of the equation

$$(m!)^x = (n!)^y$$

are given by $m = n = 1$; and $m = n,\ x = y$. Find such a proof.

(b)* Prove the same result without using Bertrand's Postulate or equivalent results from number theory.

MM Q657. **by Edward T. H. Wang**

Find all solutions to the Diophantine equation

$$1! + 2! + \cdots + n! = m^2.$$

CRUX 159. **by Viktors Linis**

Show that

$$x! + y! = z!$$

has only one solution in positive integers, and that

$$x!y! = z!$$

has infinitely many for $x > 1$, $y > 1$, and $z > 1$.

Diophantine equations: linear

CRUX 179. **by Steven R. Conrad**

The equation $5x + 7y = c$ has exactly three solutions (x, y) in positive integers. Find the largest possible value of c.

Diophantine equations: mediants

MSJ 439. **by Joseph O'Sullivan and Sidney Penner**

Joe Poorstudent believes that

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}.$$

Are there any positive integers a, b, c, and d for which his method yields the correct result?

Diophantine equations: radicals

MATYC 108. **by Gene Zirkel**

If

$$\frac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}} = \frac{\sqrt{a}+\sqrt{b}}{2},$$

where a and b are both integers, find $a+b$. Prove that this is the only solution.

Diophantine equations: solution in rationals

FQ B-337. **by Wray G. Brady**

Show that there are infinitely many points with both x and y rational on the ellipse $25x^2 + 16y^2 = 82$.

MM 968. **by Sidney Penner and H. Ian Whitlock**

A point in the plane is called rational if both of its coordinates are rational numbers. Show that $x^2 + y^2 = 2$ has an infinite number of rational solutions.

MSJ 450. **by Sidney Penner**

Let n be a positive integer and let a/b and c/d represent rational numbers in lowest terms. If $(a/b, c/d)$ is a solution of $x^2 + y^2 = n$, prove that $d = \pm b$.

CRUX PS2-2.

Determine all pairs of rational numbers (x, y) such that

$$x^3 + y^3 = x^2 + y^2.$$

FQ H-256. **by E. Karst**

Find all solutions of

(a) $x + y + z = 2^{2n+1} - 1$,

(b) $x^3 + y^3 + z^3 = 2^{6n+1} - 1$,

simultaneously for $n < 5$, given that: x, y, and z are positive rationals; $2^{2n+1} - 1$ and $2^{6n+1} - 1$ are integers; and $n = \log_2 \sqrt{t}$, where t is a positive integer.

NAvW 545. **by R. J. Stroeker**

Determine all solutions in nonzero rationals x and y of the equation

$$\left(x^2 + y\right)\left(x + y^2\right) = (x - y)^3.$$

PARAB 316.

The rational numbers 169/30 and 13/15 are such that their sum is the same as their quotient:

$$\frac{169}{30} + \frac{13}{15} = \frac{13}{2} = \frac{\frac{169}{30}}{\frac{13}{15}}.$$

Find all pairs of rational numbers which have this property.

Diophantine equations: systems of equations

AMM E2615. **by D. Rameswar Rao**

Show that the system of Diophantine equations

$$x^2 + y^2 = u^2 + v^2,$$
$$x^3 + y^3 = u^3 + v^3$$

has no solutions in positive integers with $(x, y) \neq (u, v)$. Prove the same for the system

$$x^2 + y^2 = u^2 + v^2,$$
$$x^5 + y^5 = u^5 + v^5.$$

AMM E2664. **by Robert L. Bishop**

(a) For a fixed $n \geq 3$, describe how one can construct all solutions of the system of Diophantine equations

$$\left(\sum_{i=1}^{n} x_i\right) - x_j = y_j^2, \qquad 1 \leq j \leq n.$$

(b) For $n = 10$, find a solution such that the x_i are distinct positive integers and $x_1 + \cdots + x_{10}$ is minimal.

ISMJ 11.9.

How many quadruples (a, b, c, d) of nonnegative integers are there such that $a + b = cd$ and $c + d = ab$?

NYSMTJ 71. **by Steven R. Conrad**

Find all integral solutions of the system:

$$x + yz = 6$$
$$y + xz = 6$$
$$z + xy = 6.$$

IMO 1976/5.

Consider the system of p equations in $q = 2p$ unknowns $x_1, x_2, \ldots, x_q$:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q = 0$$
$$a_{12}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q = 0$$
$$\cdots$$
$$a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pq}x_q = 0$$

with every coefficient a_{ij} a member of the set $\{-1, 0, 1\}$. Prove that the system has a solution $(x_1, x_2, \ldots, x_q)$ such that

(1) all x_i $(j = 1, 2, \ldots, q)$ are integers,

(2) there is at least one value of j for which $x_j \neq 0$, and

(3) $|x_j| \leq q$ $(j = 1, 2, \ldots, q)$.

OMG 18.2.2.

If the sum of two numbers is 8 and the product of these two numbers is 10, find the sum of the squares of these numbers.

CMB P245. **by D. Rameswar Rao**

Show that the system of equations

$$x^2 + y^2 = X^2 + Y^2$$
$$x^3 + y^3 = X^3 + Y^3$$

has no integer solutions.

Divisibility: consecutive integers

ISMJ J11-13.

Show that in any set of ten consecutive integers there is at least one integer that is not divisible by 2, 3, 5, or 7.

CRUX 212. **by Bruce McColl**

Find four consecutive integers that are divisible by 5, 7, 9, and 11 respectively.

Divisibility: cube roots

SSM 3581. **by Alan Wayne**

Find the set of natural numbers each of which is exactly divisible by the greatest integer in its cube root.

Divisibility: difference of squares

CRUX 337. **by V. G. Hobbes**

If p and q are primes greater than 3, prove that $p^2 - q^2$ is a multiple of 24.

Divisibility: exponentials

AMM E2772. **by Robert B. McNeill**

Let m be a positive integer. Find all ordered pairs of positive integers (a, b) for which $(a + b) \mid \left(a^{2m} + b^{2m}\right)$.

CMB P241. **by A. Meir and S. K. Sehgal**

Characterize those pairs of positive integers (n, α) for which

$$p \mid n^\alpha - 1 \Rightarrow p \mid n - 1.$$

CMB P275. **by J. P. Jones**

Prove that

$$10^{5^{10^{5^{10}}}} + 5^{10^{5^{10^{5}}}}$$

is divisible by 11.

JRM 422. **by David L. Silverman**

Let $A_n = n^{100} + 100^n$ $(n = 1, 2, 3, 4, \ldots)$.

(a) Prove that 3, 7, 11, and 13 are not factors of A_n for any value of n.

(b) Are there infinitely many primes that never divide A_n? Are there any greater than 13?

SSM 3635. **by Herta T. Freitag**

Prove that for all natural numbers n, dividing the expression $5^{n+2}\left[5^{n+2} + 6(2 \cdot 3^n - 5)\right] + 36 \cdot 3^n(3^n - 5)$ by 64 leaves a remainder of 31.

SSM 3603. **by Herta T. Freitag**

Let a and b be odd numbers, and let n be any natural number. Then 2^n divides $a^n - b^n$ if and only if 2^n divides $a - b$. True or false?

AMM E2643. **by Harry D. Ruderman**

Show that for no integer $n > 1$, $2^n - 1$ divides $3^n - 1$.

MM Q635. **by Erwin Just**

Prove that for any prime, p, there exists an infinite number of values of m for which p is a divisor of $2^{m+1} + 3^m - 17$.

Divisibility: factorials

TYCMJ 84. **by R. S. Luthar**

Let $p > 3$ be a prime. Prove or disprove that $\lfloor (p-2)!/p \rfloor$ divides $(p-2)! - 1$.

Divisibility: floor function

SSM 3628. **by Herta T. Freitag**

Prove or disprove that

$$\lfloor n/2 \rfloor - 3n + (-1)^n - 1$$

is always divisible by 5.

Divisibility: geometry

AMM E2653. **by Albert A. Mullin**

A lattice point $(x, y) \in \mathbb{Z}^2$ is visible if $\gcd(x, y) = 1$. Prove or disprove: Given a positive integer n, there exists a lattice point (a, b) whose distance from every visible point is at least n.

Divisibility: polynomials

TYCMJ 36. **by Aleksandras Zujus**

For each integer $n > 1$, prove that $n^n - n^2 + n - 1$ is divisible by $(n-1)^2$.

CRUX 87. **by H. G. Dworschak**

(a) If $u_n = x^{2n} + x^n + 1$, for which positive integer n is u_n divisible by u_1?

(b) For which positive integer n does $x + \frac{1}{x} = 1$ imply $x^n + \frac{1}{x^n} = 1$?

JRM 632. **by Diophantus McLeod**

Solve these two simultaneous "divisibilities" in positive integers:

$$x \,|\, (y+5); \quad y \,|\, (x+3).$$

How many solutions are there if the word "positive" is deleted?

MSJ 463.

Prove that $(21n-3)/4$ and $(15n+2)/6$ cannot both be integers for the same positive integer n.

MM 1009. **by Sidney Kravitz**

Let x, y, and n be positive integers and define $f(x) = x^2 - x + 41$ and $g(y) = y^2 - y + 68501$. Prove or disprove that n divides $g(y)$ for some y if and only if n divides $f(x)$ for some x.

JRM 591. **by Kenneth M. Wilke**

A problem is stated as follows: "Prove that there exist infinitely many pairs of positive integers x, y such that $x(x+1) \,|\, y(y+1)$, $x \nmid y$, $(x+1) \nmid y$, $x \nmid (y+1)$, $(x+1) \nmid (y+1)$ and find the least such pair." The solution gives the family: $x = 36k + 14$, $y = (12k+5)(18k+7)$, $k = 0, 1, 2 \ldots$, with $x = 14$, $y = 35$ as the least pair. The solution further states "... it is easy to show that there are no smaller numbers with the desired property."

(a) Find all solution pairs x, y with $y \le 35$. In the process you will discover that $14, 35$ is not, in fact, the smallest pair with the desired property.

(b) Find a family of solutions (x, y) in which $xy < (36k+14)(12k+5)(18k+7)$.

OSSMB 75-1.

For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?

CRUX 107. **by Viktors Linis**

For which integers m and n is the ratio

$$\frac{4m}{2m + 2n - mn}$$

an integer?

MSJ 474.

Prove that $n^2 + n + 1$ is a multiple of 19 for infinitely many integral values of n.

SPECT 8.7. **by B. G. Eke**

If m and n are odd integers, show that $8 \,|\, m^2 - n^2$.

SSM 3757. **by Charles W. Trigg**

Are there any integer values of n for which $n^2 - 17$ is exactly divisible by $5n + 33$?

JRM 467. **by Les Marvin**

What is the largest integer that can divide two successive numbers of the form $n^2 + 3$?

CRUX 81. **by H. G. Dworschak**

Which of the following are divisible by 6 for all positive integers n?

(a) $n(n+1)(n+2)$
(b) $n(n+1)(2n+1)$
(c) $n(n^2+5)$
(d) $(n+1)^{2k} - (n^{2k} + 2n + 1)$, k a positive integer.

ISMJ 12.9.

Show that, for any two integers a and b, the number $(a+b)(a-b)ab$ is a multiple of six.

CRUX 35. **by John Thomas**

Let m denote a positive integer and p a prime. Show that if $p \,|\, (m^4 - m^2 + 1)$, then $p \equiv 1 \pmod{12}$.

ISMJ J11.1.

Prove that if n is an odd number greater than 3, then $n^4 - 18n^2 + 17$ is divisible by 64.

FUNCT 1.3.3. **by Rob Saunders**

Show that, for all integers a and b,

$$30 \,|\, ab(a^2 - b^2)(a^2 + b^2).$$

CRUX 392. **by Stephen R. Conrad**

Find all natural numbers n for which $n^8 - n^2$ is not divisible by 504.

PARAB 359.

An infinitely long list is made of all the pairs of integers m, n for which $23m - 10n$ is exactly divisible by 17. Another list is made of all the pairs for which $7x + 11y$ is exactly divisible by 17. Prove that the two lists are exactly alike.

PARAB 417.

Let a and b be integers. Show that $10a + b$ is a multiple of 7 if and only if $a - 2b$ is also.

PME 404. **by Bob Prielipp**

Let x be a positive integer of the form $24n - 1$. Prove that if a and b are positive integers such that $x = ab$, then $a + b$ is a multiple of 24.

USA 1977/1.

Determine all pairs of positive integers (m, n) such that

$$1 + x^n + x^{2n} + \cdots + x^{mn}$$

is divisible by

$$1 + x + x^2 + \cdots + x^m.$$

Divisibility: powers of 2

SSM 3790. **by Anton Glaser and Karl W. Schlecker**

Let $K(n) = (3n+1)/(2^x)$ where x is the greatest integer that will still leave $K(n)$ an integer. Prove that if $n \equiv 3 \pmod{10}$, then $K(n)$ is an odd multiple of 5.

Divisibility: products

SSM 3748. **by Charles W. Trigg**

Show that the product P of the ten differences of any five integers is divisible by 288.

Divisibility: triangular numbers

TYCMJ 68. **by Sidney Penner**

Let $f(n)$ be defined as the least positive integer k such that $n \mid \sum_{i=1}^{k} i$. Prove that $f(n) = 2n-1$ if and only if there exists a nonnegative integer m such that $n = 2^m$.

Divisibility: word problems

CRUX 12. **by Viktors Linis**

There are about 100 apples in a basket. It is possible to divide the apples equally among 2, 3 and 5 children but not among 4 children. How many apples are there in the basket?

Divisors

AMM 6144.* **by Carl Pomerance**

If n is a natural number, denote by $A(n)$ the arithmetic mean of the divisors of n.

(a) Prove that the asymptotic density of the set of n, for which $A(n)$ is an integer, is 1.

(b) Show that for any N there is an integer m such that $A(n) = m$ has at least N solutions.

(c) If it exists, find the asymptotic density of the set of integers m for which $A(n) = m$ has a solution.

AMM 6190.* **by D. E. Daykin and D. J. Kleitman**

Let n be a square-free integer that is not prime. Let F be a set of divisors of n such that neither the product of two elements of F nor n^2 divided by such a product is in F. What is the maximal proportion of the divisors of n that may lie in F?

SSM 3578. **by Robert A. Carman**

Show that any number of the form $6n-1$ has factors a and b such that $a+b$ is a multiple of 6.

SSM 3623. **by Bob Prielipp**

It is known that the sum of the reciprocals of the positive integer divisors of a perfect number is 2.

(a) Find four positive integers such that the sum of the reciprocals of the positive integer divisors of each of these numbers is 3.

(b) Find four positive integers such that the sum of the reciprocals of the positive integer divisors of each of these numbers is 4.

CRUX 467. **by Harold N. Shapiro**

Let $n_1, \dots, n_k$ be given positive integers and form the vectors $(d_1, \dots, d_k)$ where, for each $i = 1, \dots, k, d_i$ is a divisor of n_i. Letting $\tau(d)$ = the number of divisors of d, the number of these vectors is $\tau(n_1)\tau(n_2)\dots\tau(n_k)$. How many of these have the property that their components are relatively prime in pairs?

SPECT 9.2. **by B. G. Eke**
ISMJ 11.18.

A changing room has n lockers numbered 1 to n and all are locked. An attendant performs the sequence of operations $T_1, T_2, \dots, T_n$, where T_k is the operation whereby the condition of being locked or unlocked is altered in the case of those lockers (and only those) whose numbers are divisible by k. Which lockers are unlocked at the end?

Equations

CRUX 307. **by Steven R. Conrad**

Find the least and greatest values of x such that

$$xy = nx + ny,$$

if n, x, and y are all positive integers.

AMM 6197.* **by Manuel Scarowsky**

Let p be a prime; let a and b be positive integers; and let (x_0, y_0) be a solution of $ax + by = p$ in positive integers with x_0 minimal, if such exists (otherwise take $x_0 = 0$). Find an estimate for $\sum_{a,b} x_0$.

Euler totient: divisors

AMM 6193. **by Robert E. Shafer**

Given that n is such that $2\phi(n) = n-1$ (ϕ is the Euler totient function), prove

(a) $3 \nmid n$;

(b) If p and q are distinct prime divisors of n, then $p \not\equiv 1 \pmod{2q}$;

(c) n has at least 11 distinct prime divisors.

AMM 6160. **by Robert E. Shafer**

(a) If m is the largest odd divisor of n, then with the exception of (b), prove that

$$2^{v(n)} m^{v(m)/2} \mid \phi(a^n + b^n)$$

for $a > b \geq 1$, where $v(n)$ is the number of divisors of n and ϕ is the Euler totient function.

(b) If $a = 2$, $b = 1$, $n = 3^d c$, c odd, $d \geq 1$, then prove that

$$2^{v(n)-1} m^{v(m)/2} 3^{d-1} \mid \phi(a^n + b^n).$$

Euler totient: fractions

AMM 6070. **by P. Erdős and C. W. Anderson**

Where $\phi(n)$ is Euler's totient function, let

$$\Phi(n) = \frac{\phi(n)}{n}.$$

$\Phi\colon \mathbb{N} \to (0,1]$ densely. For given a, demonstrate that there are only a finite number of b (coprime with a) such that $\Phi(n) = a/b$ has solutions.

MM Q645. **by R. B. Eggleton**

Prove that there are infinitely many positive integers n for which $\phi(n) = n/3$, but none for which $\phi(n) = n/4$, where ϕ is Euler's phi-function.

Euler totient: inequalities

AMM E2599. **by Bernardo Recamán**
Are there arbitrarily large positive integers N such that for all $n \geq N$, we have $\phi(n) \geq \phi(N)$ while $\phi(n) \leq \phi(N)$ when $n \leq N$?

CRUX 458. **by Harold N. Shapiro**
It is known that, for each fixed integer $c > 1$, the equation $\phi(n) = n - c$ has at most a finite number of solutions for the integer n. Improve this by showing that any such solution, n, must satisfy the inequalities $c < n \leq c^2$.

AMM E2590. **by C. A. Nicol**
A natural number $n \geq 2$ is said to be ϕ-subadditive if

$$\phi(n) \leq \phi(k) + \phi(n-k)$$

for $1 \leq k \leq n-1$ and ϕ-superadditive if

$$\phi(n) \geq \phi(k) + \phi(n-k)$$

for $1 \leq k \leq n-1$ (ϕ denotes Euler's totient function). Show that there exist infinitely many ϕ-subadditive numbers and infinitely many ϕ-superadditive numbers.

Euler totient: primes

AMM E2611. **by C. A. Nicol**
Based upon the long-standing conjecture that if $n \geq 2$ is a natural number and $\phi(n) \,|\, (n-1)$ then n is prime, show that a natural number $n \geq 2$ is prime if and only if $\phi(n) \,|\, (n-1)$ and $(n+1) \,|\, \sigma(n)$.

Euler totient: quotients

JRM 474. **by Les Marvin**
What is the necessary and sufficient condition on two integers that the totient of their quotient equal the quotient of their totients?

Euler totient: solution of equations

JRM 622. **by Les Marvin**
Let $f(k)$ be the number of solutions of the equation $\phi(n) = k$.
(a) Is 4 the only solution to the equation $f(k) = k$?
(b) Has the sequence $\{f(k)/k\}$ a limit point other than zero?

Factorials

PARAB 270.
Find all positive integers between 1 and 100 having the property that $(n-1)!$ is not divisible by n^2.

CMB P250. **by P. Erdős**
Write $n! = u_1 u_2 \cdots u_k$, $n < u_1 < \cdots < u_k$. Prove that $u_k \leq 2n$ has only a finite number of solutions. Determine them.

AMM E2623. **by Ivan Niven**
For which positive integers k is it true that there are infinitely many pairs of positive integers m and n such that

$$\frac{(m+n-k)!}{m!n!}$$

is an integer?

AMM 6121. **by Harry D. Ruderman**
For all positive integers $a_1, a_2, \ldots, a_n$, the following is always an integer:

$$\prod_{i=1}^{n} (na_i)! / \left[n \prod_{i=1}^{n} (a_i!) \right]^{n-1} \left(\sum_{i=1}^{n} a_i \right)!.$$

Prove the conjecture for $n = 3$. Is it true in general?

AMM E2799. **by Marlow Sholander**
For n a positive integer, let $n!!$ denote the superfactorial $\prod_{i=1}^{n} i!$, and let $0!! = 1$. Set

$$A_n = \frac{(2n-1)!!}{[(n-1)!!]^4}.$$

Prove that A_n is an integral multiple of $(2n-1)!$.

TYCMJ 137. **by Martin Berman**
Let $r < n$ be positive integers and define $n_1 = n!$, $n_{k+1} = (n_k)!$ $(k = 1, 2, \ldots)$, and $\binom{n}{r}_k = n_k / r_k (n-r)_k$. Must $\binom{n}{r}_k$ always be an integer?

TYCMJ 70. **by Norman Schaumberger**
Determine the least possible integer N such that for all integers $n > N$,

$$\left(\frac{n^{n+1}}{(n+1)^n} \right)^n < n! < \left(\frac{n^{n+1}}{(n+1)^n} \right)^{n+1}.$$

CRUX 146. **by Jacques Marion**
Show that there exists no rational function $R(z)$ such that $R(n) = n!$ for each natural number n.

ISMJ J11.19.
How many perfect squares appear among the numbers

$$1!, 1! + 2!, 1! + 2! + 3!, \ldots, 1! + 2! + 3! + \cdots + n!?$$

Factorizations

OMG 18.3.7.
What two whole numbers, neither containing any zeros, will multiply together to equal exactly 1,000,000,000?

CRUX 64. **by Léo Sauvé**
Decompose 10,000,000,099 into a product of at least two factors.

FUNCT 1.5.3.
Find the prime factors of

5,679,431,432,056,743,205,685,679,432.

ISMJ J10.10.
In how many ways can 720 be written as a product of three positive integers different from one?

JRM 767. **by Harry Nelson**
Find all pairs of consecutive positive integers such that neither has any prime factors other than 2 or 3.

ISMJ J11-14.
Suppose n is a positive integer whose smallest prime factor is p and $p > \sqrt[3]{n}$. Show that n/p is also a prime.

JRM 371. **by Sidney Kravitz**

What is the maximum number of distinct factors that an integer between one and one million can have, and how many integers in this range have that many factors?

AMM 6015. **by C. W. Anderson**

Let $n = q_1^{a_1} q_2^{a_2} \dots q_k^{a_k}$, $k > 1$, be the prime decomposition of the integer n, and define

$$\mathrm{Ind}(n) = \max\{a_i \mid 1 \le i \le k\}.$$

Show that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=2}^{m} \mathrm{Ind}(k) = 1 + \sum_{n=2}^{\infty} \frac{\mu(n)}{n(n-1)} = 1.705211\dots .$$

CRUX 390. **by Gali Salvatore**

Show how to find the complete factorization of $2^{38}+1$ using only pencil and paper (no computers), having given that it consists of four distinct prime factors, none repeated, one of which is 229.

Farey sequences

ISMJ 13.28.

Suppose all fractions a/b in lowest terms with $b \le 100$ and $0 < a/b < 1$ are listed in increasing order and a/b and c/d are consecutive fractions in this list. Show that $b + d > 100$.

Fermat's Last Theorem

SSM 3728. **by Richard L. Francis**

Let A be the set of positive integers not divisible by 5. Show that $x^4 + y^4 = z^4$ is not possible if $x, y, z \in A$.

AMM E2631. **by Barry Powell**

It is known that if p is an odd prime and $3^p \not\equiv 3 \pmod{p^2}$, then the equation

$$x^p + y^p = z^p$$

has no solution in positive integers x, y, and z not divisible by p.

Show that this condition is satisfied by all primes p having the form

$$p = \frac{1}{2}\left(3^{2^k} + 1\right)$$

or

$$p = \frac{1}{2}\left(3^q - 1\right)$$

with q also an odd prime.

AMM E2771. **by Robert Breusch**

Let p be a prime and $p \not\equiv 1 \pmod 8$. Prove that the equation $x^{2p} + y^{2p} = z^{2p}$ has no solution in positive integers x, y, z with $xyz \not\equiv 0 \pmod p$.

Fermat's Little Theorem

CRUX 494.* **by Rufus Isaacs**

Let r_j, $j = 1, \dots, k$, be the roots of a polynomial with integral coefficients and leading coefficient 1.

(a) For p a prime, show that

$$p \mid \sum_j (r_j^p - r_j).$$

(b) Prove or disprove that for any positive integer n,

$$n \mid \sum_j \left(\sum_{d \mid n} r_j^d \mu(n/d) \right).$$

Fermat numbers

TYCMJ 121. **by Richard L. Francis**

Prove that no Fermat prime (one of the form $2^{2^n} + 1$) can be the difference of two fifth powers of positive integers.

OMG 15.3.6.

The numbers $\Phi_n = 2^{(2^n)} + 1$ for $n = 0, 1, 2, 3, \dots$ are called Fermat numbers. Approximately how large is the 6th Fermat number in scientific notation?

OSSMB 76-11.

You can imagine the tremendous size of $F_{73} = 2^{2^{73}} + 1$. Is there enough room in all the books in all the libraries in the whole world to record this giant number? In answering this question, assume the generous estimates that there are 1 million libraries, each with 1 million books, each of 1000 pages, each containing 100 lines which can hold 100 digits apiece.

Fibonacci and Lucas numbers: arrays

FQ H-257. **by V. E. Hoggatt, Jr.**

Consider this array in which F_{2n+1}, $n = 0, 1, 2, \dots$, is written in staggered columns:

1				
2	1			
5	2	1		
13	5	2	1	
34	13	5	2	1

Show that:

(a) The row sums are F_{2n+2}.

(b) The rising diagonal sums are $F_{n+1}F_{n+2}$.

(c) If the columns are multiplied by $1, 2, 3, \dots$ sequentially to the right, then the row sums are $F_{2n+3} - 1$.

FQ H-273. **by W. G. Brady**

Consider this array in which L_{2n+1}, $n = 0, 1, 2, \dots$, is written in staggered columns:

1				
4	1			
11	4	1		
29	11	4	1	
76	29	11	4	1

Show that:

(a) The row sums are $L_{2n+2} - 2$.

(b) The rising diagonal sums are $F_{2n+3} - 1$, where L_{2n+1} is the largest element in the sum.

(c) If the columns are multiplied by $1, 2, 3, \dots$ sequentially to the right, then the row sums are $L_{2n+3} - (2n+3)$.

Fibonacci and Lucas numbers: congruences

FQ B-365. **by Philip Mana**

Show that there is a unique integer $m > 1$ for which integers a and r exist with $L_n \equiv ar^n \pmod{m}$ for all integers $n \geq 0$. Also show that no such m exists for the Fibonacci numbers.

FQ B-386. **by Lawrence Somer**

Let p be a prime and let the least positive integer m with $F_m \equiv 0 \pmod{p}$ be an even integer $2k$. Prove that $F_{n+1}L_{n+k} \equiv F_n L_{n+k+1} \pmod{p}$. Generalize to other sequences.

FQ H-280. **by Paul S. Bruckman**

Prove the congruences:

$$F_{3\cdot 2^n} \equiv 2^{n+2} \pmod{2^{n+3}};$$

$$L_{3\cdot 2^n} \equiv 2 + 2^{2n+2} \pmod{2^{2n+4}},\ n = 1, 2, 3, \dots .$$

Fibonacci and Lucas numbers: determinants

FQ H-299. **by Gregory Wulczyn**

Evaluate:

$$\text{(a)}\quad \Delta = \begin{vmatrix} F_{2r} & F_{6r} & F_{10r} & F_{14r} & F_{18r} \\ F_{4r} & F_{12r} & F_{20r} & F_{28r} & F_{36r} \\ F_{6r} & F_{18r} & F_{30r} & F_{42r} & F_{54r} \\ F_{8r} & F_{28r} & F_{40r} & F_{56r} & F_{72r} \\ F_{10r} & F_{30r} & F_{50r} & F_{70r} & F_{90r} \end{vmatrix}$$

$$\text{(b)}\quad D = \begin{vmatrix} 1 & L_{2r+1} & L_{4r+2} & L_{6r+3} & L_{8r+4} \\ 1 & -L_{6r+3} & L_{12r+6} & L_{18r+9} & L_{24r+12} \\ 1 & L_{10r+5} & L_{20r+10} & L_{30r+15} & L_{40r+20} \\ 1 & -L_{14r+7} & L_{28r+14} & -L_{42r+21} & L_{56r+28} \\ 1 & L_{18r+9} & L_{36r+18} & L_{54r+27} & L_{72r+36} \end{vmatrix}$$

$$\text{(c)}\quad D_1 = \begin{vmatrix} 1 & L_{2r} & L_{4r} & L_{6r} & L_{8r} \\ 1 & L_{6r} & L_{12r} & L_{18r} & L_{24r} \\ 1 & L_{10r} & L_{20r} & L_{30r} & L_{40r} \\ 1 & L_{14r} & L_{28r} & L_{42r} & L_{56r} \\ 1 & L_{18r} & L_{36r} & L_{54r} & L_{72r} \end{vmatrix}$$

Fibonacci and Lucas numbers: divisibility

FQ B-329. **by Herta T. Freitag**

Find r, s, and t as linear functions of n such that $2F_r^2 - F_sF_t$ is an integral divisor of $L_{n+2} + L_n$ for $n = 1, 2, \dots$.

Fibonacci and Lucas numbers: finite sums

FQ H-284. **by G. Wulczyn**

Show that

$$\text{(a)}\ \sum_{k=0}^{n} \binom{n}{k} F_{rk} L_{rn-rk} = 2^n F_{rn},$$

$$\text{(b)}\ \sum_{k=0}^{n} \binom{n}{k} L_{rk} L_{rn-rk} = 2^n L_{rn} + 2L_r^n,$$

$$\text{(c)}\ \sum_{k=0}^{n} \binom{n}{k} F_{rk} F_{rn-rk} = \frac{(2^n L_{rn} - 2L6n_r)}{D}.$$

FQ B-300. **by Verner E. Hoggatt, Jr.**

Establish a simple, closed form for

$$L_{2n+2} - \sum_{k=1}^{n} (n+3-k) F_{2k}.$$

FQ B-335. **by Herta T. Freitag**

Obtain a closed form for

$$\sum_{i=0}^{n-k} (F_{i+k}L_i + F_iL_{i+k}).$$

FQ B-368. **by Herta T. Freitag**

Obtain functions $g(n)$ and $h(n)$ such that

$$\sum_{i=1}^{n} iF_iL_{n-1} = g(n)F_n + h(n)L_n$$

and use the results to obtain congruences modulo 5 and 10.

FQ H-246. **by L. Carlitz**

Let

$$F(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} F_{i+j}F_{m-i+j}F_{i+n-j}F_{m-i+n-j}$$

and

$$L(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n} L_{i+j}L_{m-i+j}L_{i+n-j}L_{m-i+n-j}.$$

Show that

$$L(m,n) - 25F(m,n) = 8L_{m+n}F_{m+1}F_{n+1}.$$

FQ B-305. **by Frank Higgins**

Prove that

$$F_{8n} = L_{2n} \sum_{k=1}^{n} L_{2n+4k-2}.$$

FQ B-306. **by Frank Higgins**

Prove that

$$F_{8n+1} - 1 = L_{2n} \sum_{k=1}^{n} L_{2n+4k-1}.$$

Fibonacci and Lucas numbers: golden ratio

FQ H-310. **by V. E. Hoggatt, Jr.**

Let $\alpha = (1+\sqrt{5})/2$, $\lfloor n\alpha \rfloor = a_n$, and $\lfloor n\alpha^2 \rfloor = b_n$. Clearly, $a_n + n = b_n$.

(a) Show that if $n = F_{2m+1}$, then $a_n = F_{2m+2}$ and $b_n = F_{2m+3}$.

(b) Show that if $n = F_{2m}$, then $a_n = F_{2m+1} - 1$ and $b_n = F_{2m+2} - 1$.

(c) Show that if $n = L_{2m}$, then $a_n = L_{2m+1}$ and $b_n = L_{2m+2}$.

(d) Show that if $n = L_{2m+1}$, then $a_n = L_{2m+2} - 1$ and $b_n = L_{2m+3} - 1$.

Fibonacci and Lucas numbers: identities

FQ B-298. **by Richard Blazej**

Show that

$$5F_{2n+3} \cdot F_{2n-3} = L_{4n} + 18.$$

FQ B-339. **by Gregory Wulczyn**

Establish Cesàro's symbolic Fibonacci-Lucas identity: $(2u+1)^n = u^{3n}$. After the binomial expansion has been performed, the powers of u are used as either Fibonacci or Lucas subscripts.

FQ H-288. **by G. Wulczyn**

Establish the identities:

$$F_k L_{k+6r+3}^2 - F_{k+8r+4} L_{k+2r+1}^2 = (-1)^{k+1} L_{2r+1}^3 F_{2r+1} L_{k+4r+2}$$

and

$$F_k L_{k+6r}^2 - F_{k+8r} L_{k+2r}^2 = (-1)^{k+1} L_{2r}^3 F_{2r} L_{k+4r}.$$

FQ H-295. **by Gregory Wulczyn**

Establish the identities:

$$F_k F_{k+6r+3}^2 - F_{k+8r+4}^2 F_{k+2r+1} = (-1)^{k+1} F_{2r+1}^3 L_{2r+1} L_{k+4r+2}$$

and

$$F_k F_{k+6r}^2 - F_{k+8r} F_{k+2r}^2 = (-1)^{k+1} F_{2r}^3 L_{2r} L_{k+4r}.$$

FQ B-354. **by Philip Mana**

Show that

$$F_{n+k}^3 - L_k^3 F_n^3 + (-1)^k F_{n-k}[F_{n-k}^2 + 3F_{n+k}F_n L_k] = 0.$$

FQ B-355. **by Gregory Wulczyn**

Show that

$$F_{n+k}^3 - L_{3k} F_n^3 + (-1)^k F_{n-k}^3 = 3(-1)^n F_n F_k F_{2k}.$$

FQ B-313. **by Verner E. Hoggatt, Jr.**

Let

$$M(x) = L_1 x + (L_2/2)x^2 + (L_3/3)x^3 + \cdots.$$

Show that the Maclaurin series expansion for $e^{M(x)}$ is

$$F_1 + F_2 x + F_3 x^2 + \cdots.$$

FQ H-279. **by G. Wulczyn**

Show that

$$F_{n+6r}^4 - (L_{4r}+1)(F_{n+4r}^4 - F_{n+2r}^4) - F_n^4 = F_{2r}F_{4r}F_{6r}F_{4n+12r},$$

and

$$F_{n+6r+3}^4 + (L_{4r+2}-1)(F_{n+4r+2}^4 - F_{n+2r+1}^4) - F_n^4 = F_{2r+1}F_{4r+2}F_{6r+3}F_{4n+12r+6}.$$

Fibonacci and Lucas numbers: infinite series

FQ B-319. **by Wray G. Brady**

Prove that

$$\frac{1}{L_2} + \frac{1}{L_6} + \frac{1}{L_{10}} + \cdots = \frac{1}{\sqrt{5}}\left(\frac{1}{F_2} - \frac{1}{F_6} + \frac{1}{F_{10}} - \cdots\right).$$

Fibonacci and Lucas numbers: primes

FQ H-260.* **by H. Edgar**

Are there infinitely many subscripts, n, for which F_n or L_n are prime?

Fibonacci and Lucas numbers: recurrences

FQ B-392. **by Phil Mana**

Let $Y_n = (2+3n)F_n + (4+5n)L_n$. Find constants h and k such that

$$Y_{n+2} - Y_{n+1} - Y_n = hF_n + kL_n.$$

Fibonacci numbers: algorithms

JRM 728. **by Frank Rubin**

Let F_i denote the ith Fibonacci number. It can be shown by induction that $F_{p+q} = F_{p-1}F_q + F_p F_{q+1}$. Note that when $p = 2$, this reduces to the well-known recursion formula for the Fibonacci sequence. Suppose that the cost of adding or subtracting two numbers is A, and the cost of multiplying them is M. Determine the lowest-cost method for calculating F_{100} if $A = 1$ and $M = 5$, and only $F_0 = 0$ and $F_1 = 1$ are assumed known.

Fibonacci numbers: ancestors

FQ B-304. **by Sidney Kravitz**

The female bee has two parents but the male bee has a mother only. Prove that if we go back n generations for a female bee, she will have F_n male ancestors in that generation and F_{n+1} female ancestors, making a total of F_{n+2} ancestors.

Fibonacci numbers: composite numbers

FQ B-302. **by Verner E. Hoggatt, Jr.**

Prove that $F_n - 1$ is a composite integer for $n \geq 7$ and that $F_n + 1$ is composite for $n \geq 4$.

Fibonacci numbers: congruences

NYSMTJ 98. **by Norman Gore**

Let F_n be the nth Fibonacci number. Prove that, for any positive integer n,

$$F_{n+10} \equiv F_n + F_{n+5} \pmod{10}.$$

FQ B-331. **by George Berzsenyi**

Prove that

$$F_{6n+1}^2 \equiv 1 \pmod{24}.$$

FQ B-378. **by George Berzsenyi**

Prove that

$$F_{3n+1} + 4^n F_{n+3} \equiv 0 \pmod{3}$$

for $n = 0, 1, 2, \ldots$.

FQ B-351. **by George Berzsenyi**

Prove that $F_4 = 3$ is the only Fibonacci number that is a prime congruent to 3 modulo 4.

FQ B-324. **by Herta T. Freitag**

Determine a constant k such that, for all positive integers n,

$$F_{3n+2} \equiv k^n F_{n-1} \pmod{5}.$$

FQ B-379. **by Herta T. Freitag**

Prove that $F_{2n} \equiv n(-1)^{n+1} \pmod 5$ for all nonnegative integers n.

FQ B-408.* **by Lawrence Somer**

Let $d \in \{2,3,\ldots\}$ and $G_n = F_{dn}/F_n$. Let p be an odd prime and $z = z(p)$ be the least positive integer n with $F_n \equiv 0 \pmod p$. For $d = 2$ and $z(p)$ an even integer $2k$, it is known that

$$F_{n+1}G_{n+k} \equiv F_n G_{n+k+1} \pmod p.$$

Establish a generalization for $d \geq 2$.

FQ H-265. **by V. E. Hoggatt, Jr.**

Show that

$$F_{2^3 \cdot 3^{k-1}} \equiv 0 \pmod{3^k}, \text{ where } k \geq 1.$$

FQ H-286. **by P. Bruckman**

Prove the following congruences:

(a) $F_{5^n} \equiv 5^n$.

(b) $F_{5^n} \equiv L_{5^{n+1}} \pmod{5^{2n+1}}$, $n = 0,1,2,\ldots$.

FQ H-250. **by L. Carlitz**

Show that if

$$A(n)F_{n+1} + B(n)F_n = C(n) \quad (n = 0,1,2,\ldots),$$

where the F_n are the Fibonacci numbers and $A(n)$, $B(n)$, $C(n)$ are polynomials, then

$$A(n) \equiv B(n) \equiv C(n) \equiv 0.$$

Fibonacci numbers: continued fractions

FQ H-278. **by V. E. Hoggatt, Jr.**

Show

$$\sqrt{\frac{5F_{n+2}}{F_n}} = \left[3, \underbrace{\overline{1,1,\ldots,1,6}}_{n-1}\right]$$

(Continued fraction notation, cyclic part under bar).

Fibonacci numbers: determinants

FQ H-294. **by Gregory Wulczyn**

Evaluate:

$$\Delta = \begin{vmatrix} F_{2r+1} & F_{6r+3} & F_{10r+5} & F_{14r+7} & F_{18r+9} \\ F_{4r+2} & F_{12r+6} & F_{20r+10} & F_{28r+14} & F_{36r+18} \\ F_{6r+3} & F_{18r+9} & F_{36r+15} & F_{42r+21} & F_{54r+27} \\ F_{8r+4} & -F_{24r+12} & F_{40r+20} & F_{56r+28} & F_{72r+36} \\ F_{10r+5} & F_{20r+15} & F_{50r+25} & F_{70r+36} & F_{50r+45} \end{vmatrix}$$

Fibonacci numbers: digit problems

CRUX 264. **by Gilbert W. Kessler**

Find a formula that gives the number of digits in the nth Fibonacci number explicitly in terms of n.

Fibonacci numbers: divisibility

NAvW 523. **by P. J. van Albada**

Let F_i denote the ith Fibonacci number. Prove the following:

(a) For every $k \in \mathbb{N}$, there is a k' such that $k \mid F_i$ if and only if $k' \mid i$.

(b) If k is a prime, $k \equiv \pm 1 \pmod{10}$, then $k' \mid (k-1)$.

(c) If k is a prime, $k \equiv \pm 3 \pmod{10}$, then $k' \mid (k+1)$.

AMM E2539.* **by A. Vince**

Let F_n denote the nth Fibonacci number. Prove or disprove that if $m^2 \mid F_n$, then $m \mid n$.

Fibonacci numbers: Euler totient

AMM E2581. **by Clark Kimberling**

Show that $\phi(F_n)$ is divisible by 4 if $n \geq 5$.

Fibonacci numbers: finite sums

FQ B-397. **by Gregory Wulczyn**

Find a closed form for the sum

$$\sum_{k=0}^{2s} \binom{2s}{k} F_{n+kt}^2.$$

NAvW 445. **by P. C. G. de Vries**

Let F_k be the kth Fibonacci number. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} F_k = \frac{1}{2^n\sqrt{5}} \left\{ \left(3+\sqrt{5}\right)^n + \left(3-\sqrt{5}\right)^n \right\}$$

for $n = 0,1,2,\ldots$.

FQ B-299. **by Verner E. Hoggatt, Jr.**

Establish a simple, closed form for

$$F_{2n+3} - \sum_{k=1}^{n} (n+2-k)F_{2k}.$$

FQ B-343. **by Verner E. Hoggatt, Jr.**

Establish a simple expression for

$$\sum_{k=1}^{n} \left[F_{2k-1}F_{2(n-k)+1} - F_{2k}F_{2(n-k+1)}\right].$$

FQ B-356. **by Herta T. Freitag**

Let

$$S_n = F_2 + 2F_4 + 3F_6 + \cdots + nF_{2n}.$$

Find m as a function of n so that F_{m+1} is an integral divisor of $F_m + S_n$.

FQ B-398. **by Herta T. Freitag**

Is there an integer K such that

$$K - F_{n+6} + \sum_{j=1}^{n} j^2 F_j$$

is an integral multiple of n for all positive integers n?

FQ B-320. **by George Berzsenyi**

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m}.$$

FQ B-321. **by George Berzsenyi**

Evaluate the sum:

$$\sum_{k=0}^{n} F_k F_{k+2m+1}.$$

Number Theory

FQ H-298. **by L. Kuipers**

Prove that

$$F_{n+1}^6 - 3F_{n+1}^5 F + 5F_{n+1}^3 F_n^3 - 3F_{n+1} F_n^5 - F_n^6$$

$$= (-1)^n;$$

$$F_{n+6}^6 - 14F_{n+5}^6 - 90F_{n+4}^6 + 350F_{n+3}^6 - 90F_{n+2}^6 - 14F_{n+1}^6$$

$$+F_n^6 = (-1)^n 80;$$

and

$$F_{n+6}^6 - 13F_{n+5}^6 + 41F_{n+4}^6 - 41F_{n+3}^6 + 13F_{n+2}^6 - F_{n+1}^6$$

$$\equiv -40 + \frac{1}{2}(1 + (-1)^n)80 \pmod{144}.$$

Fibonacci numbers: forms

FQ B-341. **by Peter A. Lindstrom**

Prove that the product $F_{2n}F_{2n+2}F_{2n+4}$ of three consecutive Fibonacci numbers with even subscripts is the product of three consecutive integers.

FQ B-396. **by Paul S. Bruckman**

Let $G_n = F_n(F_n+1)(F_n+2)(F_n+3)/24$. Prove that 60 is the smallest positive integer m such that $10 \mid G_n$ implies $10 \mid G_{n+m}$.

FQ B-409. **by Gregory Wulczyn**

Let $P_n = F_n F_{n+a}$. Must $P_{n+6r} - P_n$ be an integral multiple of $P_{n+4r} - P_{n+2r}$ for all nonnegative integers a and r?

FQ B-318. **by Herta T. Freitag**

Prove that $F_{4n}^2 + 8F_{2n}(F_{2n} + F_{6n})$ is a perfect square for $n = 1, 2, \ldots$.

Fibonacci numbers: generating functions

FQ B-381. **by V. E. Hoggatt, Jr.**

Let $a_{2n} = F_{n+1}^2$ and $a_{2n+1} = F_{n+1}F_{n+2}$. Find the rational function that has

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

as its Maclaurin series.

Fibonacci numbers: greatest common divisor

FQ B-330. **by George Berzsenyi**

Let

$$G_n = F_n + 29F_{n+4} + F_{n+8}.$$

Find the greatest common divisor of the infinite set of integers $\{G_0, G_1, G_2, \ldots\}$.

FQ H-259. **by R. Finkelstein**

Let p be an odd prime and m an odd integer such that $m \not\equiv 0 \pmod{p}$. Let $F_{mp} = F_p \cdot Q$. Can $\gcd(F_p, Q) > 1$?

Fibonacci numbers: identities

FQ H-290. **by Gregory Wulczyn**

Show that

$$F_k F_{k+6r+3}^2 - F_{k+4r+2}^3$$
$$= (-1)^{k+1} F_{2r+1}^2 (F_{k+8r+4} - 2F_{k+4r+2})$$

and

$$F_k F_{k+6r}^2 - F_{k+4r}^3 = (-1)^{k+1} F_{2r}^2 (F_{k+8r} + 2F_{k+4r}).$$

FQ H-266. **by G. Berzsenyi**

Find all identities of the form

$$\sum_{k=0}^{n} \binom{n}{k} F_{rk} = s^n F_{tn}$$

with positive integral r, s, and t.

FQ B-384. **by Gregory Wulczyn**

Establish the identity

$$F_{n+10}^4 = 55\left(F_{n+8}^4 - F_{n+2}^4\right) - 385\left(F_{n+6}^4 - F_{n+4}^4\right) + F_n^4.$$

FQ B-367. **by Gerald E. Bergum**

Let $\alpha = (1 + \sqrt{5})/2$ and suppose $n \geq 1$. Prove that

$$F_{2n} = \lfloor \alpha F_{2n-1} \rfloor$$

and

$$F_{2n+1} = \lfloor \alpha^2 F_{2n-1} \rfloor.$$

FQ B-323. **by J. A. H. Hunter**

Prove that

$$F_{n+r}^2 - (-1)^r F_n^2 = F_r F_{2n+r}.$$

Fibonacci numbers: inequalities

FQ B-395. **by V. E. Hoggatt, Jr.**

Let $\alpha = (\sqrt{5} - 1)/2$. For $n = 1, 2, 3, \ldots$, prove that

$$1/F_{n+2} < \alpha^n < 1/F_{n+1}.$$

Fibonacci numbers: infinite series

JRM 674. **by Friend H. Kierstead, Jr.**

Let $S = 1 + 1 + 1/2 + 1/3 + 1/5 + 1/8 + \cdots + 1/F_n + \cdots$, where F_n is the nth Fibonacci number.

Prove that the series converges and find the sum.

Fibonacci numbers: Pell's equation

FQ H-247. **by G. Wulczyn**

Show that for each Fibonacci number F_r, there exist an infinite number of positive nonsquare integers, D, such that

$$F_{r+s}^2 - F_r^2 D = 1.$$

Fibonacci numbers: population problems

FUNCT 1.1.9.

A pair of rabbits is put into an enclosure. They produce one pair of offspring in the first month and they reproduce just once more, producing a second pair of offspring in the second month.

Similarly, each pair of offspring follows exactly the same pattern of reproduction, beginning to reproduce one month after birth. There is no other breeding between other pairs of rabbits.

Show that the number of pairs produced in a certain month is equal to the numbers produced during the preceding two months.

Fibonacci numbers: primes

JRM 738. **by Frank Rubin**

(a) Characterize the Fibonacci numbers for which F_n is prime and n is composite.

(b) The first composite Fibonacci number for which n is prime is $F_{19} = 4181 = 37 \cdot 113$. Find the next.

SSM 3625. **by Bob Prielipp**

Prove that every positive integer greater than 3 that is both a prime number and a Fibonacci number can be expressed as the sum of two squares of distinct Fibonacci numbers.

Fibonacci numbers: recurrences

JRM 594. **by Henry Larson**

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence with $a_1 = 1, a_5 = 5$, and $a_{12} = 144$, subject to the rule that for every n, $a_n + a_{n+3} = 2a_{n+2}$.

Prove that it is the Fibonacci sequence.

FQ B-311. **by Jeffrey Shallit**

Let k be a constant and let (a_n) be defined by

$$a_n = a_{n-1} + a_{n-2} + k, \quad a_0 = 0,\ a_1 = 1.$$

Find

$$\lim_{n\to\infty} (a_n/F_n).$$

FQ B-352. **by V. E. Hoggatt, Jr.**

Let S_n be defined by $S_0 = 1$, $S_1 = 2$, and

$$S_{n+2} = 2S_{n+1} + cS_n.$$

For what value of c does $S_n = 2^n F_{n+1}$ for all nonnegative integers n?

Fibonacci numbers: systems of equations

FQ H-306. **by V. E. Hoggatt, Jr.**

(a) Prove that the system, S,

$$a + b = F_p,\ b + c = F_q,\ c + a = F_r,$$

cannot be solved in positive integers if F_p, F_q, and F_r are positive Fibonacci numbers.

(b) Likewise, show the same for this next system, T:

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,$$
$$d + e = F_s,\ e + a = F_t.$$

(c) Show that if F_p is replaced by any positive non-Fibonacci integer, then S and T have solutions.
If possible, find necessary and sufficient conditions for the following system U to be solvable in positive integers:

$$a + b = F_p,\ b + c = F_q,\ c + d = F_r,\ d + a = F_s.$$

Fibonacci numbers: triangular numbers

FQ B-346. **by Verner E. Hoggatt, Jr.**

Establish a closed form for

$$\sum_{k=1}^{n} F_{2k} T_{n-k} + T_n + 1,$$

where T_k is the triangular number $(k+2)(k+1)/2$.

Fibonacci numbers: trigonometric functions

FQ B-374. **by Frederick Stern**

Show that

$$F_n = \frac{2^{n+2}}{5}\Big[(\cos\frac{\pi}{5})^n \cdot \sin\frac{\pi}{5} \cdot \sin\frac{3\pi}{5} + (\cos\frac{3\pi}{5})^n \cdot \sin\frac{3\pi}{5} \cdot \sin\frac{9\pi}{5}\Big]$$

and

$$F_n = \frac{(-2)^{n+2}}{5}\Big[(\cos\frac{2\pi}{5})^n \cdot \sin\frac{2\pi}{5} \cdot \sin\frac{6\pi}{5} + (\cos\frac{4\pi}{5})^n \cdot \sin\frac{4\pi}{5} \cdot \sin\frac{12\pi}{5}\Big].$$

FQ B-375. **by V. E. Hoggatt, Jr.**

Express

$$\frac{2^{n+1}}{5}\sum_{k=1}^{4}\left[\left(\cos\frac{k\pi}{5}\right)^n \cdot \sin\frac{k\pi}{5} \cdot \sin\frac{3k\pi}{5}\right]$$

in terms of a Fibonacci number, F_n.

Finite products

SPECT 10.4. **by B. G. Eke**
PARAB 312.

Suppose the integers $a_1, a_2, \ldots, a_7$ are rearranged to give $b_1, b_2, \ldots, b_7$. Show that

$$(a_1 - b_1)(a_2 - b_2)\cdots(a_7 - b_7)$$

is even.

Floor function: exponentials

PENT 274. **by R. S. Luthar**

Show that $\lfloor (2+\sqrt{2})^n \rfloor$ is odd, where n is any positive integer.

Floor function: finite sums

CRUX 166. **by Steven Conrad**

Prove that for all real x and positive integers k

$$\sum_{i=0}^{k-1}\left\lfloor x + \frac{i}{k}\right\rfloor = \lfloor kx \rfloor.$$

PME 400. **by Richard A. Gibbs**

Evaluate $\sum_{k=1}^{m}\left(\left\lfloor \frac{kn}{m}\right\rfloor + \left\lceil \frac{kn}{m}\right\rceil\right)$, where m and n are positive integers.

CRUX 216. **by L. F. Meyers**

For which positive integers n is it true that

$$\sum_{k=1}^{(n-1)^2} \lfloor \sqrt[3]{kn} \rfloor = \frac{(n-1)(3n^2-7n+6)}{4}?$$

TYCMJ 69. **by V. N. Murty**

Let $N = 2 \cdot 10^k$ in which k is an arbitrary positive integer, and set

$$S = \left\lfloor \frac{N}{6} \right\rfloor + \left\lfloor \frac{N}{8} \right\rfloor - \left\lfloor \frac{N}{24} \right\rfloor.$$

Is S/N independent of k?

MM Q628. **by Alfred Brousseau**

Derive a formula for

$$\sum_{k=1}^{m} \lfloor kn/m \rfloor$$

in terms of m, n, and $d = \gcd(m, n)$.

NAvW 530. **by J. van de Lune**

For $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, let

$$S_n(\alpha) = \sum_{k=1}^{n} (-1)^{\lfloor k\alpha \rfloor}.$$

Prove that if α is irrational, then $S_n(\alpha) = 0$ for infinitely many $n \in \mathbb{N}$.

Floor function: identities

AMM E2752. **by Clark Kimberling**

Suppose a, b, c, and d are real numbers satisfying

$$\lfloor an \rfloor + \lfloor bn \rfloor = \lfloor cn \rfloor + \lfloor dn \rfloor$$

for $n = 1, 2, \ldots$.

Prove or disprove that $a - c = d - b$ is an integer.

FQ B-301. **by Phil Mana**

Let

$$A(n) = \frac{n^2 + 6n + 12}{12} \text{ and } B(n) = \frac{n^2 + 7n + 12}{6}.$$

Does $\lfloor A(n) \rfloor + \lfloor A(n+1) \rfloor = \lfloor B(n) \rfloor$ for all integers n?

Floor function: inequalities

OMG 14.1.3.

Prove that $\lfloor 5x \rfloor + \lfloor 5y \rfloor > \lfloor 3x+y \rfloor + \lfloor 3y+x \rfloor$ for $x, y > 0$.

USA 1975/1.

(a) Prove that

$$\lfloor 5x \rfloor + \lfloor 5y \rfloor \geq \lfloor 3x + y \rfloor + \lfloor 3y + x \rfloor,$$

where $x, y \geq 0$.

(b) Using (a) or otherwise, prove that

$$\frac{(5m)!(5n)!}{m!n!(3m+n)!(3n+m)!}$$

is integral for all positive integral m and n.

CRUX 150. **by Kenneth S. Williams**

Find a function $f(k)$ such that

$$\left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor \geq f(k)$$

with $f(k)$ between $\frac{3^k - 2^k}{2^k}$ and $\frac{3^k - 2^k + 2}{2^k - 1}$.

Floor function: integrals

MM 994. **by Peter Ørno**

For n and m positive integers, evaluate

$$\int_0^1 (-1)^{\lfloor nt \rfloor} (-1)^{\lfloor mt \rfloor} \, dt.$$

Floor function: iterated functions

AMM E2604. **by E. T. H. Wang**

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and let $A \colon \mathbb{N} \to \mathbb{N}$ be defined by $A(n) = \lfloor 2n/3 \rfloor$. For $n \in \mathbb{N}_0$, let $k \in \mathbb{N}_0$ be the smallest integer such that $A^k(n) = 0$, and define $f(n) = k$. Find a formula, as simple as possible, for the function f.

PME 396. **by David R. Simonds**

Let $[m]_n$ denote $\lfloor m/n \rfloor$. Prove that

$$[m]_{n^k} = [m]_n^k$$

for all $m, n, k \in \mathbb{N}$, where $[m]_n^k$ means $[[\cdots[m]_n \cdots]_n]_n$ (k sets of brackets).

Floor function: maxima and minima

FQ H-296.* **by C. Kimberling**

Suppose x and y are positive real numbers. Find the least positive integer n for which

$$\left\lfloor \frac{x}{n+y} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor.$$

Floor function: primes

AMM 6212.* **by A. A. Mullin**

Prove that $\lfloor \pi^n \rfloor$ is prime for only finitely many positive integers n.

AMM E2766. **by I. Borosh and D. Hensley**

Let r be a positive rational number but not an integer. Prove that there are infinitely many positive integers n such that $\lfloor nr \rfloor$ is prime.

FQ B-358. **by Philip Mana**

Prove that $\lfloor n^2/3 \rfloor$ is prime for only finitely many n.

Floor function: sequences

AMM E2777. **by I. Borosh, H. Diamond, M. Gbur, and D. Hensley**

Let b/a be a reduced fraction greater than 1. Let $r = r(a, b)$ denote the number of integers relatively prime to b in the sequence

$$\left\lfloor \frac{b}{a} \right\rfloor, \left\lfloor \frac{2b}{a} \right\rfloor, \ldots, \left\lfloor \frac{(a-1)b}{a} \right\rfloor.$$

State and prove a rule for determining r as a function of a and b.

PUTNAM 1979/A.5.

Denote by $S(x)$ the sequence $\lfloor x\rfloor, \lfloor 2x\rfloor, \lfloor 3x\rfloor, \ldots$. Prove that there are distinct real solutions α and β of the equation

$$x^3 - 10x^2 + 29x - 25 = 0$$

such that infinitely many positive integers appear both in $S(\alpha)$ and in $S(\beta)$.

AMM E2726. **by Roy Streit**

Define $F(a, b)$ to be the sequence $(c_0, c_1, c_2, \ldots)$, where $c_n = \lfloor an + b\rfloor$. Which $(a, b) \in \mathbb{R}^2$ have the property that $F(x, y) = F(a, b)$ implies $(x, y) = (a, b)$?

Floor function: solution of equations

MSJ 479.

Solve the equation

$$\frac{19x + 16}{10} = \left\lfloor \frac{4x + 7}{3} \right\rfloor .$$

SSM 3687. **by Herta T. Freitag**

With n being a positive integer, solve the equation $\lfloor\sqrt{n}\rfloor = \lfloor n/2\rfloor$.

SSM 3696. **by Douglas E. Scott**

Let k be a positive integer. For what positive integer values of n does $\lfloor\sqrt{n}\rfloor = \lfloor n/k\rfloor$?

Forms of numbers: decimal representations

PENT 310. **by Kenneth M. Wilke**

Consider the sequence

$$10001,\ 100010001,\ 1000100010001,\ \ldots$$

Are there any primes in this sequence?

Forms of numbers: difference of consecutive cubes

FQ H-291. **by George Berzsenyi**

Prove that there are infinitely many squares that are differences of consecutive cubes.

Forms of numbers: difference of powers

AMM E2797. **by Barry J. Powell**

Determine whether or not there are infinitely many primes p such that for any pair of coprime odd positive integers x and y with no two of p, x, y, congruent modulo p, the multiplicity of p in the prime factorization of $x^{p-1} - y^{p-1}$ is odd.

Forms of numbers: difference of squares

TYCMJ 37. **by Louis Alpert**

Prove that the product of any four consecutive integer members of an arithmetic progression may be expressed as the difference of two integer squares.

SSM 3773. **by Fred A. Miller**

Prove:

(a) The cube of any integer may be expressed as the difference of two squares of integers;

(b) The cube of any odd integer different from 1 and -1 can be expressed in two ways as the difference of two squares of integers;

(c) The difference of the cubes of any two consecutive integers can be expressed as the difference of two squares of integers.

Forms of numbers: perfect numbers

SSM 3588. **by Bob Prielipp**

Show that every perfect number can be expressed uniquely as the sum of two or more consecutive positive integers (in increasing order).

Forms of numbers: powers of 2

TYCMJ 97. **by Richard L. Francis**

Does there exist a positive integer n for which $2^{n+1} - 1$ and $2^{n-1}(2^n - 1)$ are both cubes?

Forms of numbers: prime divisors

AMM E2725. **by Solomon W. Golomb**

Given positive integers a and b, show that there exists a positive integer c such that infinitely many numbers of the form $an + b$ (n a positive integer) have all their prime factors less than or equal to c.

Forms of numbers: product of consecutive integers

MSJ 497.

Prove that for no integer n can $49n + 5$ be the product of consecutive integers.

SPECT 8.6. **by B. G. Eke**

Show that the product of four consecutive positive integers cannot be a perfect cube.

CRUX 83. **by Léo Sauvé**

Show that the product of two, three, or four consecutive positive integers is never a perfect square.

FUNCT 3.1.2.

The product of four consecutive integers is a square. Find the integers. Do the same for the case of four consecutive odd integers.

SSM 3611. **by Robert A. Carman**
ISMJ 13.12.

Show that the product of four consecutive integers increased by 1 is a perfect square.

Forms of numbers: squares

AMM E2606. **by R. S. Luthar**
SSM 3646. **by Robert A. Carman**

Show that there are infinitely many integers n such that $2n + 1$ and $3n + 1$ are both perfect squares, and that such n's must be multiples of 40.

TYCMJ 50. **by Aron Pinker**
Let k and r be nonnegative integers with $r \leq 5$. Prove that, for some value of $m \in \{25, 225, 625\}$,

$$500(k+1)(5k+2r)+m$$

is a square.

MM Q643. **by Erwin Just**
Let $m < n$ be positive integers exactly one of which is even. Prove that the only integral value of x for which $(x^{2n}-1)/(x^{2m}-1)$ is a perfect square is zero.

Forms of numbers: sum of consecutive cubes

SSM 3782. **by Charles W. Trigg**
Show that the sum of the cubes of any k consecutive, positive integers is equal to the difference of two integer squares. Describe the squares.

PARAB 298.
Find all sets of three consecutive natural numbers such that the sum of their cubes is divisible by 18.

Forms of numbers: sum of consecutive integers

SSM 3658. **by E. D. Bender**
The number 75 is the sum of consecutive, positive integers in five ways: $75 = 37+38 = 24+25+26 = 13+14+15+16+17 = 10+11+12+13+14+15 = 3+4+5+6+7+8+9+10+11+12$. Prove that there are infinitely many positive integers each of which is a sum of consecutive, positive integers in at least three ways.

PARAB 276.
Find all sets of consecutive positive integers whose sum is 1000.

MATYC 102. **by Raymond Maruca**
Prove that if S_1 and S_2 are each sums of n consecutive positive integers, then S_1 and S_2 are not relatively prime, where $n > 2$.

CANADA 1976/5.
MATYC 90. **by Dan Aulicino**
Prove that a positive integer is a sum of at least two consecutive positive integers if and only if it is not a power of two.

SSM 3615. **by Herta T. Freitag**
If $S(m,n)$ represents the sum of m successive positive integers starting with n, how many primes are there in this sequence?

ISMJ J10.7.
Find five consecutive numbers whose sum is a perfect square less than 100. Can you find four consecutive numbers whose sum is a perfect square?

Forms of numbers: sum of consecutive odd integers

CRUX 112. **by H. G. Dworschak**
OSSMB G75.3-4.
PARAB 405.
Let $k > 1$ and n be positive integers. Show that there exist n consecutive odd integers whose sum is n^k.

Forms of numbers: sum of consecutive squares

MSJ 423. **by Nathaniel Dean**
Suppose that x is the largest integer in a set of $n+1$ consecutive positive integers, the sum of the squares of which equals the sum of the squares of the next n consecutive positive integers. Express the value of x in terms of n.

Forms of numbers: sum of cubes

AMM 6232.* **by Allan Wm. Johnson, Jr.**
Prove or disprove: Given any integer $G > 13$, there exist distinct integers $x_i > 0$ such that

$$G^3 = \sum_{i=1}^{5} x_i^3.$$

MM Q649. **by Norman Schaumberger**
Show that every rational number r may be written as the sum of four or fewer rational cubes.

Forms of numbers: sum of divisors

MM 964. **by P. Erdős**
Show that every positive integer $k, k < n!$, is a sum of fewer than n distinct divisors of $n!$.

Forms of numbers: sum of factorials

MSJ 478.
Prove that no positive integer can be expressed in two distinct ways as the sum of two factorials, $n! + m!$, where $n, m \geq 1$.

Forms of numbers: sum of squared reciprocals

JRM 586. **by Friend H. Kierstead, Jr.**
It is known that every rational number in $\left[0, \pi^2/6 - 1\right]$ can be represented as a finite sum of reciprocals of distinct squares. Find such a representation for $\frac{1}{2}$,

(a) with the least number of terms;

(b) with the smallest n, where n^2 is the largest denominator.

Forms of numbers: sum of squares

OSSMB G79.2-4.
Show that three times the sum of three squares can be expressed as the sum of four squares.

MM Q634. **by M. S. Klamkin**
CRUX PS1-1.
If a, b, c, and d are positive integers where $ab = cd$, show that $a^2+b^2+c^2+d^2$ is always composite.

MM Q641. **by Erwin Just**
If n and k are integers with $n > 2$ and $k \geq 1$, show that n^k can be expressed as the sum of the squares of exactly n positive integers.

FQ B-328. **by Walter Hansell**
Show that $6(1^2+2^2+3^2+\cdots+n^2)$ is always a sum

$$m^2+(m^2+1)+(m^2+2)+\cdots+(m^2+r)$$

of consecutive integers, of which the first is a perfect square.

Number Theory

Forms of numbers: sum of two squares

ISMJ 13.15.

Let S be the set of numbers of the form m^2+n^2 where m and n are integers. Let $s, t \in S$.

(a) Show that $st \in S$.

(b) If $t \neq 0$, show that s/t is of the form x^2+y^2 where x and y are rational numbers.

JRM 590. **by Frank Rubin**

Some numbers can be expressed as the sum of the squares of two (not necessarily distinct) positive integers in several ways.

Let $A(n)$ be the smallest number expressible as a sum of squares in greater than or equal to n ways. Thus $A(1) = 2$; $A(2) = 50$; $A(3) = 325$. Extend this list through at least $A(10)$.

MSJ 491.

Let a and b be distinct positive integers. Prove that if $(a^2 + b^2)/2$ is also an integer, then it may be expressed as the sum of the squares of two integers.

MM 1042. **by Henry Klostergaard**

Prove that any integer that is the sum of the squares of two different, nonzero integers is divisible by a prime that is the sum of the squares of two different, nonzero integers.

Forms of numbers: unit fractions

ISMJ 11.20.

Show that the reciprocal of every integer greater than 1 is the sum of a finite number of terms of the sequence

$$\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \ldots, \frac{1}{j\cdot(j+1)}, \ldots .$$

TYCMJ 73. **by Allan Wm. Johnson, Jr.**

Let N be an arbitrary positive integer. Is it always possible to express 1 as a finite sum of reciprocals of distinct positive integers, each of which is a multiple of N?

USA 1978/3.

An integer n will be called good if we can write

$$n = a_1 + a_2 + \cdots + a_k,$$

where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

Given the information that the integers 33 through 73 are good, prove that every integer ≥ 33 is good.

JRM 477. **by E. J. Barbeau**

It can be shown that any set of distinct odd positive integers whose reciprocals add up to one must contain at least nine members. If no restriction is made on the number of members of such a set, find the smallest value of n such that n is the largest denominator.

ISMJ J10.17.

The ancient Egyptians represented $\frac{23}{25}$ as $\frac{1}{2}+\frac{1}{3}+\frac{1}{15}+\frac{1}{50}$. Is this the shortest representation of $23/25$ in the form

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

where $a_1, a_2, \ldots, a_n$ is a strictly increasing sequence of positive integers? Is it unique?

CRUX 346. **by Leroy F. Meyers**

It has been conjectured that every rational number of the form $4/n$ where n is an integer greater than 1 can be expressed as the sum of three or fewer unit fractions not necessarily distinct. As a partial verification of the conjecture show that at least $23/24$ of such numbers have the required expansions.

Fractional parts

AMM 6024. **by L. Kuipers**

If α is rational and different from 0, and β is irrational, then show that the sequence $(\lfloor n\alpha \rfloor n\beta)$, $n = 1, 2, \ldots$, is uniformly distributed mod 1.

JRM 681. **by Benjamin L. Schwartz**

Let $\langle x \rangle$ denote the fractional part of x, that is, $\langle x \rangle = x - \lfloor x \rfloor$.

(a) For $1 \leq n \leq 1000000$, find the minimum and maximum nonzero values of $\langle \sqrt{n}\, \rangle$.

(b) For $1 \leq n < m \leq 1000000$, find the minimum nonzero value of $\langle \sqrt{m}\, \rangle - \langle \sqrt{n}\, \rangle$.

JRM C2. **by David L. Silverman**

Find the smallest integer $N > 30{,}739$, the fractional parts of whose square root and cube root differ by a positive number less than 0.0000151.

CRUX 360. **by Hippolyte Charles**

Let $\langle x \rangle = x - \lfloor x \rfloor$. Show directly that the set

$$\{\langle \sqrt{n} \rangle \mid n = 1, 2, 3, \ldots\}$$

is dense in the unit interval $(0, 1)$.

CRUX 269. **by Kenneth M. Wilke**

Let $\langle \sqrt{10} \rangle$ denote the fractional part of $\sqrt{10}$. Prove that for any positive integer n there exists an integer I_n such that

$$\langle \sqrt{10} \rangle^n = \sqrt{I_n + 1} - \sqrt{I_n}.$$

Fractions

PARAB 393.
ISMJ J10.16.
ISMJ J11.9.

Show that, if n is any integer greater than 2, of the fractions

$$\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n},$$

an even number are in lowest terms.

Functional equations

AMM S3. **by Albert A. Mullin**
FQ H-287. **by A. Mullin**

Prove that any strictly positive real-valued arithmetical function f satisfying the functional equation

$$\frac{f(n+1)}{n+1}+n=\frac{(n+1)f(n)}{f(n+1)}$$

for every integer n exceeding some prescribed positive integer m is necessarily asymptotic to $\pi(n)$, the number of prime numbers not exceeding n.

Gaussian integers

NAvW 457. **by G. J. Rieger**

Suppose that $a+bi$ and $c+di$ are Gaussian integers. Give a proof showing that $\gcd(a+bi, c+di)=1$ if and only if $\gcd\left(a^2+b^2, ad-bc, c^2+d^2\right)=1$.

AMM 6053. **by Raphael Finkelstein**

Let $a+bi$ be a Gaussian integer with $\gcd(a,b)=1$, and let $A+Bi=(a+bi)^p$, where p is an odd prime. Let $C=\max(A,B)$ and $D=\min(A,B)$. Can C/D approach $(1+\sqrt{5})/2$ arbitrarily closely?

Generating functions

FQ B-407. **by Robert M. Giuli**

Given that

$$\frac{1}{1-x-xy}=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_{n,k}x^ny^k$$

is a double ordinary generating function for $a_{n,k}$, determine $a_{n,k}$.

OSSMB G77.1-6.

In the series

$$-1+x+4x^2+\cdots+a_ix^i+\cdots$$

every coefficient a_i is obtained from the three preceding coefficients as follows:

$a_0=-1$, $a_1=1$, $a_2=4$,

$a_{i+3}=3a_{i+2}-3a_{i+1}+a_i$, $i=0,1,2,\ldots$.

(a) Prove that the series represents a rational function with denominator $(1-x)^3$.

(b) By expanding the function, obtain an explicit function for a_i.

Geometry: cubes

JRM 528. **by David Y. Hsu**

(a) While it is possible to place the integers 0 through 7 at the vertices of a cube in such a way as to make those on each face total the same value, a similar task with 0 through 3 on a tetrahedron is impossible. Can such a labeling be made on the three other regular polyhedra?

(b) While it is possible, as in part (a), to place the consecutive integers 0 through 7 at the vertices of a cube in such a way as to have those on each face total the same value, it is not possible to perform a similar task with 0 through 3 on a square. Can such an equal-facial-sum assignment be made on cubes in higher dimensional space?

PME 402. **by Charles W. Trigg**

The first eight nonzero digits are distributed on the vertices of a cube. Addition of the digits at the extremities of each edge forms twelve edge-sums. Find distributions such that every edge-sum is the same as the sum on the opposite (non-cofacial) edge.

Geometry: cyclic quadrilaterals

AMM E2660. **by E. Ehrhart**

Find the number of congruence classes of cyclic quadrilaterals having integral sides and given perimeter n.

AMM E2557. **by R. D. Nelson**

Find all cyclic quadrilaterals with integral sides, each of which has its perimeter numerically equal to its area.

Geometry: lattice points

AMM E2570. **by J. G. Sunday**

Let (m_1,n_1), $(m_2,n_2),\ldots,(m_k,n_k)$ be distinct lattice points with $n_i\geq 2m_i>0$ for each i, and suppose that no two of them lie on any line through the origin. Show that $\operatorname{lcm}[n_1,n_2,\ldots,n_k]\geq 2k$. When can equality occur?

PME 456. **by P. Erdős**

Is there an infinite path on visible lattice points avoiding all (u,v) where u, v are primes?

Geometry: quadrilaterals

OMG 14.2.3.

One is given a quadrilateral with two consecutive right angles. What are the lengths of the sides and diagonals if all are integral length?

Geometry: rectangles

CRUX 435. **by J. A. H. Hunter**

In rectangle $ABDF$, point C is on BD, point E is on DF, $AC=125$, $CD=112$, $DE=52$, and AB, AD, and AF are also integral. Find EF.

PME 455. **by Kenneth M. Wilke**

The perimeter of a 6×4 rectangle equals the area of a 2×10 rectangle while the area of the 6×4 rectangle equals the perimeter of the 2×10 rectangle also. Show that there are an infinite number of pairs of rectangles related in the same way and find all pairs of such rectangles whose sides are integers.

Geometry: rectangular parallelepipeds

SSM 3719. **by Robert A. Carman**

Show that if, in a rectangular solid, the lengths of all face diagonals and the lengths of all edges are positive integers, then the length of at least one edge is divisible by 11. (Under the given hypothesis, several similar conclusions can be established. State and prove as many of these as you can find.)

TYCMJ 86. **by Kay Dundas**

The volume of an open-top box has been maximized by turning up the sides of an $m\times n$ rectangle after $x\times x$ squares have been cut from each corner. Assume that m and n are integers with $\gcd(m,n)=1$. Prove that there are an infinite number of values of m and n for which x is rational.

Geometry: right triangles

PARAB 261.

In a right triangle, the shortest side has length a, the longest side has length c, and the other side has length b. If a, b, c are all integers, when does $a^2 = b + c$?

FUNCT 2.2.4.

Let n be an integer greater than 2. Prove that the nth power of the length of the hypotenuse of a right triangle is greater than the sum of the nth powers of the lengths of the other two sides.

Geometry: semicircles

PME 398. **by Richard S. Field**

A quadrilateral with consecutive sides A, B, C, $2R$ is inscribed in a semicircle of radius R with one side lying along the diameter. Find solutions in integers $A = B \neq C \neq R$ and $A \neq B \neq C = R$ for the sides of the quadrilateral. Also, find solutions in integers $A \neq B \neq C \neq R$, or prove that none exist.

Greatest common divisor

TYCMJ 34. **by Bob Jewett**

Let A, B, C, and D be integers for which

$$\gcd(A, B, C, D) = 1.$$

Prove or disprove that, for each integer n,

$$\gcd(An + B, Cn + D) = 1$$

if and only if each prime divisor of $AD - BC$ is a divisor of both A and C.

SPECT 7.6. **by B. G. Eke**

Show that, among any ten consecutive positive integers, at least one is relatively prime to all the others.

SPECT 10.6. **by L. Mirsky**

For any positive integers a, b, m, n with $\gcd(a, b) = 1$, show that

$$\gcd(a^m - b^m, a^n - b^n) = a^{\gcd(m,n)} - b^{\gcd(m,n)}.$$

CRUX 243. **by Hippolyte Charles**

(a) Find necessary and sufficient conditions for the greatest common divisor of two positive integers a and b, $a > b$, to equal their difference.

(b) Find all pairs of positive integers whose greatest common divisor equals their difference and whose least common multiple is 180.

FQ B-412. **by Phil Mana**

Find the greatest common divisor of the integers in the infinite set

$$\{2^9 - 2, 3^9 - 3, 4^9 - 4, \ldots, n^9 - n, \ldots\}.$$

AMM E2560. **by Richard Madsen**

Let $n_1, \ldots, n_k$ be natural numbers. Define $d_1 = 1$ and

$$d_i = \frac{\gcd(n_1, \ldots, n_{i-1})}{\gcd(n_1, \ldots, n_i)}$$

for $i \geq 2$. Show that the $d_1 \cdots d_k$ possible sums

$$\sum_{i=1}^{k} a_i n_i, \qquad a_i \in \{1, 2, \ldots, d_i\},$$

are all distinct modulo n_1.

Harmonic series

OSSMB 75-12.

Let S denote the sum of the terms remaining in the harmonic series upon the deletion of the terms which contain an even digit:

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{19} + \frac{1}{31} + \cdots + \frac{1}{39} + \frac{1}{51} + \cdots.$$

Prove that $S < 7$.

JRM 503. **by Les Marvin**

From the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots,$$

every term in which the denominator is divisible by a prime of two or more digits is deleted. Either sum the series that remains, or prove that it diverges.

PUTNAM 1975/B.6.

Show that if

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

then

(a) $n(n+1)^{1/n} < n + H_n$ for $n > 1$, and
(b) $(n-1)n^{-1/(n-1)} < n - H_n$ for $n > 2$.

SPECT 8.5. **by Ian D. Macdonald**

Let r, s be positive integers with $r > s$. Prove that

$$\sum_{k=0}^{2s} \frac{1}{r - s + k} > \frac{2s+1}{r},$$

and deduce that, if n is an integer greater than 1 and $m = \frac{3^n - 1}{2}$, then

$$1 + \frac{1}{2} + \cdots + \frac{1}{m} > n.$$

IMO 1978/5.

Let $\{a_k\}$ $(k = 1, 2, 3, \ldots, n, \ldots)$ be a sequence of distinct positive integers. Prove that for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

PARAB 434.

Show that if N is taken sufficiently large, the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N}$$

is larger than 100.

TYCMJ 44. **by Benjamin Burrell**

Does $\sum_{k=1}^{\infty} \frac{1}{k(1+1/2+\cdots+1/k)}$ converge?

JRM 512. **by Robert Walsh**

Let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} .$$

For each $k = 1, 2, 3, \ldots$, let $H_n(k)$ be the smallest partial sum that exceeds k and call $E(k) = H_n(k) - k$ the kth excess. The intuitive guess that $E(k)$ is monotonically decreasing is quickly negated, since $E(1) = 0.5$, $E(2) = 0.0833$, $E(3) = 0.0199$, but $E(4) = 0.0272$.

(a) Find the next counterexample $[E(n) < E(n+1)]$.

(b) Does the series of excesses converge?

Inequalities: binomial coefficients

AMM 6019. **by R. E. Shafer**

Prove for all positive integers n that

$$\frac{2^{2n}}{\sqrt{\pi}(n^2+n/2+1/8)^{1/4}} < \binom{2n}{n} < \frac{2^{2n}}{\sqrt{(n+1/4)\pi}}.$$

Inequalities: congruences

AMM 6200. **by Brian Conrey, David Leep, and Gerry Myerson**

Define $\left(\frac{a}{b}\right)_r$ to be the least positive integer x such that $bx \equiv a \pmod r$. Let k, m, n be positive integers with $\gcd(m,n) = 1$, $k < n$, $m < n$. Show that

(a) $m < \left(\frac{k}{m}\right)_n + \left(\frac{k}{n}\right)_m \le n$;

(b) $\left(\frac{1}{m}\right)_n + \left(\frac{1}{n}\right)_m = \frac{m+n}{2}$ if and only if $n - m = 2$.

Thus, for m and n prime, (b) characterizes twin primes.

Inequalities: exponentials

PARAB 269.

(a) Show that for any positive integer n

$$2 < \left(1 + \frac{1}{n}\right)^n < 3.$$

(b) Which is larger, 1000^{1000} or 1001^{999}?

AMM 6239. **by F. David Hammer**

Is the following conjecture true? Let $p(x,y)$ be any polynomial in x and y; then $|x^y - y^x| \le |p(x,y)|$ has only finitely many solutions (x,y) in unequal integers not less than 2.

Inequalities: fractional parts

AMM 6199. **by Hugh L. Montgomery**

Suppose $q \ge 1$ and $\gcd(a,q) = 1$. Set

$$\mathcal{L} = \left\{ n \,|\, 1 \le n \le q, \{an/q\} \le (n/q)^2 \right\},$$

where $\{\theta\} = \theta - \lfloor\theta\rfloor$ is the fractional part of θ. Show that

$$\sum_{n\in\mathcal{L}} n^{-2} \le 9/q.$$

Inequalities: logarithms

MM 986. **by P. Erdős**

Show that there exists a constant c such that $a + b < n + c\ln n$, for all positive integers a, b, and n for which $n!/(a!b!)$ is an integer.

NAvW 463. **by P. Erdős**

Let $q_r(n)$ be the maximum number of integers less than or equal to n such that the product of at most r of them is never a square.

(a) Determine $q_1(n)$ and $q_2(n)$;

(b) Prove that there exists a constant c such that $q_3(n) > cn$ (Conjecture: $q_3(n) \sim c'n$ for some constant c');

(c) Prove that there are constants c_1 and c_2 such that, for $r \ge 4$,

$$c_1 \frac{n}{\log n} < q_r(n) < c_2 \frac{n}{\log n} .$$

Inequalities: powers

PARAB 433.

Which is larger, 100^{300} or $300!$?

Inequalities: powers of 2

CRUX 23. **by Léo Sauvé**

Determine if there exists a sequence (u_n) of natural numbers such that

$$2^{u_n} < 2n + 1 < 2^{1+u_n}$$

for all positive integers n.

Inequalities: products

PARAB 360.

Suppose $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ are integers such that $a_1 \ge b_1 \ge 1$, $a_2 \ge b_2 \ge 1$, and so on. Let

$$a = a_1 + a_2 + \cdots + a_k,$$

and

$$b = b_1 + b_2 + \cdots + b_k.$$

(a) Prove that the product

$$[b_1(a_1-b_1)+1]\,[b_2(a_2-b_2)+1] \times\cdots\times [b_k(a_k-b_k)+1]$$

is greater than or equal to $a - b + 1$.

(b) Can you determine exactly under what conditions equality occurs?

Inequalities: radicals

CRUX 84. **by Viktors Linis**

Prove that for any positive integer n

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}}.$$

Inequalities: simultaneous inequalities

MSJ 421.

Find the smallest positive integers x, y, z, and w that satisfy the following simultaneous inequalities:

$$2x < x+y < x+z < 2y < x+w < y+z < 2z < y+w < z+w < 2w.$$

Inequalities: sum and product

ISMJ J10.3.

For what integer values of x, y, and z is it true that $x \le x+y+z \le y \le xyz \le z$?

Inequalities: sum of squared differences

IMO 1975/1.

Let x_i and y_i $(i = 1, 2, \ldots, n)$ be real numbers such that

$$x_1 \ge x_2 \ge \cdots \ge x_n \quad \text{and} \quad \mathrm{y}_1 \ge \mathrm{y}_2 \ge \cdots \ge \mathrm{y}_\mathrm{n}.$$

Let $(z_1, z_2, \ldots, z_n)$ be any permutation of $(y_1, y_2, \ldots, y_n)$. Prove that

$$\sum_{i=1}^{n}(x_i - y_i)^2 \le \sum_{i=1}^{n}(x_i - z_i)^2.$$

Infinite products

AMM 6012. **by Daniel Shanks**

Prove the following infinite products over certain sets of primes p:

$$\prod_{p=14k\pm1}\left(1+\frac{3p^2}{(p^2-1)^2}\right) = \frac{840}{817},$$

$$\prod_{p=18k\pm1}\left(1+\frac{3p^2}{(p^2-1)^2}\right) = \frac{40}{39}.$$

AMM 6240. **by Mihai Eşanu**

Let $a_n \ne 0$, $\lim_{n\to\infty} a_n = 0$. Prove that for every real number x, there exist sequences (λ_n), (μ_n) of integers such that

$$x = \sum_{n=1}^{\infty} \lambda_n a_n = \prod_{n=1}^{\infty} \mu_n a_n.$$

Irrational numbers

SSM 3627. **by Charles E. Blanchard**

Let $x = 0.1001000010000001000000001\ldots$.

(a) Express x as a sigma sum.

(b) Does $x^n - q = 0$ have a solution with q a rational number and n a positive integer?

FQ B-404. **by Phil Mana**

Let x be a positive irrational number. Let a, b, c, and d be positive integers with $a/b < x < c/d$. If $a/b < r < x$, with r rational, implies that the denominator of r exceeds b, we call a/b a good lower approximation for x. If $x < r < c/d$, with r rational, implies that the denominator of r exceeds d, c/d is a good upper approximation for x. Find all the good lower and upper approximations for $(1+\sqrt{5})/2$.

FQ B-405. **by Phil Mana**

Prove that for every positive irrational x, the good lower approximations and good upper approximations for x can be put together to form one sequence $\{p_n/q_n\}$ with

$$p_{n+1}q_n - p_n q_{n+1} = \pm 1$$

for all n. (For definitions see FQ B-404 above.)

Least common multiple

AMM E2686. **by Peter L. Montgomery**

Show that

$$(n+1) \operatorname*{lcm}_{0\le k\le n}\left[\binom{n}{k}\right] = \operatorname{lcm}[1, 2, \ldots, n+1].$$

NAvW 505. **by P. Erdős**

Let $M(n,k)$ be the least common multiple of the integers $n+1, n+2, \ldots, n+k$. Prove that, for fixed $k \ge 5$, the equation

$$M(n,k) = M(n+1,k)$$

has a solution $n > k$. Prove that there exists a number c_r such that, for fixed $k > c_r$, the equation

$$M(n,k) = M(n+r,k)$$

has a solution $n > k$. Show that both equations have a finite number of solutions. (Conjecture: $M(n,k) \ne M(m,k)$ for $m \ge n+k$.)

NAvW 417. **by P. Erdős**

Let L_n be the least common multiple of the integers not exceeding n. Let $f(n)$ be the smallest integer k for which

$$L_n = a_1 a_2 \cdots a_k,$$

with

$$a_1 \le a_2 \le \cdots \le a_k \le n.$$

Prove that

$$f(n) = \pi(n) - \left(\sqrt{2} - \frac{1}{2}\right)\pi\left(\sqrt{n}\right) + o\left(\pi\left(\sqrt{n}\right)\right),$$
$$n \to \infty.$$

AMM S21.* **by P. Erdős**

Let

$$A(n,k) = (n+1)(n+2)\cdots(n+k),$$

$$B(n,k) = \operatorname{lcm}[n+1, n+2, \ldots, n+k],$$

and

$$\alpha(n,k) = \frac{A(n,k)}{B(n,k)}.$$

(a) How many distinct values can $\alpha(n,k)$ take for fixed k?

(b) Do m, n, and k exist with $m > n+k-1$ and $\alpha(m,k) = \alpha(n,k)$?

CRUX 205. **by Steven R. Conrad**

Find the least common multiple of the numbers

$$(29!)(37!) \quad \text{and} \quad (23!)(41!).$$

ISMJ 11.4.

Prove that if the least common multiple of two numbers is equal to the square of their difference, then their highest common factor is the product of two consecutive integers.

CANADA 1979/3.

Let a, b, c, d, e be integers such that $1 \le a < b < c < d < e$. Prove that

$$\frac{1}{\operatorname{lcm}[a,b]} + \frac{1}{\operatorname{lcm}[b,c]} + \frac{1}{\operatorname{lcm}[c,d]} + \frac{1}{\operatorname{lcm}[d,e]} \le \frac{15}{16}.$$

Number Theory

Legendre symbol

CMB P271. **by Kenneth S. Williams**

Let $p > 3$ be an odd prime. Determine $N(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, the number of integers x $(1 \le x \le p-3)$ such that

$$\left(\frac{x}{p}\right) = \varepsilon_1, \quad \left(\frac{x+1}{p}\right) = \varepsilon_2, \quad \left(\frac{x+2}{p}\right) = \varepsilon_3$$

where (x/p) denotes Legendre's symbol and $\varepsilon_i = \pm 1$.

CRUX 449. **by Kenneth S. Williams**

Let p be a prime $\equiv 3 \pmod 8$ and let each of the numbers α, β, and γ have one of the values ± 1. Prove that the number $N_p(\alpha, \beta, \gamma)$ of consecutive triples x, $x+1$, $x+2$ $(x = 1, 2, \ldots, p-3)$ with

$$\left(\frac{x}{p}\right) = \alpha, \ \left(\frac{x+1}{p}\right) = \beta, \ \left(\frac{x+2}{p}\right) = \gamma$$

where $\left(\frac{x}{p}\right)$ is the Legendre symbol, is the same no matter what values are assigned to α, β, and γ.

AMM E2760. **by Kenneth S. Williams**

Let p be a prime. If $p \equiv 1 \pmod 4$ let a be the unique integer such that

$$p = a^2 + b^2, \qquad a \equiv -1 \pmod 4, \quad b \text{ even}.$$

Prove that

$$\sum_{i=0}^{p-1} \left(\frac{i^3 + 6i^2 + i}{p}\right) = \begin{cases} 2\left(\frac{2}{p}\right) a, & \text{if } p \equiv 1 \pmod 4, \\ 0, & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol.

Limits

NAvW 493. **by P. Erdős**

Let $f(y)$ denote the maximal value of $\sum_{a \in S} a^{-1}$, where S denotes a set of relatively prime integers in the interval $\left(y, y^2\right)$. Prove that

$$\lim_{y \to \infty} f(y) = \log 2.$$

AMM E2807. **by Solomon W. Golomb**

Let a and r be fixed positive constants with $r > 1$. For each positive integer k, there is a smallest positive integer $n = n(k)$ that satisfies $(n+a)^k \le rn^k$. Show that $\lim n(k)/k$ as $k \to \infty$ exists and evaluate this limit.

CRUX 382. **by Kenneth S. Williams**

Let a, b, c, and d be positive integers. Evaluate

$$\lim_{n \to \infty} \frac{a(a+b)(a+2b)\cdots(a+(n-1)b)}{c(c+d)(c+2d)\cdots(c+(n-1)d)}.$$

Lucas numbers: binomial coefficients

FQ B-327. **by George Berzsenyi**

Find all integral values of r and s for which the equality

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i L_{ri} = s^n L_n$$

holds for all positive integers n.

FQ B-414. **by Herta T. Freitag**

Let

$$S_n = L_{n+5} + \binom{n}{2} L_{n+2} - \sum_{i=2}^{n} \binom{i}{2} L_i - 11.$$

Determine all $n \in \{2, 3, 4, \ldots\}$ for which S_n is (a) prime; (b) odd.

Lucas numbers: congruences

FQ B-403. **by Gregory Wulczyn**

Let $m = 5^n$. Show that $L_{2m} \equiv -2 \pmod{5m^2}$.

FQ H-262. **by L. Carlitz**

Show that

$$L_{p^2} \equiv 1 \pmod{p^2} \text{ if and only if } L_p \equiv 1 \pmod{p^2}.$$

FQ H-263. **by G. Berzsenyi**

Prove that $L_{2mn}^2 \equiv 4 \pmod{L_m^2}$ for every $n, m = 1, 2, 3, \ldots$.

FQ B-314. **by Herta T. Freitag**

Show that $L_{2p^k} \equiv 3 \pmod{10}$ for all primes $p \ge 5$.

FQ B-366. **by Wray G. Brady**

Prove that $L_i L_j \equiv L_h L_k \pmod 5$ when $i + j = h + k$.

Lucas numbers: cubes

FQ B-342. **by Gregory Wulczyn**

Prove that

$$2L_{n-1}^3 + L_n^3 + 6L_{n+1}^2 L_{n-1}$$

is a perfect cube for $n = 1, 2, \ldots$.

Lucas numbers: digit problems

FQ B-382. **by A. G. Shannon**

Prove that L_n has the same last digit (i.e., units digit) for all n in the infinite geometric progression $4, 8, 16, 32, \ldots$.

Lucas numbers: divisibility

FQ B-317. **by Herta T. Freitag**

Prove that L_{2n-1} is an exact divisor of $L_{4n-1} - 1$ for $n = 1, 2, \ldots$.

Lucas numbers: sequences

FQ B-406. **by Wray G. Brady**

Let $x_n = 4L_{3n} - L_n^3$. Find the greatest common divisor of the terms of the sequence x_1, x_2, x_3, ...

Lucas numbers: sets

FQ H-304.* **by V. E. Hoggatt, Jr.**

(a) Show that there is a unique partition of the positive integers $\mathbb{N}$ into two sets, A_1 and A_2, such that

$$A_1 \cup A_2 = \mathbb{N}, \quad A_1 \cap A_2 = \varnothing,$$

and no two distinct elements from the same set add up to a Lucas number.

(b) Show that every positive integer, M, that is not a Lucas number is the sum of two distinct elements of the same set.

FQ B-369. **by George Berzsenyi**

For all integers $n \geq 0$, prove that the set

$$S_n = \{L_{2n+1}, L_{2n+3}, L_{2n+5}\}$$

has the property that if $x, y \in S_n$ and $x \neq y$, then $xy + 5$ is a perfect square. For $n = 0$, verify that there is no integer z that is not in S_n and for which $\{z,\ L_{2n+1}, L_{2n+3}, L_{2n+5}\}$ has this property.

Matrices

MM 1063. **by D. A. Moran**

Let M be an $n \times n$ matrix of integers whose inverse is also a matrix of integers. Prove that the number of odd entries in M is at least n and at most $n^2 - n + 1$, and that these are the best possible bounds.

AMM 6210. **by Olga Taussky**

Let A be an integral square matrix that is congruent to the unit matrix I modulo an odd prime number. Then A either is equal to I or is of infinite order. Give a proof based on the eigenvalues of A.

Maxima and minima

NAvW 528. **by P. Erdős and J. H. van Lint**

For fixed k and $1 \leq i \leq k$, let $R(n, i, k)$ denote the number of integers $m \in (n, n+k]$ such that $\gcd(m, n+i) = 1$. For $n \geq 0$, we define

$$f_k(n) = \min\left\{R(n, i, k) \;\middle|\; 1 \leq i \leq k\right\}.$$

(a) Determine

$$\liminf_{n\to\infty} f_k(n).$$

(b) Show that there are constants c_1, c_2 such that, for all k,

$$\frac{c_1 k}{\log\log k} < \max\left\{f_k(n) \;\middle|\; n \geq 0\right\} < \frac{c_2 k}{\log\log k}\ .$$

CRUX 25. **by Viktors Linis**

Find the smallest positive value of $36^k - 5^l$ where k and l are positive integers.

CRUX PS8-3.

Let n be a given natural number. Find nonnegative integers k and l so that their sum differs from n by a natural number and so that the following expression is as large as possible:

$$\frac{k}{k+l} + \frac{n-k}{n-(k+l)}.$$

IMO 1976/4.

Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

JRM 711. **by Friend H. Kierstead, Jr.**

(a) How should the number 36 be partitioned into integer summands so that the product of the summands is as large as possible? What is the maximum product?

(b) How should 36 be so partitioned into reals?

(c) Generalize to other real numbers.

NAvW 552. **by P. Erdős**

Let $1 \leq a_1 < a_2 < \cdots < a_k < n$ and for $1 \leq i < j \leq k$ let $\gcd\left(a_i, a_j\right) \neq 1$, $a_i \nmid a_j$. Determine the maximal value of k.

Means

CRUX 77. **by H. G. Dworschak**

Let A_n, G_n, and H_n denote the arithmetic, geometric, and harmonic means of the n positive integers $n+1, n+2, \ldots, n+n$. Evaluate

$$\lim_{n\to\infty} \frac{A_n}{n},\ \lim_{n\to\infty} \frac{G_n}{n},\ \lim_{n\to\infty} \frac{H_n}{n}.$$

SSM 3759. **by Alan Wayne**

Find conditions under which the harmonic mean of two distinct, positive integers is an integer.

SSM 3613. **by Alan Wayne**

Given two different positive integers, prove that the arithmetic mean of their harmonic mean and their geometric mean is less than their arithmetic mean.

Mersenne numbers

SSM 3770. **by Richard L. Francis**

Show that between any two Mersenne primes, there is a prime number.

Möbius function

AMM 6108. **by Aleksander Ivić**

Find all multiplicative functions $f(n)$ such that

$$f(n^2) = \sum_{d\,|\,n} \mu^2(d) f\left(\frac{n}{d}\right)$$

and

$$f^2(n) = \sum_{d\,|\,n} f(d^2).$$

AMM 6235. **by Robert J. Anderson and M. Ram Murty**

Let $M(x) = \sum_{n\leq x} \mu(n)$, where μ is the Möbius function. It has been conjectured and supported with numerical evidence that $\sum_{n\geq x} M(n) = O(x \log x)$. Settle this conjecture.

AMM 6035. **by Arthur Marshall**

For every natural number k, let N_k be the kth number in natural order of the sequence consisting solely of primes and the (square-free) products of (two or more) successive primes. Let μ be the Möbius function. Does the series

$$\sum_{k=1}^{\infty} \frac{\mu(N_k)}{N_k}\ \ln N_k$$

diverge (positively or negatively), converge, or oscillate?

Modular arithmetic: complete residue systems

AMM E2781. **by James Propp**

Let S be a set of n integers and $m = n(n+1)/2$. When $n \geq 3$, can $S + S$ constitute a complete residue set modulo m?

Modular arithmetic: coprime integers

MM Q653. **by L. Kuipers**

Show that if $\gcd(a,b)=1$, then the set of integers

$$\{ai \mid 0 \le i \le r-1\} \cup \{bj \mid 1 \le j \le s\},$$

where $r+s=a+b$, forms a complete set of residues mod $(a+b)$.

AMM S9. **by M. S. Klamkin and A. Liu**

(a) Determine all positive integers n such that

$$\gcd(x,n)=1$$

implies that $x^2 \equiv 1 \pmod{n}$.

(b) Determine all positive integers n such that

$$xy+1 \equiv 0$$

$\pmod{n}$ implies that $x+y \equiv 0 \pmod{n}$.

Modular arithmetic: fields

CMB P274. **by Kenneth S. Williams**

Let p be a prime congruent to 7 modulo 8, so that there are odd positive integers u and v such that $p=u^2-2v^2$. Let $T+U\sqrt{p}$ be the fundamental unit of the real quadratic field $Q(\sqrt{p})$. Prove that

$$T \equiv 0 \pmod{16} \iff u \equiv \pm 1 \pmod{8}.$$

Modular arithmetic: groups

AMM E2753. **by Haim Rose**

Let p be a prime and $g=\{r_1,r_2,\ldots,r_k\}$ be any group under multiplication modulo p, where the r_i are integers with $0<r_i<p$. Let P be the product of all the r_i and Q be the product of those r_i satisfying $0<r_i<p/2$. Prove:

(a) $P \equiv -(-1)^k \pmod{p}$.

(b) If $k=2h$, with h an odd integer, then $Q \equiv \pm 1 \pmod{p}$.

(c) If $1 \le r_i \le (p-1)/2$ for $1 \le i \le k$, then $P \equiv 1 \pmod{p}$. Can this situation actually occur?

(d) If $k=2h$, $h \ge 2$, then p^2 is an integral divisor of the numerator of the sum

$$\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_k}.$$

Modular arithmetic: permutations

MM 948. **by Bob Prielipp and N. J. Kuenzi**

Let Z_n be the ring of integers modulo n. For what values of n different from 2 do there exist permutations f and g on Z_n such that the pointwise product fg is also a permutation on Z_n?

Modular arithmetic: powers

CRUX 76. **by H. G. Dworschak**

What is the remainder when 23^{23} is divided by 53?

AMM E2673. **by Haim Rose**

Let $p=6n+1$ be a prime number, n a positive integer. An n-residue $\pmod{p}$ is an integer a such that $0<a<p$ and $a \equiv b^n \pmod{p}$ for some integer b. Prove that the product of all n-residues $\pmod{p}$ that are less than $p/2$ is congruent to $-1 \pmod{p}$.

AMM E2798. **by Doug Hensley**

Prove that there are infinitely many pairs (p,q) of primes such that $(q-1)/p$ is an integer k and 2 is a kth power modulo q.

Modular arithmetic: quadratic congruences

MM 1044. **by J. Metzger**

Let p be a prime and k a positive integer. The congruence relation $(x-a)(x-b) \equiv 0 \pmod{p^k}$ has the obvious solutions $x \equiv a \pmod{p^k}$ and $x \equiv b \pmod{p^k}$. When are these the only solutions?

AMM E2704. **by S. Collins, S. M. Reddy, and N. J. A. Sloane**

Find the number of solutions of $x^2=x$ in the ring of integers modulo n.

Modular arithmetic: reciprocals

OMG 15.3.7.

What is the reciprocal of 3 modulo 5?

Modular arithmetic: solution of equations

AMM E2773. **by Michael W. Ecker**

What is the number of solutions in $\mathbb{Z}_n$ of $x^3=x$?

Modular arithmetic: squares

CMB P254. **by D. Ž. Djoković**

Let p be a prime, $p \equiv 1 \pmod{16}$. Let a be an integer such that $2a^2 \equiv 1 \pmod{p}$; it is well known that such integers exist. Prove that $1+a$ is a square mod p.

Modular arithmetic: sum of squares

AMM 6148. **by Charles Small**

Let $s(n)$ denote the smallest r such that -1 is a sum of r squares $\pmod{n}$. Show that $s(n)$ equals:

$$\begin{cases} 1, & \text{if } 4 \nmid n \text{ and } p \nmid n \text{ for all primes } p \equiv 3 \pmod{4}, \\ 2, & \text{if } 4 \nmid n \text{ and } p \mid n \text{ for some prime } p \equiv 3 \pmod{4}, \\ 3, & \text{if } 4 \mid n \text{ but } 8 \nmid n, \\ 4, & \text{if } 8 \mid n. \end{cases}$$

Modular arithmetic: systems of congruences

ISMJ J11.18.

If the remainder when 100 is divided by d is 4 and the remainder when 90 is divided by d is 18, what is d?

Multinomial coefficients

FQ B-307. **by Verner E. Hoggatt, Jr.**

Let

$$(1+x+x^2)^n = a_{n,0}+a_{n,1}x+a_{n,2}x^2+\cdots,$$

(where, of course, $a_{n,k}=0$ for $k>2n$). Also let

$$A_n=\sum_{j=0}^{\infty} a_{n,4j}, \quad B_n=\sum_{j=0}^{\infty} a_{n,4j+1},$$

$$C_n=\sum_{j=0}^{\infty} a_{n,4j+2}, \quad D_n=\sum_{j=0}^{\infty} a_{n,4j+3}.$$

Find the relationship of A_n, B_n, C_n, and D_n to each other.

Multiplication tables

OMG 15.3.9.

Given the following multiplication table, what is the value of 7/5?

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Normal numbers

AMM 6219. **by M. J. Pelling**

Construct an uncountable class of real numbers not normal in the scales of 3 and 5.

Number of divisors

MM 983. **by Bernardo Recamán**

Are there arbitrarily long sequences of consecutive integers no two of which have the same number of prime divisors?

CMB P264. **by P. Erdős**

Determine the limit points of $d((n+1)!)/d(n!)$, where $d(m)$ is the number of divisors of m.

CMB P267. **by P. Erdős**

If t_n is an integer and $t_n > n^{1/2}$ show that

$$\lim_{n\to\infty} \frac{d((n+t_n)!)}{d(n!)} = \infty$$

where $d(m)$ denotes the number of divisors of m.

AMM E2780. **by Jim Totten**

Let $d(n)$ be the number of (positive integral) divisors of the natural number n and define $S(n)$ as $\sum d(k)$, with the sum taken over all divisors k of n. Determine the values of n for which $n = S(n)$.

NAvW 499. **by J. van de Lune**

For any positive integer n, let $\tau(n)$ denote the number of divisors of n. Since $\tau(n) \le n$, we have that, for every $n \in \mathbb{N}$, the sequence of τ-iterates of n

$$n, \tau(n), \tau(\tau(n)), \ldots$$

becomes eventually constant. Let $D(n)$ be the number of different integers in this sequence. Prove that

$$D(n) = O\left(\frac{\log n}{\log\log n}\right), \qquad n \to \infty.$$

NAvW 483. **by P. Erdős and A. Sárkőzi**

Let $1 \le a_1 < a_2 < \cdots$ be an infinite sequence of integers. Denote by $d_A(n)$ the number of divisors of n among the elements of the sequence. It is easy to see that

$$\sum_{n\le x} d_A(n) = x \sum_{a_i \le x} \frac{1}{a_i} + O(x),$$

i.e., the average value of $d_A(n)$ for $n \le x$ is

$$\sum_{a_i\le x} \frac{1}{a_i} + O(1).$$

Prove that if $\gcd(a_i, a_j) = 1$ for all pairs (i, j), $i \ne j$, then

$$\lim_{x\to\infty} \max_{1\le n\le x} d_A(n) \Big/ \sum_{a_i\le x} \frac{1}{a_i} = \infty.$$

Number representations: Fibonacci numbers

FQ B-416.* **by Gene Jakubowski and V. E. Hoggatt, Jr.**

Prove that every positive integer m has at least one representation of the form

$$m = \sum_{j=-N}^{N} \alpha_j F_j,$$

with each α_j in $\{0, 1\}$ and $\alpha_j = 0$ when j is an integral multiple of 3.

PARAB 438.

Prove that every positive integer can be written as the sum of distinct Fibonacci numbers.

Number representations: fractions

SSM 3636. **by Robert A. Carman**

Express $\frac{883}{285444}$ as the sum of two fractions whose denominators are 881 and 324.

Number representations: Lucas numbers

NYSMTJ 90. **by H. O. Eberhart**

Let $L_1 = 1$, $L_2 = 2$, and $L_n = L_{n-1} + L_{n-2}$ for $n > 2$. Show that every positive integer can be expressed as a sum of distinct L_i.

Number representations: perfect numbers

MM 954. **by Richard L. Francis**

Show that any even perfect number greater than 28 can be represented as the sum of at least two perfect numbers.

Number representations: polygonal numbers

SSM 3784. **by William J. O'Donnell**

Prove that every pentagonal number greater than one can be written as the sum of three triangular numbers, two of which are equal. Triangular numbers, T_n, are positive integers of the form $n(n+1)/2$ and pentagonal numbers, P_n, are positive integers of the form $n(3n-1)/2$.

Number representations: ratios

SSM 3574. **by Charles W. Trigg**
In a paperback book of Mathematical Puzzles and Their Solutions, the following problem appears: "The product 1/3 of 60 is represented in a certain number system by the symbol 15. How would 1/9 of 48 be represented in that number system?" The given solution in its entirety is: "Set up the proportion $60/3 : 15 = 48/9 : x$ where x is the solution. We are dealing with a system that represents the quantity 20, which is 1/3 of 60, by the symbol 15, and therefore every other quantity by a number 3/4 as large as the number normally used. Note that only 15 and x are in the new system. The solution is 4." Do you agree?

Number representations: sets

SSM 3723. **by Henry Lulli**
Given is a set of five distinct digits. Using each digit exactly once and one or more of the three operators, addition, subtraction, and juxtaposition, represent as many of the numbers from 0 to 20 as you can. Find as many five digit sets as you can for which it is possible to represent each of the numbers from 0 to 20.

Number representations: standard symbols

MSJ 454. **by Steven R. Conrad**
Use four 4's and standard mathematical symbols to represent the numbers 73 and 89.

Number representations: unit fractions

PME 371. **by I. P. Scalisi**
Write $2/n$ as the sum of 4 (or 6 or 10 or 14) distinct unit fractions.

Palindromes

CRUX 389. **by Kenneth M. Wilke**
Prove that all the numbers in the sequence

$$100001, 10000100001, 1000010000100001, \dots$$

are composite.

CRUX 31. **by Léo Sauvé**
A driver cruising on the highway observed that the odometer of his car showed 15,951 miles. He noticed that this number is palindromic: it reads the same backward and forward.

"Curious," the driver said to himself. "It will be a long time before that happens again." But exactly two hours later the odometer showed a new palindromic number. What was the average speed of the car in those two hours?

MATYC 79. **by Marvin Johnson**
Prove that a palindrome with an even number of digits is divisible by 11.

CRUX 490.* **by Michael W. Ecker**
Are there infinitely many palindromic primes?

MSJ 425. **by John Murphy**
An "odd-odd number" is a positive integer all of whose digits are odd. Find all positive integers from 1 to 10,000 that are prime palindromes but are not odd-odd.

SSM 3662. **by R. W. Crittenden**
First-class postage is now 13 cents per ounce. Using postage stamps of various denominations from 1 cent to 9 cents, there are a variety of arrangements you may use to decorate your envelopes. Two of these arrangements are noteworthy: 3 cents + 1 cent + 5 cents + 1 cent + 3 cents and 3 cents + 7 cents + 3 cents since both arrangements depict palindromic prime numbers. List all other palindromic prime representations of 13 cents worth of postage stamps. Use any denominations from 1 cent to 9 cents in any combinations.

CRUX 439. **by Ram Rekha Tiwari**
The palindromic number 252 has the property that it becomes a perfect square when multiplied (or divided) by 7. Are there any other such even palindromic numbers?

MATYC 94. **by R. W. Crittenden**
SSM 3651. **by R. W. Crittenden**
The number $698,896$ is the square of 836. Is this the only square palindrome containing an even number of digits?

Pandigital numbers

JRM 571. **by Sidney Kravitz**
The smallest pandigital number x such that $2x$, $4x$, and $8x$ are also pandigital is 0123456789 and the largest is 1234567890. What are the next-smallest and next-largest such numbers?

Partitions

PUTNAM 1979/A.1.
Find the positive integers n and $a_1, a_2, \dots, a_n$ such that

$$a_1 + a_2 + \cdots + a_n = 1979$$

and the product $a_1 a_2 \cdots a_n$ is as large as possible.

CRUX 6. **by Léo Sauvé**
(a) If n is a given nonnegative integer, how many distinct nonnegative integer solutions are there for each of the following equations?

$$x + y = n, \quad x + y + z = n, \quad x + y + z + t = n.$$

(b) Use (a) to conjecture and then prove a formula for the number of distinct nonnegative integer solutions of the equation

$$x_1 + x_2 + \cdots + x_r = n.$$

AMM 6137. **by I. J. Good**
Let $p(n)$ denote the number of partitions of n, $n = 1, 2, \dots$, and let k denote an integer greater than 3. Prove that $\Delta^k p(n)$, $n = 1, 2, \dots$, is a sequence of alternating terms.

PENT 272. **by Charles Trigg**
Let $P_k(n)$ be the number of partitions of n into k unordered parts.

Show that $[P_2(2n)]\,[P_2(2n+1)]$ is a perfect square.

CRUX 13. **by Léo Sauvé**
Prove the following: For every sum of p positive integers (not necessarily distinct) each less than or equal to q, there exist q positive integers (not necessarily distinct) each less than or equal to p, with the same sum.

FQ B-376. **by Frank Kocher and Gary L. Mullen**

Find all integers $n > 3$ such that $n-p$ is an odd prime for all odd primes p less than n.

CRUX 52. **by Viktors Linis**

The sum of one hundred positive integers, each less than 100, is 200. Show that one can select a partial sum equal to 100.

Pascal's triangle

PME 451. **by Solomon W. Golomb**

Find all instances of three consecutive terms in a row of Pascal's triangle in the ratio 1 : 2 : 3.

NAvW 432. **by H. W. Labbers, Jr.**

Let V be the set of odd natural numbers n such that a regular n-gon can be constructed with ruler and compass. The number 1 is to be included in V. Place the first 32 elements of V, each written in binary notation, into a column with the order increasing from top to bottom. Prove that this forms the first 32 rows of Pascal's triangle, reduced modulo 2.

AMM E2775. **by Ko-Wei Lih**

If we replace even integers by 0 and odd integers by 1 in the ordinary Pascal triangle, we get the following modulo 2 Pascal triangle:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&0&&1&& \\ &1&&1&&1&&1& \\ 1&&0&&0&&0&&1 \\ .&.&.&.&.&.&.&.&. \end{array}$$

Will 1101 or 1011 occur as a consecutive segment in any row of this modulo 2 Pascal triangle?

Pell numbers

FQ H-275. **by Verner E. Hoggatt, Jr.**

Let P_n denote the Pell sequence defined by $P_1 = 1$, $P_2 = 2$, and

$$P_{n+2} = 2P_{n+1} + P_n \quad (n \geq 1).$$

Consider the array below.

$$\begin{array}{ccccccccccccc} 1&&2&&5&&12&&29&&70&\cdots&(P_n) \\ &1&&3&&7&&17&&41&\cdots&& \\ &&2&&4&&10&&24&\cdots&&& \\ &&&2&&6&&14&\cdots&&&& \\ &&&&4&&8&\cdots&&&&& \\ &&&&&4&\cdots&&&&&& \end{array}$$

Each row is obtained by taking differences in the row above. Let D_n denote the left diagonal sequence in this array; i.e., $D_1 = D_2 = 1$, $D_3 = D_4 = 2$, $D_5 = D_6 = 4$, $D_7 = D_8 = 8, \ldots$.

(a) Show that $D_{2n-1} = D_{2n} = 2^{n-1} \quad (n \geq 1)$.

(b) Show that if $F(x)$ represents the generating function for $\{P_n\}_{n=1}^{\infty}$ and $D(x)$ represents the generating function for $\{D_n\}_{n=1}^{\infty}$, then

$$D(x) = \frac{1}{1+x} F\left(\frac{x}{1+x}\right).$$

Perfect numbers

SSM 3583. **by Richard L. Francis**

If x and y are even perfect numbers, show that $x+y$ cannot be perfect.

JRM 791. **by Friend H. Kierstead, Jr.**

Prove that every even perfect number except one is the sum of the cubes of the first 2^n odd integers, for some positive integer n.

NYSMTJ 84. **by Norman Gore**

Show that every even perfect number is a sum of consecutive integers beginning with unity.

Permutations: derangements

OSSMB 76-5.

The number of "derangements" of n objects is given by the formula

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

and is often denoted $!n$ and called sub-factorial n. Prove that

$$!n \equiv n! \pmod{n-1}.$$

NYSMTJ 49. **by Bruce King**

A derangement is a permutation of the set

$$S = \{1, 2, 3, \ldots, n\}$$

in which no number occupies its "natural" position. Let D_n represent the number of derangements of S. Evaluate

$$\sum_{k=0}^{n} \binom{n}{k} D_k.$$

Use $D_0 = 1$.

FUNCT 3.4.3. **by Andrew Mattingley**

Let D_n denote the number of ways of putting n letters into n addressed envelopes so that every letter goes into a wrong envelope. Derive a formula from which D_n may be calculated.

AMM 6234. **by Edward T. H. Wang**

Let D_n and M_n denote the derangement number and the ménage number, respectively. Prove or disprove that the sequence $\{M_n/D_n\}$, $n = 4, 5, 6, \ldots$, is monotonically increasing and

$$\lim_{n\to\infty} (M_n/D_n) = 1/e.$$

Permutations: fixed points

MM 979. **by Mike Chamberlain and John Hawkins**

Define $P(m, n)$ to be the number of permutations of the first n natural numbers for which m is the first number whose position is left unchanged. Clearly $P(1, n) = (n-1)!$ for all n. Show that for $m = 1, 2, \ldots, n-1$,

$$P(m+1, n) = P(m, n) - P(m, n-1).$$

Number Theory

Permutations: inequalities

SSM 3749. **by F. David Hammer**

Let $a_1, a_2, \ldots, a_n$ be any nonnegative real numbers, and let $b_1, b_2, \ldots, b_n$ be any permutation of these numbers. Show that for some integer i, $a_i(1 - b_i) \leq 1/4$.

Permutations: modular arithmetic

FQ H-309.* **by David Singmaster**

Let f be a permutation of $\{1, 2, \ldots, m-1\}$ such that the terms $i + f(i)$ are all distinct (mod m). Characterize and/or enumerate such f.

MM 1002. **by Bernardo Recamán and John Hoyt**

(a) For which values of n is it possible to find a permutation $[a_1, a_2, \ldots, a_n]$ of $[0, 1, \ldots, n-1]$ so that the partial sums

$$\sum_{i=1}^{k} a_i, \quad k = 1, 2, \ldots, n,$$

when reduced modulo n, are also a permutation of the integers $[0, 1, \ldots, n-1]$?

(b) Find the number of permutations of $[0, 1, \ldots, n-1]$ for $n \leq 12$ which solve part (a). Can a general formula for the number of solutions be found?

Permutations: order

JRM 734. **by Frank Rubin**

Let $L(n)$ be the largest possible order for a permutation of n objects.

(a) Find $L(100)$.

(b) What is the smallest value of n for which $L(n)$ is a multiple of 100?

(c) What is the largest value of n for which $L(n)$ is not a multiple of 100?

Permutations: powers

JRM 702. **by Harry L. Nelson**

A *power chain* is a sequence that is a permutation of the first n natural numbers, with $n > 1$, such that the sum of each pair of adjacent elements is a power. Thus $6-2-7-1-3-5-4$ and $8-1-7-2-6-3-5-4$ are power chains for $n = 7$ and 8, respectively.

(a) Find two other power chains.

(b) Do there exist only four power chains?

Polygonal numbers: consecutive integers

PME 359. **by Gregory Wulczyn**

Show that there is an infinitude of pairs of consecutive integers, each pair consisting of a pentagonal number and a hexagonal number.

Polygonal numbers: formulas

SSM 3571. **by Herta Freitag**

Note that triangular numbers T_n defined by

$$T_n = n(n+1)/2$$

are such that differences between successive T_i's start with 2 and increase by 1 each successive pair. That is, $T_2 - T_1 = 2$, $T_3 - T_2 = 3$, $T_4 - T_3 = 4$, etc. For square numbers S_n, $S_n = n^2$, these differences start with 3, and increase by 2 each time. In the case of the pentagonal numbers R_n defined by $R_n = n(3n-1)/2$, the first such difference is 4. This time the increase in these differences is always 3. Continue to define polygonal numbers in this manner and obtain a general formula applicable to all polygonal numbers, such that the relationships for triangular numbers, square numbers, etc., all become special cases. (Let the first polygonal number always equal 1.)

Polygonal numbers: heptagonal numbers

SSM 3764. **by W. J. O'Donnell and G. E. O'Donnell, Jr.**

Heptagonal numbers (denoted HP_n) are positive integers of the form $n(5n-3)/2$ for $n = 1, 2, 3, \ldots$. Prove that $\mathrm{HP}_n \equiv n \pmod 5$.

Polygonal numbers: hexagonal numbers

SSM 3609. **by William J. O'Donnell**

Find the smallest hexagonal number $H_n = 2n^2 - n$, such that both n and H_n are palindromes.

Polygonal numbers: modular arithmetic

FQ B-363. **by Herta T. Freitag**

Do the sequences of squares $S_n = n^2$ and of pentagonal numbers $P_n = n(3n-1)/2$ have the symmetry property of reading the same from right to left as they do from left to right for their residues modulo m?

FQ B-362. **by Herta T. Freitag**

Let n be an integer greater than one, and let R_n be the remainder when the triangular number $T_n = n(n+1)/2$ is divided by m. Show that the sequence $R_0, R_1, R_2, \ldots$ repeats in a block $R_0, R_1, \ldots, R_t$ which reads the same from right to left as it does from left to right.

Polygonal numbers: octagonal numbers

SSM 3586. **by Charles W. Trigg**

In the decimal system, show that all octagonal numbers $E_n = n(3n-2)$, having 3 as their units' digit, terminate in 33.

SSM 3745. **by William J. O'Donnell**

Prove that if an octagonal number terminates with the digit 8, it terminates in 08. Octagonal numbers are integers of the form $n(3n-2)$.

Polygonal numbers: pentagonal numbers

SSM 3619. **by Randall J. Covill**

Find two pentagonal numbers, P'' and P', such that $P'' - P' = 605$. A pentagonal number is a number of the form $n(3n-1)/2$, where n is a positive integer.

SSM 3621. **by Robert A. Carman**

Find a number that is simultaneously triangular, pentagonal, and hexagonal. A triangular number is of the form $n(n+1)/2$. A pentagonal number is of the form $n(3n-1)/2$. A hexagonal number is of the form $n(2n-1)$.

PENT 285. **by Randall J. Covill**

If $O = n(3n-2)$ is an octagonal number and $P = m(3m-1)/2$ is a pentagonal number and $m = n$, then P and O are said to be complements of each other. It can be easily shown by algebraic manipulation of the formulas for P and O that, to every difference between an octagonal number and its complementary pentagonal number, there corresponds a multiple of 3 that is a unique positive integer. Show that, for at least one multiple of 3 that is a positive integer, there is not any corresponding difference between an octagonal number and its complementary pentagonal number.

AMM E2618. **by Amy J. Phelps**

Find all natural numbers that are simultaneously triangular, square, and pentagonal.

SSM 3589. **by Robert A. Carman**

Find a pair of pentagonal numbers whose sum and difference are both pentagonal numbers. A pentagonal number is of the form $n(3n-1)/2$.

SSM 3657. **by William J. O'Donnell**

Prove that no pentagonal number ends in 3, 4, 8, or 9. Pentagonal numbers are positive integers of the form $P_n = n(3n-1)/2$.

Polyhedral numbers

SSM 3644. **by William J. O'Donnell**

Prove that there are an infinite number of tetrahedral numbers that are also dodecahedral numbers.

SSM 3616. **by Robert A. Carman**

Show that every tetrahedral number,

$$(n/6)(n+1)(n+2),$$

is a square only for $n = 2^k$, $k \geq 0$.

Polynomials: 2 variables

AMM 6028.* **by F. D. Hammer**

Is there a polynomial in two variables with integral coefficients that is a bijection from $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}$? If so, how many such polynomials are there?

Polynomials: 3 variables

FQ B-309. **by Phil Mana**

Let $z^2 = xz + y$, and let k, m, and n be nonnegative integers. Prove that:

(a) $z^n = p_n(x,y)z + q_n(x,y)$, where p_n and q_n are polynomials in x and y with integer coefficients and p_n has degree $n-1$ in x for $n > 0$.

(b) There are polynomials r, s, and t not all identically zero and with integer coefficients, such that

$$z^k r(x,y) + z^m s(x,y) + z^n t(x,y) = 0.$$

Polynomials: age problems

OSSMB 78-10.

Professor Adams wrote on the blackboard a polynomial, $f(x)$ with integer coefficients and said, "Today is my son's birthday and when we substitute x equal to his age, a, then $f(a) = a$. You will also notice that $f(0) = p$, a prime number greater than a." How old is Professor Adams' son?

Polynomials: congruences

AMM E2763. **by Lorraine L. Foster**

Let

$$f(n) = n^3 + 396n^2 - 111n + 38.$$

Prove that the congruence $f(n) \equiv 0 \pmod{3^a}$ has precisely nine solutions $\pmod{3^a}$ for all integers $a \geq 5$.

Polynomials: cyclotomic polynomials

NAvW 496. **by L. Kuipers**

Let p and q be odd distinct primes. Let n be a positive integer, $n \geq 2$. Let $F_{p^n q}(x)$ be the cyclotomic polynomial of order $p^n q$. Show that, if

$$F_{p^n q}(x) = \sum_{j=0}^{\phi(p^n q)} c_j x^j,$$

then $c_j = (-1)^\gamma$ if $j = \alpha p^2 + \beta pq + \gamma p$ uniquely and $c_j = 0$ otherwise. Here α and β are nonnegative integers and $\gamma = 0$ or 1. Find also the coefficient of the central term.

Polynomials: degree 2

CRUX 72. **by Léo Sauvé**

Determine the ordered pair (p,q) such that p and q

(a) are *the* roots of the equation $x^2 + px + q = 0$;

(b) each satisfy the equation $x^2 + px + q = 0$.

MM 923. **by Aron Pinker**

If r and s are roots of $x^2 + px + q = 0$, where p and q are integers with $q \mid p^2$, then prove that $(r^n + s^n)/q$ is an integer for $n = 2, 3, \ldots$.

PARAB 426.

Find all pairs (m,n) of integers so that $x^2 + mx + n$ and $x^2 + nx + m$ both have integer roots.

Polynomials: degree 5

CRUX 452. **by Kenneth M. Wilke**

Precocious Percy wrote a polynomial on the blackboard and told his mathematics professor: "This polynomial has my age as one of its zeros." The professor looked at the blackboard and thought to himself: "This polynomial is monic, quintic, has integral coefficients, and is truly an odd function. If I try 10, I get -29670."

Find Percy's age and the polynomial.

Polynomials: evaluations

CRUX 30. **by Léo Sauvé**

Let a, b, and c denote three distinct integers and let P denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$.

Polynomials: inequalities

FUNCT 2.5.4.
Let P be a nonconstant polynomial with integer coefficients. If $n(P)$ is the number of distinct integers k such that $[P(k)]^2 = 1$, prove that $n(P) - \deg(P) \leq 2$.

Polynomials: injections

AMM E2554. **by F. David Hammer**
Can a polynomial function with integer coefficients be one-to-one when restricted to the rationals, but not one-to-one on the reals?

Polynomials: products

AMM S7. **by George E. Andrews and Richard Askey**
Let

$$p_n(x) = (x+1)(x+q)\cdots\left(x+q^{n-1}\right),$$

$n = 1, 2, \ldots, p_0(x) = 1$. Find the coefficients $a(k, m, n)$ defined by

$$p_n(x) \cdot p_m(x) = \sum_{k=0}^{m+n} a(k, m, n) \cdot p_k(x).$$

CANADA 1977/4.
OMG 16.2.4.
Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

be two polynomials with integer coefficients. Suppose that all the coefficients of the product $p(x) \cdot q(x)$ are even but not all of them are divisible by 4. Show that one of $p(x)$ and $q(x)$ has all even coefficients and the other has at least one odd coefficient.

Polynomials: roots

FQ B-347. **by Verner E. Hoggatt, Jr.**
Let a, b, and c be the roots of $x^3 - x^2 - x - 1 = 0$. Show that

$$\frac{a^n - b^n}{a-b} + \frac{b^n - c^n}{b-c} + \frac{c^n - a^n}{c-a}$$

is an integer for $n = 0, 1, 2, \ldots$.

Powers: differences

SSM 3737. **by Alan Wayne**
Prove each of the following two propositions:
(a) If two integers differ in absolute value, then the sum or difference of their reciprocals cannot be an integer.
(b) If r and s are unequal, positive integers, then there is no integer t, other than zero and one, such that $r^t - s^t = t$.

Powers: integers

JRM 378. **by Diophantus McLeod**
Prove that if x^{1776} and x^{1975} are both integers, then so also is x.

Powers: powers of 2

ISMJ 13.8.
Show that if n is a positive integer, there are no positive integers a and k with $k \geq 2$ such that $a^k = 2^n - 1$.

CRUX 410.* **by James Gary Propp**
Are there only finitely many powers of 2 that have no zeros in their decimal expansions?

AMM E2805. **by Wells Johnson**
Let the integer $r \geq 0$ be given. Show that each of the numbers $\left(2^{2^r}\right)^n - 1$ has at least $2r+1$ distinct prime factors if $n > 2^r$, with the lone exception $r = 1$, $n = 3$, when $4^3 - 1 = 3^2 \cdot 7$.

SPECT 8.1.
Let n be a positive integer. Show that
(a) if $2^n - 1$ is prime, then n is prime,
(b) if $2^n + 1$ is prime, then n must be a power of 2.
Is the converse of (a) true?

Powers: powers of 2 and 3

CRUX 250.* **by Gilbert W. Kessler**
(a) Find all pairs (m, n) of positive integers such that

$$|3^m - 2^n| = 1.$$

(b) If $|3^m - 2^n| \neq 1$, is there always a prime between 3^m and 2^n?

Powers: radicals

TYCMJ 99. **by Alan Wayne**
Prove that if a and n are integers, then

$$\left(\frac{1}{2}\right)^n \left[\left(a + \sqrt{a^2-4}\right)^n + \left(a - \sqrt{a^2-4}\right)^n\right]$$

is an integer.

FUNCT 1.1.10.
If $(1+\sqrt{2})^n = a + b\sqrt{2}$ where a, b, and n are positive integers, then prove that a is the integer closest to $b\sqrt{2}$.
Use a computer to print a, an approximation to $b\sqrt{2}$, and the difference between a and $b\sqrt{2}$ as n increases.
Can you generalize the above problem in any way?

Powers: tetration

JRM 732. **by Frank Rubin**
Let a^*n be defined by $a^*n = a^{a^*(n-1)}$, with $a^*1 = a$. Thus $5^*4 = 5^{(5^{(5^5)})}$.
(a) What is the smallest value of n for which 10^*n exceeds $3^*(n+1)$?
(b) For each integer k, what is the largest integer k' such that k'^*n never exceeds $k^*(n+1)$?

Primes: arithmetic progressions

ISMJ J10.15.
Prove that if a, b, and c are prime numbers greater than 3 and $b - a = c - b$, then the number $b - a$ is divisible by 6.

Number Theory

Primes: complete residue systems

JRM 672. **by Bernardo Recamán**

The first two primes, 2 and 3, form a complete residue system modulo 2, and the first four primes, but not the first three, form a complete residue system modulo 3. Is there a prime $p > 2$ such that the first p primes form a complete residue system modulo p?

Primes: congruences

SSM 3634. **by Bob Prielipp**

If $p \neq 5$ and the numbers p, $p+2$, $p+6$, and $p+8$ are prime, then $p \equiv 11 \pmod{210}$ or $p \equiv 101 \pmod{210}$ or $p \equiv 191 \pmod{210}$.

Primes: digit permutations

AMM E2718. **by Gordon D. Prichett**

Find all prime numbers p that have the following two properties:

(i) All numbers obtained from p by permuting its digits are also prime.

(ii) The sum and the product of the digits of p are also prime.

Primes: digit reversals

JRM 700. **by Les Card**

The table shown lists some of the facts known about reversible primes. The final column, which represents the number of digits, times the number of reversible primes having that number of digits, divided by the total number of primes having that number of digits, has an apparent minimum of 0.769 at four digits. Is there a proof, or at least a reasonable argument, that this ratio will never be less than 0.769, regardless of the number of digits?

Primes: forms of numbers

SSM 3620. **by Bob Prielipp**

Euler established that every prime number of the form $6k+1$ can be expressed as x^2+3y^2 for some positive integers x and y. Show that a prime number of the form $6k+1$ can be expressed as x^2+3y^2 for some positive integers x and y if and only if it can be expressed as a^2+ab+b^2 for some positive integers a and b.

CRUX 302. **by Leroy F. Meyers**

Show that if p is a prime, then p^2+5 is not a prime.

PARAB 430.

Let p be a prime greater than 3. Show that p^2 is one more than a multiple of 12.

Primes: gaps

JRM 708. **by Richard L. Francis**

If p is a prime and no other primes occur in the interval $[p-2k, p+2k]$, where k is an integer, then p will be called *isolated of order* k. For example, 211 is of order 5, since all of the integers except 211 in the interval $[201, 221]$ are composite; 211 is not of order 6, however, since 199 is prime.

(a) What is the maximum number of isolated primes of order 3 that can occur in an interval of 50 consecutive integers?

(b) Such maximal sets of isolated primes actually occur. What are the elements of any such set?

(c) Same questions for an interval of 100.

JRM 654. **by Harry Nelson**

What is the most probable difference between consecutive primes?

Primes: generators

CRUX 142. **by André Bourbeau**

Find 40 consecutive positive integral values of x for which $f(x) = x^2+x+41$ will yield composite values only.

OSSMB 75-4.

Prove that, for all integers x, x^2+x+41 is never divisible by any natural number between 1 and 41.

PME 393. **by Peter A. Lindstrom**

Let $f(n) = n^2-n+41$. Find $\gcd(f(n), f(n+1))$.

FUNCT 1.5.4.

Check that x^2-x+41 is a prime for $x = 1, 2, \ldots, 40$.

JRM 714. **by Harry L. Nelson**

It is known that the formula x^2+x+41 produces primes for the forty integer values $0 \le x \le 39$ and perhaps less well known that $x^2-79x+1601$ produces primes for the eighty values $0 \le x \le 79$. Find a polynomial which produces primes for more than eighty consecutive integer values of x.

MM Q623. **by Erwin Just and Norman Schaumberger**

It is known that the range of a nonconstant polynomial function with integral coefficients cannot consist wholly of primes. The range of the polynomial $2x-1$, however, contains all the odd primes. Is there a polynomial of degree greater than 1 whose range contains all the primes?

CRUX 154.* **by Kenneth S. Williams**

Let p_n denote the nth prime, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc. Prove or disprove that the following method finds p_{n+1} given $p_1, p_2, \ldots, p_n$.

In a row list the integers from 1 to $p_n - 1$. Corresponding to each r $(1 \le r \le p_n - 1)$ in this list, say $r = p_1^{a_1} \ldots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1} \ldots p_n^{a_{n-1}}$ in a second row. Let l be the smallest odd integer not appearing in the second row. The claim is that $l = p_{n+1}$.

FQ B-334. **by Philip Mana**

Define the sequence $11, 17, 29, 53, \ldots$ by $u_0 = 11$ and $u_{n+1} = 2u_n - 5$ for $n \ge 0$. Are all the terms prime?

Primes: greatest prime factor

AMM 6135.* **by P. Erdős**

Denote by $P(n)$ the greatest prime factor of n and set

$$A(x,y) = \prod_{1 \le i \le y-x} (x+i).$$

An integer n is called exceptional if for some $x \le n \le y$, $(P(A(x,y)))^2$ divides $A(x,y)$, i.e., the greatest prime factor of $A(x,y)$ occurs with an exponent greater than 1.

Prove that the density of exceptional numbers is 0, and estimate the number $E(x)$ not exceeding x as well as you can.

Number Theory

Primes: pi function

OSSMB 76-15.

The positive integer 10 is "balanced" from the point of view that half the positive integers between 1 and it are prime numbers $(2, 3, 5, 7)$ and half are composite numbers $(4, 6, 8, 9)$. Find all such balanced numbers.

AMM 6153. **by Bernardo Mz.-Recamán**

Let $\pi(x)$ denote the number of primes that do not exceed x. Are there infinitely many integers, such as 2, 4, 6, 8, 30, 33, 100, with the property that $\pi(n)$ divides n?

Primes: polynomials

CRUX 327. **by F. G. B. Maskell**

Let p_n be the nth prime number. For which n is p_n^2+2 also prime?

CRUX 97. **by Viktors Linis**

Find all primes p such that

$$p^3 + p^2 + 11p + 2$$

is a prime.

PARAB 277.

Prove that 3 is the only prime value of p for which $p^3 + p^2 + 11p + 2$ is prime.

CRUX 296. **by F. G. B. Maskell**

Let p be prime. Show that $p^4 - 20p^2 + 4$ is composite.

NYSMTJ 42.

Find all integral values of w for which $w^4 + 4$ is prime.

Primes: powers

AMM 6110.* **by David M. Battany**

Let p and q be primes, not both even. Let m, n, and v be integers, $m, n \ge 2$, $v \ge 0$. Prove that for each value of v, there exists at most one pair of powers (p^m, q^n) such that $p^m - q^n = 2^v$.

Primes: prime chains

AMM 6189.* **by Edward T. H. Wang**

Prove or disprove that for each natural number $n \ge 2$, one can arrange the numbers $1, 2, \ldots, n$ in a sequence such that the sum of any two adjacent terms is a prime.

JRM 566. **by Henry Larson**

(a) The diagram shows for $n = 2, 3, 4$, and 5, how the integers from 1 through n can be arranged sequentially in such a way that the sum of every pair of adjacent numbers is prime. Show that "prime chains" exist for n up to 50.

(b) What is the smallest value of n for which there is no prime chain?

```
      1 — 2
    1 — 2 — 3
  1 — 4 — 3 — 2
1 — 4 — 3 — 2 — 5
```

JRM 679. **by Randall J. Covill**

A prime chain of order n is a sequence containing each of the integers from 1 to n exactly once, such that the sum of every pair of adjacent integers is a prime. A prime circle is a prime chain in which the sum of the first and last integers is a prime.

Show that any prime circle of order n can be transformed into a prime chain of order $n + 1$.

Primes: products

CRUX 246. **by Kenneth M. Wilke**

Let p_i denote the ith prime and let P_n denote the product of the first n primes. Prove that the number N defined by

$$N = \frac{P_n}{p_i p_j \cdots p_r} \pm p_i p_j \cdots p_r,$$

where $p_i p_j \cdots p_r$ are any of the first n primes, all different, or unity, is a prime whenever $N < p_{n+1}^2$.

MM 956. **by Arthur Marshall**

Let Q_m be the product of the first m primes: $Q_2 = 6$, $Q_3 = 30$, etc. Then, for $m \ge 2$, $Q_m/2$ is the product of the first $m - 1$ odd primes. Now $Q_2/2 = 2^1 + 1 = 2^2 - 1$, while $Q_3/2 = 2^4 - 1$. For $m > 3$, can $Q_m/2 = 2^j \pm 1$ for some integer j?

NAvW 466. **by H. J. J. te Riele**

Let P_k $(k \ge 1)$ be the product of the first k primes. Let

$$(a_i^{(k)})_{i=1}^{\phi(P_k)+1}$$

be the increasing sequence of positive integers less than or equal to $P_k + 1$ that are relatively prime to P_k. Let $N_k(d)$ be the number of terms $a_i^{(k)}$ for which

$$a_{i+1}^{(k)} - a_i^{(k)} = d.$$

Determine $N_k(d)$ for $d = 2$, 4, and 6, in terms of the first k primes.

Primes: recurrences

AMM E2648. **by R. P. Nederpelt, R. B. Eggleton, and John H. Loxton**

(a) Show that there is no infinite sequence of prime numbers $p_1, p_2, \ldots$ such that $p_{k+1} = 2p_k \pm 1$ for all k.

(b) Find a longest finite sequence $p_1, p_2, \ldots, p_n$ of primes such that $p_{k+1} = 2p_k + 1$ for $1 \le k \le n - 1$.

Primes: sequences

NAvW 539. **by P. Erdős**

Let $\{A_1, A_2, \ldots, A_n\}$ be a partition of the sequence of primes into n subsequences. Let A_ν^+ denote the set of integers that can be represented as a sum of distinct elements of A_ν. Show that, for at least one value of ν, the set A_ν^+ has upper density 1.

Primes: sum of primes

CRUX 249. **by Clayton W. Dodge**

The positive integers 1, 4, and 6 are not primes and cannot be written as sums of distinct primes. Prove or disprove that all other positive integers are either prime or can be written as sums of distinct primes.

Number Theory

SSM 3624. **by Charles W. Trigg**

Among the sums of three consecutive primes greater than 7 in the decimal system, locate
(a) the smallest nonsquare composite sum;
(b) the smallest multiple of 5;
(c) the integer composed of consecutive digits;
(d) a perfect cube.

Products

MM Q640. **by Peter A. Lindstrom**

For positive integers n, find the values of

$$\prod_{i=0}^{n-1}[n(n+1)-i(i+1)]$$

and

$$(n+1)\prod_{i=0}^{n-1}[n(n+2)-i(i+2)].$$

AMM E2510. **by Saul Singer**

If n is a natural number, let

$$Q(n)=\prod_{k=1}^{n-1}k^{2k-n-1}.$$

(a) Show that $Q(n)$ is an integer whenever n is prime.
(b) For which composite n, if any, is $Q(n)$ an integer?

NAvW 441. **by P. Erdős**

Consider k integers a_i with

$$1<a_1<a_2<\cdots<a_k\le x,$$

$k>\pi(x)$. Prove that the products

$$\prod_{i=1}^{k}a_i^{\alpha_i},\qquad 0\le\alpha_i,\qquad i=1,2,\dots,k,$$

cannot all be different.

AMM E2637. **by Armond E. Spencer**

If $a_0,a_1,\dots,a_{n-1}$ are integers, show that

$$\prod_{0\le i<j\le n-1}\frac{a_i-a_j}{i-j}$$

is also an integer.

CRUX 475. **by Hayo Ahlburg**

Consider the products

$$(341+\frac{2}{3})(205-\frac{2}{5})=341\cdot 205,\quad (43+\frac{2}{5})(31-\frac{2}{7})=43\cdot 31,$$

$$(781+\frac{1}{2})(521-\frac{1}{3})=781\cdot 521,\quad (57+\frac{1}{3})(43-\frac{1}{4})=57\cdot 43.$$

Find an infinite set of products having the same property.

Pythagorean triples: area

CRUX 223. **by Steven R. Conrad**

Find the smallest integer that can represent the area of two noncongruent primitive Pythagorean triangles.

SSM 3569. **by Bob Prielipp**

Prove that infinitely many primitive Pythagorean triangles have areas which are multiples of 30.

Pythagorean triples: area and perimeter

MM 1088.* **by Alan Wayne**

(a) For each integer $m\ge 1$, how many Pythagorean triangles are there that have an area equal to m times the perimeter? How many of these are primitive?

(b) Can this result be generalized to all triangles with integer sides and area equal to m times the perimeter?

SSM 3587. **by Alan Wayne**

(a) Show that, for every natural number m, there is at least one primitive Pythagorean triangle in which the area is m times the perimeter.

(b) Find the number of Pythagorean triangles in which the area is 360 times the perimeter.

Pythagorean triples: arithmetic progressions

SSM 3641. **by Irwin K. Feinstein**

Prove that the only right triangle with integral sides and with the sides and area in arithmetic progression is the $3:4:5$ triangle.

Pythagorean triples: counting problems

MM 1007.* **by Thomas E. Elsner**

It is known that given an integer n, $n\ge 0$, there is a positive integer k, such that k occurs in exactly n distinct Pythagorean triples (x,y,z), $x<y<z$, $x^2+y^2=z^2$. For example, 2^{n+1} occurs in exactly n Pythagorean triples. For each n, determine $m_n=\min\{k\,|\,k$ occurs in exactly n Pythagorean triples$\}$.

PENT 298. **by H. Laurence Ridge**

It is well known that all primitive Pythagorean triangles (PPT) are generated by the formulae

$$x=2ab$$
$$y=a^2-b^2$$
$$z=a^2+b^2$$

where a and b are positive integers of opposite parity and $\gcd(a,b)=1$.

Let N be an arbitrary positive integer. What are the necessary and sufficient conditions for N to be a leg (or hypotenuse) of exactly one PPT?

SSM 3638. **by Bob Prielipp**

Find infinitely many primitive Pythagorean triples (x,y,z) such that $z=x+2$.

Pythagorean triples: digit problems

SSM 3752. **by Robert A. Carman**

The triple $(5,12,13)$ is a primitive Pythagorean triple. So is the triple $(15,112,113)$ formed by affixing the same digit (in this case, a 1) to each member of the first triple. Prove or disprove that there are no other pairs of primitive Pythagorean triples that are related in this way.

Pythagorean triples: divisibility

CRUX 437. **by Clayton W. Dodge**

Find all Pythagorean triangles having the hypotenuse divisible by 7.

SSM 3633. **by Alan Wayne**
Show that if a triangle is primitive Pythagorean, then
(a) the length of one leg is divisible by four,
(b) the length of one leg is divisible by 3,
(c) the length of the hypotenuse is either an odd prime of the form $4k+1$ or else a product of such primes.

Pythagorean triples: Fibonacci and Lucas numbers

FQ B-402. **by Gregory Wulczyn**
Show that

$$(L_n L_{n+3},\ 2L_{n+1}L_{n+2},\ 5F_{2n+3})$$

is a Pythagorean triple.

Pythagorean triples: generators

SSM 3771. **by Bob Prielipp**
Show how to generate an infinite sequence of primitive Pythagorean triangles each having a hypotenuse of length eight more than the length of one of its legs.

SSM 3742. **by Robert A. Carman**
Let $P=(x,y,z)$ be a Pythagorean triple. Find a non-trivial 3×3 matrix T such that PT is always a Pythagorean triple.

Pythagorean triples: hypotenuse

SSM 3592. **by Bob Prielipp**
Prove that the hypotenuse of a primitive Pythagorean triangle is of the form $12k+1$ or $12k+5$.

Pythagorean triples: inequalities

MM Q625. **by A. Wilansky**
If (a,b,c) is a Pythagorean triple, prove that

$$(a+b)/2 < c/\sqrt{2}.$$

Pythagorean triples: inradius

TYCMJ 107. **by Abe Simowitz**
Prove or disprove that the radius of a circle inscribed in a Pythagorean triangle is an integral multiple of the greatest common divisor of the three sides.

Pythagorean triples: inscribed squares

MM 945. **by Alan Wayne**
Find the smallest Pythagorean triangle in which a square with integer sides can be inscribed so that an angle of the square coincides with the right angle of the triangle.

TYCMJ 64. **by Aron Pinker**
An integer-sided square is inscribed in an integer-sided right triangle so that a side of the square lies on the hypotenuse. What is the smallest possible length of the side of the square?

Pythagorean triples: odd and even

CRUX 460. **by Clayton W. Dodge**
Can two consecutive even integers ever be the sides of a Pythagorean triangle? Show how to find all such Pythagorean triangles.

MATYC 124. **by Charles W. Trigg**
Can two consecutive odd integers be the sides of a Pythagorean triangle?

Pythagorean triples: partitions

AMM E2530.* **by F. Loupekine**
(a) Show that it is possible to partition the natural numbers into three classes so that if (x,y,z) is a primitive Pythagorean triple, then x, y, and z are in different classes.
(b) Can such a partition be made if the above is to hold for all Pythagorean triples, not just primitive ones?

Pythagorean triples: primes

PME 459. **by Bob Prielipp**
Prove that every Pythagorean triple (x,y,z) where both x and z are prime numbers and $x \geq 11$ is such that 60 divides y.

SSM 3606. **by Bob Prielipp**
Prove that any Pythagorean triple (x,y,z) must be of the form $\bigl(p,[p^2-1]/2,[p^2+1]/2\bigr)$, where p is an odd prime number, whenever both x and z are prime numbers.

Pythagorean triples: reciprocals

JRM 795. **by Arnon Boneh**
Define the three positive integers a,b, and c as a *reciprocal Pythagorean triple* if $a^{-2}+b^{-2}=c^{-2}$. For example, $156, 65, 60$ is a reciprocal Pythagorean triple.
What is the minimal sum of such a triple?

Pythagorean triples: squares

CRUX 5. **by F. G. B. Maskell**
Prove that if (a,b,c) and (a',b',c') are each primitive Pythagorean triples, with $a>b>c$, and $a'>b'>c'$, then either

$$aa' \pm (bc'-cb') \quad \text{or} \quad \mathrm{aa}' \pm (\mathrm{bb}'-\mathrm{cc}')$$

are perfect squares.

Pythagorean triples: systems of equations

CRUX 86. **by Viktors Linis**
Find all rational Pythagorean triples (a,b,c) such that

$$a^2+b^2=c^2 \quad \text{and} \quad \mathrm{a}+\mathrm{b}=\mathrm{c}^2.$$

Quadratic fields

AMM 6270.* **by Kenneth S. Williams**
Let p be a prime congruent to 1 modulo 8. Let ε_{2p} denote the fundamental unit of the real quadratic field $Q\left(\sqrt{2p}\right)$, and let $h(-2p)$ denote the class number of the imaginary quadratic field $Q\left(\sqrt{-2p}\right)$. Prove that if the norm of ε_{2p} is -1, then

$$h(-2p) \equiv 0 \pmod 8, \quad \text{if } p \equiv 1 \pmod{16},$$

and

$$h(-2p) \equiv 4 \pmod 8, \quad \text{if } p \equiv 9 \pmod{16}.$$

Quadratic reciprocity

CMB P249. **by Kenneth S. Williams**

Let p be a prime congruent to 1 (mod 4), so that there are integers a and b such that

$$p = a^2 + b^2, \quad a \equiv 1 \pmod 4, \quad b \equiv 0 \pmod 2.$$

It is easily proved using the law of quadratic reciprocity for Jacobi symbols that $(\frac{a}{b}) = +1$, so that there exists an integer c such that $a \equiv c^2 \pmod p$. Determine $(\frac{c}{p})$.

Quadratic residues

NAvW 413. **by R. Tijdeman**

Let p be a prime and denote by $f(p)$ the number of pairs of quadratic residue classes that differ by 1. Compute $f(p)$ for all p.

AMM 6156. **by Herbert Knothe**

Prove that if a prime p has the form $8n+7$, then the number of even quadratic residues greater than $p/2$ is equal to $n+1$. If a prime p has the form $8n+3$, then the number of even quadratic residues less than $p/2$ is equal to n. Each residue r is restricted so that $0 \le r < p-1$.

AMM 6058. **by Larry Taylor**

(a) If $p \equiv 31$ or $39 \pmod{40}$ is prime, and if

$$a \equiv \frac{\sqrt{5}+2}{3} \qquad \text{and} \qquad b \equiv \frac{\sqrt{5}-2}{3}$$

are of even order (mod p), prove that either $a-1$, a, and $a+1$ or $b-1$, b, and $b+1$ are quadratic nonresidues of p.

(b) If $p \equiv 19 \pmod{24}$ is prime, and if

$$a \equiv \sqrt{-\frac{1}{3}}$$

is of even order (mod p), prove that $a-1$, a, and $a+1$ are quadratic nonresidues of p.

FQ H-277. **by L. Taylor**

If $p \equiv +1 \pmod{10}$ is prime and $x \equiv \sqrt{5}$ is of even order (mod p), prove that $x-3$, $x-2$, $x-1$, x, $x+1$, and $x+2$ are quadratic nonresidues of p if and only if $p \equiv 39 \pmod{40}$.

FQ H-307. **by Larry Taylor**

(a) If $p \equiv \pm 1 \pmod{10}$ is prime, $x \equiv \sqrt{5}$, and $a \equiv \frac{2(x-5)}{x+7} \pmod p$, prove that a, $a+1$, $a+2$, $a+3$, and $a+4$ have the same quadratic character modulo p if and only if $11 < p \equiv 1$ or $11 \pmod{60}$ and $(-2x/p) = 1$.

(b) If $p \equiv 1 \pmod{60}$, $2x/p = 1$, and $b \equiv \frac{-2(x+5)}{7-x}$ (mod p), then b, $b+2$, $b+3$, and $b+4$ have the same quadratic character modulo p. Prove that $(11ab/p) = 1$.

AMM E2627. **by Ron Evans**

Let m and n be fixed integers greater than 1, n odd. Suppose n is a quadratic residue modulo p for all sufficiently large prime numbers $p \equiv -1 \pmod{2^m}$. Show that n is a square.

AMM 6094. **by Francis Cald**

A pair of primes, P and Q, is said to be acquainted if the set of quadratic residues and the set of quadratic nonresidues of P are, respectively, a subset of the set of residues and the set of nonresidues of Q. Is there a positive constant C such that infinitely many pairs of acquainted primes exist for which $Q - P \le C$?

Rational expressions

CRUX 319. **by Leigh Janes**

Find necessary and sufficient conditions for the positive integer triple (A, B, C) to satisfy

$$\frac{A^3+B^3}{A^3+C^3} = \frac{A+B}{A+C}.$$

CRUX 91. **by Léo Sauvé**

If a, a', b, and b' are positive integers, show that a sufficient condition for the fraction $\frac{a+a'}{b+b'}$ to be irreducible is

$$|ab' - ba'| = 1.$$

Is this condition also necessary?

CRUX 92. **by Léo Sauvé**
PARAB 429.
PENT 277. **by Kenneth M. Wilke**

If a is a positive integer, show that the fraction

$$\frac{a^3+2a}{a^4+3a^2+1}$$

is irreducible.

Rational numbers

JRM 511. **by Steven Cook**

$E_1 : 1/2,\ 1/3,\ 2/3,\ 1/4,\ 3/4,\ 1/5,\ 2/5,\ 3/5,\ 4/5,\ 1/6,\ \dots$
$E_2 : 1/2,\ 1/3,\ 1/4,\ 2/3,\ 1/5,\ 1/6,\ 2/5,\ 3/4,\ 1/7,\ 3/5,\ \dots$

Shown above are two different enumerations of the rationals in $(0,1)$. Enumeration E_1 lists them by increasing denominator, and for equal denominators, by increasing numerator. Enumeration E_2 lists them by increasing sum of numerator and denominator and for equal sums of terms, by increasing numerator. The numbers 1/2, 1/3, and 2/5 have the same relative position in both numerations. A near miss occurs at 5/13. Find the next few occurrences of a match.

PARAB 272.

Let m and n be two relatively prime positive integers. Prove that if the $m+n-2$ fractions

$$\frac{m+n}{m}, \frac{2(m+n)}{m}, \frac{3(m+n)}{m}, \dots, \frac{(m-1)(m+n)}{m},$$
$$\frac{m+n}{n}, \frac{2(m+n)}{n}, \frac{3(m+n)}{n}, \dots, \frac{(n-1)(m+n)}{n},$$

are plotted as points on the real number line, exactly one of these fractions lies inside each of the unit intervals

$$(1,2),\ (2,3),\ (3,4), \dots, (m+n-2, m+n-1).$$

Rectangles

MSJ 424. **by John Murphy**

Find the dimensions of all integral-sided rectangles each of which has its perimeter numerically equal to its area.

Recurrences: arrays

FQ B-353. **by V. E. Hoggatt, Jr.**

For k and n integers with $0 \le k \le n$, let $A(k, n)$ be defined by $A(0, n) = 1 = A(n, n)$, $A(1, 2) = c + 2$, and

$$A(k+1, n+2) = cA(k, n) + A(k, n+1) + A(k+1, n+1).$$

Also let $S_n = A(0, n) + A(1, n) + \cdots + A(n, n)$. Show that

$$S_{n+2} = 2S_{n+1} + cS_n.$$

SIAM 78-6. **by Peter Shor**

A function $S(m, n)$ is defined over the nonnegative integers by

$$S(0,0) = 1,$$

$$S(0,n) = S(m,0) = 0 \qquad \text{for } m, n \ge 1,$$

$$S(m+1, n) = mS(m,n) + (m+n)S(m, n-1).$$

Show that

$$\sum_{n=1}^{m} S(m,n) = m^m.$$

AMM E2609. **by Glen E. Bredon**

Define integers a_{ij}, $i, j \ge 1$ by $a_{i1} = a_{1j} = 1$ and

$$a_{ij} = ia_{i,j-1} + ja_{i-1,j}, \qquad i, j \ge 2.$$

Show that

$$\sum_{i=1}^{2n-1} (-1)^{i-1} a_{i,2n-i} \equiv 1 \pmod 3.$$

AMM 6151. **by Clarence H. Best**

A two-dimensional array is defined according to the following rule:

$$a_{1,1} = 1,$$
$$a_{i,1} = a_{1,i-1}, \qquad i > 1,$$
$$a_{i,j} = a_{i+1,j-1} + a_{i,j-1}, \qquad j > 1.$$

(a) Prove that $a_{1,j}$ equals the number of distinct partitions of a j-element set.

(b) Choose an nth-order determinant D_n from the upper left corner of the array and prove

$$D_n = \prod_{0 \le i \le n-1} i!\ .$$

Recurrences: finite sums

SSM 3721. **by Herta T. Freitag**

(a) Let a sequence $\{a_i\}$ be defined by $a_0 = 0$ and $a_n = a_{n-1} + n$ for $n \ge 1$. Express $\sum_{i=0}^{n} a_i$ as a binomial coefficient.

(b) Let a sequence $\{b_i\}$ be defined by $b_0 = 1$ and $b_{n+1} = b_n + T_{n+2}$ for $n \ge 1$, where $T_k = k(k+1)/2$ is the kth triangular number. Express $\sum_{i=0}^{n} b_i$ as a binomial coefficient.

Recurrences: floor function

JRM 625. **by David L. Silverman**

Define the sequence $A = a_0, a_1, a_2, \ldots$ as follows:
(1) $a_0 = 0,\ a_1 = 1.$
(2) $a_{n+1} = a_n - \left\lfloor \frac{1}{2}(a_n + 1) \right\rfloor,\ n > 0,$
unless that number has occurred earlier in A, in which case the minus sign is replaced by a plus sign. Thus the first few elements of A are 0, 1, 2, 3, 5, 8, 4, 6, 9, 14, 7, … .

Prove that:
(a) All elements of A are distinct.
(b) Every positive integer is an element of A.

FQ B-417. **by R. M. Grassl and P. L. Mana**

Let $f(n)$ be defined by $f(0) = 1 = f(1)$, $f(2) = 2$, $f(3) = 3$, and

$$f(n) = f(n-4) + [1 + (n/2) + (n^2/12)]$$

for $n \in \{4, 5, 6, \ldots\}$. Do there exist rational numbers a, b, c, and d such that

$$f(n) = [a + bn + cn^2 + dn^3]?$$

IMO 1976/6.

A sequence $\{u_n\}$ is defined by

$$u_0 = 2, \quad u_1 = 5/2, \quad u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$$

for $n = 1, 2, \ldots$. Prove that for positive integers n,

$$\lfloor u_n \rfloor = 2^{[2^n - (-1)^n]/3}.$$

AMM E2619. **by Thomas C. Brown**

Let $a_1 = 1$ and

$$a_{n+1} = a_n + \lfloor \sqrt{a_n}\ \rfloor$$

for $n = 1, 2, \ldots$. Show that a_n is a square if and only if $n = 2^k + k - 2$ for some positive integer k.

Recurrences: fractions

PUTNAM 1979/A.3.

Let $x_1, x_2, x_3, \ldots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}} \quad \text{for n} = 3, 4, 5, \ldots\ .$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

Recurrences: generalized Fibonacci sequences

JRM 537. **by Les Marvin**

In the diagram below, each row is a Fibonacci-type sequence in which the $(n+2)$-nd term is the sum of the nth term and the $(n+1)$-st. The kth row is the sequence that begins with the numbers $1, k$. Consider the sequence of elements along the main diagonal, $1, 2, 4, 9, 17, 33, 61, \ldots$. What is the limiting ratio of successive terms?

1	1	2	3	5	8	13
1	2	3	5	8	13	21
1	3	4	7	11	18	29
1	4	5	9	14	23	37
1	5	6	11	17	28	45

FQ H-302. **by George Berzsenyi**

Let c be a constant and define the sequence (a_n) by $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} + ca_{n-2}$ for $n \geq 2$. Determine the sequence (b_n) for which

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k.$$

FQ H-248. **by F. D. Parker**

Prove that, if a sequence $\{y_0, y_1, \ldots\}$ satisfies the equation

$$y_n = y_{n-1} + y_{n-2},$$

and if y_0 and y_1 are integers, then there exists an integer N such that

$$y_n^2 - y_{n-1}y_{n+1} = N(-1)^n.$$

Furthermore show that N cannot be of the form $4k+2$, and show that $4N$ terminates in 0, 4, or 6.

FQ H-285. **by V. E. Hoggatt, Jr.**

Consider two sequences $\{H_n\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$ such that

(i) $\gcd(H_n, H_{n+1}) = 1$,
(ii) $\gcd(G_n, G_{n+1}) = 1$,
(iii) $H_{n+2} = H_{n+1} + H_n,\ n \geq 1$, and
(iv) $H_{n+1} + H_{n-1} = sG_n,\ n \geq 1$, where s is independent of n.

Show that $s = 1$ or $s = 5$.

FUNCT 1.1.7. **by Christopher Stuart**

Show that if U_n is the nth term of any Fibonacci sequence, then

$$U_n^2 - U_{n-2}^2 = U_{n-1}(2U_{n-1} + U_{n-4}).$$

FQ H-305.* **by Martin Schechter**

For fixed positive integers, m and n, define a Fibonacci-like sequence as follows:

$$S_1 = 1,\ S_2 = m,\ S_k = \begin{cases} mS_{k-1} + S_{k-2}, & \text{if } k \text{ is even,} \\ nS_{k-1} + S_{k-2}, & \text{if } k \text{ is odd.} \end{cases}$$

(a) Show that if $j \mid k$ then $S_j \mid S_k$ and in fact that $\gcd(S_q, S_r) = S_{\gcd(q,r)}$.

(b) Show that the sequences such that $(m, n) = (1, 4)$ and $(m, n) = (1, 8)$ have only the element 1 in common.

JRM 766. **by Anthon K. Whitford**

Define the generalized Fibonacci sequence by $G_{n+m} = G_n + G_{n+m-1}$, $G_1 = G_2 = \cdots = G_m = 1$.

(a) Derive an expression for the sum of the first n terms.

(b) Find the limiting value of G_{n+1}/G_n as n approaches infinity.

TYCMJ 48. **by Warren Page**

For any two real numbers, a and b, let $f_0 = 0$, $f_1 = a$, $f_2 = b$, and $f_{k+2} = f_{k+1} + f_k$ $(k = 1, 2, \ldots, 8)$. Prove that

$$\sum_{i=0}^{10} (f_i - r)^2 \geq 1430ab$$

for every real number r.

PENT 311. **by Kenneth M. Wilke**

A teacher of mathematics propounded the following addition problem: Two numbers are selected at random and each succeeding number equals the sum of the two preceding numbers until a list of ten numbers is reached; e.g., starting with 365 and 142, the list to be added by the class was

365+142+507+649+1156+1805+2961+4766+7727+12493.

Just as the teacher told the class to add these numbers, young Leslie Morely announced the sum to be 32571. Astounded, the teacher verified the correctness of Leslie's answer with a pocket calculator. Assuming that Leslie performed this feat mentally, how did he do it?

MM 1013. **by James Propp**

Let the sequence (S_n) be defined by

$$S_1 = a,\ S_2 = a + b,\ \text{and } \mathrm{S_{n+1} = S_n + S_{n-1}},\ \text{for } \mathrm{n} \geq 2,$$

where a and b are distinct positive integers.

Define a hole of (S_n) as an integer that is not expressible as a sum of distinct terms of (S_n). Find a general formula for $J(k)$, the number of holes of (S_n) between S_k and S_{k+1}.

FQ B-336. **by Herta T. Freitag**

Let $Q_0 = 1 = Q_1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$. Show that $2(Q_{2n}^2 - 1)$ is a perfect square for $n = 1, 2, 3, \ldots$.

Recurrences: inequalities

PARAB 317.

Let a and b be positive integers and define $a_1 = \sqrt{ab}$, $b_1 = \frac{1}{2}(a + b)$, $a_2 = \sqrt{a_1 b_1}$, $b_2 = \frac{1}{2}(a_1 + b_1)$, ... Thus, in general, $a_{n+1} = \sqrt{a_n b_n}$, $b_{n+1} = \frac{1}{2}(a_n + b_n)$. Show that

$$|b_n - a_n| \leq \frac{|b - a|}{2^n}$$

for each positive integer n.

Recurrences: limits

FQ B-345. **by Frank Higgins**

Let $r > s > 0$. Find $\lim_{n\to\infty} P_n$, where P_n is defined by $P_1 = r+s$ and $P_{n+1} = r+s-(rs/P_n)$ for $n = 1, 2, 3, \ldots$.

FQ B-344. **by Frank Higgins**

Let c and d be real numbers. Find $\lim_{n\to\infty} x_n$, where x_n is defined by $x_1 = c$, $x_2 = d$, and

$$x_{n+2} = (x_{n+1} + x_n)/2 \text{ for } n = 1, 2, 3, \ldots .$$

Recurrences: modular arithmetic

NAvW 420. **by Hosia W. Labbers, Jr.**

For all $n \geq 0$, $m \geq 0$, the Ackermann function $A(n, m)$ is recursively defined by the equations

$$A(0, m) = m + 1,$$
$$A(n + 1, 0) = A(n, 1),$$
$$A(n + 1, m + 1) = A(n, A(n + 1, m)).$$

Prove that for each k there exists $N(k)$ such that

$$A(n, n) \equiv A(n', n') \pmod{k}$$

for all $n, n' \geq N(k)$.

Recurrences: multiplicative Fibonacci sequences

FQ H-300.* **by James L. Murphy**

Given two positive integers A and B relatively prime, form a multiplicative Fibonacci sequence (A_i) with $A_1 = A$, $A_2 = B$, and $A_{i+2} = A_i A_{i+1}$. Now form the sequence of partial sums (S_n) where

$$S_n = \sum_{i=1}^{n} A_i.$$

(S_n) is a subsequence of the arithmetic sequence (T_n) where $T_n = A + nB$, and by Dirichlet's theorem we know that infinitely many of the T_n are prime. Does such a sparse subsequence (S_n) of the arithmetic sequence $A + nB$ also contain infinitely many primes?

Recurrences: order 1

AMM S18. **by V. E. Hoggatt, Jr. and P. L. Mana**

Let $\{a_n\}$ be defined by $a_1 = 1$, $a_{n+1} = 2 + a_n$ if n is in $A_n = \{a_1, a_2, \ldots, a_n\}$, and $a_{n+1} = 1 + a_n$ if n is not in A_n. Also let $a_0 = 0$. For integers k and n with $0 \le k \le n$, let $\left[{n \atop k}\right] = a_n - a_k - a_{n-k}$. Prove that:

(a) There are an infinite number of integers m such that $\left[{m \atop k}\right] = 1$ for $0 < k < m$.

(b) There are an infinite number of integers r such that $\left[{r-s+t \atop t}\right] = \left[{s \atop t}\right]$ for $0 \le t \le s \le r$.

MM 1079. **by James Propp**

Define $a_0 = 1$ and $a_{n+1} = (a_n - 2)/a_n$ for $n \ge 0$.

(a) Show that the set $\{a_n \,|\, n = 0, 1, 2, \ldots\}$ is unbounded.

(b) There exists a real number α such that $\{n \,|\, a_n \ge 1\} = \{\lfloor k\alpha \rfloor \,|\, k = 0, 1, 2, \ldots\}$. Find α.

(c) Find the closure of the set defined in part (a).

PARAB 388.

Let a list of integers $a_1, a_2, \ldots, a_n$ be defined in succession by

$$a_{n+1} = a_n^2 - a_n + 1 \quad \text{and} \quad a_1 = 2.$$

Show that the integers are pairwise relatively prime.

PARAB 389.

Let a list of integers $a_1, a_2, \ldots, a_n$ be defined in succession by

$$a_{n+1} = a_n^2 - a_n + 1 \quad \text{and} \quad a_1 = 2.$$

Show that for N sufficiently large,

$$\left| \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - 1 \right| < \frac{1}{10^{10}}$$

for all $n > N$.

Recurrences: order 2

TYCMJ 56. **by Joseph Rothschild**

Let the sequence of integers (a_i), $i = 1, 2, \ldots$, be defined by $a_n = a_{n-1} + 2a_{n-2} + 4n$, with $a_0 = -4$ and $a_1 = -5$. Determine a_n as a function of n.

CANADA 1976/2.

Suppose

$$n(n+1)a_{n+1} = n(n-1)a_n - (n-2)a_{n-1}$$

for every positive integer $n \ge 1$. Given that $a_0 = 1$, $a_1 = 2$, find

$$\frac{a_0}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{50}}{a_{51}}.$$

FQ H-297. **by V. E. Hoggatt, Jr.**

Let $P_0 = P_1 = 1$, $P_n(\lambda) = P_{n-1}(\lambda) - \lambda P_{n-2}(\lambda)$. Show

$$\lim_{n\to\infty} \frac{P_{n-1}(\lambda)}{P_n(\lambda)} = \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda} = \sum_{n=0}^{\infty} C_{n+1} x^n,$$

where C_n is the nth Catalan number. Note that the coefficients of $P_n(\lambda)$ lie along the rising diagonals of Pascal's triangle with alternating signs.

Recurrences: order 3

FQ B-359. **by R. S. Field**

Find the first three terms T_1, T_2, and T_3 of a Tribonacci sequence of positive integers for which

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{and}$$

$$\sum_{n=1}^{\infty} \left(\frac{T_n}{10^n} \right) = \frac{1}{T_4} \; .$$

PME 445. **by Richard S. Field**

A "Tribonacci-like" integer sequence $\{A_n\}$ is defined in which $m_1 A_i + m_2 A_{i+1} + m_3 A_{i+2} = A_{i+3}$ ($A_0 = A_1 = A_2 = 1$; m_1, m_2, m_3 are arbitrary integers).

A particular sequence of this kind is found ($m_1 = -1$, $m_2 = 5$, $m_3 = 5$) which appears to yield only perfect squares, viz.: 1, 1, 1, 9, 49, 289, 1681, ...

(a) Prove that, for this particular sequence, the successive terms continue to be perfect squares.

(b) Can other values of m_1, m_2, and m_3 be found which result in the same property; namely, a sequence of perfect squares?

Recurrences: rates of divergence

DELTA 5.2-1. **by R. C. Buck**
DELTA 6.1-1. **by R. C. Buck**

Let $x_1 = 1$ and $x_{n+1} = x_n + \left(\frac{1}{x_n}\right)^3$ for $n \ge 1$. Prove that $\lim_{n\to\infty} x_n = \infty$ and that, in fact, x_n approaches infinity more slowly than $\sqrt[3]{n}$.

Recurrences: square roots

AMM 6196. **by Daniel Shanks**

(a) Let $-5 < x_0 < 0$ and let

$$x_n = \begin{cases} \sqrt{x_{n-1} + 5}, & \text{if} \quad n \not\equiv 0 \pmod 4, \\ -\sqrt{x_{n-1} + 5}, & \text{if} \quad n \equiv 0 \pmod 4. \end{cases}$$

Identify the numbers toward which x_{4m}, x_{4m+1}, x_{4m+2}, and x_{4m+3} converge as $m \to \infty$.

(b) Let p be a prime for which $(5|p) = +1$, so that $\sqrt{5}$ exists modulo p. Show that

$$(15 \pm 6\sqrt{5} \; |p) = +1, \text{or } -1,$$

according as $p \equiv \pm 1 \pmod{15}$ or $p \equiv \pm 4 \pmod{15}$, respectively.

(c) What is the relation between problems (a) and (b)?

Recurrences: sum of digits

OSSMB 78-15.
OSSMB 79-17. **by Greg Bennett**

From an arbitrary initial positive integer a_0, a sequence $\{a_n\}$ is constructed by alternately performing the two operations:

(1) adding the digits,

(2) raising to the kth power, k an arbitrary but fixed integer ($k \geq 2$), with either (1) or (2) used first.

Determine whether every such sequence $\{a_n\}$ eventually cycles.

Recurrences: systems of recurrences

JRM 784. **by Friend H. Kierstead, Jr.**

Let $p_1 = q_1 = 1$; $p_{n+1} = p_n + q_n$; $q_{n+1} = 2p_n + q_n$. Find a relation between p_n and q_n.

Repdigits

CRUX 339.* **by Steven R. Conrad**

Is $\binom{37}{2} = 666$ the only binomial coefficient $\binom{n}{r}$ whose decimal representation consists of a single digit repeated k times for $k \geq 3$?

MM 1046. **by Daniel J. Aulicino**

For an arbitrary positive integer k, consider the decimal integer h consisting of m copies of k followed by n zeros. Show that for each positive integer x, there exist an $m, m \neq 0$, and an n such that x divides h.

SSM 3582. **by Bob Prielipp**

Prove that, given any odd number q not divisible by 5, and any single digit d, $1 \leq d \leq 9$, there is a number of the form $ddd \ldots d$ that is divisible by q.

TYCMJ 93. **by Dan Aulicino**

Let k be a nonzero decimal digit and n a positive integer. Prove that there exists an integer m such that $n \mid m$ and the decimal representation of m consists of a block of digits, each equal to k followed by a block of zeros.

PARAB 321.

The following factorizations of numbers are true:

$$12 = 3 \cdot 4;$$

$$1122 = 34 \cdot 33;$$

$$111222 = 334 \cdot 333;$$

$$11112222 = 3334 \cdot 3333.$$

Can this scheme be continued indefinitely?

JRM 756. **by Daniel P. Shine**

(a) What is the smallest integer composed of $2n$ identical digits that is the product of two n-digit integers?

(b) What is the smallest such integer that has an n-digit prime factor?

(c) Is there any such integer that has two n-digit prime factors?

JRM 676. **by Joseph D. Thompson**

Let $\mathtt{D}_n$ denote the digit $\mathtt{D}$ repeated n times. For example, $8_4 = 8888$. Let $S(n) = \mathtt{D}_1 + \mathtt{D}_2 + \mathtt{D}_3 + \cdots + \mathtt{D}_n$.

(a) Show that there is at least one nonzero value of $\mathtt{D}$ such that for infinitely many values of n, $S(n)$ contains at most two different digits.

(b) Let ϕ represent the final four digits of $S(n)$, and let $\mathtt{D} = 1$. Find all $S(n)$ such that $\phi = n$ and ϕ is divisible by 9.

Repunits

PENT 316. **by Randall J. Covill**

A Fermat number has the form $2^{2^k} + 1$ for any integer $k > 0$. Are any Fermat numbers also repunits?

PENT 320. **by Michael W. Ecker**

Define a permuted repunit pair (PRP) to be a pair of positive integers x, y with $x > y$ such that

(1) the decimal digits of x and y are permutations of one another; and

(2) $x + y =$ a repunit.

If n is the number of ones in a given repunit, for which values of n do corresponding PRP's exist? For a given integer n for which PRP's exist, find the PRP (x, y) such that the product xy is a maximum.

FUNCT 1.4.4.

Prove that no number in the sequence

$$11,\ 111,\ 1111,\ 11111,\ \ldots$$

is the square of an integer.

ISMJ 12.8.
ISMJ 14.13.

Let A be the $2n$-digit number whose digits are all 1's and let B be the n-digit number whose digits are all 2's. Show that $A - B$ is a perfect square.

Riemann zeta function

NAvW 429. **by H. Jager**

Let $f(x)$ denote the number of ordered pairs (m, n) of positive integers satisfying $1 < m < n \leq x$, $\gcd(m, n) = 1$, $m^2 \equiv 1 \pmod{n}$. Prove that

$$f(x) = \frac{1}{\zeta(2)} x \log x + \frac{1}{\zeta(2)} \left\{ 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} - \zeta(2) - \frac{1}{2} \log 2 \right\} x + O(x^{\frac{1}{2}} \log x), \qquad (x \to \infty),$$

where ζ is Riemann's zeta-function and γ is Euler's constant.

Sequences: binary sequences

AMM E2544. **by Harvey Cohn**

Consider the sequence of words formed by "Fibonaccian juxtaposition": $w_1 = 0$, $w_2 = 1$, $w_{n+2} = w_n w_{n+1}$ for $n \geq 1$. Form the sequence S by

$$S = w_1 w_2 w_3 w_4 \ldots = 010110101101 \ldots .$$

Now let

$$\alpha = \frac{1}{2}\left(\sqrt{5} - 1\right)$$

and define

$$t_n = \lfloor n\alpha \rfloor - \lfloor (n-1)\alpha \rfloor$$

for $n = 1, 2, \ldots$. Form the sequence $T = t_1 t_2 t_3 \ldots$. Show that the sequences S and T are identical. Generalize.

Sequences: binomial coefficients

AMM S1. **by George Pólya**

Consider the composite integer n and the three sequences

$$\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1},$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix}, \begin{bmatrix} n \\ 3 \end{bmatrix}, \ldots, \begin{bmatrix} n \\ n-1 \end{bmatrix},$$

$$\begin{Bmatrix} n \\ 2 \end{Bmatrix}, \begin{Bmatrix} n \\ 3 \end{Bmatrix}, \ldots, \begin{Bmatrix} n \\ n-1 \end{Bmatrix},$$

(binomial coefficients, Stirling numbers of the first and second kind, respectively).

Prove that each sequence contains a term not divisible by n.

Sequences: consecutive integers

JRM 502. **by Michael Lauder**

Let n be any positive integer. Show that there exists a sequence of consecutive positive integers such that for each element k in the sequence, at least n other elements in the sequence share divisors with k that are greater than 1.

Sequences: counts

NAvW 515. **by N. G. de Bruijn**

Let $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ be sequences of elements of $\mathbb{N}$. Show the existence of

$$a_1, a_2, \ldots \in \mathbb{N}$$

such that, for every $k \in \mathbb{N}$, there are exactly n_k values of $i \in \mathbb{N}$ with $a_i = k$ and exactly m_k values of $j \in \mathbb{N}$ with $\left|a_{j+1} - a_j\right| = k$.

Sequences: density

NAvW 473. **by J. van de Lune**

For $n \in \mathbb{N}$, let $\beta(n)$ denote the largest square-free divisor of n. Let $\alpha \in (0,1)$. Prove that the natural density of the integers m, having the property $\beta(m) \le m^{\alpha}$, is zero.

Sequences: digits

PARAB 421.

In the sequence 19796 . . . , each digit after 6 is the sum of the preceding four digits. Show that . . . 1979 . . . turns up again in the sequence, but that . . . 1980 . . . never occurs at all.

MSJ 468.

Beginning with 2 and 7, the sequence of numbers 2, 7, 1, 4, 7, 4, 2, 8, . . . is constructed by multiplying successive pairs of its members and adjoining the result as the next one or two members of the sequence depending on whether the product is a 1- or a 2-digit number. Prove that the digit 6 appears an infinite number of times in the sequence.

Sequences: divisibility

SPECT 9.8. **by B. G. Eke**

Let S_1 denote the sequence of positive integers, and define the sequence S_{n+1} in terms of S_n by adding 1 to those integers in S_n that are divisible by n. Determine those integers n with the property that the first $n-1$ integers in S_n are n.

Sequences: family of sequences

CRUX 355.* **by James Gary Propp**

Given a finite sequence $A = (a_n)$ of positive integers, we define the family of sequences

$$A_0 = A; \quad A_i = (b_r), \quad i = 1, 2, 3, \ldots,$$

where b_r is the number of times that the rth lowest term of A_{i-1} occurs in A_{i-1}.

For example, if $A = A_0 = (2,4,2,2,4,5)$, then $A_1 = (8,2,1)$, $A_2 = (1,1,1)$, $A_3 = (3)$, and $A_4 = (1) = A_5 = A_6 = \ldots$.

The *degree* of a sequence A is the smallest i such that $A_i = (1)$.

(a) Prove that every sequence considered has a degree.

(b) Find an algorithm that will yield, for all integers $d \ge 2$, a shortest sequence of degree d.

(c) Let $A(d)$ be the length of the shortest sequence of degree d. Find a formula, recurrence relation, or asymptotic approximation for $A(d)$.

(d) Given sequences A and B, define C as the concatenation of A and B. Find sharp upper and lower bounds on the degree of C in terms of the degrees of A and B.

MM 1047. **by James Propp**

Given an infinite sequence $A = (a_n)$ of positive integers, we define a family of sequences A_i, where $A_0 = A$ and $A_i = (b_r)$ for $i = 1, 2, 3, \ldots$, where b_r is the number of times that the rth lowest term of A_{i-1} occurs in A_{i-1}. For example, if $A = A_0 = \{1,2,2,3,3,3,4,4,4,4\ldots\}$, then $A_1 = \{1,2,3,4,\ldots\}$ and $A_2 = \{1,1,1,1,\ldots\}$.

(a) Find a nondecreasing sequence A such that the sequences A_i are all distinct.

(b) Let $T = (t_n)$ be the unique nondecreasing sequence containing all the positive integers which has the property that $T_1 = T_0$. Define $U = (u_n)$ and $V = (v_n)$ so that for all $n, u_n = t_{2n-1}$ and $v_n = t_{2n}$. Are the sequences U_i and V_i all distinct?

Sequences: finite sequences

OSSMB 75-16.

For certain natural numbers n, it is possible to construct a sequence in which each of the numbers $1, 2, 3, \ldots, n$ occurs twice, the second occurrence of the number r being r places beyond its first occurrence. Prove that such a sequence cannot exist unless n is congruent to 0 or 1 (mod 4).

OSSMB 77-18.

Let $S = \{a_1,\ a_2, \ldots, a_n\}$ be a sequence of length n where each a_i is chosen from $\{1,\ 2,\ 3, \ldots, k\}$, and denote by M_s the maximum term in the sequence.

Show that

$$\sum M_s = k^{n+1} - \{1^n + 2^n + 3^n + \cdots + (k-1)^n\}$$

where the sum is taken over all such sequences.

ISMJ 14.3.

Suppose $0 < n_1 < n_2 < \cdots < n_{15}$ are integers. Assume that $n_r n_s = n_{rs}$ whenever $r \neq s$ and $rs \leq 15$.

(a) Show $n_3 < n_2^2$.

(b) If $n_2 = 2$, show that $n_{15} = 15$.

JRM 619. **by Alfred H. Tannenburg**

Let points $x_1, x_2, \ldots, x_n$ be selected in $[0, 1)$ in such a way that x_1 and x_2 lie in different halves of the interval; x_1, x_2, and x_3 lie in different thirds; and in general, for $k = 2, 3, \ldots, n$, the points $x_1, x_2, \ldots, x_k$ lie in different $k$ths of $[0, 1)$. It is well known that such a selection is possible only if $n \leq 17$.

Selections $(0, 1/2)$, $(0, 2/3, 1/3)$, and $(0, 3/4, 1/2, 1/4)$ satisfy the above conditions for $n = 2, 3, 4$, respectively, and in each case the sum of the n values is minimal among all such selections. Extend this list of minimal selections up to the case $n = 17$.

Sequences: floor function

CMB P243. **by L Kuipers**

Let k be an integer ≥ 1 and let $\beta_1, \beta_2, \ldots, \beta_k$ be k irrational numbers. Let the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ and 1 be linearly independent over the rationals. Prove that the sequence

$$([n\alpha_1]\beta_1 + [n\alpha_2]\beta_2 + \cdots + [n\alpha_k]\beta_k), \quad n = 1, 2 \ldots$$

is uniformly distributed mod 1 if and only if the numbers

$$\alpha_1\beta_2 + \cdots + \alpha_k\beta_k, \alpha_1, \alpha_2, \ldots, \alpha_k, 1$$

are linearly independent over the rationals.

FQ B-349. **by Richard M. Grassl**

Let $a_0, a_1, a_2, \ldots$ be the sequence $1, 1, 2, 2, 3, 3, \ldots$; i.e., let $a_n = \lfloor 1 + (n/2) \rfloor$. Give a recursion formula for the a_n and express the generating function

$$\sum_{n=0}^{\infty} a_n x^n$$

as a quotient of polynomials.

Sequences: inequalities

JRM 673. **by David L. Silverman**

Define a *cool* sequence as a sequence of positive integers for which the sum of the square roots of the first $n+1$ terms is less than the nth term for every n. An example of a cool sequence is $3^2, 4^2, 5^2, \ldots$. One such sequence $a_1, a_2, \ldots$ will be considered *cooler* than another such sequence $b_1, b_2, \ldots$ if $a_k < b_k$ for some k and $a_i \leq b_i$ for all i.

(a) Prove that there is a *coolest* sequence, that it begins $6, 10, 14 \ldots$, and give the first ten terms.

(b) Can the sequence be represented by a closed formula?

JRM 788. **by David L. Silverman**

Define a *kool* sequence as a sequence of positive integers for which the sum of the square roots of the first $2n$ terms is less than or equal to the nth term for every n. One such sequence $a_1, a_2, \ldots$ will be considered *kooler* than another such sequence $b_1, b_2, \ldots$ if $a_k < b_k$ for some k and $a_i \leq b_i$ for all i. What is the *koolest* sequence?

Sequences: law of formation

MATYC 74. **by J. Kapoor**

Find the nth term of $16, 20, 50, 105, 196, 336, \ldots$.

OMG 14.1.2.

What is the rule of formation for the sequence: 0, 1, 10, 2, 100, 11, 1000, 3, 20, 101, 10000, 12, 100000, 1001, 110, 4, 1000000, 21, ... ?

SSM 3787. **by Charles W. Trigg**

How are the following two sequences related: 1, 2, 9, 64, 7776, ... and 1, 8, 81, 1024, 15625, 279936, ... ?

JRM 790. **by Joseph D. Thompson**

Given the sequence: $1, 484, 36926037, \ldots$

(a) Find the fourth member of the sequence.

(b) Find the number of digits in the fifth through ninth members.

CRUX 16. **by Léo Sauvé**

For $n = 1, 2, 3, \ldots$, the finite sequence S_n is a permutation of $1, 2, 3, \ldots, n$, formed according to a law to be determined. According to this law, we have

$$S_1 = (1)$$
$$S_2 = (1, 2)$$
$$S_3 = (1, 3, 2)$$
$$S_4 = (4, 1, 3, 2)$$
....
$$S_9 = (8, 5, 4, 9, 1, 7, 6, 3, 2)$$

Discover a law of formation which is satisfied by the above sequences, and then give S_{10}.

IMO 1978/3.

It shall be assumed that the set of all positive integers is the union of two disjoint subsets $\{f(1), f(2), \ldots, f(n), \ldots\}$ and $\{g(1), g(2), \ldots, g(n), \ldots\}$, where

$$f(1) < f(2) < \cdots < f(n) < \cdots,$$
$$g(1) < g(2) < \cdots < g(n) < \cdots,$$

and

$$g(n) = f(f(n)) + 1 \text{ for } n = 1, 2, \ldots .$$

Determine $f(240)$.

MM 961. **by Erwin Schmid**

The sequence $11^0, 11^1, 11^2, \ldots$ of integral powers of the number 11, reduced modulo 50 (i.e. $1, 11, 21, 31, 41, 1, \ldots$) is in both geometric and arithmetic progression. What is the law of formation for such sequences?

CRUX 326. **by Harry D. Ruderman**

If the members of the set

$$S = \{2^x 3^y | x, y \text{ are nonnegative integers}\}$$

are arranged in increasing order we get the sequence beginning

$$1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 9,\ 12,\ 16,\ 18, \ldots .$$

(a) What is the position of $2^a 3^b$ in the sequence in terms of a and b?

(b) What is the nth term of the sequence in terms of n?

FQ B-340. **by Philip Mana**

Characterize a sequence whose first 28 terms are: 1779, 1784, 1790, 1802, 1813, 1819, 1824, 1830, 1841, 1847, 1852, 1858, 1869, 1875, 1880, 1886, 1897, 1909, 1915, 1920, 1926, 1937, 1943, 1948, 1954, 1965, 1971, 1976.

Sequences: limits

AMM 6271. **by Michael Barr**

For positive integers n, define

$$a_n = \frac{n-1}{n} + \frac{(n-1)(n-2)}{n^2} + \cdots + \frac{(n-1)!}{n^{n-1}},$$
$$b_n = \frac{n}{n+1} + \frac{n^2}{(n+1)(n+2)} + \cdots + \frac{n^{n-1}}{(n+1)\cdots(2n-1)}.$$

(a) Prove that for all $n > 1$, $0 < b_n - a_n < 1$.
(b) Prove or disprove that

$$\lim_{n\to\infty}(b_n - a_n) = \frac{2}{3}$$

and that

$$b_n - a_n - \frac{2}{3} = O\left(\frac{1}{n}\right).$$

Sequences: monotone sequences

CRUX 474. **by James Propp**

Suppose (s_n) is a monotone increasing sequence of natural numbers satisfying $s_{s_n} = 3n$ for all n. Determine all possible values of s_{1979}.

NAvW 422. **by J. van de Lune**

For $n \in \mathbb{N}$ and $s \in \mathbb{C}$, we define

$$Q_n(s) = \sum_{k=0}^{n-1}(-1)^k(n-k)^s.$$

Prove that if $s \in \mathbb{N}$ and $s \geq 2$, then $n^{-s}Q_n(s)$ is decreasing (in n).

IMO 1975/2.
PARAB 378.

Let $a_1, a_2, a_3, \cdots$ be an infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many a_m which can be written in the form

$$a_m = xa_p + ya_q$$

with x and y positive integers and $q > p$.

MM 1008. **by P. Erdős and Melvyn B. Nathanson**

(a) Let (a_n) be an increasing sequence of positive integers and let $S_n = a_1 + a_2 + \cdots + a_n$. Show that if $\underline{\lim}\ a_n/n > 2 + \sqrt{2}$, then for all n sufficiently large there exists a perfect square between S_n and S_{n+1}.

(b) If $\underline{\lim}\ a_n/n = 2 + \sqrt{2}$ and the above conclusion fails, then show that $\overline{\lim}\ a_n/n = \infty$.

MATYC 133. **by Ely Stern and Leo Chosid**

Find the least positive integer N for which

$$F(n) = \frac{n+1}{n} \cdot \frac{1}{\sqrt[n]{n}}$$

is monotone increasing for $n > N$.

MM 1073.* **by James Propp**

Let A and B be the unique nondecreasing sequences of odd integers and even integers, respectively, such that for all $n \geq 1$, the number of integers i satisfying $A_i = 2n-1$ is A_n and the number of integers i satisfying $B_i = 2n$ is B_n. That is, $A = (1,3,3,3,5,5,5,7,7,7,9,9,9,9,9,\ldots)$ and $B = (2,2,4,4,6,6,6,6,8,8,8,8,\ldots)$. Is the difference $|A_n - B_n|$ bounded?

Sequences: partitions

OSSMB 75-18.

Divide $\mathbb{N}$ into groups, as follows: (1), $(2,3)$, $(4,5,6)$, $(7,8,9,10)$, $(11,12,13,14,15), \ldots$. Delete every second group. Prove that the sum of the elements in the first k groups that remain is k^4.

PENT 318. **by Charles W. Trigg**

Find the sum of the integers in the nth group, where the groups are given by: (1), $(2,3,4,5)$, $(6,7,8,9,10,11,12)$, $(13,14,15,16,17,18,19,20,21,22), \ldots$.

SSM 3763. **by Fred A. Miller**

The series of natural numbers is divided into groups 1, $(2+3+4)$, $(5+6+7+8+9)$, $(10+11+12+13+14+15+16)$, $\ldots$. Find the sum of the numbers in the nth group.

Sequences: products

FQ H-271.* **by R. Whitney**

Define the binary dual, D, as follows:

$$D = \left\{ t = \prod_{i=0}^{n}(a_i + 2i); \quad a_i \in \{0,1\}; \quad n \geq 0 \right\}.$$

Let $\overline{D}$ denote the complement of D, with respect to the set of positive integers. Form a sequence, $\{S_n\}_{n=1}^{\infty}$, by arranging $\overline{D}$ in increasing order. Find a formula for S_n.

Sequences: rational numbers

OSSMB 78-9. **by David Ash**

A sequence $\{c_n\}$ of rational numbers, $c_n = a_n/b_n$, a_n, b_n positive integers with $\gcd(a_n, b_n) = 1$ is defined as follows:

(0) $a_1 = 5$, $a_2 = 7$, $b_1 = 7$, $b_2 = 10$,
(1) for all n, $a_n b_{n+1} - a_{n+1} b_n = 1$,
(2) for all n, $a_{n+1} \geq b_n$,
(3) for all n, a_{n+1} is the smallest positive integer satisfying (1) and (2). Find the limit of the sequence $\{c_n\}$.

Sequences: runs

AMM 6281.* **by Clark Kimberling**

If $A = (1, a_1, a_2, \ldots)$ is a sequence of 1's and 2's, let $B = (1, b_1, b_2, \ldots)$, where b_n is the length of the nth maximal string of identical symbols in A. If $B = A$, then A must be $(1,2,2,1,1,2,1,2,2,1,\ldots)$. By a run is meant a finite subsequence of consecutive terms of A. Its complement is obtained by interchanging all 1's and 2's.

Prove or disprove:
(a) The complement of every run is also a run.
(b) Every run occurs infinitely many times.

Sequences: subsequences

PARAB 331.

Suppose that n^2+1 boys are lined up shoulder-to-shoulder in a straight line. Show that it is always possible to select $n+1$ boys to take one pace forward so that, going from left to right, their heights are either increasing or decreasing.

Sequences: sum of consecutive terms

MSJ 483.

A student preparing for a mathematics contest to be held in eleven weeks solves at least one problem every day but no more than twelve a week. Prove that during this preparation there is at least one set of consecutive days in which the student solves exactly twenty problems.

OSSMB 77-10.

Let $n_1, n_2, \ldots, n_{30}$ be a sequence of positive integers whose sum is at most 48. We say that an integer k is attainable if for some i and j ($i \le j$) we have $n_i + n_{i+1} + \cdots + n_j = k$. Find all k that are attainable for every such sequence of n's.

OSSMB 77-9.

A doctor wishing to test a new medication, gives a test-patient a batch of 48 pills and instructs him to take pills over a 30-day period. The patient is at liberty to distribute the pills however he likes, subject to the condition that he take at least one pill each day. Show that there is some stretch of consecutive days for which the total number of pills taken over those days is 11.

CRUX PS3-2.

Prove that from any row of n integers one may always select a block of adjacent integers whose sum is divisible by n.

IMO 1977/2.
OSSMB 77-12.
PARAB 365.

In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

Sequences: trees

ISMJ 10.12.

An increasing sequence of integers starting with 1 has the property that if n is a member of the sequence then both $3n$ and $n+7$ are also members of the sequence. Also, all the members are generated from just the first member 1. Determine all the positive integers that are not members of the sequence.

Series: alternating series

MATYC 84. **by Gary Baldwin**

Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} .$$

OSSMB G76.3-5.

Find the sum of n terms of the series

$$\frac{5}{1\cdot 2} - \frac{3}{2\cdot 3} + \frac{9}{3\cdot 4} - \frac{7}{4\cdot 5} + \frac{13}{5\cdot 6} - \frac{11}{6\cdot 7} + \frac{17}{7\cdot 8} - \cdots .$$

Series: arithmetic progressions

MATYC 111. **by Gene Zirkel**

Let a_i be the ith term of an arithmetic progression consisting of n terms, $n \ge 2$. Select an integer k, such that $0 \le k \le n-2$. Show that

$$\sum_{i=1}^{n} a_i^k (-1)^{i-1} \binom{n-1}{i-1} = 0.$$

Series: binomial coefficients

CRUX 183. **by Viktors Linis**

If $x+y=1$, show that

$$x^{m+1} \sum_{j=0}^{n} y^j C_{m+j}^{j} + y^{n+1} \sum_{i=0}^{m} x^i C_{n+1}^{i} = 1$$

holds for all $m,\ n = 0, 1, 2, \ldots$.

MM 1049. **by Edward T. H. Wang**

For nonnegative integers n, let $L_n = \binom{2n}{n}/(n+1)$. Prove that

$$\sum_{k=0}^{n} L_k L_{n-k} = L_{n+1}.$$

TYCMJ 116. **by V. N. Murty**

Assume r and n are nonnegative integers and $r \le n$. Evaluate:

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^r.$$

SSM 3758. **by Herta T. Freitag**

Prove that for each positive integer n,

$$\sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} = \sum_{k=2}^{n} \sum_{i=k}^{n} (-1)^i \binom{n}{i}.$$

FQ H-253. **by L. Carlitz**

Show that

$$\sum_{t=0}^{k} \binom{(\beta-1)n+t+1}{t} \sum_{j=0}^{n-k-1} \binom{n-k-1}{j}$$

$$\cdot \sum_{m=0}^{j} (-1)^{n+m+k+1} \binom{j}{m}$$

$$\cdot \sum_{r=0}^{n+m-t-j-1} \binom{j}{n+m-j-t-r-1} \binom{2j+r-1}{r}$$

$$= 2^{n-k-1} \binom{\beta n}{k},$$

where β is an arbitrary complex number and n and k are positive integers, $k < n$.

AMM E2685. **by Ronald Evans**

If p is an odd prime, show that

$$\sum_{i=0}^{p-1} (-1)^i \binom{p^2-p}{pi} \equiv p^{p-1} \pmod{p^p}.$$

AMM E2770. **by Warren Page**

Let n and N be fixed positive integers, and let

$$S_k = \sum_{m=1}^{n} m^k$$

for $k = 1, 2, \ldots, N$. Prove

$$\text{(a)} \quad \sum_{h=1}^{N} \sum_{k=1}^{h} \binom{h+1}{k} S_k = \frac{n+1}{n} \left[(n+1)^{N+1} - (N+1)n - 1 \right]$$

and

$$\text{(b)} \quad \sum_{h=1}^{N} \sum_{k=1}^{h} (-1)^h \binom{h+1}{k} S_k = \begin{cases} \frac{n+1}{n+2} \left[(n+1)^{N+2} + 1 \right], & \text{for odd } N, \\ \frac{(n+1)^2}{n+2} \left[(n+1)^{N} - 1 \right], & \text{for even } N. \end{cases}$$

CRUX 368. **by Lai Lane Luey**

Let a and n be integers with $a \geq n \geq 0$, c any constant, and

$$f(a) = \sum_{k=0}^{a} (-1)^k \binom{a}{k} (a-k+c)^n.$$

Prove that $f(a) = 0$ if $a > n$ and $f(n) = n!$.

MATYC 76. **by Etta Mae Whitton**

Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^n = n!\ .$$

FQ H-255. **by L. Carlitz**

Show that

$$\sum_{j=0}^{2m} \sum_{k=0}^{2n} (-1)^{j+k} \binom{2m}{j} \binom{2n}{k} \binom{2m+2n}{j+k} \binom{2m+2n}{2m-j+k} = (-1)^{m+n} \frac{(3m+3n)!(2m)!(2n)!}{m!n!(m+n)!(2m+n)!(m+2n)!}.$$

FQ B-380. **by Dan Zwillinger**

Let a, b, and c be nonnegative integers. Prove that

$$\sum_{k=1}^{n} \binom{k+a-1}{a} \binom{n-k+b-c}{b} = \binom{n+a+b-c}{a+b+1}.$$

SIAM 79-13. **by T. V. Narayana and M. Özsoyoglu**

Prove that

$$\sum_{i=1}^{n} i \frac{m-n+2i+1}{m+n+1} \binom{m+n+1}{n-i} = \sum_{i=1}^{n} i 2^{i-1} \frac{m-n+i}{m+n-i} \binom{m+n-i}{m}$$

where m, n are integral with $m > n > 0$.

AMM 6123.* **by E. G. Kundert**

Let s be any integer ≥ 2, and let ε_i be the following function defined on the integers:

$$\varepsilon_i = \begin{cases} 0, & \text{if } i \equiv 0, 6 \\ 1, & \text{if } i \equiv 2, 4, 7, 11 \\ -1, & \text{if } i \equiv 1, 5, 8, 10 \\ 2, & \text{if } i \equiv 9 \\ -2, & \text{if } i \equiv 3. \end{cases} \pmod{12}$$

Show that the following identity holds:

$$\sum_{1 \leq i,j \leq s} \varepsilon_i \varepsilon_j \binom{j+1}{s-i} \binom{s+1}{j+1} 3^{\lfloor i/2 \rfloor + \lfloor j/2 \rfloor - \lfloor (s-2)/2 \rfloor} = -3\varepsilon_s.$$

TYCMJ 124. **by Norman Schaumberger**

Assume that $a > 1$ is an integer. Prove that

$$\sum_{n=a}^{\infty} \frac{1}{\binom{n}{a}} = \frac{a}{a-1}.$$

SIAM 75-4. **by P. Barrucand**

Let

$$A(n) = \sum_{i+j+k=n} \frac{n!^2}{i!^2 j!^2 k!^2}\ ,$$

where i, j, and k are nonnegative integers, and let

$$B(n) = \sum_{m=0}^{n} \binom{n}{m}^3.$$

Prove that

$$A(n) = \sum_{m=0}^{n} \binom{n}{m} B(m).$$

FQ H-283. **by D. Beverage**

For $n \geq 0$, find a closed form for

$$\sum_{k=0}^{\infty} \binom{n+k}{n} \left(\frac{1}{2}\right)^{n+k}, \quad n \geq 0.$$

FQ H-272. **by L. Carlitz**

Show that

$$\sum_{j=0}^{m} \binom{r}{j} \binom{p}{m-j} \binom{q}{m-j} \binom{p+q-m+j}{j}$$

is symmetric in p, q, and r.

Series: congruences

MM Q656. **by Warren Page**

For each positive integer n, show that either

$$\sum_{k=1}^{n} k \equiv 1 \pmod 5 \quad \text{or} \quad \sum_{k=1}^{n} k^2 \equiv 0 \pmod 5.$$

Number Theory

Series: digit problems

AMM E2533. **by E. S. Pondiczery**

Calculate to an accuracy of 1% the sum of the reciprocals of the 8,877,690 positive integers whose decimal representations contain no repeated digits.

AMM E2675. **by R. P. Boas**

If n is a positive integer, let $f(n)$ be the number of zeros in the decimal representation of n. For which values of $a > 0$ is the following series convergent:

$$\sum_{n\geq 1} \frac{a^{f(n)}}{n^2} \ ?$$

CRUX 377. **by Michael W. Ecker**

For $n = 1, 2, 3, \ldots$, let $f(n)$ be the number of zeros in the decimal representation of n, and let

$$F(p) = \sum_{n=1}^{\infty} \frac{f(n)}{n^p}.$$

Find the real values of p for which the series $F(p)$ converges.

AMM E2529. **by S. W. Golomb**

Let $N_k(n)$ denote the number of "digits" in the base k representation of the natural number n. Show that if

$$S_k = \sum_{n=1}^{\infty} \frac{1}{n\,(N_k(n))^2},$$

then $S_k \sim A \log k$ for some constant A. Find A and estimate the error term.

Series: divisibility

TYCMJ 67. **by Richard Johnsonbaugh**

Let x and a_i $(i = 0, 1, 2, \ldots, k)$ be arbitrary integers. Prove or disprove that $\sum_{i=0}^{k} a_i(x^2+1)^{3i}$ is divisible by $x^2 \pm x + 1$ if and only if $\sum_{i=0}^{k}(-1)^i a_i$ is divisible by $x^2 \pm x + 1$.

Series: factorials

TYCMJ 66. **by B. Bernstein**

Find a closed form expression for

$$\sum_{n=1}^{\infty} \frac{n^3}{(2n+1)!}.$$

SSM 3762. **by Fred A. Miller**

Find the sum of the following infinite series:

$$\frac{2}{1!} + \frac{3}{2!} + \frac{6}{3!} + \frac{11}{4!} + \frac{18}{5!} + \cdots.$$

SSM 3785. **by William T. Bailey**

Find the sum of the following infinite series:

$$1 + \frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + \frac{4^2}{5!} + \cdots.$$

MM 985. **by Jeffrey Shallit**

Let

$$Q_k = \frac{1}{(k+2)!} + \frac{2}{(k+3)!} + \frac{3}{(k+4)!} + \cdots.$$

Show that Q_k is transcendental for all positive integers k, but rational for $k = 0$.

OSSMB G78.2-2.

Find the value of the series

$$2 + \frac{5}{2!3} + \frac{5\cdot 7}{3!3^2} + \frac{5\cdot 7\cdot 9}{4!3^3} + \frac{5\cdot 7\cdot 9\cdot 11}{5!3^4} + \cdots$$

by first getting an equivalent binomial form $(1-a)^b$.

FUNCT 2.5.2.

Observe that the value of

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$$

is $\frac{1}{2}$, $\frac{5}{6}$, $\frac{23}{24}$, for $n = 1, 2, 3$, respectively. Guess the general law and prove your guess.

PARAB 293.

Determine all positive integers n such that

$$1! + 2! + 3! + \cdots + n!$$

is a perfect square.

FQ H-270. **by L. Carlitz**

Sum the series

$$\sum_{a,b,c} \frac{x^a y^b z^c}{(b+c-a)!(c+a-b)!(a+b-c)!},$$

where the summation is over all nonnegative a, b, c such that

$$a \leq b+c, \quad b \leq c+a, \quad c \leq a+b.$$

MM 999. **by Joseph Silverman**

Let (a_i) and (b_i), $i = 1, 2, \ldots, k$, be natural numbers arranged in nondecreasing order. For which values of k is it true that

$$\sum_{i=1}^{k}(a_i!) = \sum_{i=1}^{k}(b_i!)$$

implies $a_i = b_i$ for all i?

What is the corresponding result if the two sequences are strictly increasing?

Series: floor function

FQ B-350. **by Richard M. Grassl**

Let $a_n = \lfloor 1 + (n/2) \rfloor$. Find a closed form for

$$\sum_{k=0}^{n} a_{n-k}(a_k + k)$$

(a) in which n is even, and
(b) in which n is odd.

CRUX 160. **by Viktors Linis**

Evaluate

$$\left\lfloor \sum_{n=1}^{10^9} n^{-2/3} \right\rfloor .$$

MM Q661. **by J. Phipps McGrath**

A professor wishes to add up the integral parts of the numbers $\ln n$ for the first 10^9 positive integers n. Show the professor how to simply evaluate his sum.

AMM E2758. **by Bruce C. Berndt and Ronald J. Evans**

Let c and d be relatively prime positive integers of opposite parity and define

$$F(d,c) = \sum_{j=1}^{c-1} (-1)^{j+1+\lfloor dj/c \rfloor}.$$

Prove that $F(d,c) + F(c,d) = 1$.

FQ B-377. **by Paul S. Bruckman**

For all real numbers $a \geq 1$ and $b \geq 1$, prove that

$$\sum_{k=1}^{\lfloor a \rfloor} \left\lfloor b\sqrt{1-(k/a)2} \right\rfloor = \sum_{k=1}^{\lfloor b \rfloor} \left\lfloor a\sqrt{1-(k/b)2} \right\rfloor .$$

Series: geometric series

SSM 3720. **by N. J. Kuenzi**

Prove that, for each real number q and for each positive integer n,

$$1 + q + q^2 + \cdots + q^{n-1} = \sum_{i=1}^{n} \binom{n}{i} q^{n-i}(1-q)^{i-1}.$$

MM Q615. **by Joseph A. Wehlen**

Let q be any positive integer except an integral power of 10. Let 10^a be the integral power of 10 satisfying the inequality

$$10^a > q > 10^{a-1}.$$

Expand $1/q$ as the sum of an infinite geometric series whose first term and ratio depend on only q and 10^a.

Series: identities

SSM 3778. **by Herta T. Freitag**

Verify that for each positive integer n

$$\left(\sum_{i=1}^{n} (2i-1) \right)^2 = \sum_{i=1}^{n} (4i^3 - 6i^2 + 4i - 1).$$

Series: inequalities

PME 339. **by P. Erdős**

Let $a_1 < a_2 < \cdots$ be a sequence of integers such that $\gcd(a_i, a_j) = 1$ and $a_{i+2} - a_{i+1} \geq a_{i+1} - a_i$. Prove that $\sum \frac{1}{a_i} < \infty$.

AMM 6247. **by Mihály Bencze**

Let $\alpha > 1$, $m > 1$, and $n > 1$ with m and n integers. Prove that

$$\sum_{k=1}^{n^m-1} \alpha^k \left\lfloor \sqrt[m]{k} \right\rfloor \leq (n-1)\frac{\alpha^{n^m} - \alpha^{(n/2)^m}}{\alpha - 1}.$$

CRUX 459. **by V. N. Murty**

If n is a positive integer, prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \leq \frac{\pi^2}{8} \cdot \frac{1}{1-2^{-2n}}.$$

CMB P273. **by Mihály Bencze**

If $s > 1$, show that

$$\sum_p \frac{1}{p^s} \leq -s\frac{\zeta'(s)}{\zeta(s)} \leq \sum_p \frac{1}{p^{s/2}}$$

where the sum extends over all positive prime numbers and ζ is the Riemann zeta function.

FQ H-258. **by L. Carlitz**

Sum the series

$$S = \sum x^a y^b z^c t^d,$$

where the summation is over all nonnegative a, b, c, d such that

$$\begin{aligned} 2a &\leq b + c + d, \\ 2b &\leq a + c + d, \\ 2c &\leq a + b + d, \\ 2d &\leq a + b + c. \end{aligned}$$

CRUX 108. **by Viktors Linis**

Prove that, for all integers $n \geq 2$,

$$\sum_{k=1}^{n} \frac{1}{k^2} > \frac{3n}{2n+1}.$$

Series: infinite series

FQ H-282. **by H. W. Gould and W. E. Greig**

Prove that

$$\sum_{n=1}^{\infty} \frac{\alpha^{2n}}{\alpha^{4n} - 1} = \sum_{k=1}^{\infty} \frac{1}{\alpha^{2k} - 1} ,$$

where k is odd and $\alpha = (1+\sqrt{5})/2$, and determine which series converges the faster.

MM 1048. **by P. Erdős**

Let (a_k) be an increasing sequence of positive integers with $a_{k+1}/a_k \to 1$ as $k \to \infty$. Prove that

$$\sum_{k=1}^{\infty} \frac{(a_k - 1)^2}{a_1 \cdots a_k}$$

is irrational. What happens if it is not assumed that $a_{k+1}/a_k \to 1$ but the series converges?

PME 360. **by P. Erdős and Ernst Straus**

Denote by A_n the least common multiple of the integers from 1 to n, and denote by $d(n)$ the number of divisors of n.

(a) Prove that $\sum_{n=1}^{\infty} \frac{1}{A_n}$ is irrational.

(b) Prove that $\sum_{n=1}^{\infty} \frac{d(n)}{A_n}$ is irrational.

(c) Prove that $\sum_{n=1}^{\infty} \frac{f(n)}{A_n}$ is irrational, where $f(x)$ is a polynomial with integer coefficients.

Series: least common multiple

NAvW 551. **by P. Erdős**

If $1 < a_1 < a_2 < \cdots$ is a sequence of integers such that, for some $c \in (0,1)$ and all x,

$$\operatorname{card}\{a_i \mid a_i \le x\} > cx,$$

then show that

$$\sum_{k=1}^{\infty} \frac{\operatorname{lcm}[a_1, a_2, \ldots, a_k]}{a_1 a_2 \cdots a_k}$$

is irrational.

Series: limits

MM 974. **by John P. Hoyt**

Let $n^{\underline{i}} = n(n-1)\cdots(n-i+1)$. For k a positive integer, evaluate

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n^{\underline{i}}}{(2n+k)^{\underline{i}}}.$$

AMM E2784. **by F. S. Cater**

For each positive integer n and each positive number x, let $F_n(x) = 0$ if $x < (n+1)^{-1}$, and let

$$F_n(x) = k^{-1}\left[(n+1)^{-1} + 2(n+2)^{-1} + \cdots + k(n+k)^{-1}\right]$$

if $x \ge (n+1)^{-1}$, where k is the largest integer satisfying

$$(n+1)^{-1} + (n+2)^{-1} + (n+3)^{-1} + \cdots + (n+k)^{-1} \le x.$$

Let

$$F(x) = \lim_{n\to\infty} F_n(x)$$

for $x > 0$. Determine the function $F(x)$. Is the convergence of $F_n(x)$ uniform in x? Find $\sup F(x)$ and $\sup [F(x)/x]$.

Series: logarithms

NAvW 538. **by P. Erdős**

Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers such that

$$\sum_{n=1}^{\infty} n^{-2} \log\log a_n$$

is convergent. Let $\sigma(a_k)$ be the sum of the reciprocals of the divisors of a_k that do not divide any a_i with $i < k$.

Show that

$$\sum_{k=1}^{\infty} k^{-1} \sigma(a_k)$$

is convergent.

Series: multinomial coefficients

FQ H-289. **by L. Carlitz**

Show that

$$\sum_{r+s+t=\lambda} (r,s,t)(m-2r, n-2s, p-2t)$$
$$= \sum_{i+j+k+u=\lambda} (-2)^{i+j+k}(i,j,k,u) \times (m-j-k, n-k-i, p-i-j)$$

whenever $m+n+p \ge 2\lambda$, where

$$(m_1, m_2, \ldots, m_k) = \frac{(m_1+m_2+\cdots+m_k)!}{m_1! m_2! \cdots m_k!}.$$

Series: multiples

CRUX 53. **by Léo Sauvé**

Show that the sum of all positive integers less than $10n$ and relatively prime to 2 and 5 equals $20n^2$.

PARAB 285.
ISMJ J11.10.

Find the sum of all the numbers from 1 to 300 that are multiples of 3 or 5 or 7.

Series: permutations

CRUX 69. **by Léo Sauvé**

Does there exist a permutation $n \mapsto a_n$ of the natural numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}$$

converges?

Series: polynomials

AMM 6010. **by L. Carlitz**

Coefficients $c_{m,n}^{(k)}$ are defined by means of

$$(1+x)^m (1-x)^n = \sum_{k=0}^{m+n} c_{m,n}^{(k)} x^k \qquad (m \ge 0, n \ge 0).$$

Show that

$$\sum_{k=0}^{m+n} \left(c_{m,n}^{(k)}\right)^2 = \frac{(2m)!(2n)!}{m!n!(m+n)!}.$$

Series: power series

FQ H-301. **by Verner E. Hoggatt, Jr.**

Let $A_0, A_1, A_2, \ldots, A_n, \ldots$ be a sequence such that the nth differences are zero (that is, the Diagonal Sequence terminates). Show that, if

$$A(x) = \sum_{i=0}^{\infty} A_i x^i,$$

then

$$A(x) = \frac{1}{1-x} \cdot D\left(\frac{x}{1-x}\right),$$

where

$$D(x) = \sum_{i=0}^{\infty} d_i x^i \text{ and } d_i = \Delta^i A_0.$$

Series: powers

FUNCT 3.5.3. **by Y.-T. Yu**

Set

$$S_r(n) = \sum_{k=1}^{n} k^r.$$

When does n divide $S_1(n)$? $S_2(n)$? $S_3(n)$? Can the result be generalized?

Series: powers of 2

CRUX 447. **by Viktors Linis**

The number

$$\sum_{k=1}^{n} \frac{2^k}{k}$$

is represented as an irreducible fraction p_n/q_n.

(a) Show that p_n is even.

(b) Show that if $n > 3$ then p_n is divisible by 8.

(c) Show that for every natural number k there exists an n such that all the numbers $p_n, p_{n+1}, \ldots$ are divisible by 2^k.

Series: primes

AMM 6016.* **by C. J. Moreno**

Let $D(n)$ be the function defined by $D(n) = \prod p$, where the product runs over those primes p such that $p-1$ divides $2n$. Find an asymptotic formula for the function

$$\sum_{n \le x} D(n).$$

PME 384. **by R. S. Luthar**

Discuss the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{p_n^2}$$

where p_n is the nth prime.

Series: Stirling numbers

FQ H-268. **by L. Carlitz**

Put

$$S_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k,$$

where $\left\{ {n \atop k} \right\}$ denotes the Stirling number of the second kind defined by

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)\cdots(x-k+1).$$

Show that

$$\begin{cases} xS_n(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} S_{j+1}(x) \\ S_{n+1}(x) = x \sum_{j=0}^{n} \binom{n}{j} S_j(x). \end{cases}$$

More generally, evaluate the coefficients $c(n,k,j)$ in the expansion

$$X^k S_n(x) = \sum_{j=0}^{n+k} c(n,\ k,\ j) S_j(x) \qquad (k, n \ge 0).$$

Series: subseries

MM 1025. **by W. C. Waterhouse**

Let (a_n) be a sequence of positive real numbers with $\sum a_n = \infty$ and $\sum a_n^2 < \infty$. For a given $C > 0$, the sequence (m_i) of positive integers is such that $\sum a_n > C$, the sum being over those n such that $m_i < n \le m_{i+1}$.

(a) Prove that there is a sequence (p_i) with $m_i < p_i \le m_{i+1}$, such that $\sum a_{p_i} < \infty$.

(b) Show by an example that $\sum a_{p_i}$ need not converge for all such (p_i).

Series: sum of squares

OSSMB G76.2-6.

Sum the series

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + - \cdots + (-1)^{n-1} n^2$$

without recourse to the formula for the sum of the squares of the natural numbers.

Series: unit fractions

CRUX 49. **by H. G. Dworschak**

Evaluate

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \cdots + \frac{1}{6n-5} - \frac{1}{6n-1} + \cdots.$$

AMM 6194. **by Erwin Just and Norman Schaumberger**

Let N be an arbitrary integer larger than 6, and let $\{a_i\}$, $i = 1, 2, \ldots, m$, denote the set of positive composite integers less than N that are not powers of primes. Prove that

$$\sum_{i=1}^{m} \frac{1}{a_i}$$

is not an integer.

NAvW 516. **by P. Erdős**

Let $a_1, a_2, \ldots$ be an increasing sequence of positive integers such that $\sum a_i^{-1}$ is convergent. Prove that for every i there are infinitely many sets of a_i consecutive integers that are not divisible by any a_j with $j > i$.

OSSMB 76-17.

In the evaluation of

$$\frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \cdots$$

as a decimal, what is the digit in the 37th decimal place?

PUTNAM 1978/B.2.

Express

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2 n + m n^2 + 2mn}$$

as a rational number.

SSM 3774. **by Fred A. Miller**

Find the sum of the following infinite series:

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \cdots.$$

JRM 762. **by R. Robinson Rowe**

Evaluate

$$\Big[\sum_{n=0}^{\infty} \frac{1}{(2n)!}\Big]^2 - \Big[\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\Big]^2.$$

IMO 1979/1.

Let p and q be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

SIAM 76-5. **by D. J. Newman**

To determine positive integers $a_1, a_2, \ldots, a_n$ such that

$$S_n \equiv \sum_{i=1}^{n} \frac{1}{a_i} < 1$$

and S_n is a maximum, it is conjectured that at each choice one picks the smallest integer still satisfying the inequality constraint. Is this conjecture true?

AMM E2719. **by John S. Lew**

For a fixed positive integer m, let S_m be the sum of the series

$$\pm 1 \pm \frac{1}{3} \pm \frac{1}{5} \pm \frac{1}{7} \pm \frac{1}{9} \pm \cdots,$$

where the first m terms have sign $+$, the next m terms have sign $-$, then the succeeding m terms have sign $+$, etc.

Evaluate S_2 and S_3.

AMM E2743. **by Peter Ungar**

Evaluate

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{(-1)^{i+j}}{i+j}.$$

AMM 6105. **by Harry D. Ruderman**

Prove that the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor n\sqrt{2}\rfloor}}{n}.$$

Estimate its value.

JRM 652. **by E. J. Barbeau**

Call the product of two distinct primes a *semiprime*. Unity can be represented as the sum of distinct unit fractions with semiprime denominators. Find the shortest such representation.

MM 1015.* **by Allan W. Johnson, Jr.**

Show that for $n \geq 5$ there are $2n+1$ distinct, positive, odd, square-free integers whose reciprocals add to one.

SSM 3643. **by John Hudson Tiner**

Does the infinite series

$$\frac{1}{1} + \frac{1}{1+3} + \frac{1}{1+3+5} + \frac{1}{1+3+5+7} + \cdots$$

converge or diverge?

Sets: arithmetic means

PARAB 354.

Prove that it is possible to select 2^k different numbers $a_1, \ldots, a_{2^k}$ from the set $\{0, 1, 2, \ldots, 3^k - 1\}$ in such a way that none of the a's is the arithmetic mean of any other two.

Sets: arithmetic progressions

AMM E2730. **by R. L. Graham**

Describe all finite sets A of real numbers with the property that any two elements of A belong to some three-term arithmetic progression in A.

PME 389.* **by P. Erdős**

Find a sequence of positive integers $1 \leq a_1 < a_2 < \cdots$ that omits infinitely many integers from every arithmetic progression (in fact it has density 0), but which contains all but a finite number of terms of every geometric progression. Prove also that there is a set S of real numbers that omits infinitely many terms of any arithmetic progression, but contains every geometric progression (disregarding a finite number of terms).

Sets: closed under product

NAvW 392. **by F. Beukers**

Let S be a subset of $\mathbb{N}$. A number $p \in S$, $p \neq 1$, is called an S-prime when p cannot be written as the product of two smaller elements of S. Let A be the set of multiplicatively closed subsets S of $\mathbb{N}$ such that every element of S has a unique factorization in S-primes (up to the order of the factors).

Prove or disprove:

$$\forall_{S_1\in A}\forall_{S_2\in A}\,[S_1 \cap S_2 \in A]\,.$$

Sets: density

TYCMJ 111. **by Michael W. Ecker**

Define the density of a subset A of the natural numbers by $d(A) = \lim_{n\to\infty} A_n/n$ (provided this limit exists), where A_n is the number of elements of A which do not exceed n. What is the range of d?

AMM 6217.* **by M. J. Pelling**

Let B be a subset of the nonnegative integers having positive density. Is it always true that there is an infinite subset X of B and an infinite sequence $k_1 < k_2 < \cdots$ of integers such that all the translates $X + k_i \subseteq B$?

Sets: divisibility

IMO 1977/3.
PARAB 366.

Let n be a given integer > 2, and let V_n be the set of integers $1 + kn$, where $k = 1, 2, \ldots$. A number $m \in V_n$ is called indecomposable in V_n if there do not exist numbers $p, q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Products which differ only in the order of their factors will be considered the same.)

MSJ 462. **by P. Erdős**
ISMJ 13.26.

Prove that in any selection of 51 of the first 100 positive integers, there exists at least one pair of integers for which one member of the pair divides the other. Prove that 51 cannot be replaced by any smaller number.

OSSMB 79-12.

(a) Prove that, given any 52 integers, there exist two whose sum or whose difference is divisible by 100.

(b) Prove that, given a set of 100 integers with none divisible by 100, there exists a subset, the sum of whose elements is divisible by 100.

MSJ 492.

Let S and T be subsets of $\{1, 2, 3, \ldots, n\}$ such that the number of elements of S plus the number of elements of T is greater than n. Prove that some member of S is relatively prime to some member of T.

CRUX 26. **by Viktors Linis**

Given n integers. Show that one can select a subset of these numbers and insert plus or minus signs so that the number obtained is divisible by n.

MSJ 495.

Let N be a nonempty set consisting of n positive integers. Prove that there exists a nonempty subset M of N such that the sum of the elements of M is divisible by n.

MSJ 500.

Let N be a set containing n positive integers. What is the smallest value of n which will ensure that one can always pick four elements of N whose sum is divisible by 4?

NYSMTJ OBG2. **by Erwin Just**

Let k be a positive integer, $n = 2^k$, and let S be a set consisting of $2n - 1$ integers. Prove that there is a subset T of S, such that T has exactly n elements and the sum of the elements of T is divisible by n.

MM Q620. **by Sidney Penner**

Let S be the set of the first n positive integers, let r be an integer and let $T = S \cup \{r\}$. Prove that there exists an integer in T such that its removal results in a set in which the sum of its elements is divisible by n.

Sets: family of sets

MM 1037. **by James Propp**

Let $n \geq 3$ and let $A_1, A_2, \ldots, A_n$ be nonempty sets of positive integers with the property that $a \in A_i$ and $b \in A_{i+1}$ implies $a + b \in A_{i+2}$, where we identify A_{n+1} as A_1 and A_{n+2} as A_2.

(a) If $1 \in A_1$ and $2 \in A_2$, find an integer that belongs to at least two of the sets.

(b) Is it possible for $A_1, A_2, \ldots, A_n$ to be pairwise disjoint?

Sets: irrational numbers

AMM 6161. **by Clark Kimberling**

For $0 < r < 1$, let $S(r)$ be the set of integers n such that one and only one integer lies in the open interval $(nr, nr+r)$. Prove or disprove that r is irrational if and only if, for every positive integer M, the set $S(r)$ contains a complete residue system modulo M.

Sets: maxima and minima

AMM E2638. **by Robert McNaughton**

Call a set of positive integers a clique if no two of its elements are relatively prime. Call a member of a clique a leader if it is not a proper multiple of another member of the clique. Construct a maximal clique with infinitely many leaders. (The set of all cliques is partially ordered by inclusion.)

Sets: n-tuples

AMM E2546. **by Richard Stanley**

Let n be a positive integer, and let S be a set of n-tuples of nonnegative integers with the property that if $(a_1, \ldots, a_n) \in S$ and if $0 \leq b_i \leq a_i$ for $i = 1, 2, \ldots, n$, then $(b_1, \ldots, b_n) \in S$. Let $H(m)$ be the number of elements of S whose coordinates sum to m. Prove that $H(m)$ is a polynomial in m for m sufficiently large.

Sets: partitions

JRM 651. **by David L. Silverman**

(a) What is the largest value of n such that the integers $1, 2, \ldots, n$ can be partitioned into disjoint sets in such a way that if a, b, and c are in arithmetic progression, then a, b, and c are neither in the same set nor all in different sets?

(b) What is the largest value of n if the condition that a, b, and c be in arithmetic progression is replaced by the condition that $a + b = c$?

SPECT 8.4.

Is it possible to partition the integers $1, 2, \ldots, 13$ into two subsets such that neither subset possesses three integers in arithmetic progression?

PARAB 294.

Given a positive integer n, find (in terms of n) the largest N such that the set of integers

$$S = \{n,\ n+1,\ n+2, \ldots, N\}$$

can be split up into two subsets A and B such that $A \cup B = S$; and the difference $x - y$ between any two elements x, y of one of the sets A, B is in the other set.

NYSMTJ 41. **by Norman Schaumberger and Erwin Just**

A set S consists of 14 integers, not necessarily distinct. Whenever any one of the integers is deleted from the set, the remaining 13 integers can be partitioned into three subsets in such a manner that the subsets have equal sums.

(a) Prove that each member of S is divisible by 3.

(b) Is it possible that some member of S is not equal to zero?

CRUX 226. **by David L. Silverman**

The positive integers are divided into two disjoint sets A and B. A positive integer is an A-number if and only if it is the sum of two different A-numbers or of two different B-numbers. Find A.

Number Theory

CRUX 342.* **by James Gary Propp**

For fixed $n \geq 2$, the set of all positive integers is partitioned into the (disjoint) subsets $S_1, S_2, \ldots, S_n$ as follows: for each positive integer m, we have $m \in S_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the n subsets.

Prove that $m \in S_n$ for all sufficiently large m.

CRUX 473.* **by A. Liu**

The set of all positive integers is partitioned into the (disjoint) subsets $T_1, T_2, T_3, \ldots$ as follows: for each positive integer m, we have $m \in T_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the subsets. Prove that each T_k is finite.

JRM 567. **by David L. Silverman**

(a) Prove that the positive integers can be divided into two disjoint sets such that the sum of two members of the same set is never prime.

(b) Prove that the above division is unique.

(c) Prove that the positive integers have a unique division into two disjoint sets with the property that a positive integer is a Fibonacci number if and only if it is not the sum of two distinct members of the same set.

Sets: polynomials

AMM E2804.* **by Harry D. Ruderman**

Let k be a positive integer and S_k be the set of integers j expressible in the form

$$j = k|ab| + a + b,$$

where a and b run through the nonzero integers. Find the cardinality of the set of positive integers not in S_k.

CRUX 294. **by Harry D. Ruderman**

Prove that there are infinitely many integers that cannot be expressed in the form $3ab + a + b$, where a and b are nonzero integers.

CRUX 403. **by Kenneth S. Williams**

Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and set

$A_1 = \{3m^2 + 6mn + 3n^2 + 2m + 3n + 1 : m, n \in \mathbb{N}_0\}$,
$A_2 = \{3m^2 + 6mn + 3n^2 + 4m + 5n + 2 : m, n \in \mathbb{N}_0\}$,
$A_3 = \{3m^2 + 6mn + 3n^2 + 5m + 6n + 3 : m, n \in \mathbb{N}_0\}$,
$A_4 = \{3m^2 + 6mn + 3n^2 + 6m + 7n + 4 : m, n \in \mathbb{N}_0\}$,
$A_5 = \{3m^2 + 6mn + 3n^2 + 7m + 8n + 5 : m, n \in \mathbb{N}_0\}$,
$A_6 = \{3m^2 + 6mn + 3n^2 + 9m + 10n + 8 : m, n \in \mathbb{N}_0\}$,

so that

$A_1 = \{1, 6, 7, 17, 18, 19, 34, 35, 36, 37, 57, 58, 59, 60, 61, \ldots\}$,
$A_2 = \{2, 9, 10, 22, 23, 24, 41, 42, 43, 44, \ldots\}$,
$A_3 = \{3, 11, 12, 25, 26, 27, 45, 46, 47, 48, \ldots\}$,
$A_4 = \{4, 13, 14, 28, 29, 30, 49, 50, 51, 52, \ldots\}$,
$A_5 = \{5, 15, 16, 31, 32, 33, 53, 54, 55, 56, \ldots\}$,
$A_6 = \{8, 20, 21, 38, 39, 40, 62, 63, 64, 65, \ldots\}$.

Prove or disprove that

(a) the elements of A_i are all distinct for $1 \leq i \leq 6$;

(b) $A_i \cap A_j = \varnothing$ for $1 \leq i < j \leq 6$;

(c) $\{0\} \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = \mathbb{N}_0$.

MSJ 487.

On a mathematics examination, each participant's score, S, will be calculated by the formula $S = 4C - W + 30$, where C is the number of correct and W is the number of wrong answers marked on the 30 multiple choice problems. (Answers left blank are not penalized). Find the six scores between 0 and 150 that are impossible to attain on the exam.

Sets: prime divisors

AMM E2644. **by Solomon W. Golomb and Lloyd R. Welch**

Let $A_n = ru^n + sv^n$, $n \geq 0$, where r, s, u, and v are integers, $psuv \neq 0$, $u \neq \pm v$, and let P_n be the set of prime divisors of A_n. Show that the union of all the P_n is infinite.

SPECT 11.1. **by H. J. Godwin**

The prime factorizations of $r + 1$ positive integers ($r \geq 1$) together involve only r primes. Prove that there is a subset of these integers whose product is a perfect square.

Sets: subsets

ISMJ 12.24.

What is the largest subset of the 1,000 numbers between 1 and 1,000 that has no relatively prime pair?

OSSMB 78-4.

A sequence $\{b_n\}$ is defined by requiring that b_n is the number of subsets of $\{1, 2, \ldots, n\}$ having the property that any two different elements of the subset differ by more than 1. Show that for all n, $b_{n+2} = b_{n+1} + b_n$ and then determine b_{10}.

Sets: sum of elements

PUTNAM 1978/A.1.
OSSMB 79-3.

Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

SSM 3602. **by William J. O'Donnell**

(a) Find two sets of three consecutive prime numbers such that the sum of the elements of the sets are the squares of two consecutive prime numbers.

(b) Can you find three sets of three consecutive prime numbers such that the sums of the elements of the sets are the squares of three consecutive prime numbers?

AMM E2526. **by Paul Smith**

Call a set $\{a_1, \ldots, a_n\}$ of positive integers sum-distinct if the 2^n possible sums $\sum \varepsilon_i a_i$ (with $\varepsilon_i = 0$ or 1) are all distinct. Clearly, for any n, the set $\{1, 2, 4, \ldots, 2^{n-1}\}$ is an n-element sum-distinct set. Do n-element sum-distinct sets exist with $a_i < 2^{n-1}$ for every i?

PARAB 403.
CRUX 3. **by H. G. Dworschak**
OSSMB 78-14.

Given any set of ten distinct, positive integers each less than 100, show that there are two subsets of this set having no elements in common such that the sums of the numbers in the subsets are equal.

JRM 383. **by Victor G. Feser**

What is the largest possible number of distinct integers (not necessarily positive) such that the sum of every pair is prime (also not necessarily positive)? How many such maximal sets are there?

CRUX 85. **by Viktors Linis**

Find n natural numbers such that the sum of any number of them is never a square.

MM 934. **by Erwin Just**

From the first kn positive integers, choose a subset, K, consisting of $(k-1)n+1$ distinct integers. Prove that at least one member of K is the sum of k members (not necessarily distinct) of K.

PME 356. **by Erwin Just**

From the set of integers contained in $[1, 2n]$, a subset K consisting of $n+2$ integers is chosen. Prove that at least one element of K is the sum of two other distinct elements of K.

TYCMJ 91. **by Sidney Penner**

Let A be an arbitrary subset of $\mathbb{N}$, and define $\overline{A} = A \cup \{a_i + a_j | a_i, a_j \in A\}$. Prove or disprove that for any $B \subset \mathbb{N}$, there exists a nonempty $A \subset \mathbb{N}$ such that $\overline{A} \subset B$ or $\overline{A} \subset \mathbb{N} \backslash B$.

Sets: triples

NAvW 428. **by P. Erdős**

Let $\{A_k \mid k \in \mathbb{N}\}$ be a system of triples on the integers such that every pair occurs in at most one triple, i.e., $|A_k| = 3$ and $|A_i \cap A_j| \leq 1$ if $i \neq j$. Denote by $f(n)$ the number of triples contained in $\{1, 2, \ldots, n\}$. It is known that there is such a system for which

$$\limsup_{n\to\infty} n^{-2} f(n) = \frac{1}{6} .$$

Prove that there is a constant c such that for all systems

$$\liminf_{n\to\infty} n^{-2} f(n) \leq \frac{1}{6} - c.$$

Sets: unit fractions

AMM E2689. **by L.-S. Hahn**

Is there a nonempty finite set S of positive integers that satisfies the following properties?

(i) $n \in S \Rightarrow n-1 \in S$ or $n+1 \in S$;
(ii) $\sum_{n\in S} 1/n$ is an integer.

Square roots

PME 452. **by Tom M. Apostol**

Given integers $m > n > 0$, let

$$a = \sqrt{m} + \sqrt{n}, \qquad b = \sqrt{m} - \sqrt{n}.$$

If $m - n$ is twice an odd integer, prove that both a and b are irrational.

MSJ 473.

Prove that there are no positive integers x and y for which $\sqrt{1978} = \sqrt{x} + \sqrt{y}$.

PME 427. **by Jackie E. Fritts**

If a, b, c, and d are integers, with $u = \sqrt{a^2+b^2}$, $v = \sqrt{(a-c)^2 + (b-d)^2}$, and $w = \sqrt{c^2+d^2}$, then prove that

$$\sqrt{(u+v+w)(u+v-w)(u-v+w)(-u+v+w)}$$

is an even integer.

Squares

OSSMB G78.1-1.

(a) Find the sequence of square numbers which when divided by 7 leave a remainder 4.

(b) Find a natural number that is greater than 3 times the integral part of its square root by 1. Show that only two such numbers exist.

OSSMB 77-3.

Find all perfect squares that differ by 1 from a power of 2.

OSSMB 77-5.

Prove that $n = 13$ is the greatest integer n that makes $4^8 + 4^{11} + 4^n$ a perfect square.

FUNCT 1.1.2.

Find a rational number for which the square, when increased or decreased by 5, remains a square.

CRUX 111. **by H. G. Dworschak**

Prove that, for all distinct rational values of a, b, and c, the expression

$$\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2}$$

is a perfect square.

Sum and product

ISMJ 12.27.

The integers a and b are relatively prime. Prove that $a+b$ and ab are also relatively prime. Is it true that if $a+b$ and ab are relatively prime, then a and b also are?

CRUX 172. **by Steven R. Conrad**

Find all sets of five positive integers whose sum equals their product.

NYSMTJ 39. **by Alan Wayne**

Show that, for every integer $n > 1$, there exist n positive integers whose sum equals their product.

Sum of consecutive odd integers

SSM 3699. **by Herta T. Freitag**

Let n be a given positive integer. If S_1 denotes the sum of the odd, positive integers smaller than n, and S_2 represents the analogous sum comprised of even, positive integers, determine $|S_1 - S_2|$.

Sum of divisors: almost perfect numbers

AMM E2571.* **by Sidney Kravitz**

A number n is perfect-plus-one (pp1) if $\sigma(n) = 2n - 1$. It is known that if $n = 2^k$, then n is pp1, but it is not known if there are any other pp1 numbers.

Discuss the situation for pp2 numbers, i.e., numbers n for which $\sigma(n) = 2n - 2$.

PUTNAM 1976/B.6.

Let $\sigma(N)$ denote the sum of all the positive integral divisors of N, including 1 and N. A positive integer N is called quasiperfect if $\sigma(N) = 2N + 1$. Prove that every quasiperfect number is the square of an odd integer.

Sum of divisors: density

AMM 6020.* **by C. W. Anderson and Dean Hickerson**

A pair of distinct numbers (k, m) is called a friendly pair (k is a friend of m) if $\Sigma(k) = \Sigma(m)$, where $\Sigma(n) = \sigma(n)/n$, where $\sigma(n)$ is the sum of the divisors of n. Show that almost all numbers have friends, i.e., the natural (asymptotic) density of numbers with friends is unity. Show that the density of solitary numbers (numbers without friends) is zero.

AMM 6065. **by C. W. Anderson**

Where $\phi\colon \mathbb{N} \to \mathbb{N}$ is Euler's totient function, it is known that the natural density of $\phi(\mathbb{N}) \subset \mathbb{N}$ is zero — in symbols, $d[\phi(\mathbb{N})] = 0$. Where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is the sum of the divisors function, demonstrate that $d[\sigma(\mathbb{N})] = 0$.

Sum of divisors: divisibility

MATYC 73. **by James M. Thelen**
MATYC 77. **by James Thelen**

Let n be the product of k distinct odd primes. Prove that the sum of the divisors of n is divisible by 2^k.

Sum of divisors: evaluations

MM Q614. **by Rod Cooper**

Find the sum of all distinct positive divisors of the number 104,060,401.

Sum of divisors: iterated functions

AMM 6064. **by H. W. Lenstra, Jr.**

For a nonnegative integer m, let $s(m)$ denote the sum of those divisors d of m for which $1 \leq d < m$. Prove that for every integer $t \geq 1$ there exists an m such that

$$m < s(m) < s^2(m) < \cdots < s^t(m).$$

Here $s^2(m) = s(s(m))$, and so on.

Sum of divisors: number of divisors

AMM E2543. **by C. W. Anderson**

Show that there exists a constant $k > 0$ such that if $x = \sigma(n)/n$ is sufficiently large, then

$$\tau(n) > 2^{\exp kx}.$$

Show also that there exist n for which $\tau(n)$ is arbitrarily large, but for which $\sigma(n)/n$ is arbitrarily close to unity.

CRUX 465. **by Peter A. Lindstrom**

For positive integer n, let $\sigma(n) =$ the sum of the divisors of n and $\tau(n) =$ the number of divisors of n. Show that if $\sigma(n)$ is a prime then $\tau(n)$ is a prime.

AMM 6048.* **by H. M. Edgar**

A positive integer n is said to be harmonic if the ratio

$$\frac{n\tau(n)}{\sigma(n)}$$

is again integral.

(a) Are there any harmonic numbers other than the number 1 that are perfect squares?

(b) Do there exist infinitely many harmonic numbers?

Sum of divisors: perfect numbers

PME 349. **by R. Sivaramakrishnan**

If 2^n ($n \geq 1$) is the highest power of 2 dividing an even perfect number m, prove that $\sigma(m^2) + 1 \equiv 0 \pmod{2^{n+1}}$.

AMM 6036. **by Carl Pomerance**

If n is a natural number, let $\sigma(n)$ denote the sum of the divisors of n, $S(n)$ the set of prime divisors of n, and $\omega(n)$ the cardinality of $S(n)$. Clearly, if n is an even perfect number, then $S(n) = S(\sigma(n))$ and $\omega(n) = 2$. Prove the converse.

Sum of divisors: prime factorizations

CRUX 187. **by André Bourbeau**

If $m = 2^n \cdot 3 \cdot p$, where n is a positive integer and p is an odd prime, find all values of m for which $\sigma(m) = 3m$.

Sum of divisors: products

FQ B-303. **by David Singmaster**

What relation holds between $\sigma(mn)$ and $\sigma(m)\sigma(n)$?

FQ B-326. **by David Zeitlin**

Prove that

$$\sigma(mn) > 2\sqrt{\sigma(m)\sigma(n)} \quad \text{for } m > 1 \text{ and } n > 1.$$

Sum of divisors: sets

MM 982. **by Roy DeMeo Jr.**
AMM 6107. **by Roy E. DeMeo Jr.**

Let $\sigma(n)$ be the sum of all the positive divisors of the positive integer n, including 1 and n. Let A denote the set of all rational numbers of the form $\sigma(n+1)/\sigma(n)$. Determine the closure of A in the set of real numbers.

Sum of powers

PARAB 373.

For which values of n is $1^n + 2^n + 3^n + 4^n$ divisible by 5?

PENT 295. **by Kenneth M. Wilke**

Let $S_k = 1^k + 2^k + \cdots + n^k$ where n is an arbitrary positive integer and k is an odd positive integer. Under what conditions is S_k divisible by

$$S_1 = \frac{n(n+1)}{2}$$

for all positive integers n?

NAvW 485. **by L. Kuipers**

Let p and $4p+1$ be odd primes. If, for nonzero pairwise prime integers a, b, and c, we have $a^p + b^p + c^p = 0$, then precisely one of the integers a, b, c is divisible by $4p+1$. Prove this, and also that precisely one of the integers a, b, c is divisible by p.

Triangles: 60 degree angle

MM Q654. **by George Berzsenyi**

Find all acute triangles with integral sides and a 60° angle in which the sides adjacent to the 60° angle differ by unity.

Triangles: 120 degree angle

AMM E2566. **by Edvard Kramer**

A triplet (a, b, c) of natural numbers is an obtuse Pythagorean triplet if a, b, and c are the sides of a triangle ABC with $\angle C = 120^\circ$. Such a triplet is primitive if $\gcd(a, b, c) = 1$.

(a) Show that each positive integer except 1, 2, 4, and 8 can appear as the smallest member of an obtuse Pythagorean triplet.

(b) What positive integers can appear in primitive obtuse Pythagorean triplets?

Triangles: area

CRUX 290. **by R. Robinson Rowe**

Find a 9-digit integer A representing the area of a triangle of which the three sides are consecutive integers.

JRM 495. **by Michael R. W. Buckley**

Define an artful number as an integer that can be the area of a rational triangle. Thus 1, 2, and 3 are artful, being the areas, respectively, of a $(3/2, 5/3, 17/6)$, a $(5/6, 29/6, 5)$, and a $(5/2, 5/2, 3)$ triangle. What, if any, is the smallest art*less* number?

Triangles: area and perimeter

SSM 3669. **by Alan Wayne**

Find all of the acute triangles whose sides are positive integers and whose area is four times the perimeter.

Triangles: base and altitude

AMM E2687. **by Ronald Evans**

Does there exist a triangle with rational sides whose base is equal to its altitude?

Triangles: consecutive integers

MM 1023. **by Steven R. Conrad**

Call a triangle super-Heronian if it has integral sides and integral area, and the sides are consecutive integers. Are there infinitely many distinct super-Heronian triangles?

Triangles: counting problems

SSM 3649. **by Alan Wayne**

Find formulas for:

(a) the number of triangles with integer sides and perimeter $12k - 4$, and

(b) the number of such triangles which are isosceles.

SSM 3700. **by Douglas E. Scott**

Find an algorithm or formula that will give the total number of isosceles triangles (an equilateral triangle counts as isosceles) having integer sides and a given perimeter P.

PARAB 325.

The sides of a triangle have lengths a, b, c, where a, b, c are integers and $a \le b \le c$. If c is given, show that the number of different triangles is $(c+1)^2/4$ or $c(c+2)/4$ according to whether c is odd or even, respectively.

MM 1077. **by Henry Klostergaard**

Show that the number of integral-sided right triangles whose ratio of area to semiperimeter is p^m, where p is a prime and m is a positive integer, is $m+1$ if $p = 2$ and $2m+1$ if $p \ne 2$.

Triangles: geometric progressions

ISMJ 11.7.

For what values of the common ratio can three successive terms of a geometric progression of positive numbers be the lengths of the sides of a triangle?

Triangles: isosceles triangles

TYCMJ 131. **by Alan Wayne**

The lengths of the sides of an isosceles triangle are integers, and its area is the product of the perimeter and a prime. What are the possible values of the prime?

Triangles: nonisosceles triangles

AMM E2668. **by Ron Evans and I. Martin Isaacs**

Find all nonisosceles triangles with two or more rational sides and with all angles rational (measured in degrees).

Triangles: obtuse triangles

SSM 3727. **by Douglas E. Scott**

Find two or more obtuse triangles such that

(a) their sides have integral length;

(b) their perimeters are the same;

(c) their areas are the same; and

(d) the perimeter in each instance is one-fourth the area.

Triangles: perimeter

SSM 3703. **by Douglas E. Scott**

Find an algorithm or a formula that will give the sides of the

(a) most acute

(b) most obtuse, and

(c) equilateral or most nearly equilateral isosceles triangles having integer sides and a given perimeter.

Triangles: primes

JRM 630. **by Les Marvin**

"Pay attention," said the Wizard to his three apprentices. "Each of you has noted that numbers have been inked on the foreheads of the other two. I have penned on each of your foreheads a prime number, and the three numbers form the sides of a triangle with prime perimeter. The first apprentice to deduce his number will succeed me as Wizard when I retire. In an hour I will return and ask if anyone can tell me his number. If no one can, you may use that information an hour later when I return a second time. I'll rematerialize every hour until my successor has proved himself."

But the Wizard was mistaken. After several returns his apprentices were still producing only frustrated, baffled looks. Impatiently he asked his Familiar how many additional returns would be needed.

"You can keep returning forever," purred the Familiar, "and you will still not know your successor. How did you happen to choose those primes?"

"All were chosen randomly among the primes less than 100,000. Why do you ask?" said the Wizard.

"Very curious. Since one of the numbers is 5, the other two happen to be the smallest that make it impossible for any of the apprentices to deduce his number."

What were the other two numbers?

Triangles: right triangles

JRM 494. **by Michael R. W. Buckley**

An APT number is an integer that can be the area of a Pythagorean rational triangle, i.e., a right triangle with rational sides. What is the smallest APT number?

Triangles: scalene triangles

SSM 3722. **by Richard L. Francis**

Can a scalene triangle have side measures, each of which is an even perfect number?

Triangles: similar triangles

JRM 595. **by Archimedes O'Toole**

Call two triangles "almost congruent" if two sides of one triangle are equal, respectively, to two sides of the other, and the triangles are similar, but not congruent. If a pair of almost congruent triangles has sides all of which are integers, what is the smallest possible value for the least of these integers?

Triangular numbers: counting problems

FQ B-385. **by Herta T. Freitag**

Let $T_n = n(n+1)/2$. For how many positive integers n does one have both $10^6 < T_n < 2 \cdot 10^6$ and $T_n \equiv 8 \pmod{10}$?

Triangular numbers: forms of numbers

ISMJ 10.11.

Show that any triangular number can be expressed as the difference between two triangular numbers.

PUTNAM 1975/A.1.

Supposing that an integer n is the sum of two triangular numbers,

$$n = \frac{a^2+a}{2} + \frac{b^2+b}{2},$$

write $4n+1$ as the sum of two squares, $4n+1 = x^2+y^2$, and show how x and y can be expressed in terms of a and b. Show that, conversely, if $4n+1 = x^2+y^2$, then n is the sum of two triangular numbers. [Of course, a, b, x, y are understood to be integers.]

Triangular numbers: identities

MATYC 82. **by John M. Samoylo**

If x is any positive integer and y is the sum of the integers from 1 to x, show that $x^3 = y^2 - (y-x)^2$.

FQ B-393. **by V. E. Hoggatt, Jr.**

Let $T_n = \binom{n+1}{2}$, $P_0 = 1$, $P_n = T_1T_2\cdots T_n$ for $n > 0$, and $\left[{n \atop k}\right] = P_n/P_kP_{n-k}$ for integers k and n with $0 \le k \le n$. Show that

$$\left[{n \atop k}\right] = \frac{1}{n-k+1}\binom{n}{k}\binom{n+1}{k+1}.$$

Triangular numbers: palindromes

SSM 3572. **by Robert A. Carman**

Show that the triangular number $T_{111\ldots}$ ($2n$ ones) is always a palindrome.

Triangular numbers: polynomials

SSM 3640. **by Herta T. Freitag**

What, if anything, do the following have in common:

(a) The number of terms in $(a+b+c)^m$, m a nonnegative integer.

(b) The number of differently shaped rectangles (or squares) of integral sides that may be drawn on an n by n checkerboard.

(c) Triangular numbers.

(d) The sum of the cubes of the first n consecutive positive integers.

Triangular numbers: series

FQ B-371. **by Herta T. Freitag**

Let

$$S_n = \sum_{k=1}^{F_n}\sum_{j=1}^{k} T_j,$$

where T_j is the triangular number $j(j+1)/2$. Does each of $n \equiv 5 \pmod{15}$ and $n \equiv 10 \pmod{15}$ imply that $S_n \equiv 0 \pmod{10}$?

FQ B-372. **by Herta T. Freitag**

Let

$$S_n = \sum_{k=1}^{F_n}\sum_{j=1}^{k} T_j.$$

Does $S_n \equiv 0 \pmod{10}$ imply that n is congruent to either 5 or 10 modulo 15?

SSM 3729. **by Herta T. Freitag**

When $f(i) = 0$ and $k = 1$, $\sum_{i=1}^{n}(-1)^{f(i)}i^k = \sum_{i=1}^{n} i = n(n+1)/2$ which is the nth triangular number. Determine $f(i)$ so that $\sum_{i=1}^{n}(-1)^{f(i)}i^2$ will be a triangular number.

SSM 3647. **by Gregory Wulczyn**

Prove that $\sum_{r=1}^{n}(2r-1)^3$ is a triangular number for each positive integer n.

FQ B-388. **by Herta T. Freitag**

Let $T_n = n(n+1)/2$. Show that

$$T_1 + T_2 + T_3 + \cdots + T_{2n-1} = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$$

and express these equal sums as a binomial coefficient.

Triangular numbers: squares

MM Q633. **by J. D. Baum**

The sum of the first eight positive integers is 36, a perfect square. Are there any other values of k for which the sum of the first k positive integers is a perfect square? Are there infinitely many k?

Triangular numbers: sum of squares

FQ B-400. **by Herta T. Freitag**

Let $T_n = n(n+1)/2$. For which positive integers n is $T_1^2 + T_2^2 + \cdots + T_n^2$ an integral multiple of T_n?

Twin primes

JRM 797. **by Sidney Kravitz**

Three consecutive positive integers are of the form $p, 2^a n, q$, with p, n, and q prime.

(a) What is the value of n?

(b) For what values of a less than 48 does this occur?

PME 340. **by Charles W. Trigg**

The arithmetic mean of the twin primes 17 and 19 is the heptagonal number 18. Heptagonal numbers have the form $n(5n-3)/2$. Are there any other twin primes with a heptagonal mean?

MATYC 96. **by Charles W. Trigg**

The arithmetic mean of the twin primes 3 and 5 is the tetrahedral number 4. Tetrahedral numbers have the form $n(n+1)(n+2)/6$. Are there any other twin primes that have a tetrahedral mean?

SSM 3735. **by Richard L. Francis**

A pair of twin primes is two prime numbers which differ by two, and an initial prime is a prime having 1 as its leftmost digit. Show that if one number of a set of twin primes is an initial prime, so is the other.

MM Q648. **by J. D. Baum**

Show that if p and q are twin primes $(q = p+2)$ and if $pq - 2$ is also prime, then p is uniquely determined.

MATYC 78. **by John M. Samoylo**

For all pairs of twin primes other than 3 and 5, show that the sum of the number between any pair of twin primes and the primes themselves is divisible by 18.

Arrays

CRUX 387. **by Harry D. Ruderman**

A group of N people lock arms to dance in a circle the traditional Israeli Hora. After a break they lock arms to dance a second round. Let $P(N)$ be the probability that for the second round no dancer locks arms with a dancer previously locked to in the first round. Find $\lim_{N\to\infty} P(N)$.

Bingo

MATYC 70. **by Gene Zirkel**

During an evening, 23 games of bingo were played. Each game ended when exactly nine of the 75 numbers had been called. Of the 207 numbers called that evening, what is the probability that only 74 different numbers had been called; i.e., that exactly one number was never called.

PME 419. **by Michael W. Ecker**

Seventy-five balls are numbered 1 to 75, and are partitioned into sets of 15 elements each as follows:

$$B = \{1, \dots, 15\}, \quad I = \{16, \dots, 30\}, \quad N = \{31, \dots, 45\},$$
$$G = \{46, \dots, 60\}, \quad \text{and} \quad O = \{61, \dots, 75\},$$

as in Bingo.

Balls are chosen at random, one at a time, until one of the following occurs: At least one from each of the sets B, I, G, O has been chosen, or four of the chosen numbers are from the set N, or five of the numbers are from one of the sets B, I, G, O.

Find the probability that, of these possible results, four N's are chosen first.

Biology

JRM 384. **by Michael R. Buckley**

The Anableps is a South American flatfish with a couple of curious characteristics. Its two eyes are really four — divided horizontally so that it has binocular vision above and below water level simultaneously. But its advantage in reconnaissance capability is offset by the frustration it experiences in locating a mate, for its two sexes are also really four. A left-handed male can mate only with a right-handed female and a right-handed male only with a left-handed female. But that is the Anableps' problem. Here are yours:

(a) How many Anableps must one capture to ensure at least a 50% chance of having a mateable pair?

(b) What is the probability of having at least m separate and disjoint mateable pairs in n randomly chosen Anableps?

(c) What is the probability of having at least m mateable pairs in n randomly chosen Anableps if polygamy and polyandry are permitted?

AMM E2636. **by D. E. Knuth**

A pair of microbes was recently discovered that reproduce in a very peculiar way. The male microbe (a diphage) has two receptors on its surface, and the female (a triphage) has three receptors. When a culture of diphages and triphages is irradiated with a psi-particle, exactly one of the receptors absorbs the particle (each receptor being equally likely). If it was a diphage, it changes to a triphage; but if it was a triphage, it splits into two diphages.

Give a simple formula for the average number of diphages present if we begin with a single diphage and irradiate the culture n times with psi-particles.

Birthdays

CRUX 195. **by John Karam**

(a) How many persons would have to be in a room for the odds to be better than 50% that three persons in the room have the same birthday?

(b) In the Quebec-based lottery Loto-Perfecta, each entrant picks six distinct numbers from 1 to 36. If, at the draw, his six numbers come out in some order (dans le désordre) he wins a sum of money; if his numbers come out in order (dans l'ordre), he wins a larger sum of money. How many entries would there have to be for the odds to be better than 50% that two persons have picked the same numbers (i) dans le désordre, (ii) dans l'ordre?

Cards

JRM C3. **by David L. Silverman**

The 52 playing cards are shuffled and dealt out in a row. What is the probability that no three adjacent cards are of the same suit?

AMM E2645. **by Jerrold W. Grossman**

A deck of N cards is shuffled according to the following scheme. The cards, labeled 1 through N, are placed in order in a row. Independent random integers $r_1, \dots, r_N$ are chosen successively, $1 \le r_i \le N$, and after the choice of each r_i the card then in position i is interchanged with the card then in position r_i. What is the probability that card s ends up in position t after the shuffle is complete?

MM 1022. **by Joe Dan Austin**

We have n cards numbered 1 though n. Find the expected number of drawings needed to put the cards in order by each of the following strategies:

(a) The shuffled cards are drawn without replacement until card 1 is drawn. The remaining $n-1$ cards are shuffled and drawn without replacement until card 2 is drawn. This process is continued until all the cards are drawn and put in linear order.

(b) A card, say card k, is drawn from the shuffled deck. The remaining cards are shuffled and drawn without replacement until either card $k-1$ or card $k+1$ is drawn. We identify card $1-1$ as card n and card $n+1$ as card 1. This process is continued until all the cards are drawn and put in circular order.

JRM 782. **by R. S. Johnson**

"Here is a deck of cards for you to examine," I said to Harry. After a moment he replied, "They're not all here, but all of the suits seem to be well represented. What comes next?" I instructed him to spread the cards face-down on the table. "Now," I said, "I'll bet two hundred dollars against one of yours that you can't pick at random, and turn over, four hearts in succession." When Harry's first two picks were hearts, I began to worry, especially since the odds for the third card by itself were only four to one against him. However, he flipped a spade and paid me a dollar. Checking the discards, Harry observed, "I see that you removed almost a quarter of the deck and that you were on pretty safe ground."

How many cards did I remove, how many were hearts, and what were the total odds against Harry for the entire exercise?

Probability

USA 1975/5.

A deck of n playing cards, which contains three aces, is shuffled at random (it is assumed that all possible card distributions are equally likely). The cards are then turned up one by one from the top until the second ace appears. Prove that the expected (average) number of cards to be turned up is $(n+1)/2$.

Cauchy distribution

AMM 6164. **by Ignacy I. Kotlarski**

Let the random variable $Z_1 = X$ follow the Cauchy distribution with the probability density function

$$f(x) = \left[\pi(1+x^2)\right]^{-1},$$

$x \in \mathbb{R}$. Show that for $n = 2, 3, \dots$, the random variables

$$Z_2 = \frac{2X}{1-X^2}\ ,$$

$$Z_3 = \frac{3X-X^3}{1-3X^2}\ ,$$

$$Z_4 = \frac{4X-4X^3}{1-6X^2+X^4}\ ,$$

$$\vdots$$

$$Z_n = \frac{\binom{n}{1}X - \binom{n}{3}X^3 + \binom{n}{5}X^5 - \cdots}{1 - \binom{n}{2}X^2 + \binom{n}{4}X^4 - \binom{n}{6}X^6 + \cdots}\ , \dots$$

also follow the same Cauchy distribution.

Coin tossing

FUNCT 3.2.6.

I toss three coins. I argue that the probability of them all falling heads is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. The probability that they all fall tails is also $\frac{1}{8}$ so that the probability of them all falling alike is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. My friend argues differently. If three coins are thrown up, at least two must come down alike; the probability that the third coin comes down the same as the other two is $\frac{1}{2}$, as it has an equal chance of being like or unlike. Who is correct?

SIAM 77-11. **by Danny Newman**

If one tosses a fair coin until a head first appears, then the probability that this event occurs on an even numbered toss is exactly 1/3. For this procedure, the expected number of tosses equals 2. Can one design a procedure, using a fair coin, to give a success probability of 1/3 but have the expected number of tosses be less than 2?

TYCMJ 103. **by Richard Johnsonbaugh**

A fair coin is flipped n times. Let E be the event "a head is obtained on the first flip", and let F_k be the event "exactly k heads are obtained". For which pairs (n,k) are E and F_k independent?

CRUX 265. **by David Wheeler**

A game involves tossing a coin n times. What is the probability that two heads will turn up in succession somewhere in the sequence of throws?

PME 370. **by David L. Silverman**

Able, Baker, and Charlie take turns cyclically, in that order, tossing a coin until three successive heads or three successive tails appear. With what probabilities will the game terminate on Able's turn? On Baker's?

Coloring problems

AMM 6229.* **by David W. Erbach**

Suppose that the plane is tiled with regular hexagons in the customary manner. Color each black or white independently with probability $1/2$. What is the expected size of a connected monochromatic component? What is the probability that there is an infinite component?

Conditional probability

PARAB 306.

I post a letter to a friend. There is a probability of 4/5 that the letter will reach its destination. If he received the letter, he would send me a reply. What is the probability that he received the letter if I receive no reply?

FUNCT 1.3.7.

Of three prisoners, Mark, Luke, and John, two are to be executed, but Mark does not know which two. He therefore asks the jailer, "Since either Luke or John are certainly going to be executed, you will give me no information about my own chances if you give me the name of one man, either Luke or John, who is going to be executed."

Accepting this argument, the jailer truthfully replied "Luke will be executed." Thereupon, Mark felt happier because, before the jailer replied, his own chances of execution were 2/3; but afterwards there are only two people, himself and John, who could be the one not to be executed, and so his chance of execution is only 1/2.

Is Mark right to feel happier?

JRM 530. **by Les Marvin**

"Will the weather be good tomorrow, or should I postpone my next labor?" asked Hercules of the Oracle of Apollo, after completing his twelfth labor. "Good," said the Oracle, who had established a 2/3 record of accuracy. "Good," agreed the Oracle of Hermes, who had a 5/8 accuracy record. "Bad," said the Oracle of Zeus with a 7/9 accuracy record. Confused, Hercules postponed his 13th labor, as it turned out, forever. Did he act properly in light of the predictions, which, you may assume, were entirely independent of each other, as were the records of the three oracles? What was the probability of good weather, given that good weather and bad weather were equally likely at that time of year?

Density functions

SIAM 79-6.* **by L. B. Klebanov**

Let $f(x)$, $g(x)$ be two probability densities on R^1 with $g(x) > 0$. Suppose that the condition

$$\int_{-\infty}^{\infty} (u-c) \prod_{j=1}^{n} f\left(x_j - u\right) g(u)\, du = 0$$

holds for all $x_1, x_2, \dots, x_n$ such that $\sum_{j=1}^{n} x_j = 0$ where $n \geq 3$ and c is some constant. Prove that

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}.$$

Dice problems: independent trials

FUNCT 1.3.5.

A die is thrown until a 6 is obtained. What is the probability that 5 was not thrown, meanwhile?

TYCMJ 136. **by Michael W. Ecker**

Assume that it is required to throw a pair of dice and obtain a total of 5, or a 5 on at least one of the dice. What is the expected number of throws required for this to occur?

Dice problems: loaded dice

MM 1011. **by Richard A. Gibbs**
CRUX 118. **by Paul Khoury**
FUNCT 3.3.1.

Is it possible to load a pair of dice so that the probability of rolling each possible sum is 1/11?

JRM 588. **by Ray Lipman**

It is known that no tampering with a pair of dice (assuming that it affects them independently) can make the totals $2, 3, \ldots, 12$ occur with equal frequency. By assigning arbitrary probabilities to the twelve faces (the two dice need not be loaded in the same way), determine the loading that makes the frequencies of the various totals "most equal": For purposes of this problem an assignment A_1 of probabilities to the twelve faces is considered to make the frequencies of the totals "more equal" than another assignment A_2 if, among the eleven relative frequencies of the various totals, the largest discrepancy from 1/11 under A_1 is less than that under A_2.

Dice problems: matching problems

PME 407. **by Ben Gold, John M. Howell, and Vance Stine**

Two sets of n dice are rolled ($n = 1, 2, 3, 4, 5, 6$). What is the probability of k matches ($k = 0, 1, \ldots, n$)?

Dice problems: n-sided dice

JRM 506. **by Osias Bain**

Let $p(k, n)$ be the probability that in k tosses of a fair n-sided die, each face that has come up at all has come up at least twice.

(a) Determine $P_1(n)$, the least K such that $p(k, n)$ is strictly increasing in k for $k \geq K$.

(b) Determine $P_2(n)$, the least K such that $p(k, n) \geq 1/2$ for all $k \geq K$.

USA 1979/3.

Given three identical n-faced dice whose corresponding faces are identically numbered with arbitrary integers. Prove that if they are tossed at random, the probability that the sum of the bottom three face numbers is divisible by three is greater than or equal to 1/4.

Dice problems: number of occurrences

CRUX PS4-1.

What is the probability of an odd number of sixes turning up in a random toss of n fair dice?

Dice problems: octahedral dice

SSM 3598. **by Charles W. Trigg**

Two octahedrons are converted into octahedral dice by distributing the digits from 1 to 8 on the faces of each one and in the same order. As with cubical dice, when they are rolled, the 'point' made is the sum of the digits on the two uppermost faces when the dice come to rest on a flat horizontal surface.

Construct a table showing the probabilities of occurrence of the various 'points' that can be made with two octahedral dice, assuming that both dice are symmetrical and have uniform density.

Digit problems

SIAM 76-16. **by A. Feldstein and R. Goodman**

Fix an integer $\beta \geq 2$, and let A be a positive normalized floating point number represented in base β. For an integer $n \geq 2$, consider the probability $p_n(a)$ that the nth digit of A is the integer a, where $0 \leq a \leq \beta - 1$. Let A be chosen at random from the logarithmic distribution, then

$$p_n(a) = \log_\beta \prod_{m=\beta^{n-2}}^{\beta^{n-1}-1} \frac{\beta m + a + 1}{\beta m + a} \qquad \text{for } n \geq 2$$

and $\lim_{n\to\infty} p_n(a) = 1/\beta$.

It is of interest to analyze Δ_n, the deviation from uniform distribution, given for $n \geq 2$ by

$$\Delta_n \equiv \max_{0 \leq a \leq \beta-1} \left| p_n(a) - \frac{1}{\beta} \right| = \max\left(p_n(0) - \frac{1}{\beta}, \frac{1}{\beta} - p_n(\beta - 1) \right).$$

We conjecture that:

(a) $\Delta_n = p_n(0) - \frac{1}{\beta}$,

(b) $\lim_{n\to\infty} \beta^n \Delta_n$ exists and equals $\frac{(\beta-1)^2}{2\beta \ln \beta}$.

SSM 3570. **by Charles W. Trigg**

(a) If a 10-digit integer in the decimal system is chosen at random, what is the probability that it will contain ten distinct digits?

(b) If an r-digit integer in the scale of notation with base r is chosen at random, what is the probability that it will contain r distinct digits?

CRUX 50. **by John Thomas**

(a) Show that 2^n can begin with any sequence of digits.

(b) Let N be an r-digit number. What is the probability that the first r digits of 2^n represent N?

Distribution functions

AMM 6115. **by L. Franklin Kemp, Jr.**

Let $F_1(x_1), F_2(x_2), \ldots, F_n(x_n)$ be n probability distribution functions (d.f.'s). If $H(x_1, x_2, \ldots, x_n)$ is any n-dimensional d.f. with marginals $F_i(x_i)$, then an earlier solution showed that $\min_i F_i(x_i)$ is an n-dimensional d.f. such that $H(x_1, x_2, \ldots, x_n) \le \min_i F_i(x_i)$. What is the n-dimensional d.f. that bounds any $H(x_1, x_2, \ldots, x_n)$ from below?

SIAM 78-4.* **by C. L. Mallows**

Find the symmetric cumulative distribution function $G(x)$ satisfying $dG(0) = \alpha$, $0 < \alpha < 1$ that minimizes the integral

$$I_f = \int_{-\infty}^{\infty} \frac{\left(f'(x)\right)^2}{f(x)}\, dx,$$

where $f(x)$ is the convolution

$$f(x) = \int_{-\infty}^{\infty} \phi(x-u)\, dG(u),$$

with $\phi(u)$ the standard Gaussian density

$$\phi(u) = (2\pi)^{-1/2} \exp\left[-\frac{1}{2}u^2\right].$$

It is believed that G is a step function, so that

$$f(x) = \sum p_j \phi\left(x - g_j\right),$$

with $g_{-j} = -g_j$, $p_{-j} = p_j > 0$, $p_0 = \alpha$.

Distribution problems

SSM 3601. **by Joe Dan Austin**

Ten people enter an elevator that is to make 13 stops. Assume that the 10 people select their exit independently and that each stop has the same probability of being selected. Find

(a) the probability that at least two people exit at the same stop and

(b) the probability that exactly two people exit at the same stop.

AMM E2515. **by C. L. Mallows**

A careless file clerk has documents $D_1, D_2, \ldots, D_d$ that should go respectively into files F_1, $F_2, \ldots, F_d$; instead he places them independently, at random, into a total of f files ($f \ge d \ge 1$) so that each of the f^d possible arrangements is equally likely. Show that the event that some nonempty subset S of the files F_1, $F_2, \ldots, F_d$ can be made to have the correct contents by redistributing within S the union of their contents, has probability d/f.

Examinations

OSSMB 79-10.

In answering general knowledge questions, all answerable with yes or no, the teacher's probability of being correct is α and a student's probability of being correct is β or γ according to whether the student is male or female. If the probability of a randomly chosen student's answer agreeing with the teacher's is $1/2$, find the ratio of the number of males to females in the class.

Gambler's ruin

JRM 631. **by David L. Silverman**

Three players compete in a game of chance in which on each play, each player's chance of winning is directly proportional to his current holding. The stake on each play is the holding of the player with the smallest current (positive) number of chips. The game continues until two players have been eliminated. If the players start with respective holdings of 1, 2, and 3 chips, what is each player's chance of emerging with all six chips?

JRM 423. **by David L. Silverman**

Gamblers B and C, holding a total of 7 chips between them, are engaged in a series of games that will terminate when one player has won all 7 chips. Each game offers each player a $50-50$ chance of success, and the stakes for each game are determined by the poorer (in chips) player — from one chip apiece to a mutual wager of the poorer player's entire fortune. At any juncture, if B holds the majority of chips, C will make the mutual wager on the next game one chip. But if C holds the majority of chips, B will wager all his chips on the next game. At a stage when the bold player B holds n chips ($n = 1, 2, 3, 4, 5, 6$), what is the probability P_n that he will eventually ruin the cautious player C? Generalize.

Game theory: card games

MM 1066. **by Eric Mendelsohn and Stephen Tanny**

Consider the following children's game ("clock"): k copies of well-shuffled cards numbered $1, 2, 3, \ldots, L$ are distributed in boxes labeled $1, 2, 3, \ldots, L$, with exactly k cards per box. At the start of the game, the top card in box 1 is drawn. If the value of this card is j ($j = 1, 2, 3, \ldots, L$) we proceed to box j, draw the top card and go to the box so numbered, draw the top card, and so on. The objective of the game (a 'win') is to draw all cards from every box before being directed to an empty box. Characterize all winning distributions of cards, and find the probability of a win.

SPECT 11.3.

The rules for the card game of Clock Patience are as follows:

Shuffle the pack of cards and deal them into thirteen piles of four labeled A,2,3,4,5,6,7,8,9,10,J,Q,K. To play, take away the top card of the K pile (say it is 5), then the top card of the 5 pile (say it is J), then the top card of the J pile, and so on. The game proceeds until the fourth K is taken, and the game is said to 'come out' if, when the fourth K is taken, all the original piles are empty.

(a) What is the probability that a game comes out?

(b) Assume that, after dealing, the bottom cards on the piles form a rearrangement of A,2,3,4,5,6,7,8,9,10,J,Q,K. Show that this game comes out if and only if this rearrangement is a cyclic rearrangement.

Game theory: coin tossing

SPECT 7.4. **by T. J. Fletcher**

Two players, A and B, begin with capital of p and q units respectively. They gamble by tossing a coin. At each toss, one unit of capital is transferred from the loser to the winner of that toss. The game continues until one or the other is bankrupt. Compare A's and B's chances of winning.

JRM 463. **by Fred Walbrook**

Two players each have two coins, one of which is foreign and is considered in terms of its exchange value in cents. Each tosses both his coins, and the point total he receives is the value, if any, of his coins that come up heads. The player with the larger point total wins his opponent's coins. In the event of a tie, the players toss again. The game is fair in the sense that each player's chance of winning is proportional to his holdings. One player has a nickel, the other a penny. The two foreign coins have the same value. What is that value?

JRM 675. **by J. Sennetti**

The St. Petersburg game is played between a player and a banker. An admission fee is charged to the player, who then tosses a coin until heads comes up. If heads first appears on the nth toss, the banker pays the player 2^n dollars.

(a) What admission fee should be charged to make the game fair?

(b) Suppose the rules are modified so that the game cannot exceed k tosses; i.e. the banker pays 2^k after the kth toss, regardless of the outcome. What then is the fair admission price?

Game theory: dice games

MM 1071. **by Joseph Browne**

Player A rolls $n+1$ dice and keeps the highest n. Player B rolls n dice. The higher total wins, with ties awarded to Player B.

(a) For $n = 2$, show that Player A wins and find his probability of winning.

(b) Find the smallest value of n for which Player B wins.

CRUX 333. **by R. Robinson Rowe**

The World War I COOTIE, lousy vector of trench fever, popularized a simple but hilarious game by that name in the early 1920's. Five or more players each with pad and pencil, cast a single die in turn. Rolling a 6, a player sketched a "body" on the pad and on later turns added a head with a 5, four legs with a 4, the tail with a 3. Having the head, he could add two eyes with a 2 and a proboscis or nose with a 1. Having all six he yelled "COOOOOTIEEEEE!" and raked in the pot.

What was the probability of capturing a COOTIE in just six turns?

MATYC 92. **by Michael Brozinsky**

Find the expected number of throws in the game of craps.

CRUX 409. **by L. F. Meyers**

In a certain bingo game for children, each move consists in rolling two dice. One of the dice is marked with the symbols B, I, N, G, O, and *, and the other die is marked with 1, 2, 3, 4, 5, and 6. A disadvantage of this form of bingo, in comparison with the adult form of the game, is that a combination (such as B3) may appear repeatedly. What is the expected number of the move at which the first repetition occurs in each of these cases:

(a) all 36 combinations (B1 through *6) are considered to be different (and equally likely)?

(b) all 36 combinations (B1 through *6) are considered to be equally likely, but the six combinations containing * are considered to be the same?

Game theory: selection games

PME 403. **by David L. Silverman**

Two players play a game of "Take It or Leave It" on the unit interval $(0, 1)$. Each player privately generates a random number from the uniform distribution, and either keeps it as his score or rejects it and generates a second number which becomes his score. Neither player knows, prior to his own play, what his opponent's score is or whether it is the result of an acceptance or a rejection.

The scores are compared and the player with the higher score wins $1 from the other.

(a) What strategy will give a player the highest expected score?

(b) What strategy will give a player the best chance of winning?

(c) If one player knows that his opponent is playing so as to maximize his score, how much of an advantage will he have if he employs the best counterstrategy?

JRM 499. **by David L. Silverman**

There are two players, Giver and Taker. There are two unlocked strongboxes to which only Giver has the key. The boxes are such that their contents are visible, but whether they are locked or not can only be determined by trying to open them.

The game begins with both boxes empty. Unseen by Taker, Giver has selected one box which he has locked, leaving the other unlocked. Taker then tries to open one of the boxes. If he picks the unlocked box, he receives its contents, and the game is over. If he picks the locked box, Giver adds a dollar to the unlocked box. Taker leaves the room, and Giver again causes one of the boxes to be locked and one unlocked. Taker reenters the room and chooses again. The game continues until the unlocked box is chosen.

Assuming that Taker plays optimally to maximize, and Giver to minimize Taker's winnings, how much, on the average, will Taker win?

Game theory: TV game shows

JRM 769. **by Harry Nelson**

On the hypothetical TV game show *Stump the Panel*, three panelists try to match four husbands with their four wives. Those couples whom the first panelist identifies correctly are eliminated and receive nothing. Those remaining are then matched up by the second panelist. Those correctly matched receive $50 and are eliminated. Finally the third panelist tries to match those left. If they are now correctly matched, they receive $100, and if not, $1,000.

Assuming pure guessing on the part of the panelist among those arrangements still possible, what is the probability that the $1,000 prize will be won? What are a couple's expected winnings?

PME 355. **by John M. Howell**

On the TV game show called "Who's Who?", four panelists try to match the occupations of four contestants with signs marking their occupations. If the first panelist matches correctly, the contestants get nothing and the game is over. If the second panelist succeeds in matching correctly, the contestants get $25. If the second panelist fails but the third succeeds, the contestants get $50. If the fourth panelist matches after the third fails, the contestants get $75. If there is no match, the contestants win $100. What is the expected value of the contestants' winnings? Assume pure guessing and that no panelist repeats a previous arrangement.

Geometry: boxes

SSM 3783. **by Stephen J. Ruberg**

Suppose that the volume V of a box with a square base is chosen randomly from the chi-square distribution with two degrees of freedom. In addition, let the length S of a side of the base be chosen randomly and independent of V according to the following procedure: $S = |T|$, where T is chosen from the normal distribution with mean zero and standard deviation one. What is the probability that the height H of the box exceeds 400?

Geometry: circles

MATYC 98. **by Philip Cheifetz**

A large number of people are asked to draw a chord at random inside a circle of given radius R. After the chords are drawn, an equilateral triangle is inscribed in the circle. Deduce from probability considerations what percentage of the chords may be expected to be longer than the side of the triangle.

OSSMB G76.1-6. **by T. C. Simmonds**

(a) Three tangents, no two parallel, are drawn at random to a given circle. Show that the odds are 3 to 1 against the circle being inscribed (as opposed to being escribed) in the triangle formed by the tangents.

(b) If a triangle is formed by joining three points taken at random on the circumference of a circle, with the restriction that no pair of them be diametrically opposite one another, prove that the odds are 3 : 1 against it being acute angled.

Geometry: concyclic points

JRM 509. **by Les Marvin**

Points are selected at random on the circumference of the unit circle until the inscribed polygon that they determine encloses the center of the circle. It is known that the average (or statistically expected) number of points selected is five. Verify this with a Monte Carlo program, using the simplest program gimmick you can devise to enable the computer to tell when the center of the circle has been "netted".

Geometry: convex hull

AMM 6230. **by Gérard Letac**

Let $X(t)$ be the perimeter length of the convex hull of $b(s)_{0\le s\le t}$, where b is the standard brownian motion in the Euclidean plane. Compute $E(X(t))$.

Geometry: discs

OSSMB 77-2.

A disc of diameter 1 is tossed at random onto a coordinate plane. What is the probability that it covers a lattice point?

Geometry: point spacing

SIAM 76-4. **by Iwao Sugai**

Two points are chosen at random, uniformly with respect to area, one each from the two plane regions

$$0 \le x^2 + y^2 \le a^2$$

and

$$(a-b)^2 \le x^2 + y^2 \le a^2,$$

respectively. Find the probability that the distance between the two points is at most b $(0 < b < a)$.

AMM E2629. **by David P. Robbins**

Two points are chosen at random (uniform distribution) in the box $|x| \le a$, $|y| \le b$, $|z| \le c$ of $\mathbb{R}^3$. What is the expected distance between them?

SIAM 78-8. **by Timo Leipälä**

Determine
(a) the probability density,
(b) the mean, and
(c) the variance
for the Euclidean distance between two points which are independently and uniformly distributed in a unit cube.

NAvW 556. **by J. van de Lune**

For any $n \in \mathbb{N}$, let $\rho(n)$ be the mathematical expectation of the distance between two independent random points in the n-dimensional unit cube. Determine

$$\lim_{n\to\infty} n^{-\frac{1}{2}}\rho(n).$$

MM 946. **by M. H. Hoehn**

Two points are selected at random on the boundary of a unit square. What is the expected value of the length of the line segment joining the points?

Geometry: polygons

AMM E2594.* **by David P. Robbins**

Suppose that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are vectors corresponding to the edges of an oriented regular polygon. Since their sum is 0, an object undergoing displacements by each of these vectors in some order traces out a closed polygon. If this order is chosen at random, what is the probability that the polygon does not intersect itself?

Geometry: polyhedra

CRUX 499. **by Jordi Dou**

A certain polyhedron has all its edges of unit length. An ant travels along the edges and, at each vertex it reaches, chooses at random a new edge along which to travel (each edge at a vertex being equally likely to be chosen). The expected (mean) length of a return trip from one vertex back to it is 6 for some vertices and 7.5 for the other vertices.

Calculate the volume of the polyhedron.

AMM 6149. **by Gérard Letac**

A bug runs along the edges of a regular dodecahedron with constant speed: one edge per unit of time. At time 0 the bug is on some vertex A; at time n (n an integer) it chooses randomly one of the three possible edges. If p_n is the probability that the bug is on A at time n, then it is trivial to compute $p_0 = 1$, $p_1 = 0$, $p_2 = 1/3$, $p_3 = 0$, $p_4 = 5/27, \ldots$. Determine the generating function

$$\sum_{n=0}^{\infty} p_n s^n$$

of the sequence (p_n).

Geometry: quadrilaterals

NAvW 452. **by O. Bottema**

Let $P_1 < P_2 < P_3$ be three distinct points selected at random from the open interval $(0,1)$. Find the probability that a convex, inscribable (in a circle) quadrilateral can be formed having side lengths P_1, $P_2 - P_1$, $P_3 - P_2$, and $1 - P_3$. Also find the probability without the convexity restriction.

Geometry: rectangles

JRM 713. **by William C. Reil**

The lengths and widths of two rectangles are chosen randomly in the interval $(0,1)$.

(a) What is the probability that one will fit completely within the other?

(b) What is the probability that the one with the smaller area has the larger perimeter?

Geometry: squares

JRM 620. **by Susan Laird**

Four points are selected at random in a unit square. What is the probability that they are the vertices of a convex quadrilateral?

JRM 683. **by Daniel P. Shine**

A point is chosen at a random position in a unit square. On the average:

(a) How far is it from the center of the square?

(b) How far is it from the lower left corner of the square?

(c) How far is it from the *nearest* corner of the square?

(d) Substitute a circle of unit area for the square. How far is it from the center of the circle?

Geometry: triangles

SSM 3767. **by N. J. Kuenzi and Bob Prielipp**

Suppose an isosceles triangle with two sides of length a is formed by randomly selecting the length of the third side from the set of all possible lengths. Find the probability that the triangle formed is obtuse.

Independent trials

PME 395. **by Joe Dan Austin**

Assume that n independent Bernoulli experiments are made with $p = P$[success], $1-p = P$[failure], and $0 < p < 1$. Intuitively it seems that P[success on the first trial | exactly one success] is always less than P[success on the first trial | at least one success]. Verify directly that this is indeed the case.

MM 1070. **by Thomas E. Elsner and Joseph C. Hudson**

Let $p_1 + p_2 + \cdots + p_k = 1$ be a sum of $k \geq 2$ probabilities and let M_n for $n = 1, 2, \ldots$, be the multinomial distribution based on these probabilities and n trials. Event A_n occurs if, during the n trials, no possible outcome of the experiment occurs in two consecutive trials. Find the sum

$$\sum_{n=1}^{\infty} P(A_n).$$

What are the convergence criteria for this sum to exist?

AMM E2705. **by Clark Kimberling**

For an experiment having m equally probable outcomes, find the expected number of independent trials for k consecutive occurrences of at least one of these outcomes.

Inequalities

CRUX 484. **by Gali Salvatore**

Let A and B be two independent events in a sample space, and let X_A, X_B be their characteristic functions. If $F = X_A + X_B$, show that at least one of the three numbers

$$a = P(F = 2), \quad b = P(F = 1), \quad c = P(F = 0)$$

is not less than 4/9.

TYCMJ 152. **by Daniel Gallin**

Let E_i $(i = 1, 2, \ldots, n)$ be events in a probability space. Prove that $\max\left\{0, \sum_{i=1}^{n} P(E_i) - n + 1\right\}$ gives the best lower bound for $P\left(\bigcap_{i=1}^{n} E_i\right)$, given any n prescribed values $p_i = P(E_i)$, $0 \leq p_i \leq 1$, $(i = 1, 2, \ldots, n)$.

PUTNAM 1976/B.3.

Suppose that we have n events $A_1, \ldots, A_n$, each of which has probability at least $1 - a$ of occurring, where $a < 1/4$. Further suppose that A_i and A_j are mutually independent if $|i - j| > 1$, although A_i and A_{i+1} may be dependent. Assume as known that the recurrence $u_{k+1} = u_k - a u_{k-1}$, $u_0 = 1$, $u_1 = 1 - a$, defines positive real numbers u_k for $k = 0, 1, \ldots$. Show that the probability of all of $A_1, \ldots, A_n$ occurring is at least u_n.

SIAM 78-16. **by L. A. Shepp and A. M. Odlyzko**

Let $X_1, X_2, \ldots, X_n$ be independent random variables and let $Y_i = f_i(X_i)$ where $f_i(x) \uparrow$, $i = 1, 2, \ldots, n$. Prove or disprove that if

$$A = \{X_1 + X_2 + \cdots + X_i \geq 0, \quad i = 1, 2, \ldots, n\}$$

and

$$B = \{Y_1 + Y_2 + \cdots + Y_i \geq 0, \quad i = 1, 2, \ldots, n\},$$

then $P(A \mid B) \geq P(A)$.

AMM 6050. **by D. E. Knuth**

Suppose $X_1, X_2, Y_1, \ldots, Y_{m+n}$ are independent random variables, where X_1 and X_2 have common distribution F and the random variables $Y_1, \ldots, Y_{m+n}$ have common distribution G. Prove that

$$P\big[X_1 + \max(Y_1, \ldots, Y_m) \leq X_2 + \max(Y_{m+1}, \ldots, Y_{m+n})\big]$$

lies in the closed interval $[1/2, n/(m+n)]$ when $m \leq n$ and G is differentiable.

Jury decisions

FUNCT 3.1.1.

Two people of a 3-person jury each independently arrive at a correct decision with probability p. The third person flips a coin. The decision of the majority is final. What is the probability of the jury's reaching a correct decision?

Number theory

TYCMJ 57. **by Martin Berman**

From a set of n counters numbered $0, 1, 2, \ldots, n-1$, $(n \geq 2)$, a counter is removed at random, replaced, and then a counter is removed a second time at random.

(a) What is the probability that the numbers on the two counters satisfy the congruence $x + y \equiv xy \pmod{n}$?

(b) Show that the maximum probability occurs when $n = 2$.

TYCMJ 135. **by Gino T. Fala**

Let k be a positive integer consisting of $n-1$ digits. For $m \geq n$ let S_m be the set of positive integers consisting of m digits. A number is chosen at random from S_m. Denote by $P_m(k)$ the probability that the selected number is divisible by k.

(a) Determine $P_m(k)$.

(b) Determine $\lim_{m\to\infty} P_m(k)$.

CRUX 43. **by André Bourbeau**

In a 3×3 matrix, the entries a_{ij} are randomly selected integers such that $0 \leq a_{ij} \leq 9$. Find the probability that

(a) the three-digit numbers formed by each row will be divisible by 11;

(b) the three-digit numbers formed by each row and each column will be divisible by 11;

JRM 559. **by Diophantus McLeod**

A positive integer n is selected at random. What is the probability that n is a factor of $1^3 + 2^3 + \cdots + n^3$ but not of $1^2 + 2^2 + \cdots + n^2$?

MM 970. **by Martin Berman**

A plus or minus sign is assigned randomly to each of the numbers $1, 2, 3, \ldots, n$. What are the probabilities that the sum of the signed numbers is positive, negative, and zero?

PENT 306. **by Kenneth M. Wilke**

If n is a positive integer selected at random, what is the probability that

$$\frac{(2n+1)(3n^2+3n-1)}{15}$$

is an integer?

Order statistics

ISMJ 10.10.

Using the methods described in an article about high jumping, find for any number y the probability that the runner-up (second highest jump) in n attempts is larger than y.

Permutations

TYCMJ 54. **by John P. Hoyt**

Let $(i_1, i_2, \ldots, i_n)$ be a random rearrangement of the first n natural numbers 1, 2, ... n, where $n \geq 3$. What is the probability that, for each k, $i_k \geq k-3$?

Random variables

AMM 6195. **by Andreas N. Philippou**

For $j = 1, 2, \ldots$ and $n \geq j$, let X_{nj} and X_j be random variables defined on a probability space (Ω, A, P). Assume that $\sup_j \mathcal{E}|X_j|^r < \infty$ $(r > 0)$, where $\mathcal{E}$ denotes expectation under P. Show that

$$\max\left[\mathcal{E}|X_{nj} - X_j|^r, \quad 1 \leq j \leq n\right] \to 0,$$

if and only if

$$\max\left\{P\left[|X_{nj} - X_j| > \varepsilon\right], \quad 1 \leq j \leq n\right\} \to 0$$

and

$$\max\left[\mathcal{E}|X_{nj}|^r - \mathcal{E}|X_j|^r, \quad 1 \leq j \leq n\right] \to 0.$$

AMM 6031. **by I. I. Kotlarski**

Let ϕ be a periodic function on $\mathbb{R}$ with period 2π, given by

$$\phi(t) = 1 - \sqrt{\frac{|t|}{\pi}\left(2 - \frac{|t|}{\pi}\right)}, \qquad t \in [-\pi, \pi].$$

Prove that ϕ is a characteristic function of a real random variable X, and find its probability structure.

Let $X_1, X_2, \ldots, X_n, \ldots$ be a collection of independent identically distributed random variables, all distributed according to the characteristic function given above. Let

$$Y_n = \frac{(X_1 + X_2 + \cdots + X_n)}{n}$$

and

$$Z_n = \frac{(X_1 + X_2 + \cdots + X_n)}{n^2}.$$

Show that Y_n does not have a limiting distribution, while the limit distribution of Z_n is the stable distribution with exponent $1/2$.

SIAM 76-18. **by A. A. Jagers**

Let ϕ be a nonnegative function defined on $[0, \infty)$ with $\phi(0) = 0$. Let $\mathcal{F}$ be the class of all nonnegative random variables X such that $0 < E(X^s) < \infty$ for all $s > 0$. For $X \in \mathcal{F}$, $s > 0$, put

$$q_x(s) = \left[\frac{E\left(\phi(X)^s\right)}{E(X^s)}\right]^{1/s}.$$

(a) Prove that if ϕ is convex, then q_x is nondecreasing for each $X \in \mathcal{F}$.

(b) Determine a necessary and sufficient condition on ϕ in order that q_x is nondecreasing for each $X \in \mathcal{F}$.

AMM 6103. **by Gérard Letac**

Let $(X_n)_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables, valued in a real vector space E of finite dimension d. Let Y be a random linear form on E such that $\lim_{n\to\infty} Y(X_n)$ exists almost surely. If $d = 1$, it is easily proved that either $Y = 0$ almost surely or $X_n =$ constant almost surely. What happens if $d > 1$?

AMM 6114. **by R. M. Norton**

Let $Z_1 = XY$, where X and Y are independent standard normal random variables, and let Z_1 and Z_2 be independent and identically distributed. Derive the density function $f(x)$ of $Z_1 + Z_2$.

AMM 6104. **by Leonard W. Deaton**

Let X and Y be independent normal random variables with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Let

$$f(z) = \frac{\sigma_1\sigma_2}{\pi\left(\sigma_1^2 + \sigma_2^2 z^2\right)}, \qquad -\infty < z < \infty.$$

Is f the probability density function of $Z = X/Y$?

AMM 6030. **by David Griffeath**

Let X and Y be jointly distributed real random variables. Consider the following conjecture: If X, Y, $X+Y$, and $X-Y$ are all identically distributed, then $X = 0$ almost surely. Prove or disprove the conjecture in the following cases:

(a) if X is square-integrable;
(b) if X is integrable;
(c) in general.

AMM 6174. **by Barthel W. Huff**

A family $\mathcal{F} = \{X_\lambda | \lambda \in \Lambda\}$ of random variables is said to be uniformly integrable if

$$\lim_{\alpha\to\infty} \sup_\lambda E|X_\lambda| \cdot I_{[|X_\lambda| \geq \alpha]} = 0,$$

where I_A is the indicator function of the event A. One sufficient condition for uniform integrability is that there exists a random variable Y such that $|X_\lambda| \leq Y$ a.s., for all λ, and $EY < \infty$. A weaker sufficient condition is that there exists a nonnegative random variable Y such that

$$P\left[|X_\lambda| \geq \alpha\right] \leq P\left[Y \geq \alpha\right],$$

for all $\alpha > 0$, for all λ and $EY < \infty$. Is the converse to the weaker condition true?

Random vectors

AMM 6175. **by Ignacy I. Kotlarski**

Let

$$(X_1, X_2, \ldots, X_n)$$

be an n-dimensional real random vector. Consider the random polynomial of order n, $n = 2, 3, \ldots$, on the complex plane

$$P_n(\lambda) = (\lambda - X_1)(\lambda - X_2)\cdots(\lambda - X_n), \qquad \lambda \in \mathbb{Z},$$

and define

$$Z_n = \frac{1}{i}\frac{P_n(i) - (-1)^n P_n(-i)}{P_n(i) + (-1)^n P_n(-i)}, \qquad n = 2, 3, \ldots .$$

Show that if one of the X_k is independent from the others and follows the Cauchy distribution

$$P(X_k \leq x) = \frac{1}{2} + \frac{1}{\pi}\arctan x, \qquad x \in \mathbb{R},$$

then all the Z_n are real random variables having the same Cauchy distribution.

AMM 6207. **by Ignacy I. Kotlarski**

Let X and Y be two independent $(2n+2)$-dimensional normal random vectors with means 0 and positive definite variance covariance matrices C and C^{-1}, respectively ($n = 0, 1, \ldots$). Find the distribution of their inner product $Z = X \cdot Y$.

Relative motion

SIAM 76-13. **by L. K. Arnold, L. Dodson, and L. Rosen**

Two ships A and B are cruising along straight line paths in a planar ocean at constant speeds u and v, respectively. If B's direction is a random variable uniformly distributed over $(0, 2\pi]$, then the expected speed of B relative to A is given by

$$z = \int_0^{2\pi} \left\{u^2 + v^2 - 2uv\cos\theta\right\}^{1/2} \frac{d\theta}{2\pi}.$$

We have found that

$$\overline{z} = x + .27y^2/x,$$

where $x = \max(u, v)$, $y = \min(u, v)$, is a fair approximation to z. More precisely, $|z - \overline{z}| \leq .25$ knots for $u, v = 0(.25)40$. Prove or disprove the latter error bound for all u, v between 0 and 40.

Selection problems: distribution problems

JRM 379. **by Harry L. Nelson**

Admiring the wit of his court jester, the King decided to exercise it by subjecting him to a test. Four bags would be brought up from the treasury, containing four gold, four silver, four copper, and four zinc coins, respectively.

The King would randomly pick four of the sixteen coins, unobserved by the Jester, put these four coins in his pocket, summon the Jester and hand him two of the coins, randomly pulled from his pocket. The Jester, after looking at the two sample coins, would then attempt to guess the nature of the two coins left in the King's pocket (that is, "Zinc, Zinc" or "Gold, Zinc," with the order of the two metals in the latter case irrelevant).

"What do I get, Sire, if I guess right?" asked the Jester.

"All four of the coins from my original random selection."

"And what do I get if I miss completely or guess half right?"

"The chance to repeat the experiment a year from now."

"What random process will you use, Sire, to obtain the four coins?"

"A random selection is a random selection, Sirrah!"

"Yes, Sire, but two different selection schemes, both random, could result in quite different probability distributions. Two natural schemes that occur to me are to mix all 16 coins together and pull out four at random, or to choose randomly four times one of the four bags, with replacement of the chosen bag after each of the first three picks. The four coins would be determined by the number of times the various bags were picked. Thus if you picked the gold coin bag once and the zinc coin bag three times, your pocket sample would consist of one gold and three zinc coins. Another scheme that occurs to me . . ."

"You try our patience, Jester. You may choose between your two schemes. Tell the Chancellor of the Exchequer to fetch the bags."

Which scheme should the Jester choose in order to maximize his probability of making a correct guess?

If the King had decreed instead that the Jester need only guess correctly the number of different metals represented in the pocket sample of four, which scheme would give the Jester the better chance of guessing correctly?

In both the above variants and under each of the two schemes, what are the Jester's optimal guesses if the relative values of gold, silver, copper, and zinc coins are 4:3:2:1?

OMG 17.2.1.

A prisoner is given ten green balls, ten red balls, and two identical boxes. He is told that an executioner, while blindfolded, will randomly select one of the two boxes and will then randomly withdraw one ball from the box selected. If the ball drawn is green, the prisoner will be set free; but if the ball drawn is red, the prisoner will die. If the prisoner arranges the balls in the boxes so that he will have the best chance for survival, find the probability of his survival.

Selection problems: horse racing

AMM 6041. **by S. W. Golomb**

There are n horses in a "random" horse race, in which all $n!$ orders of finish are equally probable a priori. A gambler is allowed to select k horses, to finish first, second, …, kth.

(a) What is the probability $Q_n(k,i)$ that exactly i of his k selections will finish among the first k?

(b) What is the probability $P_n(k,i)$ that exactly i of his k selections will finish in the precise positions predicted for them?

Selection problems: limits

NAvW 509. **by J. H. van Lint and N. J. A. Sloane**

After serving French fries for many years, Heinz has become an expert. If you ask for, say, 27 French fries, he reaches in and scoops out almost exactly 27. He is so good now that he always gets within one of the number you want. It turns out that if one asks for a number of French fries, then Heinz picks uniformly from the admissible subsets. Assume that there are n French fries, and you ask for a number picked at random between 0 and n. Let P_n be the probability that Heinz scoops out exactly the number you ask for. Determine $\lim_{n\to\infty} P_n$.

Selection problems: points

NAvW 480. **by O. Bottema**

Two points are chosen at random and independently, on a line segment s. The distribution of each random point is uniform with respect to length. Determine the probability that an acute triangle can be constructed whose sides are equal to the three parts into which s is divided.

JRM 650. **by Daniel P. Shine**

Four birds land at random positions on a finite length of wire. Upon landing, each bird looks at its nearest neighbor.

(a) What is the probability that a bird picked at random is looking at another bird that is looking at it?

(b) What is the probability if there are n birds, with $n > 4$?

TYCMJ 96. **by Milton H. Hoehn**

On a line segment of length l, n points are selected at random. What is the expected value of the sum of the distances between all pairs of these points?

Selection problems: sets

AMM 6155. **by Milton P. Eisner**

Let $\{x_1, x_2, \ldots, x_k\}$ be a set of numbers. Define the width of the set to be $\min_{i\neq j}\{|x_i - x_j|\}$. Suppose the k numbers are selected at random from the set $\{1, 2, \ldots, n\}$. Find the expected value of the width of the resulting set if the numbers are chosen without replacement.

Selection problems: socks

JRM 621. **by Friend H. Kierstead, Jr.**

Individual socks from N distinguishable pairs are removed one by one from a dryer. Every time the second member of a pair is removed, it is matched immediately with its mate and the two are rolled together and set aside. What is the expected maximum number of unmatched socks?

Selection problems: sum of squares

AMM 6187. **by Ronald Evans**

Let $X_1, X_2, \ldots$ be a sequence of random numbers, uniformly distributed in $[0,1]$, and let N be minimal such that

$$\sum_{1\le i\le N} X_i^2 > 1.$$

Show that the expected value of N is

$$e^{\pi/4}\left(1 + \int_0^1 e^{-\pi t^2/4}\,dt\right).$$

Selection problems: sums

AMM E2696. **by William P. Wardlaw**

(a) If numbers are drawn randomly (using uniform distribution with replacements) from the set $\{1, 2, \ldots, n\}$ until their sum first exceeds n, what is the expected number of draws?

(b) Solve the same problem for numbers selected from the set $\{0, 1, \ldots, n-1\}$, until their sum exceeds $n-1$.

MATYC 122. **by Gene Zirkel**

A sequence of real numbers, $x_1, x_2, x_3, \ldots, x_n$, are picked at random from the interval $[0,1]$. This random selection is continued until their sum exceeds one and is then stopped. It is known that the expected value of the number N of reals chosen is given by $EN = e$.

What is EN if we instead continue until the sum exceeds two?

Selection problems: unit interval

PME 429. **by Richard S. Field**

Let P denote the product of n random numbers selected from the interval $(0,1)$. Is the expected value of P greater or less than the expected value of the nth power of a single number randomly selected from the interval $(0,1)$?

Selection problems: urns

SSM 3648. **by Wayne Wild**

An urn contains r red balls and b blue ones. The numbers r and b are such that if two balls are chosen simultaneously at random, the probability that they will be of opposite color is $1/2$. Characterize the numbers r and b.

CRUX 117. **by Paul Khoury**

The sultan said to Ali Baba: "Here are two urns, a white balls and b black balls. Distribute the balls in the urns, then I shall make the urns indistinguishable. To save your life, you must select one black ball." How can Ali Baba maximize his chances?

AMM E2722.* **by Clark Kimberling**

A ball is drawn from an urn containing one red ball and one green ball. If it is red, it is returned to the urn with one additional red ball and one additional green ball, but if it is green, no balls are put into the urn. After the first drawing, subsequent drawings take place following the same rules. Find the probability that the urn always contains at least one green ball.

JRM 623. **by Mark Wetzel**

An urn starts with one red and one green marble, and successive random samples are taken from it consisting of one marble, the sampling process terminating when a red marble is drawn. Determine the expected number of samples and the most probable number of samples if, after a green marble is drawn, it is replaced, together with another:

(a) green marble;
(b) red marble.

AMM E2724. **by Harry Lass**

An urn contains k_1 white balls, k_2 red balls, and k_3 blue balls. The balls are withdrawn one at a time at random without replacements until all balls of one color (red, white, or blue) have been removed.

(a) Determine the probability that all white balls are removed first.

(b) Determine the expected number of trials until all balls of some one color have been removed.

FUNCT 3.2.4.

A bag contains three red balls and five white ones. Balls are drawn at random from the bag without replacement, until all have been withdrawn. Show that the probability of getting a red ball on any particular draw is 3/8.

ISMJ 12.4.

Two boxes each contain three beads, one has 2 white and 1 red, the other 1 white and 2 red. A player chooses a box at random and a bead is taken at random from it. Having observed the color of the bead, the player may choose a second bead at random from the same box (without replacing the first) or from the other box. Find the probability that the second bead is red for each of the four strategies described in the article in the issue containing this problem.

Sequences

NAvW 489. **by A. J. Bosch and D. A. Overdijk**

Given are n different symbols. Let β be a finite sequence of these symbols. A machine produces these symbols successively such that every symbol has probability $1/n$ to be produced. The machine operates such that it stops as soon as a tail piece of the produced sequence equals β. The expectation of the length of the produced sequence will be denoted by μ and the variance by σ^2.

Prove:

(a) $\mu = \sum_{\ell \in L} n^\ell$,

(b) $\sigma^2 = \mu^2 - \sum_{\ell \in L} (2\ell - 1)n^\ell$,

where the set L is defined as follows: $L = \{\ell \in \mathbb{N} \mid$ there exists an initial piece of β with length ℓ that is also a tail piece of $\beta\}$.

AMM 6146. **by Edward J. Wegman and Anton Glaser**

Sir Francis Bacon assigned the 24 letters of the alphabet (j and u were absent) to the first 24 five-bit strings from 00000 to 10111. The word "Bacon" would appear as

00001 00000 00010 01101 01100

and this in turn could be hidden in a covertext of at least 25 letters, such as

00	00	10	000	00	00	1001	10	101	100
↓↓	↓↓	↓↓	↓↓↓	↓↓	↓↓	↓↓↓↓	↓↓	↓↓↓	↓↓↓
To	be,	*or*	not	to	be,	*that*	*is*	*the*	*q*uestion.

Here, "0" was replaced by one type style (in this case Roman) and "1" by another (in this case, italic). Thus, the 4,500,000 letters of the *First Folio* may be interpreted as a string of 4,500,000 binary digits.

What is the probability that the message "Bacon wrote this" appears in the *First Folio* "by accident" if

(a) the probability of a letter's being Roman is $1/2$?
(b) the probability of a letter's being italic is $1/10$?

Sets

AMM 6248. **by Milton P. Eisner**

Let the set $S = \{1, 2, \ldots, mn\}$, where m and n are positive integers, be partitioned randomly into n subsets each with m elements. For $0 \le k \le n$, what is the probability $P(m, n, k)$ that exactly k of these subsets have the property of consisting of m consecutive integers?

Slide rules

JRM 592. **by Les Marvin**

Two randomly selected numbers are to be multiplied together on a slide rule. What is the probability that, in lining up the index, the C scale must be moved to the left rather than to the right?

Sports

JRM 441. **by Sidney Kravitz**

Two baseball teams play in a World Series in which the first team to win four games wins the series. If the teams have an equal likelihood of winning any game, what is the probability that the series will run 4 games, 5 games, 6 games, and 7 games?

JRM 573. **by Harry Nelson**

Suppose two opposing pitchers both throw balls and unhittable strikes randomly but with ball or strike equally probable. Because of the low expectation of a run in any half inning, chances are high that the game will end with a score of $1-0$ after extra innings. In any event, determine both the expected and the most probable number of innings. When a game ends with t outs in the home half of the nth inning ($n > 8$), the game is considered to have lasted $n - \frac{1}{2} + \frac{t}{6}$ innings.

PME 373. **by Joe Dan Austin**

Assume that the number of shots at the goal in a hockey game is a random variable Y that has a Poisson distribution with parameter λ. Each shot is either blocked or is a goal. Assume each shot is independent of the other shots and $p = P$[a shot is blocked] for each shot. Find the probability that there are exactly k goals in a game for $k = 0, 1, 2, \ldots$.

JRM 387. **by Travis Fletcher**

Server, against his perennial tennis opponent, has two serves in his arsenal — a hard one that is very effective if it lands and a soft one that lands with greater reliability but is not as effective when it does. Assume that the hard serve lands fairly with probability 1/2 and when it does, Server has a 3/4 chance of winning the point. Assume that the soft serve lands fairly 3/4 of the time, and let the probability be p that if the soft serve lands safely, Server will win the point.

With the usual tennis allowance of two serves, Server has four possible serving strategies, unambiguously denoted HH, HS, SH, and SS.

For what ranges of p are each of the four serving strategies optimal?

Statistics

MATYC 115. **by Ronald McCuiston**

Prove that if the X-scores are all the same, then $r_s = 0.5$ where r_s is the Spearman Rank Correlation Coefficient.

MSJ 467.

Two teachers, working independently of one another, communicate to the publisher that their classes found 64 and 55 mistakes respectively in the same textbook. Comparison shows that exactly 40 of the mistakes were found by both classes. Estimate the number of mistakes that remained unnoticed by either of them.

JRM 376. **by Richard S. Field Jr.**

In 1900 the man-made, land-locked, freshwater Lake Stochastica was stocked with fish of several different species, no two species from the same "family" (in the sense of common usage, as opposed to the taxonomic sense). In other words, if any trout were stocked, they were all of the same species. Unfortunately, no records were kept as to the number of different species stocked, so the only upper bound we may assume in estimating the number of species presently in the lake is 750, which we will suppose is the number of distinct freshwater fish "families" on the planet. We know nothing about the relative proportions with which the original fish were stocked, nor do we know about the probable rate of growth of one species vis-a-vis another in the unusual environment of Lake Stochastica.

We do have reason to believe that the fish population has reached stability, both in absolute size and in relative proportion of species, because the Bureau of fisheries, using valid mark-sampling techniques in 1950, 1960, and 1970, found on each occasion that the estimated total fish population as well as the relative proportion of the various species was the same. Unfortunately, all records about the distinct species observed were lost, and all that remains is the Bureau's estimate that the total fish population of Lake Stochastica between 1950 and 1970 had peaked out at one million. Fortunately, the Lake has been protected from pollution, so the estimate of one million is still valid.

Recently, a research team from *Piscator Magazine* took a random sample of 1000 fish from the Lake, using methods that ensured that every one of the million fish, regardless of age, size, or habits peculiar to its species, had an equal chance of being netted. The sample consisted of 300 bass, 350 catfish, 200 gar, 150 perch, and 100 salmon. Had the sample been ten times as large, perhaps more than five species would have been netted, but *Piscator's* budget is limited. Using the data at hand, is there a statistically valid method for arriving at a "best" estimate of the total number of different fish species in Lake Stochastica?

MATYC 117. **by Richard Gibbs**

Suppose the scores on an exam are ranked as follows:

score	100	95	94	93	91	89	86	83	80
rank	1	3	5.5	7	8.5	11.5	14	15.5	18.5

How many scored 83? How many scored higher than 83?

Stochastic processes

SIAM 78-7. **by John Haigh**

Given a Poisson process P of rate λ whose successive points are $P_1, P_2, \ldots$, construct a process Q as follows. Let $U_1, U_2, \ldots$ be a sequence of independent identically distributed random variables taking values in $[0, 1]$ and independent of P, and let the point Q_n of Q be placed in the interval $[P_n, P_{n+1}]$ so as to divide the interval in the ratio $U_n : 1 - U_n$. Find necessary and sufficient conditions for Q to be a Poisson process.

JRM 480. **by David L. Silverman**

A point moves on an infinite rectangular lattice. At each stage, it moves with equal probability to one of the rookwise adjacent vertices that has not been previously occupied. With probability one it will eventually be stymied, although the potential number of moves prior to the stymie is obviously unbounded. Estimate the average duration of such a random walk.

Student's t-distribution

AMM 6092. **by Ignacy Kotlarski**

Let X_1 and X_2 be two independent Student distributed random variables with 1 and 3 degrees of freedom, respectively. Define

$$Y = \frac{1}{2}X_1\sqrt{3} + \frac{1}{2}X_2.$$

Show that the probability density functions of X_1, X_2, Y satisfy the relation

$$f_Y(y) = \frac{1}{2}f_{X_1\sqrt{3}}(y) + \frac{1}{2}f_{X_2}(y)$$

almost everywhere on $\mathbb{R}$.

Can this analogy be generalized to

$$Y = a_1X_1 + a_2X_2 + \cdots + a_nX_n,$$

where $X_1, \ldots, X_n$ are independent, Student distributed with $1, 3, \ldots, 2n-1$ degrees of freedom, and $a_1, a_2, \ldots, a_n$ are constants?

Tournaments

JRM 568. **by Michael Lauder**

Al, Bob, Carl, and Don were the four quarterfinalists in a Pong Tournament. They were paired off for the two semifinal matches. Then the two winners played in the finals for the championship, while the two semifinal losers played a consolation match for third place. I had a hunch that Al would meet Bob in the tournament and would beat him, but when I asked Al if my hunch had been right, he said no. Assuming he told me the truth, what is the probability that Bob won the tournament?

Transportation

CRUX 68. **by H. G. Dworschak**

It takes 5 minutes to cross a certain bridge and 1000 people cross it in a day of 12 hours, all times of day being equally likely. Find the probability that there will be nobody on the bridge at noon.

SIAM 75-8. **by L. J. Dickson**

The city has only one hospital and all ambulances are based there. Streets are so crowded that any ambulance on duty, whether going or returning, strikes and injures pedestrians at an average of one per mile. More precisely, an ambulance traveling over a stretch of Δx miles always has the probability $\exp(-\Delta x)$ of injuring nobody and the probability zero of hitting more than one person at once. Each downed pedestrian is picked up by a different ambulance.

A man at a distance d miles from City Hospital has a heart attack and calls for an ambulance.

(a) Find the probability $P_k(d)$ that exactly k pedestrians will be injured by all of the ambulances.

(b) What is the expected value of k?

MM 1034. **by Marlow Sholander**

We are familiar with the standard clover-leaf interchange [CLI] which has, inside the four ramps for making right-hand turns, the arrangement whereby left-hand turns are achieved by turning right into lanes which outline the four leaf clover. Your car approaches the CLI from the south. A mechanism has been installed so that at each point where there exists a choice of directions, the car turns to the right with fixed probability r.

(a) If $r = 1/2$, find P(emerge from CLI going west).

(b) Which r maximizes P in (a)?

Waiting times

PENT 313. **by Michael W. Ecker**

Joe and Moe plan to meet for lunch at the pizza parlor between noon and 1:00 PM but they can't decide what time to meet. Joe suggested that whoever arrives first should wait 10 minutes for the other before leaving.

Moe likes Joe's suggestion but he wonders if a 10-minute wait will guarantee that they will have at least an even chance of meeting for lunch. Assuming each of Joe's and Moe's times of arrival are random, what is the minimum time the first to arrive must wait to guarantee that their probability of having lunch together is at least 1/2?

Alphametics: animals

JRM 514. **by Michael R. W. Buckley**

Solve the alphametic:

$$\texttt{CLEAR} + \texttt{LAKES} + \texttt{LURE} + \texttt{LARGE} = \texttt{DRAKES}.$$

where DRAKES is as large as possible.

Alphametics: chess moves

JRM 639. **by Michael Keith**

Solve the chess alphametic

$$(\texttt{B}-\texttt{B6}) + (\texttt{K}-\texttt{R6}) + (\texttt{B}-\texttt{B5}) + (\texttt{K}-\texttt{R7}) + (\texttt{B}-\texttt{Q5}) + (\texttt{K}-\texttt{R8}) + (\texttt{O}-\texttt{O}) = \texttt{MATE}$$

where the numbers which already appear may be reused, the dash stands for a digit, (O − O) represents the castling move, and $\texttt{Q} \geq 3 \times \texttt{B}$. Can anyone come up with a chess alphametic having DRAW as its solution?

JRM 721. **by Michael Keith**

Solve the chess alphametic

$$(\texttt{K}-\texttt{f4}) + (\texttt{P}-\texttt{c5}) + (\texttt{P}-\texttt{d5}) + (\texttt{P}-\texttt{f5}) + (\texttt{K}-\texttt{g5}) + (\texttt{P}-\texttt{f4}) + (\texttt{P}-\texttt{f3}) = \texttt{QZAP}$$

where the numbers which already appear may be reused and the dash stands for a digit.

Alphametics: Christmas

JRM 413. **by Sidney Kravitz**

Solve the alphametic:

$$\texttt{TOYS} + \texttt{NOEL} + \texttt{SANTA} = \texttt{CLAUS}$$

where the prime interest of many children will be in their TOYS.

Alphametics: congruences

JRM 456. **by Randall J. Covill**

Find the smallest positive integral values of S, P, O, T, and I such that the following is true (the * indicates multiplication):

$$\texttt{S} * \texttt{P} * \texttt{O} * \texttt{T} \equiv 0 \pmod{462}$$
$$\texttt{I} * \texttt{S} \equiv 0 \pmod{12}$$
$$\texttt{T} * \texttt{O} * \texttt{P} * \texttt{S} \equiv 3 \pmod{5}.$$

Alphametics: constructions

JRM 707. **by Saburo Tamura**

There are 2,401 unique-solution decimal cryptarithms of the form 3 digits + 3 digits = 4 digits. For example, EEL + OWL = ODDY.

(a) How many decimal cryptarithms are there of the form 3 digits + 1 digit = 4 digits?

(b) How many of them have unique solutions?

(c) How many of these are realizable with English words?

Alphametics: cubes

PME 381. **by Clayton W. Dodge**

Solve the following alphametics:

$$\texttt{ICE}^3 = \texttt{ICYWHEEE}$$
$$\texttt{ICE}^3 = \texttt{ICYOHOH}.$$

Alphametics: division

JRM 403. **by R. S. Johnson**

Solve the alphametic: NEST/EDEN = .UNSAIDUNSAID...

OSSMB 77-8.

Each letter below represents a different digit, where THA − TZE = TB, TBY − TZE = BK, BKE − TZE = EKZ, and EKZR − EAZY = EBB. Find the digits.

$$\frac{\texttt{THAYER}}{\texttt{IRA}} = \texttt{EEEY}$$

SSM 3645. **by Alan Wayne**

Solve the long division alphametic:

```
MY)JUG(IS
   NO
   MUG
   MUG.
```

SSM 3654. **by Alan Wayne**

Solve the long division alphametic:

```
ED)OIL(UP
   AN
   EEL
   EEL.
```

Alphametics: doubly true

JRM 436. **by Sidney Kravitz**

Solve the alphametic:

$$\texttt{EEN} + \texttt{TVEIR} + \texttt{TRE} + \texttt{VIER} = \texttt{DIECI}$$

where the sum of the digits of TRE is 10.

JRM 414. **by Herman Nijon**

Solve the alphametic:

$$\texttt{TWINTIG} + \texttt{TWINTIG} + \texttt{DERTIG} + \texttt{DERTIG} = \texttt{HONDERD}$$

where TWINTIG is not divisible by 3, but DERTIG is divisible by 3.

JRM 415. **by Herman Nijon**

Solve the alphametic:

$$\texttt{TWENTY} + \texttt{TWENTY} + \texttt{THIRTY} + \texttt{THIRTY} = \texttt{HUNDRED}.$$

CRUX 491. **by Alan Wayne**

Solve the alphametic:

$$\texttt{UN} + \texttt{DEUX} + \texttt{DEUX} + \texttt{DEUX} + \texttt{DEUX} + \texttt{DEUX} = \texttt{ONZE}.$$

JRM 749. **by Masazumi Hanazawa**

Solve the alphametic:

$$\texttt{ONE} + \texttt{TWO} + \texttt{TWO} + \texttt{THREE} + \texttt{THREE} = \texttt{ELEVEN}.$$

CRUX 481. **by Herman Nyon**

Solve the alphametic

$$\texttt{DEUX} + \texttt{DEUX} + \texttt{DEUX} + \texttt{TROIS} + \texttt{TROIS} = \texttt{DOUZE}$$

where DEUX is divisible by 2 and DOUZE is divisible by 8.

NYSMTJ 91. **by Alan Wayne**

Solve the alphametic:

$$12(\texttt{UNIT}) = \texttt{TWELVE}.$$

JRM 368. **by Michael R. Buckley**

Solve the alphametic:

$$\texttt{ONE} + \texttt{ONE} = \texttt{TWO}.$$

What is the lowest base in which this alphametic has a unique solution?

SSM 3618. **by Charles W. Trigg**

In each of two different bases, there is a solution of the doubly-true cryptarithm:

$$\texttt{ONE} + \texttt{ONE} = \texttt{TWO}$$

in which the digits uniquely represented by the letters, in some order, are consecutive. Find the two solutions and show that there are no others involving consecutive digits in any base.

JRM 584. **by Michael Keith**

Solve the alphametic:

$$\texttt{UNO} + \texttt{UNO} = \texttt{DOS}.$$

This alphametic has a unique solution in one and only one positive base b. Find b and show that it is indeed unique.

CRUX 341. **by Herman Nyon**

Solve the alphametic

$$\texttt{TROIS} + \texttt{TROIS} + \texttt{SEPT} + \texttt{SEPT} = \texttt{VINGT}$$

where SEPT is divisible by 7.

JRM 398. **by Steven Kahan**

Solve the alphametic:

$$\texttt{ELEVEN} + \texttt{THREE} + \texttt{TWO} + \texttt{TWO} + \texttt{TWO} = \texttt{TWENTY}.$$

JRM 399. **by Steven Kahan**

Solve the alphametic:

$$\texttt{ELEVEN} + \texttt{THREE} + \texttt{TWO} + \texttt{TWO} + \texttt{ONE} + \texttt{ONE} = \texttt{TWENTY}$$

where THREE is divisible by 3.

JRM 400. **by Steven Kahan**

Solve the alphametic:

$$\texttt{ELEVEN}+\texttt{THREE}+\texttt{ONE}+\texttt{ONE}+\texttt{ONE}+\texttt{ONE}+\texttt{ONE}+\texttt{ONE} = \texttt{TWENTY}$$

where THREE is divisible by 3.

JRM 437. **by Sidney Kravitz**

The following alphametic was published earlier:

```
  NINE
 EIGHT
 THREE
------
TWENTY
```

Solve the alphametic where these numbers are vertical:

```
   T
  ETW
 NIHE
 IGRN
 NHET
----
 ETEY
```

In both alphametics, the sum of the digits of TWENTY is 20 and only nine of the ten digits are used; the missing digit is the same in both cases.

JRM 486. **by Steven Kahan**

Solve the alphametic:

$$\texttt{SEVEN} + \texttt{FIVE} + \texttt{FIVE} + \texttt{ONE} + \texttt{ONE} + \texttt{ONE} = \texttt{TWENTY}$$

where SEVEN is divisible by 7.

JRM 576. **by R. S. Johnson**

Solve the alphametic:

$$\texttt{EIGHT} + \texttt{THREE} + \texttt{THREE} + \texttt{THREE} + \texttt{THREE} = \texttt{TWENTY}$$

where EIGHT, THREE, and TWENTY are all divisible by 83.

JRM 612. **by Herman Nijon**

Solve the alphametic:

$$\texttt{CUATRO} + \texttt{CUATRO} + \texttt{CUATRO} + \texttt{CUATRO} + \texttt{CUATRO} = \texttt{VEINTE}.$$

JRM 691. **by Masazumi Hanazawa**

Solve the alphametic:

$$\texttt{THREE} + \texttt{THREE} + \texttt{THREE} + \texttt{ELEVEN} = \texttt{TWENTY}.$$

CRUX 281. **by Alan Wayne**
JRM 725. **by Gordon S. Lessells**

Solve the alphametic

$$\texttt{HUIT} + \texttt{HUIT} + \texttt{HUIT} = \texttt{DOUZE} + \texttt{DOUZE}.$$

JRM 719. **by Dave Millar**

Solve the alphametic:

$$\texttt{TEN} + \texttt{NINES} + \texttt{LESS} + \texttt{SIXTY} = \texttt{THIRTY}$$

where NINES is divisible by 9.

PME 391. **by Clayton W. Dodge**

Solve the alphametic:

$$\texttt{TWELVE} + \texttt{NINE} + \texttt{NINE} = \texttt{THIRTY}$$

where NINE is divisible by 9.

JRM 693. **by Herman Nijon**

Solve the alphametic:

$$\texttt{ZEVEN} + \texttt{DRIE} + \texttt{TIEN} + \texttt{TIEN} = \texttt{DERTIG}.$$

JRM 578. **by Michael R. W. Buckley**
TYCMJ 142. **by Alan Wayne**
Solve the alphametic:

$$\texttt{COUPLE} + \texttt{COUPLE} = \texttt{QUARTET}.$$

NYSMTJ 37. **by Alan Wayne**
Solve the simultaneous cryptarithms:

$$\texttt{TWO} + \texttt{TWO} = \texttt{FOUR}$$

$$\texttt{TWO} \times \texttt{W} = \texttt{FOUR}.$$

JRM 418. **by Steven Kahan**
Solve the alphametic:

$$\texttt{FOURTEEN} + \texttt{SEVEN} + \texttt{TEN} + \texttt{TEN} = \texttt{FORTYONE}.$$

JRM 458. **by Steven Kahan**
Solve the alphametic:

$$\texttt{FOURTEEN} + \texttt{FIVE} + \texttt{FIVE} + \texttt{FIVE} + \texttt{FIVE}$$

$$+ \texttt{FIVE} + \texttt{ONE} + \texttt{ONE} = \texttt{FORTYONE}$$

where FOURTEEN is divisible by 7.

JRM 665-3. **by Jay Stevens**
Solve the alphametic:

$$\texttt{FOURTEEN} + \texttt{ELEVEN} + 16(\texttt{ONE}) = \texttt{FORTYONE}.$$

JRM 543. **by Herman Nijon**
Solve the alphametic:

$$7(\texttt{SEVEN}) = \texttt{FORTY9}$$

where SEVEN is divisible by 7.

JRM 665-2. **by Dave Millar**
Solve the alphametic:

$$\texttt{FORTY} + \texttt{NINE} + \texttt{IS} + \texttt{SEVEN} = \texttt{SEVENS}.$$

JRM 727. **by Frank Rubin**
Solve the alphametic

$$\texttt{SEVEN} \times \texttt{SEVEN} = \texttt{FORTYNINE}$$

in the smallest possible base.

JRM 774. **by Hans Havermann**
Solve the alphametic:

$$\texttt{ONE} + \texttt{ONE} + \texttt{ONE} + \texttt{ONE} = \texttt{FOUR} + \texttt{ONE} = \texttt{FIVE}.$$

JRM 409. **by Brian R. Barwell**
Solve the alphametic:

$$25(\texttt{TWO}) = \texttt{FIFTY}.$$

JRM 459. **by Steven Kahan**
Solve the alphametic:

$$\texttt{SEVEN} + \texttt{SEVEN} + \texttt{THREE} + \texttt{THREE} + \texttt{TEN} + \texttt{TEN} + \texttt{TEN} = \texttt{FIFTY}$$

where SEVEN is divisible by 7.

JRM 636. **by Dave Millar**
Solve the alphametic:

$$\texttt{FIVE} + \texttt{TIMES} + \texttt{TEN} + \texttt{IS} = \texttt{FIFTY}$$

where FIVE is divisible by 5 and TEN is divisible by 10.

JRM 641. **by Alf D. Seider**
Solve the alphametic:

$$2(\texttt{NINETEEN}) + \texttt{FIVE} + \texttt{FIVE} + \texttt{SIX} + \texttt{ONE} + \texttt{ONE} = \texttt{FIFTYSIX}$$

where SIX is even.

JRM 432. **by Steven Kahan**
Solve the alphametic:

$$6(\texttt{EIGHT} + \texttt{TWO}) = \texttt{SIXTY}$$

where EIGHT is divisible by 8.

JRM 746. **by Herman Nijon**
Solve the alphametic:

$$\texttt{DOZEN} + \texttt{DOZEN} + \texttt{DOZEN} + \texttt{DOZEN} + \texttt{DOZEN} = \texttt{SIXTY}$$

where DOZEN is divisible by 12, and so is its digital sum.

JRM 583. **by Jay Stevens**
Solve the alphametic:

$$2(\texttt{FIFTEEN}) + 3(\texttt{SEVEN}) + \texttt{TEN} = \texttt{SIXTYONE}.$$

JRM 544. **by Herman Nijon**
Solve the alphametic:

$$8(\texttt{EIGHT}) = \texttt{SIXTY4}.$$

JRM 492. **by Anton Pavlis**
Solve the alphametic:

$$\texttt{WE} + \texttt{ADD} + \texttt{3AND4} = \texttt{SEVEN}.$$

CRUX 451.
Solve the alphametic

$$\texttt{TWENTY} + \texttt{TWENTY} + \texttt{THIRTY} = \texttt{SEVENTY},$$

where THIRTY is divisible by 30.

JRM 364. **by S. Kahan**
Solve the alphametic:

$$\texttt{SIXTEEN} + \texttt{THIRTY} + \texttt{SIX} + \texttt{SIX} + \texttt{SIX} + \texttt{SIX} = \texttt{SEVENTY}$$

where SEVENTY is even.

JRM 525. **by Steven Kahan**
Solve the alphametic:

$$\texttt{TWENTY} + 4(\texttt{ELEVEN}) + \texttt{THREE} + \texttt{THREE} = \texttt{SEVENTY}.$$

JRM 526. **by Steven Kahan**
Solve the alphametic:

$$\texttt{TWENTY} + 4(\texttt{ELEVEN}) + \texttt{SIX} = \texttt{SEVENTY}.$$

JRM 608. **by Michael R. W. Buckley**
Solve the alphametic:

$$5(\texttt{ELEVEN}) + \texttt{SEVEN} + \texttt{SEVEN} + \texttt{HALF} + \texttt{HALF} = \texttt{SEVENTY}.$$

JRM 726. **by Alf D. Seider**
Solve the alphametic:

$$\texttt{FIFTEEN} + \texttt{FIFTEEN} + 10(\texttt{THREE}) + \texttt{TEN} = \texttt{SEVENTY}.$$

JRM 750. **by Kenneth Vasa**
Solve the alphametic:

$$\texttt{SIXTEEN} + \texttt{TWENTY} + 3(\texttt{TEN}) + \texttt{TWO} + \texttt{TWO} = \texttt{SEVENTY}$$

where SEVENTY is even.

JRM 516. **by Frank Rubin**
FQ B-312. **by J. A. H. Hunter**
Solve the alphametic:

$$\texttt{THREE} + \texttt{TWO} + \texttt{ONE} + \texttt{ONE} + \texttt{ONE} = \texttt{EIGHT}.$$

JRM 517. **by Frank Rubin**
Solve the alphametic:

$$\texttt{THREE} + \texttt{TWO} + \texttt{TWO} + \texttt{ONE} = \texttt{EIGHT}.$$

FQ B-316. **by J. A. H. Hunter**
Solve the alphametic

$$\texttt{TWO} + \texttt{THREE} + \texttt{THREE} = \texttt{EIGHT}$$

where none of the digits is an 8.

JRM 369. **by A. G. Bradbury**
Solve the alphametic:

$$\texttt{THEN} + \texttt{TEST} + \texttt{EIGHT} + \texttt{TIMES} + \texttt{TEN} = \texttt{EIGHTY}$$

where EIGHTY is even.

JRM 611. **by Masazumi Hanazawa**
Solve the alphametic:

$$\texttt{TWENTY} + \texttt{TWENTY} + \texttt{TWENTY} + \texttt{TEN} + \texttt{TEN} = \texttt{EIGHTY}.$$

JRM 640. **by T. Rosler**
Solve the alphametic:

$$16(\texttt{FIVE}) = \texttt{EIGHTY}.$$

JRM 665-1. **by Masazumi Hanazawa**
Solve the alphametic:

$$\texttt{TEN} + \texttt{TEN} + \texttt{TEN} + \texttt{TEN} + \texttt{TWENTY} + \texttt{TWENTY} = \texttt{EIGHTY}.$$

JRM 775. **by Masazumi Hanazawa**
Solve the alphametic:

$$\texttt{THREE} + \texttt{THREE} + \texttt{FIVE} + \texttt{NINE} + \texttt{THIRTY} + \texttt{THIRTY} = \texttt{EIGHTY}.$$

CRUX 261. **by Alan Wayne**
Solve the alphametic

$$\texttt{UN} + \texttt{DEUX} + \texttt{DEUX} + \texttt{DEUX} + \texttt{DEUX} = \texttt{NEUF}.$$

JRM 613. **by Edwin Floyd**
Solve the alphametic:

$$\texttt{TPIA} + \texttt{TPIA} + \texttt{TPIA} = \texttt{ENNEA}.$$

JRM 692. **by Edwin E. Floyd**
Solve the alphametic:

$$\texttt{TRIA} + \texttt{DVO} + \texttt{DVO} + \texttt{DVO} = \texttt{NOVEM}.$$

JRM 433. **by Steven Kahan**
Solve the alphametic:

$$\texttt{ELEVEN} + \texttt{SIXTY} + \texttt{SEVEN} + \texttt{SEVEN} + \texttt{FIVE} = \texttt{NINETY}.$$

JRM 485. **by Steven Kahan**
Solve the alphametic:

$$\texttt{FIFTY} + \texttt{SEVEN} + \texttt{SEVEN} + \texttt{EIGHT} + \texttt{EIGHT} + \texttt{TEN} = \texttt{NINETY}$$

where SEVEN is divisible by 7.

JRM 776. **by Jay Stevens**
Solve the alphametic:

$$\texttt{SIXTY} + 3(\texttt{EIGHT}) + \texttt{THREE} + \texttt{THREE} = \texttt{NINETY}$$

where NINETY is even.

Alphametics: elements

JRM 520. **by Herman Nijon**
Solve the alphametic:

$$\texttt{OXYGEN} + \texttt{XENON} + \texttt{ARGON} + \texttt{NEON} = \texttt{NATURE}.$$

Alphametics: equations

OSSMB 77-15. **by Stephen Maulsby**
Each letter in the arithmetic operations below represents a different digit. Find these digits.

$$\begin{array}{ccccc} \texttt{ABCD} & - & \texttt{DGB} & = & \texttt{AKJA} \\ \div & & + & & - \\ \texttt{AE} & \times & \texttt{EJ} & = & \texttt{DCCH} \\ = & & = & & = \\ \texttt{FGH} & + & \texttt{DBK} & = & \texttt{ABC} \end{array}$$

Alphametics: food

CRUX 351. **by Sidney Kravitz**
Solve the alphametic

$$\texttt{GRAPE} + \texttt{APPLE} = \texttt{CHERRY}.$$

CRUX 381. **by Sidney Kravitz**
Solve the alphametic

$$\texttt{BETTE} + \texttt{TOMATE} = \texttt{OIGNON}.$$

JRM 717. **by Sidney Kravitz**
Solve the alphametic:

$$\texttt{PEPPER} + \texttt{PARSNIP} = \texttt{SPINACH}.$$

JRM 720. **by Ronald J. Lancaster**
Solve the alphametic:

$$\texttt{A} + \texttt{TREAT} + \texttt{CARTER} + \texttt{PEANUT} = \texttt{BUTTER}.$$

Alphametics: letters

JRM 723. **by Herman Nijon**
Solve the alphametic:

$$\texttt{ALPHA} + \texttt{BETA} + \texttt{GAMMA} = \texttt{OMEGA}$$

where OMEGA is largest.

Alphametics: money

JRM 779. **by Gordon S. Lessells**
Solve the alphametic:

$$5(\texttt{POUND}) = \texttt{DOLLAR} + \texttt{DOLLAR}.$$

Alphametics: multiplication

CRUX 321. **by Alan Wayne**
Solve the alphametic

ONE × ONE = BYGONE.

CRUX 471. **by Alan Wayne**
Solve the alphametic:

```
  WE
  DO
 TAM
HAT
TRIM
```

ISMJ 14.8.
Solve the alphametic:

9 × HATBOX = 4 × BOXHAT.

JRM 615. **by Alan Wayne**
Solve the alphametic:

```
WE)GOT(US
   AN
   OUT
   OUT
```

JRM 669. **by Rosann Hyler**
It might seem that over 8000 different solutions to the following alphametic should be possible. However, the alphametic is

(NOT)(8∗∗∗) = VALUED

when zero is allowed only as an asterisk and 8 may be reused.

JRM 752. **by Frank Rubin**
Solve the alphametic:

NINE × FOR × EVER = GOGOGOGOGO.

SSM 3673. **by Alan Wayne**
Solve the following two alphametics:
(a) A × DIVA = AVID
(b) I × SPOT = TOPS.

SSM 3739. **by Alan Wayne**
What might be said about De Moivre:

HE × IS = BIG + FOR = TRIG.

Regard the preceding pattern as an arithmetic multiplication of integers in the decimal system in which each digit has been replaced by one and only one letter, with different digits being replaced by different letters. Restore the digits.

SSM 3750. **by Alan Wayne**
Solve the alphametic:

DEED × DEED = EDUCATOR.

JRM 695. **by T. Rosler**
Solve the alphametic:

RUM × RUM = DRINKS.

JRM 522. **by T. Marlow**
Solve the alphametic:

EAST × S = WEST.

JRM 523. **by T. Marlow**
Solve the alphametic:

WEST × S = EAST.

PENT 280. **by Kenneth M. Wilke**
Solve the cryptarithm:

THAT = (AH)(HA).

CRUX 241. **by John J. McNamee**
Solve the alphametic

(HE)(EH) = WHEW.

Alphametics: names

JRM 718. **by A. G. Bradbury**
Solve the alphametic:

THEN + THE + LION + ATE + LITTLE = ALBERT.

FQ B-322. **by Sidney Kravitz**
Solve the alphametic

ARKIN + ALDER + SALLE = ALLADI.

where 6 does not appear.

JRM 402. **by Anton Pavlis**
Solve the alphametic:

SEND + ONE + TO = ANTON.

CRUX 105. **by Walter Bluger**
INA BAIN declared once at a meeting
That she'd code her full name (without cheating),
Then divide, so she reckoned,
The first name by the second,
Thus obtaining five digits repeating.

JRM 374. **by Walter Bluger**
When asked for her phone number and date of birth, Ina Bain replied: "If each distinct letter stands for a particular but different digit, then my phone number is given by the seven letters of my name. If you divide INA by BAIN, a fraction in its lowest terms, you get a decimal with a 5-digit repeating cycle which shows the day, month, and the last two digits of the year of my birth, in that order."
What was Ina's birth date?

JRM 548. **by Fred Pence**
Solve the alphametic:

```
 BAT-
  MAN
  AND
ROBIN
```

where ROBIN is prime, and the - represents a digit.

JRM 404. **by R. S. Johnson**
Solve the alphametic:

THE + MATH + BIBLE + BY + MR + ALBERT = BEILER.

JRM 747. **by Ronald J. Lancaster**
Solve the alphametic:

MEET + A + PEANUT + FARMER + MR = CARTER.

JRM 770b. **by Eva L. Milbouer**

Solve the alphametic:

$$\texttt{MANET} + \texttt{MONET} = \texttt{COROT}$$

where COROT is as great as can be.

JRM 515. **by Leslie E. Card**

Solve the alphametic:

$$\texttt{FORD} + \texttt{AND} + \texttt{SADAT} = \texttt{CONFER}.$$

CRUX 236. **by Viktors Linis**

Solve the alphametic

$$\texttt{GAUSS} - \texttt{DIED} = 1855.$$

CRUX 238. **by Clayton W. Dodge**

Solve the alphametic

$$\texttt{CARL} + 1777 = \texttt{GAUSS}$$

where both 1 and 7 are represented among the letters.

CRUX 239. **by Clayton W. Dodge**

Solve the alphametic

$$\texttt{CARL} + 1777 + 1855 = \texttt{GAUSS}$$

where each of the digits 1, 5, 7, and 8 is represented by a letter.

CRUX 240. **by Clayton W. Dodge**

Solve the alphametic

$$\texttt{CARL} \times \texttt{F} = \texttt{GAUSS}.$$

JRM 610. **by A. G. Bradbury**

Solve the alphametic:

$$\texttt{SOON} + \texttt{HOLMES} + \texttt{WE} + \texttt{TEST} = \texttt{WATSON}.$$

CRUX 301. **by Herman Nyon**

Solve the alphametic

$$\texttt{HUNTER} - \texttt{TRIGG} = \texttt{DIGITS}.$$

where there are two solutions and the sum of the digits of HUNTER and TRIGG in one solution are equal, respectively, to the sum of the digits of TRIGG and HUNTER in the other solution.

JRM 686. **by A. G. Bradbury**

Solve the alphametic:

$$\texttt{AYE} + \texttt{AYE} + \texttt{CARRY} + \texttt{ON} = \texttt{JEEVES}$$

where "CARRY ON" may sound a little odd to some ears, but these are not odd words. Any fan of Bertie Wooster in P. G. Wodehouse's many stories would readily confirm this!

CRUX 431. **by Alan Wayne**

The following decimal alphametic is dedicated to Erwin Just, Problem Editor of the Two-Year College Mathematics Journal, who modestly refused to publish it in his own journal:

$$\texttt{YES} + \texttt{YES} + \texttt{JUST} = \texttt{ERWIN}.$$

ERWIN is, of course, unique.

JRM 429. **by Leslie E. Card**

Solve the alphametic:

$$\texttt{RECMATH} + \texttt{SALUTES} = \texttt{MADACHY}$$

in base 12.

JRM 449. **by Steven Kahan**

Solve the alphametic:

$$\texttt{NELSON} + \texttt{STARTS} + \texttt{AS} + \texttt{AN} = \texttt{EDITOR}.$$

JRM 663. **by A. G. Bradbury**

Solve the alphametic:

$$\texttt{OMARS} + \texttt{RUBY} + \texttt{HAT} + \texttt{A} = \texttt{BEAUTY}.$$

where the use of base 11 is recommended, and the largest possible RUBY is sought.

JRM 455. **by Leslie E. Card**

Solve the alphametic:

$$\texttt{POLK} + \texttt{TAFT} + \texttt{FORD} + \texttt{FOOL} = \texttt{PROOF}.$$

NYSMTJ 65. **by Janet Locke**

Solve the following alphametic:

$$\texttt{FOR} + \texttt{EITHER} + \texttt{FORD} + \texttt{OR} + \texttt{CARTER} = \texttt{ACHEER}.$$

JRM 581. **by Alan Wayne**

Solve the alphametic:

$$\texttt{ALAN} + \texttt{WAYNE} = \texttt{SOLVER}$$

where SOLVER is odd.

Alphametics: numbers

JRM 577. **by Herman Nijon**

Solve the alphametic:

$$\texttt{SQUARE} + \texttt{SQUARE} + \texttt{CUBE} + \texttt{CUBE} + \texttt{CUBE} = \texttt{NUMBERS}.$$

JRM 609. **by Peter MacDonald**

Solve the alphametic:

$$\texttt{ONES} + \texttt{ZEROES} = \texttt{BINARY}.$$

Alphametics: phrases

CRUX 251. **by Robert S. Johnson**

Solve the alphametic

$$\texttt{SPRING} + \texttt{RAINS} + \texttt{BRING} + \texttt{GREEN} = \texttt{PLAINS}.$$

JRM 416. **by Anton Pavlis**

Solve the alphametic:

```
    ON
  MOON
    NO
 GREEN
------
CHEESE
```

JRM 417. **by Frank Rubin**

Solve the alphametic:

```
 HE
 IS
---
HUB
OF
---
LAB
```

JRM 743. **by Ronald J. Lancaster**
Solve the alphametic:

$$\texttt{COME} + \texttt{ONE} + \texttt{COME} + \texttt{ALL} + \texttt{HAVE} + \texttt{A} = \texttt{BALL}$$

where BALL is prime.

JRM 616. **by Les Marvin**
Solve the alphametic:

$$\texttt{ANOTHER} + \texttt{ROTTEN} + \texttt{ENCODED} = \texttt{ADDITION}.$$

In what bases, if any, can this be solved?

PME 433. **by Clayton W. Dodge**
Solve the alphametic:

$$\texttt{PAY} + \texttt{MY} = \texttt{BILL}$$

where BILL is divisible by 4.

JRM 405. **by Derrick Murdoch**
Solve the alphametic:

$$\texttt{BRIDE} + \texttt{RIDES} + \texttt{UNDER} = \texttt{BRIDGE}.$$

JRM 428. **by A. G. Bradbury**
Solve the alphametic:

$$\texttt{CHOOSE} + \texttt{CHESS} + \texttt{OR} = \texttt{BRIDGE}$$

where neither game is ODD.

JRM 450. **by A. G. Bradbury**
Solve the alphametic:

$$\texttt{BITTER} + \texttt{SWEET} + \texttt{WISE} = \texttt{CHOICE}$$

where CHOICE is odd.

JRM 662. **by Ronald J. Lancaster**
Solve the alphametic:

$$\texttt{SMOKE} + \texttt{MAKES} + \texttt{ME} = \texttt{CHOKE}.$$

JRM 524. **by A. G. Bradbury**
Solve the alphametic:

$$\texttt{DO} + \texttt{NOT} + \texttt{SAY} = \texttt{DIE}$$

where the three-letter words form a regular magic square.

JRM 745. **by A. G. Bradbury**
Solve the alphametic:

$$\texttt{LET} + \texttt{THREE} + \texttt{LITTLE} + \texttt{MAIDS} = \texttt{DISMISS}.$$

MATYC 95. **by Sarah Brooks**
Solve the alphametic

$$\texttt{PEACE} + \texttt{HERE} + \texttt{ON} = \texttt{EARTH}.$$

JRM 753. **by Robert Gladman**
Solve the alphametic:

$$\texttt{ABLE} + \texttt{WAS} + \texttt{I} + \texttt{ERE} + \texttt{I} + \texttt{SAW} = \texttt{ELBA}$$

(a) Find a solution in which ELBA is prime both forward and backward.
(b) Find a solution in which more than four of the seven words are prime.
(c) Find a solution in which both WAS and SAW are prime.

JRM 435. **by Donna Kossy**
Solve the alphametic:

$$\texttt{LAZY} - \texttt{WEEK} = \texttt{END}.$$

JRM 547. **by R. S. Johnson**
Solve the alphametic:

$$\texttt{SCIENTIFIC} + \texttt{AMERICAN} + \texttt{MASTER} + \texttt{CREATES} + \texttt{FRENETIC} + \texttt{INTEREST} + \texttt{IN} + \texttt{IMF} + \texttt{METRIC} + \texttt{TENS} + \texttt{STATE} = \texttt{FANTASTICA}.$$

JRM 687. **by Michael R. W. Buckley**
Solve the alphametic:

$$\texttt{SCIENCE} + \texttt{FACT} + \texttt{SCIENCE} = \texttt{FICTION}.$$

JRM 614. **by Hank Venetas**
Solve the alphametic:

$$\texttt{THAT} + \texttt{THAT} + \texttt{THAT} + \texttt{THATS} + \texttt{ALL} = \texttt{FOLKS}.$$

where FOLKS is largest.

JRM 549. **by Michael R. W. Buckley**
Solve the alphametic:

$$\texttt{THESE} + \texttt{THREE} + \texttt{FLEAS} = \texttt{FREEZE}.$$

where FLEAS is odd.

JRM 582. **by J. A. H. Hunter**
Solve the alphametic:

$$\texttt{ALORS} + \texttt{ALORS} + \texttt{NOUS} + \texttt{NOUS} = \texttt{LAVONS}.$$

JRM 484. **by R. S. Johnson**
Solve the alphametic:

$$\texttt{PASSES} + \texttt{AT} + \texttt{AHHS} + \texttt{LASSIES} + \texttt{PLEASE} + \texttt{LASSIES} + \texttt{WITH} = \texttt{GLASSES}.$$

JRM 488. **by Eva L. Milbouer**
JRM 489.
JRM 490.
Solve the independent alphametics:

(1) ADAM + AND + EVE + ATE + THE = SNAKE,
(2) THE + SNAKE + IS + IN + THE = GRASS,
(3) ALAS + ALAS + ALAS + CRAB = GRASS.

Clue: the number A increases in value from (1) to (2) to (3).

JRM 688. **by Anton Pavlis**
Solve the alphametic:

$$\texttt{DOG} + \texttt{EATS} + \texttt{DOG} + \texttt{IS} = \texttt{GREED}.$$

JRM 406. **by Derrick Murdoch**
Solve the alphametic:

$$\texttt{GROOM} + \texttt{GOES} + \texttt{UNDER} = \texttt{GROUND}$$

where BRIDE is odd.

JRM 366. **by A. G. Bradbury**
Solve the alphametic:

$$\texttt{CATCH} + \texttt{THE} + \texttt{STOLEN} = \texttt{LAUNCH}.$$

JRM 574. **by John W. Harris**
Solve the alphametic:

MIX + FUN + AND = MATH

where FUN is largest.

CRUX 441. **by Sunder Lal and Léo Sauvé**
Solve the alphametic

ASHA + GOT + THE = MEDAL.

JRM 439. **by Anton Pavlis**
Solve the alphametic:

THE + BEST + SYSTEM = METRIC.

JRM 551. **by Frank Rubin**
Solve the alphametic:

DAD + SEND + MORE = MONEY.

JRM 552. **by Frank Rubin**
Solve the alphametic:

I + SENT + HIM + MORE = MONEY.

JRM 553. **by Frank Rubin**
Solve the alphametic:

SEND + YET + MORE = MONEY.

JRM 460. **by Sidney Kravitz**
Solve the alphametic:

741776 + THE + BIRTH + OF + A + FREE = NATION.

JRM 689. **by J. A. H. Hunter**
Solve the alphametic:

SO + SEEMS + NO + END + TO + MANS = NEEDS

where NEEDS is prime.

CRUX 401. **by Herman Nyon**
Solve the alphametic

HAPPY + NEW + YEAR = ∗1979

where the eight letters and the asterisk represent nine distinct nonzero digits and YEAR is divisible by 7.

JRM 606. **by Frank Rubin**
Solve the alphametic:

A + STITCH + IN + TIME = SAVES9.

JRM 605. **by Ronald J. Lancaster**
Solve the alphametic:

JRM + THE + FUN = ONE.

JRM 744. **by A. G. Bradbury**
Solve the alphametic:

HEAR + YE + HEAR + YE + SAVOY = OPERAS.

JRM 754. **by H. Everett Moore**
Solve the alphametic:

CAMP + DAVID = PEACE

where PEACE is greatest.

JRM 518. **by Paul E. Boymel**
Solve the alphametic:

CALM + AREA + LESS + MASS = MAGIC

where $E = MC^2$.

JRM 519. **by Herman Nijon**
Solve the alphametic:

EARTH + AIR + FIRE + WATER = NATURE.

JRM 491. **by Anton Pavlis**
Solve the alphametic:

POLICE + ARREST + ASSIST = PEOPLES.

JRM 575. **by Peter MacDonald**
Solve the alphametic:

ALPHA + METIC = PLEASE

where PLEASE is odd.

JRM 408. **by Michael R. W. Buckley**
Solve the alphametic:

EVEN + ODD = PRIME.

There are nine different digits, so find a solution in base 9 where ODD is odd, EVEN is even, and PRIME is prime.

JRM 482. **by Michael R. W. Buckley**
Solve the alphametic:

VERY + EASY = PUZZLE

where the power of negative thinking will help to solve this PUZZLE in the greatest base possible. This provides an opportunity for comments on modern methods of mathematics instruction.

JRM 431. **by R. S. Johnson**
Solve the alphametic:

TO + BE + OR + NOT + TO + BE + THAT + IS + THE = ????

Each ? stands for the same digit which may already be represented by one of the other letters in the alphametic.

CRUX 391. **by Allan Wm. Johnson Jr.**
Solve the alphametic

A + SUN + DRIED + GRAPE = RAISIN

where P > U.

JRM 633. **by A. G. Bradbury**
Solve the alphametic:

MEN + SEE + MINI + SKIRTS = RETURN.

JRM 401. **by Anton Pavlis**
Solve the alphametic:

SEND + SIX + RED = ROSES.

JRM 742. **by Patrick Costello**
Solve the alphametic:

STAR + WARS + WHAT + A = SIGHT.

JRM 367. **by Anton Pavlis**
Solve the alphametic:

FULL + VALUE + FOR = SILVER.

JRM 461. **by Anton Pavlis**
Solve the alphametic:

SOME + SAW + A + HOLY = SMOKE.

JRM 661. **by Ronald J. Lancaster**
Solve the alphametic:

OK + NO + JOKE + DO + NOT = SMOKE

where SMOKE is our prime objective.

JRM 542. **by J. A. H. Hunter**
Solve the alphametic:

NOT + TOO + EASY + TO = SOLVE.

JRM 690. **by Ronald J. Lancaster**
Solve the alphametic:

HATE + TO + DO + 55 + NEED + TO = SPEED

where the digit 5 may *not* be used again.

JRM 778. **by W. A. Robb**
Solve the alphametic:

NOSY + PORTER + TOO + NOSY + WRITES + WRY = STORIES

where the PERSON is odd.

JRM 660. **by Bob Vinnicombe**
Solve the alphametic:

FISN + N + CHIPS = SUPPER.

JRM 580.
Solve the alphametic:

MARS + TRIP + HIS + PRIME = TARGET.

CRUX 331. **by J. A. H. Hunter**
Solve the alphametic

WELL + WELL + A + NEW = TITLE

where TITLE is odd.

JRM 365. **by J. A. H. Hunter**
Solve the alphametic:

PETER + PETTLE + PEDDLES + PEWTER = POODLES

where POODLES are odd.

JRM 453. **by Michael R. W. Buckley**
Solve the alphametic:

BLOKE + SMOKES + BLOKE = CROAKS

where the Surgeon General has determined that alphametic solving is addictive (but not as odd as these SMOKES must be).

JRM 550. **by Steven Kahan**
Solve the alphametic:

PETER + PIPER + PICKS + PICKLED = PEPPERS

where PETER is odd.

JRM 773. **by A. G. Bradbury**
Solve the alphametic:

NO + PIANO + TUNA + IS + NOT + A + FISH = SOPHIA.

SSM 3576. **by Alan Wayne**
In the addition

THIS + ADDS + TO = TOTAL

each letter uniquely represents a decimal digit. What is the TOTAL?

JRM 481. **by A. G. Bradbury**
Solve the alphametic:

WE + END + THE + NEW + MATH = TREND

where TREND is prime.

JRM 487. **by Michael Keith**
Solve the alphametic:

DOUBLE + DOUBLE + TOIL = TROUBLE.

JRM 771. **by John A. McCallum**
Solve the alphametic:

JOHN + DONNE + AND + ANNE + DONNE + ARE = UNDONE.

JRM 635. **by Herman Nijon**
Solve the alphametic:

THE + STATE + OF + THE = UNION.

JRM 748. **by Anton Pavlis**
Solve the alphametic:

MORE + POWER + MORE = WORRY.

Alphametics: places

JRM 452. **by Michael R. W. Buckley**
Solve the alphametic:

UNION + SOUTH = AFRICA

in base 11.

JRM 397. **by Alister W. Macintyre**
Solve the alphametic:

FIFTY + STATES = AMERICA.

JRM 451. **by Michael R. W. Buckley**
Solve the alphametic:

UNITED + STATES = AMERICA

in base 11.

JRM 634. **by Gordon S. Lessells**
Solve the alphametic:

LAGOS + CAIRO = ACCRA.

where CARGO is the largest that can be transported from LAGOS to CAIRO via ACCRA.

CRUX 421. **by Sidney Kravitz**
Solve the independent alphametics

$$\texttt{UNITED} + \texttt{STATES} = \texttt{CANADA},$$

$$\texttt{UNITED} + \texttt{STATES} + \texttt{AND} = \texttt{CANADA},$$

$$\texttt{THE} + \texttt{UNITED} + \texttt{STATES} + \texttt{AND} = \texttt{CANADA},$$

$$\texttt{LES} + \texttt{ETATS} + \texttt{UNIS} + \texttt{ET} = \texttt{CANADA}.$$

JRM 777. **by Sidney Kravitz**
Solve the alphametic:

$$\texttt{FINLAND} + \texttt{IRELAND} = \texttt{DENMARK}.$$

JRM 668. **by Peter H. Mabey**
Solve the alphametic:

$$(\texttt{IRELAND} + \texttt{IRA}) \div 2 = \texttt{STRIFE}.$$

JRM 438. **by Anton Pavlis**
Solve the alphametic:

$$\texttt{CANOE} + \texttt{RIDE} + \texttt{TO} + \texttt{SCENIC} = \texttt{ONTARIO}.$$

CRUX 461. **by R. Robinson Rowe**
Solve the alphametic

$$\texttt{C} + \texttt{DODGE} + \texttt{MAINE} = \texttt{ORONO},$$

where DODGE is largest.

JRM 638. **by Anton Pavlis**
Solve the alphametic:

$$\texttt{QUEBEC} + \texttt{ELECTED} = \texttt{TROUBLE}.$$

CRUX 311. **by Sidney Kravitz**
Solve the alphametic

$$\texttt{OTTAWA} + \texttt{CALGARY} = \texttt{TORONTO}.$$

Alphametics: planets

JRM 724. **by Peter J. Martin**
Solve the alphametic:

$$\texttt{PLUTO} + \texttt{SATURN} + \texttt{URANUS} + \texttt{NEPTUNE} = \texttt{PLANETS}.$$

Alphametics: radicals

CRUX 277. **by R. Robinson Rowe**
Solve the simultaneous alphametics

$$\sqrt{\texttt{EUREKA}} = \texttt{UEA}$$

$$\sqrt[3]{\texttt{EUREKA}} = \texttt{RT}$$

and find the value of

$$\sqrt[4]{\texttt{EUREKA}}.$$

CRUX 411. **by Alan Wayne**
Solve the alphametic

$$\sqrt{\texttt{PASSION}} = \texttt{KISS}.$$

Alphametics: simultaneous alphametics

JRM 412. **by Michael Keith**
Solve the alphametic:

$$\texttt{READ} + \texttt{J} + \texttt{REC} + \texttt{MATH} = \texttt{NEAT!}$$

where READ + T = an integral power of N.

MSJ 433. **by Alan Wayne**
Solve the simultaneous alphametics:

$$\texttt{ONE} + \texttt{ONE} + \texttt{W} = \texttt{TWO}$$

$$\texttt{E} \times \texttt{ONE} = \texttt{TWO}.$$

SSM 3607. **by Alan Wayne**
Solve the Arabic-Roman cryptarithmic system:

$$\texttt{TWO} + \texttt{TWO} + \texttt{TWO} = \texttt{SIX}$$

$$\texttt{VI} + \texttt{VI} = \texttt{XII}.$$

Each letter represents just one decimal digit, and different letters represent different digits.

JRM 666. **by Michael R. W. Buckley**
JRM 667.
Solve the simultaneous alphametics:

$$\texttt{PENNY} + \texttt{PENNY} + \texttt{PENNY} + \texttt{PENNY} + \texttt{PENNY} = \texttt{NICKEL}.$$

$$\texttt{PENNY} \times \texttt{V} = \texttt{NICKEL}.$$

Alphametics: squares

CRUX 201. **by Clayton W. Dodge**
Solve the alphametic

$$\texttt{LEO}^2 = \texttt{SUAVE}.$$

CRUX 211. **by Clayton W. Dodge**
Solve the alphametic

$$\texttt{FGB}^2 = \texttt{MASKEL}$$

where FGB is divisible by 9.

CRUX 221. **by Clayton W. Dodge**
Solve this alphametic

$$\texttt{CW}^2 = \texttt{TRI.GG},$$

where the solution does not contain the digit 1.

PENT 297. **by Charles W. Trigg**
The number RETIRE is a perfect square in the decimal system with $-\texttt{RE} + \texttt{TI} = \texttt{RE}$. Each letter represents a different digit and the sum of three digits equals the fourth. What is this square number?

Alphametics: states

JRM 454. **by Leslie E. Card**
Solve the alphametic:

$$\texttt{OHIO} + \texttt{IOWA} + \texttt{UTAH} = \texttt{GUAM}.$$

JRM 483. **by Leslie E. Card**
Solve the alphametic:

$$\texttt{SAMOA} + \texttt{IDAHO} = \texttt{TEXAS}.$$

MATYC 105. **by Pat Boyle**
Solve the alphametic:

$$\texttt{CAL} + \texttt{ORE} + \texttt{WASH} + \texttt{WEST} = \texttt{COAST}$$

where CAL and ORE are prime.

JRM 457. **by R. S. Johnson**
Solve the alphametic:

$$\texttt{DEVIL} + \texttt{AS} + \texttt{NEW} + \texttt{EVE} + \texttt{IS} + \texttt{ALIVE} + \texttt{AND}$$
$$+ \texttt{WELL} + \texttt{IN} + \texttt{VEGAS} = \texttt{NEVADA}.$$

Alphametics: story problems

JRM 643. **by R. S. Johnson**
My name is OTO TOTA and my good friend INA BAIN encouraged me to pose this problem. I live in LA, which happens to be a factor of both my names. My first name divided by my surname produces the repeating decimal fraction $.\overline{\texttt{TATLO}}$. Coincidentally, this repeating group gives the day, month, and year of my young son Tatlo's birth. Can you discover the date?

Alphametics: words

ISMJ 14.6.
Solve the cryptarithm:

$$\texttt{Y} + \texttt{Y} + \texttt{Y} = \texttt{MY}.$$

ISMJ 14.7.
Solve the alphametic:

$$\texttt{ON} + \texttt{ON} + \texttt{ON} + \texttt{ON} = \texttt{GO}.$$

JRM 411. **by R. S. Johnson**
Solve the alphametic:

$$\texttt{SAD} + \texttt{DAD} + \texttt{DAL} + \texttt{JIM} + \texttt{NUN} + \texttt{LAM} + \texttt{SIN} = \texttt{SHIN}.$$

where both NUN and SIN are prime.

JRM 521. **by R. S. Johnson**
Solve the alphametic:

$$\texttt{HOOOO} + \texttt{ALIPHATIC} + \texttt{LITHAEMIA} + \texttt{PIECEMEAL} + \texttt{HEMATITIS}$$
$$+ \texttt{APATHETIC} + \texttt{MALACHITE} + \texttt{EPILEPTIC} + \texttt{TIMELIMIT}$$
$$+ \texttt{IMPLICATE} + \texttt{CLIMACTIC} = \texttt{ALPHAMETIC}.$$

JRM 637. **by Underwood Dudley**
Solve the alphametic:

$$\texttt{STABLE} + \texttt{TABLE} + \texttt{ABLE} = \texttt{ATBEST}.$$

JRM 642. **by Michael R. W. Buckley**
Show that there is a unique solution in a unique base when

$$\texttt{LILS} + \texttt{OILS} = \texttt{SPOIL}.$$

JRM 644. **by Ronald J. Lancaster**
Consider the alphametic:

$$\texttt{HELL} + \texttt{HELL} + \cdots + \texttt{HELL} = \texttt{HEAVEN}.$$

How many HELL's must be endured before one arrives at HEAVEN?

JRM 670. **by Frank Rubin**
Solve the alphametic:

$$(\texttt{HER})^{(\texttt{OLD})} = (\texttt{SHY})^{(\texttt{DOG})}$$

JRM 694. **by Peter MacDonald**
Solve the alphametic:

$$18(\texttt{ADD}) = \texttt{TOTAL}.$$

JRM 697. **by Hank Venetas**
A beau, hoping to kindle the passions of his lady, deposits half a dozen rare red ROSES on her doorstep each day. Assuming that rare quantities are indeed odd, how many must be delivered in order to insure a total ROMANCE? That is, solve

$$\texttt{ROSES} + \texttt{ROSES} + \cdots + \texttt{ROSES} = \texttt{ROMANCE}$$

where the number of summands, currently unspecified, must be a multiple of six.

JRM 716. **by Michael R. W. Buckley**
Solve the alphametic:

$$\texttt{COLOR} + \texttt{COLOR} + \texttt{COLOR} + \texttt{COLOR} = \texttt{ENOUGH}$$

where no zeros are allowed.

NYSMTJ 99. **by Alan Wayne**
Restore the digits in the decimal alphametic to answer the question, "Where was it done?"

$$\texttt{MADE} + \texttt{MEAD} = 2961.$$

SSM 3691. **by Alan Wayne**
This might be a possible Dutch treat:

$$\texttt{MADE} + \texttt{MEAD} = \texttt{EDAM}.$$

Interpret the above to be an addition problem in the decimal system, where each letter corresponds uniquely to a digit and conversely. Restore the digits.

SSM 3726. **by Al White**
Solve the alphametic:

$$\texttt{MADE} - \texttt{MEAD} = \texttt{EDAM}$$

in base b. For which values of b does this problem have exactly one solution?

PENT 287. **by Randall J. Covill**
Solve the alphametic:

$$\texttt{SUBTEND} + \texttt{ADDEND} = \texttt{ANSWERS}$$

in base 14 where $\texttt{E} \neq 0$.

SSM 3718. **by Alan Wayne**
Solve the alphametic:

$$\texttt{TITHE} + \texttt{TITHE} = \texttt{FIFTH}.$$

SSM 3780. **by Alan Wayne**
"What is the sound?"

$$\texttt{ON} + \texttt{ONE} + \texttt{NOTE} + \texttt{ONE} = 6943.$$

Regard the preceding as an addition in which each letter corresponds in a one-to-one manner with a decimal digit. Restore the digits and the letters in order to answer the question.

SSM 3708. **by Alan Wayne**

Solve the alphametic:

$$\mathtt{LIVE} + \mathtt{VILE} = \mathtt{EVIL}.$$

TYCMJ 88. **by Alan Wayne**

Solve the independent alphametics:

(a) ETNA + NEAT = ANTE, and

(b) ANTE + NEAT = ETNA.

JRM 407. **by Sidney Kravitz**

Solve the alphametic:

$$\mathtt{PISCES} + \mathtt{TAURUS} = \mathtt{SCORPIO}.$$

JRM 430. **by Garry Crum**

Solve the alphametic:

$$\mathtt{BISHOP} + \mathtt{BISHOP} = \mathtt{KNIGHTS}.$$

CRUX 361. **by R. Robinson Rowe**

Find MATH in the two-stage alphametic:

```
MH · M · AT/H = MATH
                MATH
                Axxx
              xxxxx
             xxxxx
            xxxxx
            xxMATHxx
```

in which the x's need not be distinct from M, A, T, or H.

JRM 607. **by Fred Pence**

Solve the alphametic:

$$\mathtt{NBC} + \mathtt{ABC} = \mathtt{CBS}.$$

where NBC is the PRIME network in this problem, but each network manages to avoid a zero.

JRM 722. **by Martinus Ngantung**

Solve the alphametic:

$$\mathtt{MAN} + \mathtt{WOMAN} = \mathtt{CHILD}$$

where CHILD is as small as possible, and his birthdate is C/HI/LD.

JRM 546. **by A. G. Bradbury**

Solve the alphametic:

$$\mathtt{DING} + \mathtt{DONG} + \mathtt{DING} + \mathtt{DONG} + \mathtt{BELLS} = \mathtt{SOUND}$$

where SOUND is as small as possible.

JRM 545. **by Les Marvin**

Solve the alphametic:

$$\mathtt{QUARK} + \mathtt{QUARK} + \mathtt{QUARK} = \mathtt{BARYON}$$

where these tiny particles are, of course, as small as they can be!

Arrays

JRM 443. **by David L. Silverman**

Remove the $A, 2, 3, \ldots, 9$ of spades, hearts, and diamonds from a pack of playing cards. Counting the ace as one, is it possible to arrange these 27 cards in nine groups of three in such a way that each group of three

(a) contains a spade, a heart, and a diamond *and*

(b) has a square total?

Is it possible if requirement (a) is removed?

SSM 3650. **by E. J. Ulrich**

The letters a, b, c, d, e, f, g, h, n, m are arranged around a pentagon. Replace each letter by a number from 1 through 10 (using each number but once) so that the totals of the three numbers on each of the five sides of the pentagon will all be the same. Call this common total T.

(a) What is the minimum value for T?

(b) What is the maximum value for T?

(c) Are solutions possible for all integers between these two?

JRM 420. **by P. MacDonald**

Using some or all of the calculator numerals 0, 1, 2, 5, 6, 8, 9, create a 3×3 array such that:

(a) the center row contains no zeros;

(b) when the top number (3 digits) is added to the middle number (3 digits), the result equals the bottom number;

(c) when the page is turned upside down, (b) is true again.

PME 377. **by Charles W. Trigg**

From the following square array of the first 25 positive integers, choose five, no two from the same row or column, so that the maximum of the five elements is as small as possible.

2	13	16	11	23
15	1	9	7	10
14	12	21	24	8
3	25	22	18	4
20	19	6	5	17

CRUX 22. **by H. G. Dworschak**

Show how to make the row-sums equal by moving just two of the numbers in the matrix

$$\begin{pmatrix} 1 & 2 & 7 & 9 \\ 3 & 4 & 5 & 8 \end{pmatrix}.$$

Chess tours

PARAB 283.

A king moves on an 8×8 chessboard so that in 64 moves it goes through all squares, on the last move returning to its original position. Furthermore, if the circuit is drawn by joining the center points of consecutive positions with straight line segments, the path obtained does not cross itself. Prove that at least 28 of the moves have been either horizontal or vertical.

OMG 14.2.2.

Is it possible for a knight in chess to start at one corner of a chessboard and move to the opposite corner landing exactly once on each square of the board?

Chessboard problems: coloring problems

NYSMTJ 68. **by Alvin J. Paullay and Sidney Penner**

Each square of a 4×6 chessboard is colored black or white so that the four distinct corner squares of every rectangle formed by the horizontal and vertical lines of the board are not the same color. Show that any such coloring has the same number — namely 12 — of squares of each color.

USA 1976/1.
OMG 15.1.3.

(a) Suppose that each square of a 4×7 chessboard is colored either black or white. Prove that with any such coloring, the board must contain a rectangle (formed by the horizontal and vertical lines of the board) whose four distinct unit corner squares are all of the same color.

(b) Exhibit a black-white coloring of a 4×6 board in which the four corner squares of every rectangle are not all of the same color.

AMM 6211.* **by Alvin J. Paullay and Sidney Penner**

Suppose that each square of an $n \times n$ chessboard is colored either black or white. A square, formed by the horizontal and vertical lines of the board, will be called chromatic if its four distinct corner squares are all of the same color.

(a) Exhibit a black and white coloring of a 9×9 board in which every such square, as described above (there are 204) is not chromatic.

(b) Find the smallest n, say s, such that with any such coloring, every $s \times s$ board must contain a chromatic square.

PARAB 292.

The plane is divided (like a chessboard) into congruent squares. A finite number of squares are colored black, the others (infinitely many) remain white. After 1 second, the squares change their color according to the following rule:

If the upper and right neighbors of a given square, S, have the same color, then S takes this color (irrespective of whether it had this color already or not); if they have opposite colors, then the color of S remains unchanged. This process is repeated after 2 seconds, 3 seconds, ...

Describe the eventual coloring of all squares and prove your assertion.

Chessboard problems: counting problems

JRM 703. **by Sidney Kravitz**

A typesetter who works for a chess magazine sets up chessboard diagrams by placing square type molds in an 8×8 array. He has a mold that shows a black king on a black square, another for a black king on a white square, etc. He also has molds for unoccupied black squares and unoccupied white squares. Taking account of all the possibilities, however unusual, allowed by chess rules, how many molds must he have so that he can compose any chess diagram arising from a legitimate game?

AMM 6096. **by Jan Mycielski**

A set of cells of a chessboard is called connected if a rook can visit the whole set without moving over cells that are not in the set. Set $s = a_n n^2$ and let 2^s be the number of connected subsets for a chessboard of size $n \times n$. Prove that the sequence $a_1, a_2, \ldots$ converges and estimate its limit.

Chessboard problems: covering problems

ISMJ 14.5.

A rectangle m inches by n inches is drawn where m and n are odd integers. The rectangle is divided into mn one inch boxes that are alternately colored red and black, like a chessboard. The four corners are colored black. We have $\frac{(mn-1)}{2}$ 1-inch $\times$ 2-inch dominoes and one 1-inch $\times$ 1-inch square half-domino.

(a) Show that if the half-domino is on a red square, it is not possible to cover the rest of the rectangle with dominoes.

(b) Show that if the half-domino is placed on a black square, then it is possible to cover the rest of the rectangle with dominoes, regardless of which black square we start with.

Chessboard problems: deleted squares

AMM E2665. **by Sidney Penner**

A partial chessboard is a chessboard from which squares have been removed so that

(a) it is impossible to place even one domino on the remaining board; and

(b) the replacement of a single deleted square, regardless of its location, makes it possible to place a domino on the board. (A domino covers two squares having a common side.)

It is easy to see that for an 8×8 partial chessboard, the minimum number of deleted squares is 32. What is the maximum number?

Chessboard problems: distribution problems

PARAB 415.

Thirty-two counters are placed on a chessboard so that there are four in every row and four in every column. Show that it is always possible to select eight of them so that there is one of the eight in each row and one in each column.

JRM C6. **by Ray Lipman**

Although there are actually six different ways of placing two checkers on different squares of a 2×2 board, if we consider two arrangements the same if they are reflections and/or rotations of each other, there are only two arrangements: rookwise adjacent and bishopwise adjacent. Similarly, instead of nine ways of placing a single checker on a 3×3 board, there are, topologically, only three: middle, corner, or side. The (k, N)-entry in the matrix below (which has two questionable entries) gives $f(k, N)$, the number of topologically distinct ways of placing k checkers on different squares of an $N \times N$ board ($k = 1, 2, 3, 4$ and $N = 1, 2, 3, 4$):

$$\begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & 2 & 8 & 20 \\ 0 & 1 & 16 & 43? \\ 0 & 1 & 23 & 77? \end{pmatrix}.$$

Devise a program to extend the matrix to values of both k and N up to 10.

Chessboard problems: maxima and minima

AMM E2605. **by Andreas P. Hadjipolakis**

Consider a chessboard of odd order n ($n \geq 5$). Assign label m to a cell of the chessboard if it can be reached by the knight in m steps starting from the central cell, and this m is minimal. Determine the number $K(n;m)$ of cells labeled m.

Chessboard problems: n queens problem

AMM E2698. **by Paul Monsky**

Let A_n be an $n \times n$ chessboard. The n queens problem (placing n counters on A_n so that no two lie in any row, column, or diagonal) admits solutions for all $n \neq 2$ or 3.

Let B_n be the "chessboard" obtained from A_n by identifying opposite sides so that the resulting surface is a torus. (Now, every diagonal of B_n consists of n squares.)

(a) For which values of n does there exist a solution of the n queens problem on B_n?

(b) If n satisfies (a), then a solution of (a) gives, by cyclic permutation, n superimposable solutions to the n queens problem on A_n. Do there exist n superimposable solutions (for A_n) for other values of n?

Chessboard problems: paths

TYCMJ 145. **by Sidney Penner**

A checker is placed in the upper left-hand corner of an $(n+1) \times n$ checkerboard. It begins a tour by making moves diagonally until it reaches an edge. At this point, it makes a right-angle turn and the process continues until the checker reaches a corner, after which the tour is complete. What is the number of moves for a complete tour?

Chessboard problems: probability

CRUX 446. **by R. Robinson Rowe**

An errant knight stabled at one corner of an $N \times N$ chessboard is "lost", but home happens to be at the diagonally opposite corner. If he moves at random, what is the probable number of moves he will need to get home (a) if $N = 3$ and (b) if $N = 4$?

JRM C7. **by Les Marvin**

A knight starts at the corner of a standard 8×8 chessboard and moves successively, at each stage randomly and with equal probability choosing his next square from the ones legally available. Let E equal the expected number of moves required to visit each of the 64 squares at least once. The best bounds I have at the moment for E are $64 < E < \infty$. Determine E to 3-place decimal accuracy.

JRM 425. **by David L. Silverman**

A white knight and a black knight are situated on diagonally opposite corners of a 3×3 square. In turn, starting with White, they move randomly until (inevitably) Black captures White. What is the expected number of Black moves to achieve capture?

Cryptarithms: alphabet

SSM 3593. **by Alan Wayne**

Restore the following addition in which each letter represents precisely one decimal digit, and different letters represent different digits.

$$\texttt{ABCDEFGHIJ} + \texttt{ABCDEFGHIJ} = \texttt{BDFIACEGHJ}.$$

SSM 3622. **by Alan Wayne**

Solve the system

$$(\texttt{CJ})^{\texttt{F}} = \texttt{ABCD}$$
$$(\texttt{CJ})^{\texttt{E}} = \texttt{EFGHIJ}$$

in which each letter represents one and only one decimal digit, and different letters represent different digits.

Cryptarithms: chess moves

JRM 434. **by Mike Keith**

The ten distinct digits are distributed among the ten symbols

$$\texttt{P}, -, \texttt{K}, \texttt{B}, \texttt{R}, \texttt{Q}, \texttt{x}, \texttt{I}, \texttt{W}, \texttt{N}$$

(the 3's and 4's already given can be reused) and the chess moves given by the alphametic — in proper order as shown — yield a unique, legal game, ending, as is indicated by the total, in white checkmate.

$$(\texttt{P}-\texttt{K4}) + (\texttt{P}-\texttt{K4}) + (\texttt{B}-\texttt{B4}) + (\texttt{P}-\texttt{R4})$$
$$+ (\texttt{Q}-\texttt{B3}) + (\texttt{P}-\texttt{R4}) + (\texttt{QxP}) = (\texttt{IWIN})$$

Cryptarithms: encrypted messages

CRUX 215. **by David L. Silverman**

Convert the expression given below from mathematics to English, thereby obtaining the perfect scansion and rhyme scheme of a limerick:

$$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + 5(11) = 9^2 + 0.$$

Cryptarithms: hand codes

OMG 17.3.2.

Some children developed the following coding system for numbers: both hands down = 0; one hand up = 1; both hands up = 2; one hand up, both hands down = 3; one hand up, one hand up = 4; etc. What number would be represented by one hand up, one hand up, both hands up, one hand up?

Cryptarithms: powers

SSM 3639. **by Alan Wayne**

Find the five-digit decimal integer `ABCCA` whose `C`th power is the fifteen-digit integer `CCCCCDEBFEGFGFA`.

Cryptarithms: products

NYSMTJ 70.

In the cryptarithm $\texttt{A(BC)} = \texttt{D(CB)}$, each letter represents a distinct decimal digit. If $\texttt{A} < \texttt{D}$, find all solutions.

Cryptarithms: skeletons

JRM 579. **by A. G. Bradbury**

Solve the skeleton:

```
TAJ)MAHAL(AT
    ****
     ****
     AG*R
       *A
```

JRM 698. **by Nobuyuki Yoshigahara**

Solve the skeleton

$$\frac{******}{***} = 1978$$

where the digits $*$ are to be distinct. Find an appropriate alphametic to fit the skeleton.

JRM 780. **by A. G. Bradbury**

Solve the skeleton:

```
FOUR)SIXTEEN(FOUR
     IS**
     *****
     *****
       ****
       FO*R
       **FUN
       *****
         ***
```

JRM 410. **by Michael R. W. Buckley**

Solve the skeleton:

```
   YES
   YES
   ***
  ***
 ***
 BREED
```

JRM 585. **by Frank Rubin**

Solve the skeleton:

```
        ***A***
    ×  PROBLEM
        ******P
       ******R
      ******O
     ******B
    ******L
   ******E
  *POMPOM
  **************
```

JRM 617. **by J. A. H. Hunter**

Solve the skeleton:

```
     ****
   × **7*
    *****
    ****
  ****
  ********
```

where the digit 7 appears but once.

JRM 664. **by J. A. H. Hunter**

Solve the skeleton:

```
   RUN
   RUN
  ****
  SEE
 ***
 BRAWL
```

JRM 781. **by Frank Rubin**

In base 14, solve the skeleton:

```
  EIGHT
    TWO
  *****
 *****
*****
SIXTEEN
```

CRUX 371. **by Charles W. Trigg**

Solve the skeleton:

```
               EASY
        MATHEMATICS
             M***S
            A***C
           H***I
          S***T
         Y***A
        I***M
       T***E
      C***H
     S***T
    Y***A
   I***M
   I*************S
```

JRM 696. **by Frank Rubin**

Solve the skeleton:

```
      *   *
  √ * * * *
    * *
        * *
        * *
```

Cryptarithms: tournaments

MENEMUI 1.1.3. **by K. Unsworth**

Four football teams A, B, C, and D are going to play each other once. The figures written in the incomplete table below give part of the situation when some of the matches have been played. The digits from 0 to 9 are replaced by letters, and each letter stands for the same digit wherever it appears; different letters stand for different digits.

	Played	Won	Lost	Drawn	Goals For	Goals Against	Pts.
A	x			k	h	p	
B		h			m	m	
C	p	x	h	k	t		m
D	k						

Two points are given for a win, and one point to each side in a drawn match. List the matches played and the score in each match.

Logic puzzles: Caliban puzzles

OMG 18.2.3.

The manager, the accountant, the teller and the auditor at a local bank are Mr. Smith, Mr. Brown, Mr. Jones and Mr. Foster, but I can never remember who is who. I do know that:

1) Mr. Brown is taller than the auditor or the teller.
2) The manager lunches alone.
3) Mr. Jones plays bridge with Mr. Smith.
4) The tallest of the four plays basketball.
5) Mr. Foster lunches with the auditor and the teller.
6) Mr. Smith is older than the auditor.
7) Mr. Brown plays no sports.

Determine which job each man performs.

PARAB 335.

A school held a special examination to decide which student in year 12 was best overall in the subjects of English, History, French, Mathematics, and Science. Five students — Alan, Barbara, Charles, David, and Evonne — sat for five papers, one in each of these five subjects. To simplify matters, the top student in a paper was given 5 marks, the next student was given 4 marks, and so on; the last student in a paper being awarded 1 mark (fortunately, no two students tied in any of the papers). When the marks for each student were collected, the following facts were noted:

(1) Alan had an aggregate mark of 24.

(2) Charles had obtained the same mark in four out of the five subjects.

(3) Evonne, the mathematician, had topped Mathematics, although she only came in third in Science.

(4) The students' aggregate marks were in alphabetical order, and no two students had the same aggregate.

What we want to know is:

(a) What was Barbara's mark in Mathematics?

(b) How many of the five students obtained the same mark in at least four out of the five subjects? (Charles was one of these!)

PARAB 384.

When the fire alarm went off, the six patrons in the restaurant all hurriedly seized a coat. Safely outside, they discovered that no one had his own. The coat that Alf had belonged to the man who had seized Bert's. The owner of the coat grabbed by Colin held a coat which belonged to the man who was holding Dave's coat. If the man who had seized Ern's coat was not the owner of that grabbed by Fred, who borrowed Alf's coat? Whose coat did Alf seize?

Logic puzzles: incomplete information

PENT 309. **by Richard A. Gibbs**

Once upon a time in a far away kingdom there lived many married couples. It came to the attention of the King (himself unmarried) that there were some unfaithful wives in his kingdom and he issued the following decree:

"It has come to my attention that there are unfaithful wives in my kingdom. If a husband discovers that his wife is unfaithful, he may slay her without punishment provided he does so on the day of the discovery."

Now, it so happens that if a man's wife were unfaithful he would be the only husband not to know it. Further, husbands never talk among themselves about the fidelity of their wives, and an unfaithful wife is clever enough not to be caught by her husband.

Well, following the King's decree, a month passed without incident. Then, on the 40th day, 40 unfaithful wives were slain; all that were in the kingdom.

The King was amazed! He summoned his Math Wizard for consultation and told him what had happened. The Wizard said, "That's not at all amazing." Prove that the Wizard knew that all unfaithful wives in the kingdom would be slain on the same day.

CRUX 357. **by Leroy F. Meyers**

In a certain multiple-choice test, one of the questions was illegible, but the choice of answers was clearly printed. Determine the true answer(s).

(a) All of the below.
(b) None of the below.
(c) All of the above.
(d) One of the above.
(e) None of the above.
(f) None of the above.

MM 1051. **by A. K. Austin**

A game involves a quizmaster and two players, X and Y. The quizmaster chooses an ordered pair of real numbers (x, y) and tells x to player X and y to player Y. The quizmaster also tells the players that (x, y) is in the set $A = \{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$. The quizmaster then asks X and Y alternately if they know (x, y). Find a characterization of the set A that guarantees that either X or Y will eventually know (x, y).

FUNCT 1.2.6.

A person A is told the product xy and a person B is told the sum $x + y$ of two integers x and y, where $2 < x$ and $y < 200$. Person A knows that B knows the sum, and B knows that A knows the product. The following dialogue develops:

A: I do not know (x, y).
B: I could have told you so!
A: Now I know (x, y).
B: So do I.

What is (x, y)?

MM 977. **by David J. Sprows**

Let x and y be two integers with $1 < x < y$ and $x+y \leq 100$. Suppose Ms. S. is given the value of $x+y$ and Mr. P. is given the value of xy.

(a) Mr. P. says: "I don't know the values of x and y."

(b) Ms. S. replies: "I knew that you didn't know the values."

(c) Mr. P. responds: "Oh, then I do know the values of x and y."

(d) Mr. S. exclaims: "Oh, then so do I."

What are the values of x and y?

ISMJ 13.19.

I have two different integers larger than 1. I inform Sam and Pam of this fact and I tell Sam the sum of my two numbers and I tell Pam their product. The following dialog then occurs:

Pam: I can't determine the numbers.
Sam: The sum is less than 23.
Pam: Now I know the numbers.
Sam: Now I know the numbers, too.

What are the numbers?

CRUX 400. **by Andrejs Dunkels**

In the false bottom of a chest which had belonged to the notorious pirate Captain Kidd was found a piece of parchment with instructions for finding a treasure buried on a certain island. The essence of the directions was as follows.

"Start from the gallows and walk to the white rock, counting your paces. At the rock turn left through a right angle and walk the same number of paces. Mark the spot with your knife. Return to the gallows. Count your paces to the black rock, turn right through a right angle and walk the same distance. The treasure is midway between you and the knife."

However, when the searchers got to the island they found the rocks but no trace of the gallows remained. After some thinking they managed to find the treasure anyway. How?

JRM 685. **by David L. Silverman**

"Bah! There are less than 100 doubloons here. How do you propose to divide them?"

"Let one of us take ϕ of them, then the other takes ϕ of what's left, and so on until they are all taken," said Silverbeard.

"Fine, but who gets the first pick?"

"Oh, you take it," said Silverbeard, knowing that he would get the most. How many doubloons were there?

JRM 469. **by David K. Orndoff**

In the Secret Word game, one player writes down a word of five different letters and his opponent attempts to guess it by posing candidate words of the same type which are scored according to the number of letters they have in common with the secret word (not necessarily in the same position). After 12 cracks at my opponent's word, he has scored me as follows: VOICE LYMPH DWARF JUMPY TWIGS CHAMP and EQUIP all scored 1, JUNKS SWORD BLACK and FLUNK got 2, and BOXED scored zero. I have sufficient information now. What is the word?

Logic puzzles: labeled boxes

FUNCT 1.4.5.

A repair shop has three boxes, one containing left-foot bicycle pedals, another containing right-foot bicycle pedals, and a third containing both left- and right-foot pedals. Labels describe the contents of the boxes. A naughty customer changed all the labels around. You are allowed to inspect one pedal from one box. Which box should you choose from in order to identify which box is which?

Logic puzzles: liars and truthtellers

FUNCT 1.2.4.

Three golfers named Tom, Dick, and Harry are walking to the clubhouse. The first man in line says, "The guy in the middle is Harry." The man in the middle says, "I'm Dick." The last man says, "The guy in the middle is Tom." Tom, the best golfer of the three, always tells the truth. Dick sometimes tells the truth, while Harry, the worst golfer, never does. Figure out who is who.

JRM 392. **by Victor Reyes**

On the Island of Kyensahbay the natives are divided into two tribes — the Blues, who always tell the truth, and the Reds, who never do. Twelve of them met me at the jetty, and because their names were too exotic to remember I labeled them with the letters from A to L. Since none wore his tribal colors, I made it a point to ask each during the first day of my governorship about the composition of my staff. I received the following replies:

A: H and I are Blues.
B: A and L are Blues.
C: B and G are Blues.
D: E and L are Blues.
E: C and H are Blues.
F: D and I are Blues.
G: E and J are Reds.
H: F and K are Reds.
I: G and K are Reds.
J: A and C are Reds.
K: D and F are Reds.
L: B and J are Reds.

I was quickly able to determine which of them to doubt and which to believe. Can you do likewise?

JRM 792. **by Randall J. Covill**

The police have detained three suspects who know each other very well. The police know that one of the suspects always lies, one sometimes lies, and one never lies. How can they most easily determine which is which?

Logic puzzles: relationships

ISMJ 13.7.

No family has more than four children in a certain school. One day when all the boys are in school, they answer a questionnaire that includes the question: How many brothers have you in the school? In the report put out by the school this statement appears:

"216 boys have no brother or two brothers in the school.

195 boys have one or three brothers in the school."

Can this statement be correct?

OMG 16.1.10.

If in a certain diagram "$\rightarrow$" means "-" is brother of "-", how is A related to B?

Logic puzzles: statements

FUNCT 2.4.1.

Simplify the following statement:

"If Monday is a public holiday, then I will not go to the beach, or I will stay at home, or I will neither stay at home nor go to the beach."

FUNCT 1.1.6.

The blackboard has been filled with 100 statements, as follows:

"Exactly one of these statements is incorrect.
Exactly two of these statements are incorrect.

$\vdots$

Exactly one hundred of these statements are incorrect."

Which (if any) of the 100 statements is correct?

MM 931. **by Alan Wayne**

In a list of n statements, the rth statement is, for $r = 1, 2, \ldots, n$, "The number of false statements in this list is greater than r." Determine the truth value of each statement.

Logic puzzles: switches

AMM S17. **by Leonard Gillman**

When the upstairs switch is in one position, the downstairs switch turns the stairway light on and off as it should, but when the upstairs switch is in the other position, the stairway light remains off irrespective of the position of the downstairs switch. Which is the defective switch?

Logic puzzles: transportation

OMG 15.2.1.

A man must transport a fox, a goose, and a sack of corn over a river in a boat that will hold only him and one of the fox, goose, or sack of corn. He cannot leave the fox and goose, or the goose and corn alone together or one will eat the other. How does he accomplish his task?

Logic puzzles: yes or no questions

PARAB 345.

During a trial, three different witnesses A, B, and C were called one after the other, and asked the same questions. In each case, each witness answered "yes" or "no", and the following facts were noted:

(1) All questions answered "yes" by both B and C were also answered "yes" by A;

(2) Every question answered "yes" by A was also answered "yes" by B;

(3) Every question answered "yes" by B was also answered "yes" by at least one of A and C.

Show that the witnesses A and B agreed in their answers to all questions.

Magic configurations: gnomon magic squares

SSM 3629. **by Charles W. Trigg**

In a 3×3 array there is a 2×2 array in each corner. The other five cells form an L-shaped gnomon. If the sum of the elements in each of the four corners is the same, this sum is said to be the magic sum of the gnomon magic square. Such a square is

$$\begin{matrix} 1 & 6 & 7 \\ 8 & 5 & 2 \\ 3 & 4 & 9 \end{matrix}$$

with a magic sum of 20.

Rearrange the nine digits to form a gnomon magic square with magic sum of 16.

Magic configurations: hexagons

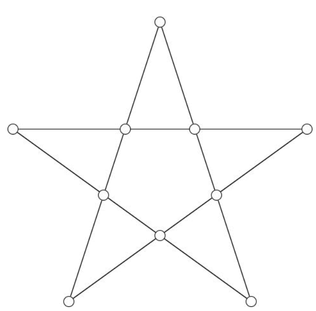
NYSMTJ 79. **by Bernard G. Hoerbelt**

The minor diagonals of a regular hexagon are drawn, forming a figure with 13 regions. Six dashed lines are drawn through the center of the hexagon, parallel to the sides and diagonals of the hexagon. Six more dashed lines are drawn connecting the midpoints of adjacent edges of the hexagon. Place the integers from 1 to 13 in the 13 regions so that the sum of every triple along each of the dashed lines is 21.

Magic configurations: magic pentagrams

JRM 385. **by Vance Revennaugh**

Can the ten vertices of the pentagram shown be labeled with the integers from 1 to 10 in such a way that the sum of the four labels along each of the five edges is the same, thus qualifying it to be called a *pentacle*, that is, a magic pentagram?



CRUX 145. **by Walter Bluger**

A pentagram is a set of 10 points consisting of the vertices and the intersections of the diagonals of a regular pentagon with an integer assigned to each point. The pentagram is said to be magic if the sums of all sets of 4 collinear points are equal.

Construct a magic pentagram with the smallest possible positive primes.

Magic configurations: magic squares

ISMJ 14.23.

Nine numbers are placed in a 3×3 array to form a magic square (the three row sums, three column sums, and two diagonal sums are each equal to some number S). Prove that S is three times the central number. Show that the conditions of the problem cannot be met if the nine numbers are all of the numbers from 1 to 10 except 7.

MSJ 430. **by Donald Baker**

Fill in the five missing entries in the following 3×3 array to form an additive magic square.

$$\begin{matrix} 17 & - & 22 \\ - & - & - \\ 13 & - & 19 \end{matrix}$$

SSM 3632. **by Bob Prielipp**

An $n \times n$ magic multiplication square is an $n \times n$ array in which the product of the entries of each diagonal, in each row, and in each column are all the same. Prove that there are infinitely many 3×3 magic multiplication squares, all of whose entries are positive integers.

PARAB 301.

The numbers 27, 20, 25, 22, 24, 26, 23, 28, and 21 are arranged in a 3×3 magic square.

$$\begin{matrix} 27 & 20 & 25 \\ 22 & 24 & 26 \\ 23 & 28 & 21 \end{matrix}$$

By moving the digits (and using no other operation), find 9 numbers in 3 rows and 3 columns such that, when the numbers in any row or column or diagonal are multiplied together, you get the same answer.

CRUX 359. **by Charles W. Trigg**

Construct a third-order additive magic square that contains three prime elements and has a magic constant of 37.

MM 943. **by Charles W. Trigg**

Early in his reign as Emperor of the West, Charlemagne ordered a pentagonal fort to be built at a strategic point of his domain. As good luck charms, he had a third order magic square with all prime elements engraved on each wall. The five magic squares were different from each other, but they had the same magic constant — the year in which the fort was completed. The fort proved its ability to resist attack midway through his reign.

On this evidence, reconstruct the magic squares.

PENT 319. **by Charles W. Trigg**

Use the basic nine-digit third-order magic square to generate eight other third-order magic squares that have a common magic constant. Each new square is to have nine distinct elements, and at least three elements are to be prime in five of the new squares.

PME 364. **by Charles W. Trigg**

Show that there is only one third-order magic square with positive prime elements and a magic constant of 267.

CRUX 399. **by Gilbert W. Kessler**

A prime magic square of order 3 is a square array of 9 distinct primes in which the three rows, three columns, and two main diagonals all add up to the same magic constant. What prime magic square of order 3 has the smallest magic constant

(a) when the 9 primes are in arithmetic progression;

(b) when they are not.

JRM 569. **by Greg Fitzgibbon**

Sidney Kravitz coined the term and the concept "talisman square" to designate a square array of numbers such that any two neighboring elements (i.e., kingwise adjacent elements) differ by at least some constant. There exist 4×4 squares whose elements are the numbers $1, 2, \ldots, 16$ that are both magic and talismanic with talisman constant 2.

(a) Display all 24 such squares.

(b) What is the maximum possible talismanic constant for an $n \times n$ magic talisman square consisting of the numbers $1, 2, \ldots, n^2$?

CRUX 482. **by Allan Wm. Johnson Jr.**

Construct a fourth-order magic square composed of distinct 2-digit primes, four of which are situated as shown:

–	17	89	–
–	–	–	–
–	–	–	–
–	19	79	–

Magic configurations: triangles

OSSMB 76-13.

The numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 are arranged in a triangular pattern, as shown:

$$\begin{array}{ccccccc} & & & a & & & \\ & & b & & i & & \\ & c & & & & h & \\ d & & e & & f & & g. \end{array}$$

If the sum of the numbers along each side of the triangle is 20, prove that the number 5 must go in a corner.

Mazes

OMG 14.3.1.

In a maze of connected straight lines, how can you tell if a point is inside or outside a closed figure without tracing a path?

AMM 6163. **by John Myhill**

Devise an algorithm for escaping from a connected, countably infinite, locally finite maze. "Countably infinite" means the number of edges and nodes is $\aleph_0$, "locally finite" means only a finite number of edges meet at each node. "Algorithm" means this: You are lost in the middle of the maze, having no idea of where the exit is. Your only possibility of escape, therefore, is to devise a tour that will take you through every node of the maze after a finite number of steps. In order to keep track of your route, you are given an everlasting pencil and an infallible eraser; at each node, and at the roadside of each road near the node, is a board on which you can write and erase. However, you have only a finite alphabet to write with, and there is a fixed bound on how many characters you can write on the boards. (In particular, then, you cannot keep on any board a record of how many times you have passed it.) Your field of vision is limited to being able to see, from any node, what is written on the board at that node and what is written on the nearby roadside boards.

Can this procedure be altered to solve the locally infinite case?

Polyominoes: coloring problems

JRM 386. **by C. R. Gossett**

Using various combinations of Red, White, and Blue (R, W, B), there are 18 linear Union Jack (or, if you prefer, "Old Glory") tromino types. One of each type has been placed in the 6×9 diagram shown. Determine the placement of each of the 18 trominoes, using, not a trial-and-error approach, but a direct line of inference that will ensure your solution is unique.

R	R	W	B	R	W	B	W	B
W	B	W	B	W	W	W	B	R
B	W	R	R	R	W	W	W	W
R	B	W	R	B	B	R	R	R
W	W	W	R	R	B	R	B	B
B	B	W	R	R	B	B	B	R

Polyominoes: dominoes

CRUX 328. **by Charles W. Trigg**

A set of $2k(k+1)$ dominoes each 2×1, can be arranged to form a square with an empty 1×1 space in the center.

(a) Show that for all k there is an arrangement such that no straight line can divide the ensemble into two parts without cutting a domino.

(b) Is it always possible to arrange the dominoes so that the ensemble can be separated into two parts by a straight line that cuts no domino?

ISMJ 12.31.

Show that if 18 1×2 dominoes are arranged to form a 6×6 square, then there is a line that divides the square into two rectangles without cutting any domino.

Polyominoes: maxima and minima

CRUX 276. **by Sidney Penner**

How many unit squares must be deleted from a 17×22 checkerboard so that it is impossible to place a 3×5 polyomino on the remaining portion of the board?

CRUX 282. **by Erwin Just and Sidney Penner**

On a 6×6 board we place 3×1 trominoes until no more trominoes can be accommodated. What is the maximum number of squares that can be left vacant?

CRUX 429. **by M. S. Klamkin and A. Liu**

On a $2n \times 2n$ board we place $n \times 1$ polyominoes (each covering exactly n unit squares of the board) until no more $n \times 1$ polyominoes can be accommodated. What is the maximum number of squares that can be left vacant?

NYSMTJ 77. **by Erwin Just and Sidney Penner**

On a 5×5 board, we place 3×1 triominoes until no more triominoes can be accommodated.

(a) What is the minimum number of squares that can be left vacant?

(b) What is the maximum number of squares that can be left vacant?

Polyominoes: pentominoes

JRM 470. **by Makoto Arisawa**

Let a pair of pentominoes, juxtaposed to form a decomino, be called a doublet. If two doublets are congruent, let the constituent pentomino pairs be connected by the double arrow — thus $VY \leftrightarrow PT$. There are many amusing and challenging exercises based on the doublet concept. In the present one, let us ignore congruent doublets in which the two doublets share a common pentomino or in which one or both doublets use the same pentomino twice. Using only congruences in which four distinct pentominoes are involved and, otherwise, employing as many or as few of the 66 distinct doublets as you wish, but including each of the 12 pentominos at least once, what is the shortest closed doublet chain of the form $AB \leftrightarrow CD \leftrightarrow EF \leftrightarrow \ldots \leftrightarrow AB$ you can construct?

JRM 391. **by Michael Keith**

The twelve different pentominoes are divided into two sets of six each and, placing each pentomino on a square grid, the two sets are arranged to form two congruent, connected figures having a single hole.

Diagrams using this division of the pentominoes or other divisions, with two sets of six each, can be used to create alternative patterns sharing a hole area of 17, but no arrangement has been discovered to date that yields a larger hole. Is 17 the maximum possible?

JRM 426. **by Michael Keith**

The twelve different pentominoes are divided into two sets of six each and, placing each pentomino on a square grid, the two sets are arranged to form two congruent, connected figures having one or more holes. What is the maximum number of holes that these figures can contain?

Polyominoes: tiling

PME 358. **by Sidney Penner and H. Ian Whitlock**

From a $2n + 1 \times 2n + 1$ checkerboard, in which the corner squares are black, two black squares and one white square are deleted. If the deleted white square and at least one of the deleted black squares are not edge squares, prove that the reduced board can be tiled with 2×1 dominoes.

MM 969. **by Veit Elser**

A cube can be unfolded into a polyomino of order six in the form of a Latin cross.

(a) Show that five congruent Latin crosses can cover the surface of the cube without overlap.

(b) Can the surface of the cube be covered with seven congruent polygons?

JRM 600. **by Andrew L. Clarke**

What is the smallest rectangle that can be tiled using only U-shaped and T-shaped pentominoes?

MSJ 477.

Consider an $n \times n$ chessboard whose four corner squares have been removed. For what values of n can this board be covered by "L"-shaped pieces having 3 squares on the long side and 2 squares on the short side?

PARAB 336.

You are given an 8×8 chessboard and 16 tiles in the shape of a "T" where each of the four squares in the T-shape is the same size as the squares of the chessboard.

(a) Can the chessboard be completely covered with these tiles?

(b) If one of the T-shaped tiles were replaced by a square tile which just covers four of the chessboard squares, can the chessboard be completely covered by these 16 tiles?

In each case, you must either show how to cover the board, or prove that it is impossible.

TYCMJ 78. **by Sidney Penner**

Assume that a single square is deleted from a $2n \times 2n$ checkerboard in which $3 \nmid n$. Prove that it is possible to tile the resultant board with right trominoes.

JRM 381. **by Mark A. Ricci**

Can a patio of dimensions 10 feet $\times$ 11 feet, from one of whose ten-foot sides two 1-square-foot areas have been removed at the corners, be tiled with 36 1-foot $\times$ 3-foot stones?

AMM E2595. **by Sidney Penner**

Consider $(2n+1)^2$ hexagons arranged in a "diamond" pattern, the kth column from the left and also from the right consisting of k hexagons, $1 \le k \le 2n+1$. Show that if $n \not\equiv 1 \pmod 3$ and the center hexagon is deleted, then the remaining hexagons can be tiled by trominoes.

Puzzles: block puzzles

PARAB 338.

You are given 216 blocks, each of dimensions $1 \times 1 \times 8$. Is it possible to build a cube of dimensions $12 \times 12 \times 12$ with these blocks?

JRM 759. **by Makoto Arisawa**

Most recreational mathematicians are familiar with the two cubes whose faces are numbered so as to be able to display any day of the month from 01 to 31. How should the faces of four cubes be numbered so as to be able to display as many years as possible from 1979 on?

AMM E2596.* **by Mark A. Spikell**

Given is a collection of Cuisenaire rods having dimension $1 \times 1 \times a$, where the length a belongs to a finite set A of positive integers and the number of rods of length a may be supposed to be unlimited for each $a \in A$. For which s can a $1 \times s \times s$ square be constructed from this collection?

Puzzles: crossnumber puzzles

JRM 473. **by Michael R. W. Buckley**

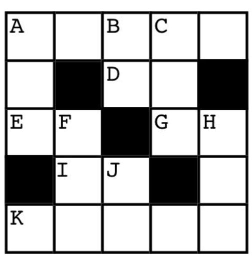
Solve the crossnumber puzzle below.

Across:

A. The square of H down.
D. E across plus a factor of H down.
E. The sum of all the digits in the puzzle.
G. The sum of the digits in C down.
I. The sum of the digits in K across.
K. The square of C down.

Down:

A. The root mean square of C down and H down.
B. A multiple of G across.
C. A prime.
F. H down less one of its factors.
H. A palindrome.
J. The sum of the digits in A across.



JRM 704. **by Harry L. Nelson**

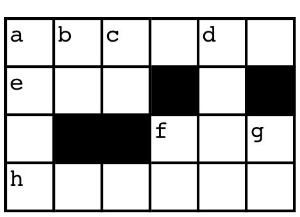
Within the puzzle shown, all numbers formed from the various digits reading across are in the decimal system, while all numbers reading down are in the octal system. Capital letters in the clues represent positive integers.

ACROSS (decimal)

a. A^B
b. C^E
f. $D \cdot Z^Z + B$
h. $F^G + E$

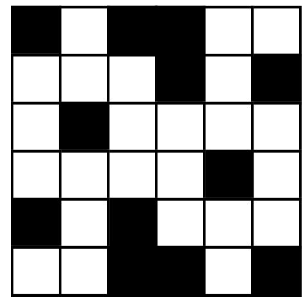
DOWN (octal)

a. $G \cdot (A+Z)^E + Z \cdot F^E + X$
b. $E + F + G$
c. $Y + Y$
d. $E^D + E$
f. $C + B$
g. Y



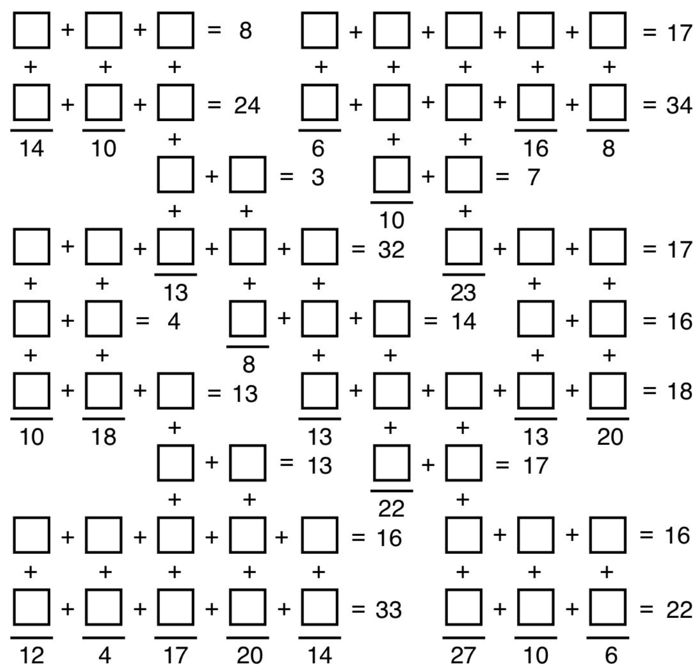
JRM 798. **by Nobuyuki Yoshigahara**

Fill in the diagram shown with distinct 2-, 3-, and 4-digit numbers that are perfect squares, none of which start with 0.



JRM 678. **by Sidney Kravitz**

In this puzzle, the 63 squares are to be filled with one decimal digit each. Each horizontal group sums to the number to its right, and each vertical group sums to the number shown below it. Each sum is made up of distinct nonzero summands.



Puzzles: peg solitaire

MM 952. **by F. D. Hammer**

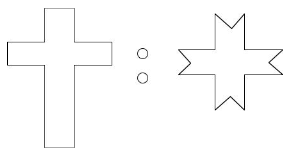
The object of a familiar puzzle is to interchange the positions of n white and n black pegs on a linear board of $2n+1$ positions, where the empty position initially separates all the white pegs from all the black pegs. One is allowed to jump pegs of opposite color, but never of the same color. A white (black) peg may move to the right (left) to an adjacent empty position.

Show that the transfer is always possible and establish a lower bound on the number of moves that is less than $2n(n+1)$.

Puzzles: picture puzzles

PME 458. **by Charles W. Trigg and Leon Bankoff**

Translate the following sketch into a mathematical term. [Other similar puzzles appear in this problem.]



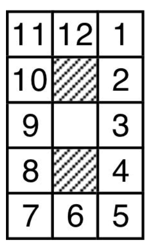
Puzzles: sliding tile puzzles

JRM 471. **by Makoto Arisawa**

The sliding puzzle shown admits two interpretations:

(a) Show that if it is interpreted as consisting of one unengraved tile and two vacant cells, all positions fall into the same class.

(b) Assuming, on the other hand, that there are two untraversible spaces and one vacant cell, determine the number of positional classes.



Riddles

CRUX 151. **by Léo Sauvé**

Identify the speaker and thereby solve the riddle:

METAPHORS

I'm a riddle in nine syllables,
An elephant, a ponderous house,
A melon strolling on two tendrils.
O red fruit, ivory, fine timbers!
This loaf's big with its yeasty rising.
Money's new-minted in this fat purse.
I'm a means, a stage, a cow in calf.
I've eaten a bag of green apples,
Boarded the train there's no getting off.

SYLVIA PLATH (1932 – 1963)

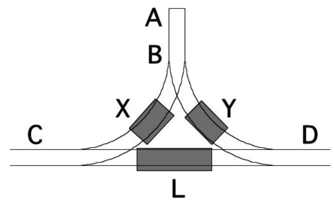
From *Crossing the Water*.

Shunting problems

PARAB 275.

A straight railway line has two sidings with part AB, common to both sidings, long enough to contain either of the two wagons X and Y but not both at once. The locomotive L is too long to go on AB.

The wagons X and Y are initially uncoupled, one on each siding. How can the positions of X and Y be interchanged? (The couplings can be connected or disconnected only while the locomotive and wagons are stationary.)



PARAB 333.

Two trains A and B are traveling in opposite directions on a line with a single track and wish to pass with the help of a siding. The siding will only take one car or one engine at a time and can only be entered from the right. If train A to the left of the siding has 3 cars and one engine and train B to the right of the siding has 4 cars and one engine, how can they pass with the minimum number of moves?

Word problems

JRM 656. **by Harry Nelson**

If the integers from 1 to 5000 are listed in equivalence classes according to the number of characters (including blanks and hyphens) needed to write them out in full in correct English, there are exactly 40 such nonempty classes. For example, class 4 contains FOUR, FIVE, and NINE. Similarly, class 42 contains the nine members 3373, 3377, 3378, 3773, 3777, 3778, 3873, 3877, and 3878.

There is only one such class that contains exactly one member. What is it?

Words

CRUX 61. **by Léo Sauvé**

Find autological adjectives other than those given in the article on page 55 of this issue.

JRM 751. **by Michael R. W. Buckley**

Four words, each an anagram of the same set of five different letters, are missing from the following rhyme: Should her renown as a cook be at − − − − − , − − − −'− − − − − − − − − − − the cake. The same four words, when appropriately arranged, form an alphametic which has a unique solution. Find that solution.

Set Theory

Chains

AMM 6220. **by Mohammad Ismail**

A collection K of sets is called a chain (resp. antichain) if for any $A, B \in K$, either $A \subseteq B$ or $B \subseteq A$ (resp. for any $A, B \in K$, $A \not\subset B$ and $B \not\subset A$). Let ω_1 be the first uncountable ordinal. Does there exist a family $\mathcal{P} = \{K_\alpha : \alpha < \omega_1\}$ of collections of subsets of a set X satisfying the following conditions?

(a) Each K_α is an infinite countable antichain.

(b) If $\alpha < \beta < \omega_1$, then every member of K_β is contained in some member of K_α and no member of K_α is contained in any member of K_β.

(c) If $\mathcal{P}^* = \cup_{\alpha<\omega_1} K_\alpha$, then every chain and every antichain in $\mathcal{P}^*$ is countable.

Mappings

AMM 6128. **by Martin Schechter and Peter Borwein**

Let 2^ω be the set of all sequences with entries 0 or 1, and let N^ω be the set of all sequences with entries from the nonnegative integers. Can one construct a bijection f from 2^ω onto N^ω with the property that for any sequence X in 2^ω, one can compute the first n entries of $f(X)$ given only the first m entries of X (where m may depend on X and n)?

FQ B-333. **by Phil Mana**

Let S_n be the set of ordered pairs of integers (a, b) with both $0 < a < b$ and $a + b \leq n$. Let T_n be the set of ordered pairs of integers (c, d) with both $0 < c < d < n$ and $c + d \leq n$. For $n \geq 3$, establish at least one bijection between S_n and T_{n+1}.

AMM 6266. **by Leopoldo Nachbin**

It is easily shown that every countable set S has the following property:

(P) Given any function $f\colon S \times S \to \mathbb{R}^+$, there exists a function $g\colon S \to \mathbb{R}^+$ such that $f(x, y) \leq g(x)g(y)$ for all $x, y \in S$.

It can be shown that (P) fails if the cardinal number of S is at least equal to that of the continuum. Can it be shown without the Continuum Hypothesis that (P) fails when S is uncountable?

Power set

MATYC 86. **by Joseph Griffin**

Given are sets A and B, with B having 24 more subsets than A. How many elements are in each set?

Relations

OSSMB 79-7.

Let S be a set and let $\mathbf{R}$ be a relation holding or not holding between every ordered pair of distinct elements of S. Suppose $\mathbf{R}$ satisfies the following conditions:

(a) If a, b are distinct elements of S, then $a\mathbf{R}b$ or $b\mathbf{R}a$ holds, but not both.

(b) If a, b, c are distinct elements of S such that $a\mathbf{R}b$ and $b\mathbf{R}c$ hold, then $c\mathbf{R}a$ holds.

Find the maximum number of elements in S.

Subsets

MM 924. **by J. Michael McVoy and Anton Glaser**

How many n-tuples, $(S_1, \ldots, S_n)$, exist with

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq V,$$

where V is a set of k elements?

CRUX 82. **by Léo Sauvé**

Let E be a finite set containing n elements. The following facts are well known and easy to prove.

(a) The number of subsets of E is 2^n.

(b) The number of relations of the form $A \subseteq B$, where $A \subseteq E$ and $B \subseteq E$, is $(2^n)^2 = 4^n$.

How many of the relations in (b) are true?

AMM E2666. **by Peter Frankl**

Let S be a finite set, and let $\mathcal{P}$ be the set of all subsets of S. For $\mathcal{A} \subset \mathcal{P}$ and $\mathcal{B} \subset \mathcal{P}$, define $\mathcal{A} * \mathcal{B}$ to be the subset of $\mathcal{P}$ consisting of subsets $X \subset S$ such that $X \subset A \cup B$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

If $|\mathcal{A}| + |\mathcal{B}| > 2^k$, prove that $|\mathcal{A} * \mathcal{B}| \geq 2^k$.

CMB P242. **by P. Frankl**

Let $\mathcal{A}$ be a set of subsets of $\{1, 2, \ldots, n\}$ such that

$$|A_1 \cup A_2 \cup A_3 \cup A_4| \leq n - 2$$

whenever $A_1, A_2, A_3, A_4 \in \mathcal{A}$. Prove that $|\ \mathcal{A}\ | \leq 2^{n-2}$.

AMM E2792. **by Robert Patenaude**

Let U be a finite set. Characterize those collections C of subsets of U with the following property: There is a unique subset R of U such that the number of sets in C which R intersects is odd.

AMM 6022. **by Neal Felsinger**

Given a collection X of subsets of S, no one containing another, let $C(X)$ consist of all minimal subsets of S that intersect every member of X. Show that if S is infinite, $C(X)$ does not necessarily exist.

ISMJ 14.4.

Let X and Y be subsets of a finite set F.

(a) Show that $X \triangle Y = X \triangle Z$ implies $Y = Z$, where $X \triangle Y = (X \cup Y) \setminus (X \cap Y)$.

(b) Suppose $\mathcal{F}$ is a family of subsets of F that is closed under $\triangle$ (i.e., $X \triangle Y$ is in $\mathcal{F}$ whenever both X and Y are; thus $X \triangle X \in \mathcal{F}$ for each $X \in \mathcal{F}$). Given $X \in \mathcal{F}$ and $a \in X$, show that a is in exactly half of the elements of $\mathcal{F}$.

Symbolic logic

AMM 6272. **by P. Olin and Kenneth W. Smith**

Given: There is a complete, $\aleph_1$-categorical theory T of first-order logic such that the direct product $T \times T$ is not $\aleph_1$-categorical. Are there complete first-order theories T_1, T_2 with T_1 the theory of a finite model, T_2 $\aleph_1$-categorical, and $T_1 \times T_2$ not $\aleph_1$-categorical? If so, find such a pair T_1, T_2 with the cardinality of the model of T_1 as small as possible.

AMM 6139. **by D. P. Munro**

Consider a first-order predicate calculus and all the relational structures appropriate to that calculus.

(a) Let $P_1, \ldots, P_k$ be a finite collection of mutually exclusive and exhaustive axiomatizable properties (so every relational structure has exactly one of the properties P_i). Must any of the P_i be in fact finitely axiomatizable, and if so, how many?

(b) As for (a), but with a countably infinite collection of mutually exclusive and exhaustive axiomatizable properties.

NAvW 391. **by J. F. A. K. van Benthem**

Let L be a first-order language with the usual logical signs (but without identity). Identify all logically equivalent sentences in L; let $f(L)$ be the cardinality of the set thus obtained. Let $m(L)$ be the cardinality of the set of all complete L-theories.

(a) Give the values of f and m for the following logics:
L_1^n: only unary predicate-letters, viz. $A_1, \ldots, A_n$ ($n \geq 1$).
L_2: only unary predicate-letters, countably many.
L_3: exactly one (and binary) predicate-letter (R).
L_4: only the identity-sign ($=$).

(b) Which connections exist between $f(L)$ and $m(L)$ for arbitrary logics L?

Analytic geometry

PARAB 319.

A rectangular box has sides of length x, y, and z, where x, y, z are different numbers. The perimeter of the box is $p = 4(x + y + z)$, its surface area is

$$s = 2(xy + yz + zx),$$

and the length of its main diagonal is $d = \sqrt{x^2 + y^2 + z^2}$. Show that the length of the shortest side is less than $\frac{\frac{1}{4}p - \sqrt{d^2 - \frac{1}{2}s}}{3}$ and the length of the longest side is greater than $\frac{\frac{1}{4}p + \sqrt{d^2 - \frac{1}{2}s}}{3}$.

NYSMTJ 64. **by Robert Exner**

Let a solid cube have one vertex at $(0, 0, 0)$ and let its diagonal from that vertex be along the positive z-axis. If the positive z-axis passes through the sun, what shadow does the cube cast on the xy-plane?

AMM E2576. **by Robert L. Helmbold**

Given any unit vector $\mathbf{n} = (n_1, n_2, n_3)$, what is the area $A(\mathbf{n})$ of the orthogonal projection of the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

onto a plane perpendicular to $\mathbf{n}$?

TYCMJ 132. **by R. S. Luthar**

Let k be a nonzero constant, and let P be the set of planes with the property that the sum of the reciprocals of the x, y, and z intercepts equals k. Prove or disprove that the members of P contain a common point.

SSM 3761. **by Gregory Wulczyn**

Find positive numbers r, s, and t that maximize the volume of the rectangular parallelepiped having one vertex at the origin and opposite diagonal vertex at (r, s, t), subject to the constraint that (r, s, t) lies on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ with a, b, and c all positive.

NYSMTJ 86. **by Robert Exner**

If one looks at the paraboloid

$$z = ax^2 + by^2, \qquad a > 0, \quad b > 0,$$

from a viewpoint in the first octant, what kind of curve on the paraboloid outlines the silhouette?

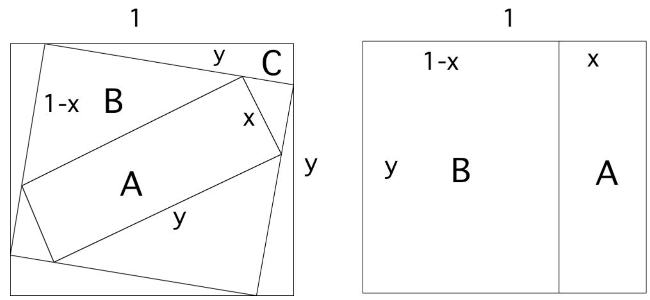
AMM E2563. **by J. Th. Korowine**

Let f_1 and f_2 be nonnegative periodic functions of period 2π, and let $h > 0$. Let $P_1(\theta)$ and $P_2(\theta)$ be the points whose cylindrical coordinates are $(f_1(\theta), \theta, 0)$ and $(f_2(\theta), \theta, h)$, respectively. Find integrals for the volume and surface area of the solid bounded by the planes $z = 0$, $z = h$, and the lines $P_1(\theta)P_2(\theta)$.

Boxes

JRM 390. **by R. Robinson Rowe**

Given are three rectangular boxes that can be nested in two ways. Box A just fits diagonally in Box B, which just fits diagonally in Box C. Alternately, using the same storage space, Boxes A and B fit snugly side by side in Box C, which is one meter long. Neglecting thickness of the box material, find the length and breadth of each box.



Complexes

AMM E2584. **by H. S. M. Coxeter**

Describe an infinite complex of congruent isosceles triangles extending systematically throughout 3-dimensional Euclidean space in such a way that each side of every triangle belongs to just two other triangles.

Convexity

AMM E2617. **by Eugene Ehrhart**

A convex body is cut by three parallel planes. If the three sections thus produced have the same area, show that the portion of the body lying between the two outside planes is a cylinder. Does the same conclusion follow if instead we are given that the three sections have the same perimeter?

JRM 507. **by Susan Laird**

Without consulting Man, the two most advanced races in the universe divided it up between them. The Riss staked out the part they wanted, taking care to consolidate their empire by claiming the line AB whenever the points A and B lay in their territory. The more peaceful Prott were content with the large, unclaimed portion left to them, even though the Riss got the larger share in the sense that a randomly traveling spaceship was more likely, at any given time, to find itself in Riss rather than Prott territory.

Accepting this as true metahistory, prove that space is finite.

Covering problems

PUTNAM 1975/B.2.

In 3-dimensional Euclidean space, define a slab to be the open set of points lying between two parallel planes. The distance between the planes is called the thickness of the slab. Given an infinite sequence $S_1, S_2, \ldots$ of slabs of thicknesses $d_1, d_2, \ldots$, respectively, such that $\sum_{i=1}^{\infty} d_i$ converges, prove that there is some point in the space which is not contained in any of the slabs.

Cubes

ISMJ 12.26.

Consider a line ℓ joining the midpoints of opposite edges of a cube. A cube has four diagonals. Show that ℓ is perpendicular to two of them.

SSM 3693. **by Charles W. Trigg**

An antiprism is a polyhedron with two regular n-gons (the parallel bases) connected by $2n$ isosceles triangles. The regular octahedron is an antiprism with $n = 3$. The regular icosahedron consists of an antiprism capped on each base with a pentagonal pyramid.

(a) Show that a cube can be viewed as an antiprism with two pyramidal caps.

(b) Find the relative volumes of the antiprism and the cube.

(c) Find the relative surface areas of the antiprism and the cube.

(d) Describe the midsection of the antiprism and find its area in terms of an edge of the cube.

Curves

CRUX 367. **by Viktors Linis**

(a) A closed polygonal curve lies on the surface of a cube with edge of length 1. If the curve intersects every face of the cube, show that the length of the curve is at least $3\sqrt{2}$.

(b) Formulate and prove similar theorems about (i) a rectangular parallelepiped, (ii) a regular tetrahedron.

Cylinders

AMM E2728. **by J. G. Mauldon**

Let a, b, c, and d be radii of four mutually externally tangent right circular cylinders whose axes are parallel to the four principal diagonals of a cube. Characterize all quadruples (a, b, c, d) that arise in this way.

NYSMTJ 46.

Prove that the intersection of a right circular cylinder and a plane, neither parallel with nor perpendicular to the axis of the cylinder, is an ellipse.

OMG 17.1.3.

A roller has an outer casing of circumference 150 cm and an inner casing of circumference 125 cm. It contains a small cylinder of circumference 60 cm which is free to roll around inside. How many revolutions will the small cylinder make if the roller is pushed a distance of 18 m?

JRM 629. **by Archimedes O'Toole**

A sphere rests on the bottom of a cylindrical container of radius r. What is the minimum volume of liquid required to immerse the sphere?

Dissection problems

FUNCT 2.2.2.

You can clearly cut a $3 \times 3 \times 3$ cube up into 27 cubes, each $1 \times 1 \times 1$, using 6 cuts. What is the smallest number of cuts that you can use to achieve the same result, perhaps by rearranging the parts after each cut?

JRM 783. **by Harry L. Nelson**

(a) Is there a solid from which 27 one-inch cubes can be cut in less than five cuts?

(b) What is the minimum largest dimension of a solid out of which 27 one-inch cubes can be cut in less than six cuts?

JRM 787. **by Scott Kim**

A *torus* is defined to be any shape which has exactly one hole through it. A *rectangular* torus is a right rectangular prism with a rectangular hole drilled through it. The hole need not be centered, but the edges of the hole must be parallel to the edges of the prism.

(a) Cut a cube into exactly two *unlinked* tori.

(b) Cut a cube into exactly five rectangular tori.

SSM 3672. **by William K. Viertel**

Using a plane parallel to the base, show how to cut a hemisphere into two parts of equal volume.

JRM 498. **by Robert Walsh**

The seam of a baseball divides the spherical surface into two congruent regions. Prove or disprove: A great circle is the only curve on a spherical surface having the above property and the additional property that the shortest route on the sphere between any two points of the same region lies entirely in that region.

Lattice points

MM 927. **by Roy Dubisch**

Pick's formula for the area of polygonal regions whose vertices are lattice points is $\frac{1}{2}b+i-1$ where b is the number of lattice points on the boundary and i is the number of lattice points in the interior. Show that no such formula exists for the volume of polyhedra whose vertices are lattice points even if we allow as variables, in addition to b and i, $e =$ the number of edges, $f =$ the number of faces, and $i' =$ the number of lattice points in the interior of the faces.

Lines

AMM E2769. **by Harry D. Ruderman**

Let λ and λ' be (not necessarily coplanar) lines in space. On each of these lines, set up a real number coordinate system, with possibly different units of length. Let XX' be the line segment joining a point X on λ to the point X' on λ' with the same coordinate. Describe how to obtain X such that XX' has minimal length for all such segments.

Locus

NAvW 414. **by O. Bottema**

Determine the locus of the points with equal distances to three skew edges of a cube.

IMO 1978/2.

Let P be a point inside a given sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V and W; Q denotes the vertex diagonally opposite to P in the parallelepiped determined by PU, PV and PW. Find the locus of Q for all such triads of rays from P.

CRUX 497. **by Ferrell Wheeler**

Given is a cube of edge length a with diagonal CD, face diagonal AB, and edge CB. Points P and Q start at the same time from A and C, respectively, move at constant rates along AB and CD, respectively, and reach B and D, respectively, at the same time. Find the area of the surface swept out by segment PQ.

Maxima and minima

CRUX 113. **by Léo Sauvé**

If $\vec{u} = (b, c, a)$ and $\vec{v} = (c, a, b)$ are two nonzero vectors in Euclidean 3-space, what is the maximum value of the angle between $\vec{u}$ and $\vec{v}$? When is this value attained?

AMM E2757.* **by Harry D. Ruderman**

Let a, b, and c be three lines in $\mathbb{R}^3$. Find points A, B, and C on a, b, and c, respectively, such that $AB+BC+CA$ is a minimum.

PME 367. **by R. Robinson Rowe**

A box of unit volume consists of a square prism topped by a pyramid. Find the side of the square base and heights of the prism and pyramid to minimize the surface area.

OSSMB 77-13.

Find the maximum volume and minimum total edge length among all rectangular solids having fixed surface area S. Show that these extreme values are attained by, and only by, the cube.

OMG 16.1.3.

If a cable spool is 1 meter wide, has an inner radius of 1/2 meter and an outer radius of 1 meter, what is the maximum possible length of nylon rope of radius 1 cm that can be wound on the spool?

CRUX 394. **by Harry D. Ruderman**

A wine glass has the shape of an isosceles trapezoid rotated about its axis of symmetry. If R, r, and h are the measures of the larger radius, smaller radius, and altitude of the trapezoid, find $r : R : h$ for the most economical dimensions.

USA 1976/4.

If the sum of the lengths of the six edges of a trirectangular tetrahedron $PABC$ is S, determine its maximum volume.

Octahedra

MM 929. **by Charles W. Trigg**

Show that there are only two octahedrons with equilateral triangular faces.

MM Q632. **by F. David Hammer**

A tetrahedron and an octahedron are built from a common stock of equilateral triangles. The tetrahedron holds a quart; what does the octahedron hold?

Packing problems

TYCMJ 100. **by Sidney Penner**

A 3-brick is a $3 \times 1 \times 1$ rectangular parallelepiped. Assume that a $7 \times 7 \times 7$ cube has been packed with 3-bricks and a single unit cube which is not located on the periphery. Prove that the unit cube must be located in the center.

IMO 1976/3.

A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.

JRM 646. **by Harry Nelson**

A rectangular box with integer dimensions $H \le W \le L$ can be packed with more than WL cylindrical cans of height 5 and diameter 1. What is the smallest possible volume? What is the smallest possible surface area? Other such boxes can be packed with more than HWL spheres of diameter 1. Find those with the smallest possible volume and surface area.

PARAB 361.

A number of blocks, each 2 cm$\times$2 cm$\times$1 cm, have been fitted snugly together to make a solid 20 cm high. (The top dimensions of the solid are, say, m cm $\times$ n cm.) A straight line, parallel to the 20 cm sides, pierces the solid from top to bottom. Prove that the straight line cannot pierce exactly one of the blocks.

JRM 733. **by Frank Rubin**

A cube is inscribed in a sphere. A second sphere is tangent externally to the cube at the center of one face and internally to the first sphere. A set of n identical spheres are tangent to the face of the cube, the first sphere, and the second sphere. What is the maximum value of n?

Paper folding

PME 460. **by Barbara Seville**

The dihedral angle of a cube is 90°. The other four Platonic solids have dihedral angles which are approximately $70^\circ 31' 43.60''$, $109^\circ 28' 16.3956''$, $116^\circ 33' 54.18''$, and $138^\circ 11' 22.866''$. How closely can these angles be constructed with straightedge and compass? Can good approximations be accomplished by paper folding? If so, how?

CRUX PS2-3.

Three unequal disjoint circles are given on a large (planar) card. If the centers of the circles are collinear, show that it is always possible to fold the card along two straight lines such that the three circles lie on a common sphere.

CRUX 375. **by M. S. Klamkin**

A convex n-gon P of cardboard is such that if lines are drawn parallel to all the sides at distances x from them so as to form within P another polygon P', then P' is similar to P. Now let the corresponding consecutive vertices of P and P' be $A_1, A_2, \ldots, A_n$ and $A'_1, A'_2, \ldots, A'_n$ respectively. From A'_2, perpendiculars A'_2B_1, A'_2B_2 are drawn to A_1A_2, A_2A_3 respectively, and the quadrilateral $A'_2B_1A_2B_2$ is cut away. Then quadrilaterals formed in a similar way are cut away from all the other corners. The remainder is folded along $A'_1A'_2, A'_2A'_3, \ldots, A'_nA'_1$ so as to form an open polygonal box of base $A'_IA'_2 \ldots A'_n$ and of height x. Determine the maximum volume of the box and the corresponding value of x.

AMM E2630. **by Edward T. Ordman**
Suppose that a polyhedral model (made, say, of cardboard) is slit along certain edges and unfolded to lie flat in the plane. The cuts may not be made so as to disconnect the figure. Now suppose that the resulting plane figure is again folded up to make a polyhedron (folding is allowed only on the original lines). The new polyhedron is not necessarily congruent to the original one. Find some interesting examples.

CRUX 140. **by Dan Pedoe**
A paper cone is cut along a generator and unfolded into a plane sheet of paper. What curves in the plane do the originally plane sections of the cone become?

ISMJ J10.13.
For what tetrahedra is it true that if the three faces are folded out and down to lie flat in the plane of the base the resulting plane figure is a triangle?

Pentahedra

CRUX 182. **by Charles W. Trigg**
A framework of uniform wire is congruent to the edges of the pentahedron in the previous problem. If the resistance of one side of the square is 1 ohm, what resistance does the framework offer when the longest edge is inserted in a circuit?

CRUX 181. **by Charles W. Trigg**
A polyhedron has one square face, two equilateral triangular faces attached to opposite sides of the square, and two isosceles trapezoidal faces, each with one edge equal to twice a side, e, of the square. What is the volume of this pentahedron in terms of a side of the square?

Plane figures

PME 420. **by Herbert Taylor**
Given four lines through a point in 3-space (no three of the lines in a plane), find four points, one on each line, forming the vertices of a parallelogram.

PUTNAM 1977/B.2.
Given a convex quadrilateral $ABCD$ and a point O not in the plane of $ABCD$ locate point A' on line OA, point B' on line OB, point C' on line OC, and point D' on line OD so that $A'B'C'D'$ is a parallelogram.

PUTNAM 1975/A.6.
Let P_1, P_2 and P_3 be the vertices of an acute-angled triangle situated in 3-dimensional space. Show that it is always possible to locate two additional points P_4 and P_5 in such a way that no three of the points are collinear and so that the line through any two of the five points is perpendicular to the plane determined by the other three.

Points in space

CANADA 1976/6.
NYSMTJ 75. **by Sidney Penner**
If A, B, C and D are four points in space, such that

$$\angle ABC = \angle BCD = \angle CDA = \angle DAB = \pi/2,$$

prove that A, B, C and D lie in a plane.

USA 1975/2.
Let A, B, C and D denote four points in space and AB the distance between A and B, and so on. Show that

$$AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2.$$

Polyhedra: combinatorial geometry

JRM 763. **by Frank R. Bernhart**
A simple polyhedron is a polyhedron on which exactly three faces meet at every vertex. Prove that if every face of a simple polyhedron is a $3n$-gon, the number of vertices is divisible by four.

OSSMB 75-8. **by Murray Klamkin**
Show that in every simple polyhedron there always exist two pairs of faces that have the same number of edges.

ISMJ 12.21.
Euler proved his formula by considering how many edges and faces were added to a polyhedron (or its map) when a vertex was added. Can you reproduce his proof?

CRUX 336. **by Viktors Linis**
Prove that if in a convex polyhedron there are four edges at each vertex then every planar section which does not pass through any vertex is a polygon with an even number of sides.

Polyhedra: convex polyhedra

PARAB 385.
Let v be the number of vertices of a convex polyhedron, e the number of edges, and f the number of faces.

(a) Show that, for any convex polyhedron, $3f \leq 2e$ and $3v \leq 2e$.

(b) Is it possible to cut a potato into a convex polyhedron having exactly seven edges?

CRUX 93. **by H. G. Dworschak**
Is there a convex polyhedron having exactly seven edges?

CRUX 121. **by Léo Sauvé**
For which n is there a convex polyhedron having exactly n edges?

Polyhedra: pentagons

CRUX 73. **by Viktors Linis**
Is there a polyhedron with exactly ten pentagons as faces?

Polyhedra: spheres

PME 352. **by Charles W. Trigg**
The edges of a semi-regular polyhedron are equal. The faces consist of eight equilateral triangles and six regular octagons. In terms of the edge e, find the diameters of the following spheres:

(a) the sphere touching the octagonal faces,
(b) the circumsphere, and
(c) the sphere touching the triangular faces.

Polyhedra: squares

AMM E2740.* **by Victor Pambuccian**
Show that if P is a convex polyhedron, one can find a square all of whose vertices are on some three faces of P, as well as a square whose vertices are on four different faces of P.

Projective geometry

NAvW 460. **by O. Bottema**

In a projective 3-space S, a tetrahedron T and a plane U, not passing through any vertex of T, are given. To define a Euclidean metric in S, the plane U is taken as the plane at infinity and a suitable conic K in U as the isotropic conic. How should K be chosen so that T is

(a) orthocentric;
(b) equifacial;
(c) regular?

NAvW 469. **by O. Bottema**

In a projective 3-dimensional space, a (nonsingular) quadric Q is given. Determine the (nonsingular) tetrahedra with their vertices on Q and their faces tangent to Q.

NAvW 491. **by O. Bottema and J. T. Groenman**

In 3-dimensional projective space, a tetrahedron

$$A = A_1A_2A_3A_4$$

and two points P and Q are given. The line A_iP intersects the opposite face of A at B_i; the line B_iQ intersects the opposite face of B $(= B_1B_2B_3B_4)$ at C_i. Show that the four lines A_iC_i are generators of a hyperboloid.

NAvW 536. **by O. Bottema**

In projective 3-spaces, a tetrahedron $A_1A_2A_3A_4$ and four points P, Q, R, S, not on one of its faces, are given. Let A_iP, A_iQ, A_iR, A_iS meet the opposite face at P_i, Q_i, R_i, S_i $(i = 1, 2, 3, 4)$. The planes $P_2P_3P_4$, $P_3P_4P_1$, $P_4P_1P_2$, and $P_1P_2P_3$ are denoted by U_1, U_2, U_3, and U_4 respectively, and the planes analogously associated with Q, R, and S by V_i, W_i, and T_i.

If U_1, V_1, W_1, T_1 pass through one point, show that the same holds for U_i, V_i, W_i, T_i $(i = 2, 3, 4)$. Show furthermore that this takes place if and only if P, Q, R, S are on a Cayley cubic surface with double points at A_i.

NAvW 546. **by O. Bottema**

In a projective 3-space, the tetrahedron $A_1A_2A_3A_4$ is taken as the fundamental tetrahedron of a projective coordinate system; α_i is the face opposite A_i. In α_i, the point $B_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4})$, $i = 1, 2, 3, 4$, is given, such that $a_{ij} + a_{ji} = 0$, $a_{ij} \neq 0$ if $i \neq j$, and

$$a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 0.$$

Show that the quadruple of lines A_iB_i is parabolic, and determine the Plücker coordinates of the unique transversal.

Pyramids

MM Q621. **by Charles W. Trigg**

Show that in a square pyramid with all edges equal, a dihedral angle formed by two triangular faces is twice a dihedral angle at the base.

KURSCHAK 1979/1.

The base of a convex pyramid is a polygon with an odd number of sides, its lateral edges are all of the same length, and the angles between its neighboring lateral faces are also equal. Prove that the base of the pyramid is a regular polygon.

Rectangular parallelepipeds

ISMJ J11.4.

If the sides of a rectangular box are increased by 2, 3, and 4 inches, respectively, it becomes a cube and its volume is increased by 827 cubic inches. Find the dimensions of the box.

SSM 3584. **by Robert A. Carman**

If (a, b, c) is a Pythagorean triple, $a^2 + b^2 = c^2$, then prove that the rectangular solid with edges $(ab)^2$, $(ac)^2$, and $(bc)^2$ has its major diagonal equal to $c^4 - a^2b^2$.

TYCMJ 134. **by Norman Schaumberger**

Let A be the surface area of a rectangular parallelepiped, V be the volume, and d be the diagonal. Prove that $2d^2 \geq A \geq 6V^{2/3}$.

Regular tetrahedra

CRUX 245. **by Charles W. Trigg**

Find the volume of a regular tetrahedron in terms of its bimedian b. (A bimedian is a segment joining the midpoints of opposite edges.)

USA 1978/4.

(a) Prove that if the six dihedral angles of a given tetrahedron are congruent, then the tetrahedron is regular.

(b) Is a tetrahedron necessarily regular if five dihedral angles are congruent?

CRUX PS5-3.

In a regular (equilateral) triangle, the circumcenter O, the incenter I, and the centroid G all coincide. Conversely, if any two of O, I, G coincide, the triangle is equilateral. Also, for a regular tetrahedron, O, I, and G coincide. Prove or disprove the converse result that if O, I, and G all coincide for the tetrahedron, the tetrahedron must be regular.

PME 425. **by Charles W. Trigg**

Without using its altitude, compute the volume of a regular tetrahedron by the prismoidal formula.

Right circular cones

FUNCT 1.1.3. **by A. Nesbit**

Prove that the volume of a frustum of a cone is obtained by either of the rules:

(a) To the areas of the two ends of the frustum add the square root of their product; multiply the result by 1/3 of the perpendicular height.

(b) To the product of the diameters of the two ends, add the sum of their squares; multiply this sum by the height, and again by 0.2618.

JRM 785. **by R. Robinson Rowe**

The Burr brothers, Tim and Lum, felled a tree and slashed off the branches, leaving a cone-shaped log, then lopped off the spindle top, leaving the trunk truncated where its diameter was six inches. Guessing it was his share, Tim cut off the next 19 feet, but when Lum cut off an equal volume from the butt end, it was only 7 feet long. This left a two-foot-long chunk, which they split up for stove wood. What was the diameter of the log at the butt end?

PARAB 410.

Mount Zircon is shaped like a perfect cone whose base is a circle of radius 2 miles, and the straight line paths up to the top are all 3 miles long. From a point A at the southernmost point of the base, a path leads to B, a point on the northern slope and $2/5$ of the way to the top. If AB is the shortest path on the mountainside joining A to B, find

(a) the length of the whole path AB, and

(b) the length of the path between P and B, where P is a point on the path at which it is horizontal.

CANADA 1977/5.
OMG 16.2.5.

A right circular cone of base radius 1 cm and slant height 3 cm is given. Suppose P is a point on the circumference of the base and the shortest path from P around the cone and back to P is drawn. What is the minimum distance from the vertex V to this path?

Skew quadrilaterals

MM Q630. **by M. S. Klamkin and M. Sayrafiezadeh**

Suppose a skew quadrilateral $ABCD$, with diagonal AC perpendicular to diagonal BD, is transformed into the quadrilateral $A'B'C'D'$ so that the corresponding lengths of the sides are preserved. Prove that $A'C'$ is perpendicular to $B'D'$.

USA 1977/4.

Prove that if the opposite sides of a skew quadrilateral are congruent, then the line joining the midpoints of the two diagonals is perpendicular to these diagonals, and conversely, if the line joining the midpoints of the two diagonals of a skew quadrilateral is perpendicular to these diagonals, then the opposite sides of the quadrilateral are congruent.

Solids of revolution

CRUX 436. **by R. Robinson Rowe**

The following method is used to approximate an oval using four circular sectors. Two nonoverlapping sectors that are symmetric about the horizontal diameter of a given circle are each translated vertically towards one another by equal distances small enough to allow their four bounding radii to continue to extend past their intersection by amounts R_1 and R_2 on the left- and right-hand sides, respectively. The other two sectors in the approximation have radii R_1 and R_2.

Determine the radii of the four circular sectors in terms of the angles these radii make with the horizontal and the lengths of the horizontal and vertical diameters of the constructed figure.

Space curves

AMM 6087. **by Nathaniel Grossman**

A loxodrome on a Riemannian surface is a curve meeting members of a specified one-parameter family of curves at a constant angle. For example, a torus has two special families, the meridians and the parallels, each defining the same family of loxodromes. Prove that a loxodrome on a torus is either periodic or dense.

MM 962. **by Curt Monash**

Consider the space curve, $C(t)$, defined by

$$C(t) = (t^k, t^m, t^n) \text{ for } t \geq 0 \text{ and } k, m, \text{ and } n \text{ integers.}$$

(a) Show that if (k, m, n) equals $(1, 2, 3)$ or $(-2, -1, 1)$, then $C(t)$ does not contain four coplanar points.

(b) Show that for (k, m, n) equal to $(1, 3, 4)$, $C(t)$ does contain four coplanar points.

(c) Find a characterization of (k, m, n) so that $C(t)$ does not contain four coplanar points.

MM 981. **by Steven Jordan**

Show that if a smooth curve in $\mathbb{R}^3$ has the property that each principal normal line passes through a fixed point, then the curve must be an arc of a circle.

Spheres

OMG 16.1.9.

A hole of length 6 m is drilled through a sphere of radius greater than 3 m. What is the volume of the remaining material?

CRUX 453. **by Viktors Linis**

In a convex polyhedron each vertex is of degree 3 (i.e. is incident with exactly 3 edges) and each face is a polygon which can be inscribed in a circle. Prove that the polyhedron can be inscribed in a sphere.

PENT 303. **by Charles W. Trigg**

Show that the ratio of the volume of a sphere to the volume of its inscribed regular octahedron is π.

AMM E2694. **by I. J. Schoenberg**

Let Π be a prism inscribed in the sphere S of unit radius and center O. The base of Π is a regular n-gon of radius r. For each face F of Π, drop a directed perpendicular from O and let A_F be the point where it intersects S. Let Π^* be the polyhedron obtained by adding to Π, for each face F, the pyramid of base F and apex A_F.

For which values of r is Π^* convex?

CRUX PS6-2.

Given are two points, one on each of two given skew lines. Prove that there exists a unique sphere tangent to each of the two given points.

CRUX 500. **by H. S. M. Coxeter**

Let 1, 2, 3, 4 be four mutually tangent spheres with six distinct points of contact 12, 13, …, 34. Let 0 and 5 be the two spheres that touch all the first four. Prove that the five "consecutive" points of contact 01, 12, 23, 34, 45 all lie on a sphere (or possibly a plane).

Spherical geometry

PARAB 305.

An airplane leaves a town of latitude 1° S, flies x km due South, then x km due East, then x km due North. He is then $3x$ km due East of his starting point. Find x.

USA 1979/2.

Let S be a great circle with pole P. On any great circle through P, two points A and B are chosen equidistant from P. For any spherical triangle ABC (the sides are great circle arcs), where C is on S, prove that the great circle arc CP is the angle bisector of angle C.

Solid Geometry

Surfaces

OMG 16.1.8.

If a 3 m high statue takes 5 liters of paint to cover, how much will be needed to cover a 30 cm high copy?

Tetrahedra: altitudes

NAvW 513. **by O. Bottema**

It is known that the four altitudes of a tetrahedron T are generators of a hyperboloid. Determine their cross ratio in terms of the six dihedral angles of T.

Tetrahedra: dihedral angles

CANADA 1979/2.

Prove that the sum of the dihedral angles of a tetrahedron is not constant.

Tetrahedra: faces

CRUX 478. **by Murray S. Klamkin**

Prove that if the circumcircles of the four faces of a tetrahedron are mutually congruent, then the circumcenter O of the tetrahedron and its incenter I coincide.

CRUX 330. **by M. S. Klamkin**

It is known that if any one of the following three conditions holds for a given tetrahedron then the four faces of the tetrahedron are mutually congruent (i.e., the tetrahedron is isosceles):

1. The perimeters of the four faces are mutually equal.
2. The areas of the four faces are mutually equal.
3. The circumcircles of the four faces are mutually congruent.

Does the condition that the incircles of the four faces be mutually congruent also imply that the tetrahedron is isosceles?

Tetrahedra: family of tetrahedra

NAvW 514. **by O. Bottema**

A tetrahedron $A_1A_2A_3A_4$ with $A_2A_3 = a$, $A_3A_1 = b$, $A_1A_2 = c$, $A_1A_4 = a_1$, $A_2A_4 = b_1$, $A_3A_4 = c_1$ is called harmonic (or isodynamic) if $aa_1 = bb_1 = cc_1 = k$. Given a nonequilateral triangle $A_1A_2A_3$ with sides a, b, and c, show that there exists a set of harmonic tetrahedra $A_1A_2A_3A_4$, and determine the upper and lower bound of k.

Tetrahedra: incenter

NAvW 526. **by O. Bottema and J. T. Groenman**

A tetrahedron $A_1A_2A_3A_4$ is given; α_i is the face opposite A_i, I is the center of the inscribed sphere, and B_i its tangent point on α_i $(1 \le i \le 4)$. The point P_i on IB_i is defined by $IP_i = d$, where d is given $(-\infty \le d \le \infty)$.

Show that the four lines A_iP_i are hyperbolic.

Tetrahedra: inscribed spheres

MM Q616. **by C. W. Trigg**

The faces of a tetrahedron and a hexahedron (triangular dipyramid) are congruent equilateral triangles. What is the ratio of the radii of their inscribed spheres?

Tetrahedra: maxima and minima

JRM 532. **by R. S. Field Jr.**

Of all plane sections of a regular tetrahedron, which one has the maximum perimeter?

Tetrahedra: octahedra

PME 386. **by Charles W. Trigg**

Show that the volume of Kepler's *Stella Octangula* (a compound of two interpenetrating tetrahedrons) is three times that of the octahedron that was stellated.

Tetrahedra: opposite edges

AMM S12. **by M. S. Klamkin**
CRUX PS4-3.

If a, a_1; b, b_1; and c, c_1 denote the lengths of the three pairs of opposite sides of an arbitrary tetrahedron, prove that $a+a_1$, $b+b_1$, and $c+c_1$ satisfy the triangle inequality.

CRUX 94. **by H. G. Dworschak**

If, in a tetrahedron, two pairs of opposite edges are orthogonal, is the third pair of opposite sides necessarily orthogonal?

Tetrahedra: planes

NAvW 451. **by O. Bottema**

Let $T = A_1A_2A_3A_4$ be a given tetrahedron, and let M_{ij} denote the midpoint of A_iA_j. Determine the convex polyhedron P bordered by the twelve planes $A_iM_{jk}M_{j\ell}$, where i, j, k, ℓ is a permutation of $1, 2, 3, 4$. Determine the volume of P if that of T is unity.

Tetrahedra: triangular pyramids

SPECT 10.2.

A pyramid on a triangular base has the length of each sloping side 1 and the length of each base side $\sqrt{2}$. The point P is a point on the base, distance d_1, d_2, d_3 from the base vertices. Determine the distance of P from the apex of the pyramid.

Triangles

AMM E2727. **by David P. Robbins**

Two triangles $A_1A_2A_3$ and $B_1B_2B_3$ in $\mathbb{R}^3$ are equivalent if there exist three different parallel lines p_1, p_2, and p_3 and rigid motions σ and τ such that $\sigma(A_i)$ and $\tau(B_i)$ lie on p_i $(i = 1, 2, 3)$.

Find necessary and sufficient conditions for equivalence of two triangles.

Banach spaces

NAvW 440. **by D. van Dulst**

Let X be a Banach space, and let B denote its unit ball. If X is nonreflexive, show that there exists an $\varepsilon > 0$ with the property that, for no weakly compact set $K \subset X$, we have $B \subset K + \varepsilon B$.

AMM 6283. **by Gordon R. Feathers**

It is well known that a strongly closed convex subset of a Banach space is weakly closed. Is the same true of a strongly closed star-shaped subset?

Cantor set

AMM 6213. **by C. G. Mendez**

Let G be an open dense subset of the Cantor set C. Is the boundary ∂G of G countable?

Compactifications

AMM 6124.* **by Thomas E. Elsner**

Let Y be a compactification of a completely regular space X. Is there a base B for Y such that the smallest algebra of sets containing B has no element in $Y \setminus X$?

Composed operations

AMM 6260. **by Eric Langford**

If X is a subset of a topological space S, then it is known that there can be formed at most six new sets by repeated formations of closures and interiors iterated in any order. It is also known that if we further allow the formation of unions, then no more than six new sets can be generated for a maximum total of thirteen. Given that we start with X and the additional six sets described in the first sentence, what is the minimum number of new sets that can occur when we further allow unions?

Connected sets

JRM 445. **by Michael R. W. Buckley**

Define a *tetrad* as the union of four closed, simply connected regions such that each of the six pairs of regions shares a boundary of positive measure. It is simple to construct a tetrad that is, itself, simply connected. It is also simple to exhibit tetrads in which the four component regions are congruent, in which, however, the tetrads themselves are not simply connected.

(a) Can a simply connected tetrad with four congruent component regions be constructed?

(b) Failing this, can a sequence of tetrads with four congruent components be exhibited in which the ratio of hole area to tetrad area approaches zero?

(c) Failing this, what is the greatest lower bound on the ratio of hole area to tetrad area for a tetrad with four congruent components?

JRM 684. **by Frank Rubin**

Define a *tetrad* as the union of four closed, simply connected regions such that each of the six pairs of regions shares a boundary of positive measure. It is possible to construct tetrads that are simply connected and which are composed of four congruent component regions. Is it possible to construct such a tetrad which in addition is convex?

MM 932. **by R. A. Struble**

Is there a topology for the set of real n-tuples, other than the Euclidean topology, relative to which the family of connected sets is exactly the usual one?

CRUX 186. **by Leroy F. Meyers**

Let A, B, C, and D be the subsets of the plane $\mathbb{R}^2$ having, respectively, both coordinates rational, both coordinates irrational, exactly one coordinate rational, and both or neither coordinates rational. Which of these sets is/are connected?

Euclidean plane

AMM 6122. **by Albert A. Mullin**

Does there exist a compact set $S \subset E^2$ such that for each $x \in E^2 \setminus S$, there exist precisely two nearest points of S? Clearly, S cannot be convex.

Function spaces

AMM 6093. **by Richard Johnsonbaugh**

Let X be a completely regular Hausdorff space, and let $C(X)$ denote all real-valued continuous functions on X with the topology of uniform convergence on compact sets. Let F be a continuous nonzero linear functional on $C(X)$. Prove that there exists a smallest compact set K with the property that if $f = 0$ on K, then $F(f) = 0$.

AMM 6113. **by Claudia Simionescu**

Let X be a compact metric space, and let F be a real finitely additive set function not of bounded variation. Let T_F be the set of Riemann-Stieltjes integrable functions. Then T_F is of first category in $C(X)$. Can this result be improved to show that T_F is nowhere dense?

AMM 6257. **by Jan Mycielski**

Let X be the space of continuous nondecreasing functions $f\colon [0,1] \to [0,1]$ having $f(0) = 0$ and $f(1) = 1$ and with the distance function $d(f,g) = \max|f(x) - g(x)|$ over $0 \le x \le 1$. Let Y be the subset of all f in X such that f is strictly increasing and the length of f is 2. Prove that $X \setminus Y$ is meager in X.

Functions

AMM 6181.* **by J. M. Arnaudies**

Let n be an integer larger than 2, and $A_0, A_1, \ldots, A_n$ be n single-valued real functions defined and continuous on a given topological Hausdorff space T. Suppose that for all $t \in T$, the 2-form

$$A_0x^n + A_1x^{n-1}y + \cdots + A_ny^n$$

(where the A_i take their values for t) defines n real distinct lines in the two-dimensional real projective space.

Give a characterization of those spaces T such that for any choice of the A_i, there necessarily exists a system of continuous functions $(P_1, Q_1, P_2, Q_2, \ldots, P_n, Q_n)$, real-valued, defined on T, satisfying the formal equality,

$$\begin{aligned} &A_0x^n + A_1x^{n-1}y + \cdots + A_ny^n \\ &\qquad = (P_1x + Q_1y)(P_2x + Q_2y)\cdots(P_nx + Q_ny). \end{aligned}$$

Graph of a function

AMM 6255. **by Adam Riese**

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function whose graph, considered as a subset of $\mathbb{R}^2$, is both closed and connected. Prove that f is continuous. What can be said when $f\colon \mathbb{R}^m \to \mathbb{R}^n$?

Hilbert spaces

CMB P257. **by S. Zaidman**

Let V and H be two Hilbert spaces and V a vector subspace of H with $v \neq H$. Suppose the inclusion map $i : V \to H$ is continuous and that V is dense in H. Then there is a function $v : [-1, 1] \to V$ such that $i \circ v : [-1, 1] \to H$ is continuous, but v itself is not continuous.

NAvW 554. **by A. A. Jagers and H. Th. Jongen**

Let H be a separable Hilbert space and P be the set of all positive, semidefinite Hermitian operators from the whole of H into H. To each convex cone F in P, we may associate a topology τ_F on H such that (H, τ_F) is a locally convex topological vector space: Take

$$\left\{ \left\{ x \in H \;\middle|\; (Ax, x) < 1 \right\} \;\middle|\; A \in F \right\}$$

as a neighborhood base at 0 for τ_F. If

$$F = \left\{ A \in P \;\middle|\; \dim A(H) < \infty \right\},$$

then τ_F is the weak topology on H; if F consists of all nuclear $A \in P$, then τ_F is called the S-topology; if $F = P$, then τ_F is just the norm topology. Prove that τ_F is equal to the bounded weak topology if F consists of all compact $A \in P$.

Knots

JRM 444. **by Horace W. Hinkle**

It is simple enough to tie up a rectangular box with string and then tie the loose ends together. Is it possible, however, to do the job with a single rubber band of suitable size with each of the six faces of the box having two segments of the band intersecting in an over-and-under knot rather than in a simple cross-over?

Locally convex spaces

AMM 6029.* **by P. P. Carreras**

Let $E[t]$ be a linear space provided with a separated locally convex topology t. Show that $E[t]$ is bornological if and only if every absolutely convex bornivorous and algebraically closed subset of $E[t]$ is a t-neighborhood of the origin.

NAvW 471. **by D. van Dulst**

Give an example of a locally convex Hausdorff space that is separable but contains a nonseparable linear subspace.

Metric spaces

FUNCT 1.2.3.

If X is a Cartesian plane and, for all points $P, Q \in X$,

$$d(P, Q) = \begin{cases} 0, & \text{if } P = Q, \\ 1, & \text{if } P \neq Q, \end{cases}$$

verify that d is a metric on X. Describe the open balls $B((0, 0);\ 2)$ and $B((0, 0);\ 1/2)$ in this metric space. Verify that every subset of the metric space is open.

FUNCT 1.2.2.

If X is a Cartesian plane and, for all points $P, Q \in X$, $d(P, Q)$ is defined as $|x - u| + |y - v|$, where (x, y) are the coordinates of P and (u, v) those of Q, verify that d is a metric on X. Draw the open ball $B((0, 0);\ 1)$ in this metric space.

AMM S8. **by R. Johnsonbaugh**

Call a function f from a metric space (M, d) into itself a weak contraction map if whenever $x, y \in M$ with $x \neq y$, we have

$$d\left(f(x), f(y)\right) < d(x, y).$$

(a) Give an example of a weak contraction map on a complete metric space with no fixed point.

(b) Show that even on a compact metric space a weak contraction map need not be a contraction map; i.e., it need not satisfy $d\left(f(x), f(y)\right) \le c d(x, y)$ for $0 < c < 1$.

(c) Prove that a weak contraction map on a compact metric space has a unique fixed point.

AMM 6081. **by T. Šalát**

Let (X, d) be a metric space. We call $f\colon X \to \mathbb{R}$ quasicontinuous at x_0 if for each positive ε and δ there exists an open sphere

$$S(x_1, \delta_1) = \{x : d(x, x_1) < \delta_1\} \subset S(x_0, \delta),$$

such that

$$f\left[S(x_1, \delta_1)\right] \subset \left(f(x_0 - \varepsilon), f(x_0 + \varepsilon)\right).$$

Does there exist a metric space of first category and with no isolated points that allows a quasicontinuous function that is nowhere continuous?

AMM 6063. **by H. J. Marcum**

Let S be the set of all circles in the plane provided with the Hausdorff metric ρ induced by the usual Euclidean metric d, i.e.,

$$\rho(A, B) = \max\left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

where $d(a, B) = \inf\{d(a, b) \,|\, b \in B\}$ denotes the distance from the point a to the set B.

Let $z\colon S \to \mathbb{R}^2$ be the function that to each circle A assigns its center $z(A)$. Prove that $d(z(A), z(B)) \le \rho(A, B)$ for all $A, B \in S$.

AMM 6126. **by Harold Reiter**

Let X be a metric space, and let $(2^X, D)$ be the associated space of compact subsets, with the Hausdorff metric. Let S be a zero-dimensional collection of compact zero-dimensional sets. Prove or disprove that $\cup\{C \,|\, C \in S\}$ is zero-dimensional.

AMM S16. **by I. J. Schoenberg**

Characterize the closed sets S of the complex plane such that $d(z + w) \le d(z) + d(w)$ for all complex numbers z and w, where $d(z)$ denotes the Euclidean distance from z to S.

AMM 6275. **by S. Foldes and E. Howorka**

Let r be a metric on $\mathbb{R}^n$ giving the same topology as the usual Euclidean metric d. Let $I(r)$, $I(d)$ denote their groups of isometries. The following conjecture has not yet been settled: If $I(r)$ contains an isomorphic copy of $I(d)$, then $I(r) \cong I(d)$. Show that $I(d) \subseteq I(r)$ implies $I(r) = I(d)$.

Topology

AMM 6025. **by S. F. Wong and B. B. Winter**
Let (X, d) be a metric space, T an arbitrary subset of X, and t an arbitrary element of T. As usual,

$$d(t, A) = \inf \{d(t, a) \mid a \in A\}$$

is $-\infty$ if $A = \varnothing$; ∂T and T^c are, respectively, the boundary and the complement of T.

(a) Is it always true that $d(t, x) < d(t, \partial T)$ implies $x \in T$? If not, find a condition on (X, d) that is necessary and sufficient for the validity of this implication.

(b) Is it always true that $d(t, \partial T) = d(t, T^c)$? If not, find a condition on (X, d) that is necessary and sufficient for the validity of this equality.

Product spaces

AMM 6023. **by S. J. Sidney**
If for each k in the uncountable index set K, I_k denotes a copy of $[0, 1]$ and U_k denotes the copy of $(0, 1]$ contained therein, prove or disprove that $\prod_k U_k$ is a Borel set in the compact space $\prod_k I_k$.

Separation properties

AMM E2806. **by F. S. Cater**
AMM 6274. **by F. S. Cater**
Let S denote a topological space in which every compact set is closed, and let x and y be distinct points of S.

(a) Prove that x and y have disjoint neighborhoods if each of x and y has a countable local base.

(b) Show by example that x and y need not have disjoint neighborhoods if each element of S, other than x, has a countable local base.

Sets

AMM E2614. **by Frank Siwiec**
A set $A \subset \mathbb{R}^n$ is called a g-set if there is a countable family $\{U_n | n = 1, 2, \ldots\}$ of open sets containing A with the property that for each open set $G \supset A$, there is a U_n with $A \subset U_n \subset G$. Which subsets of $\mathbb{R}^n$ are g-sets?

AMM 6188. **by F. S. Cater**
Do there exist complementary subsets A and B of the set of irrational numbers such that for any open intervals I and J in the real line,

(a) $A \cap I$ and $B \cap J$ are not homeomorphic in the Euclidean topology;

(b) there is a one-to-one continuous function mapping $A \cap I$ onto $B \cap J$?

AMM 6014. **by C. H. Kimberling**
Does there exist an uncountable set of real numbers all of whose closed subsets are countable?

AMM 6261. **by Hugh Noland**
Let S be an uncountable set of real numbers, and let A be a countable subset of S. Must there exist an open set U, containing A, such that $S \setminus U$ is uncountable?

CRUX 59. **by John Thomas**
Find the shortest proof to the following proposition: every open subset of R is a countable disjoint union of open intervals.

AMM E2613. **by D. E. Knuth and the Mayagüez Problems Group**
Partition the real line $\mathbb{R}$ into a countable union of compact subsets.

PUTNAM 1975/B.4.
Does there exist a subset B of the unit circle $x^2+y^2 = 1$ such that

(1) B is topologically closed, and

(2) B contains exactly one point from each pair of diametrically opposite points on the circle?

Subspaces

PME 372. **by Sidney Penner**
Prove the following theorem: Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let f be a function from a subset of X_1 into X_2. The function f is continuous in the relative topology on its domain if and only if, for every $a \in \tau_2$, there exists $b \in \tau_1$ such that

(1) $\mathrm{Dom}\, f \cap b \subset f^{-1}(a)$, and

(2) If $c \subset a \cap \mathrm{Range}\, f$, then $f^{-1}(c) \subset \mathrm{Dom}\, f \cap b$.

AMM 6147. **by Richard Johnsonbaugh**
Can a normal, separable space possess a closed, uncountable, discrete subspace?

Surfaces

AMM 6141.* **by Dennis Johnson and Herbert Taylor**
Can the Borromean rings be drawn without crossing on a surface of genus 2?

AMM E2585. **by Jan Mycielski**
Prove that for every triangulation of a two-dimensional closed surface, the average number of edges meeting at a vertex approaches 6 in the limit as the number of triangles used approaches infinity.

Topological groups

AMM 6246. **by L. Washington and W. Parry**
Let G be a compact Hausdorff topological group. Show that the only group homomorphism (not assumed continuous) from G to the integers is the trivial one.

Topological vector spaces

AMM 6009. **by J. A. Goldstein**
Let X be a finite dimensional topological vector space whose topology is given by a metric d. Let T be a surjective isometry on X such that $T0 = 0$. If d is invariant, i.e., if

$$d(p, q) = d(p - q, 0)$$

for all $p, q \in X$, so that X is a Fréchet space, then T is necessarily linear. Must T still be linear if the assumption that d is invariant is dropped? What if $\dim X = 1$?

Unit interval

AMM E2768. **by Jim Fickett**
Is there a subset E of $[0, 1]$ with E and $[0, 1] \setminus E$ homeomorphic?

AMM 6282. **by David P. Robbins**
Exhibit a homeomorphism between the metric space of rational numbers r with $0 < r < 1$ and that of rationals t with $0 \le t \le 1$.

Trigonometry

Approximations

AMM E2693. **by Alexandru Lupaş**

Find a rational function

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials with integral coefficients of degree at most 6, which is a good approximation to $\arctan x$ on $[0,1]$. More precisely, we want

$$g(x) = \arctan x - f(x)$$

to satisfy $0 \le g(x) < \varepsilon$ for $x \in [0,1]$ and ε to be small (such approximations exist if $\varepsilon = 0.000033$).

FUNCT 1.2.7.

A very good approximate method of calculating $\sin x$ for x between 0 and $\pi/2$ is by means of the formula

$$\sin x \approx x(1 - 0.16605x^2 + 0.00761x^4).$$

Use a calculator or a computer to make your own table of $\sin x$, and compare it with published tables.

OSSMB G77.2-4.

Show that $\tan \pi/10$ is a root of the equation

$$5x^4 - 10x^2 + 1 = 0.$$

Hence calculate $\tan \pi/10$ to two decimal places.

Calculator problems

NYSMTJ 55. **by Bruce King**

Suppose you are using a calculator and need to find $\tan^{-1} x$, but the calculator will give only $\sin^{-1} x$ and $\cos^{-1} x$. How can you find $\tan^{-1} x$?

Determinants

AMM E2589. **by Joe Sunday**

Let $d_1, \ldots, d_n$ be distinct integers > 1. If

$$a_{ij} = \sin^2\left(\frac{j\pi}{d_i}\right)$$

for $1 \le i, j \le n$, show that $\det(a_{ij}) \ne 0$.

CRUX 462. **by Hippolyte Charles**

Let A, B, and C be the angles of a triangle. Show that

$$\begin{vmatrix} \tan\frac{A}{2} & \cos A & 1 \\ \tan\frac{B}{2} & \cos B & 1 \\ \tan\frac{C}{2} & \cos C & 1 \end{vmatrix} = 0.$$

Fallacies

FUNCT 2.3.4.

Spot the fallacy: Since

$$\cos^2 x = 1 - \sin^2 x,$$

it follows that

$$1 + \cos x = 1 + (1 - \sin^2 x)^{\frac{1}{2}};$$

that is,

$$(1 + \cos x)^2 = \left\{1 + (1 - \sin^2 x)^{\frac{1}{2}}\right\}^2.$$

In particular, when $x = \pi$, we have

$$(1-1)^2 = \left\{1 + (1-0)^{\frac{1}{2}}\right\}^2,$$

$$0 = (1+1)^2 = 4.$$

Identities: constraints

CRUX 234. **by Viktors Linis**

If $\sin\frac{2^n\pi}{13} = \pm\sin\frac{\pi}{13}$, prove that

$$\cos\frac{\pi}{13}\cos\frac{2\pi}{13}\cos\frac{4\pi}{13}\cdots\cos\frac{2^{n-1}\pi}{13} = \pm\frac{1}{2^n}$$

CRUX 103. **by H. G. Dworschak**

If

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = 1,$$

prove that

$$\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} = 1.$$

DELTA 5.1-1. **by R. S. Luthar**

If $\cos\theta + \cos\phi + \cos\psi = \sin\theta + \sin\phi + \sin\psi = 0$, evaluate

$$\cos 3(\theta - \phi) + \cos 3(\phi - \psi) + \cos 3(\psi - \theta).$$

OSSMB G76.2-3.

(a) Given that

$$(1 + \sin A)(1 + \sin B)(1 + \sin C) = \cos A \cos B \cos C \ne 0,$$

evaluate

$$(1 - \sin A)(1 - \sin B)(1 - \sin C).$$

(b) Given that in $\triangle ABC$

$$(a + b + c)(b + c - a) = 3bc,$$

find $\angle A$.

TYCMJ 120. **by K. R. S. Sastry**

Let $\sin A + \cos B = P$ and $\cos A + \sin B = Q$, where P and Q are not both zero and $P^2 + Q^2 \le 4$. Express in terms of P and Q the values of

(a) $\sin(A + B)$,
(b) $\cos(A + B)$,
(c) $\sin(A - B)$, and
(d) $\cos(A - B)$.

OSSMB G79.2-5.

Given $A = \tan^{-1}\frac{1}{7}$, $B = \tan^{-1}\frac{1}{3}$ (A, B acute), show that $\cos 2A = \sin 4B$.

OSSMB G78.1-6.
If $\tan A$ and $\tan B$ are the roots of the equation

$$x^2 + cx + d = 0,$$

show that

$$\sin^2(A+B) + c\sin(A+B)\cos(A+B) + d\cos^2(A+B) = d.$$

Identities: cos

PME 397. **by J. S. Frame**
If $c_j = 2\cos(j\pi/n)$, prove that

$$\prod_{j=1}^{n}(1+3c_j^4) = (3^n - 3^{n/2}\cdot 2\cos(5\pi n/6)+1)^2,$$

and more generally that

$$\prod_{j=1}^{n}(t^4+c_j^4) = (x^n + x^{-n} - z^n - z^{-n})^2 = F_n^2(t),$$

where $F_n(t)/F_1(t)$ is a polynomial in t^2 with integral coefficients, and

$$x = u\bar{u} \ge 1, \quad z = u/\bar{u}, \quad \text{and} \quad u + u^{-1} = te^{\pi i/4}.$$

Identities: inverse trigonometric functions

MATYC 132. **by Warren Page**
Show that

$$\csc^{-1}(\sqrt{n+2}) + \sec^{-1}(\sqrt{n+1}) + \tan^{-1}(\sqrt{n+1}) - \tan^{-1}(\sqrt{n}) = 2\left[\cot^{-1}(2) + \cot^{-1}(3)\right]$$

for every natural number n.

PME 399. **by Jack Garfunkel**
Show that

$$\arcsin\left(\frac{x-3}{3}\right) + 2\arccos\sqrt{\frac{x}{6}} = \frac{\pi}{2}\ , \qquad (3 \le x \le 6).$$

Identities: multiple angles

OSSMB 76-9.
If f denotes the function that gives $\cos 17x$ in terms of $\cos x$, that is,

$$\cos 17x = f(\cos x),$$

show that

$$\sin 17x = f(\sin x).$$

Identities: sin

OSSMB 78-8. **by Neal E. Reid**
Let $\theta_n = \pi/(n+1)$. Prove that, for any positive integer n,

$$\sin\theta_n \cdot \sin 2\theta_n \cdot \sin 3\theta_n \cdot \cdots \cdot \sin n\theta_n = \frac{\sqrt{2n+1}}{2^n}.$$

OSSMB G79.1-4.
(a) Prove that for all A and B,

$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B.$$

(b) Prove that for any triangle ABC,

$$(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0.$$

Identities: sin and cos

OMG 17.3.8.
Given that $\cos\theta = A\cos^3\theta + B\cos\theta$ holds for every real number θ, determine the values of A and B.

OMG 18.1.6.
Prove the identity

$$(1+\tan A + \cot A)^2 = \frac{(\sin A\cdot\cos A + 1)^2}{\sin^2 A\cdot\cos^2 A}\ .$$

Identities: tan

CRUX 222. **by Bruce McColl**
Prove that

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11} = \sqrt{11}.$$

ISMJ 11.5.
Prove that

$$\tan\frac{\pi}{20} - \tan\frac{3\pi}{20} + \tan\frac{5\pi}{20} - \tan\frac{7\pi}{20} + \tan\frac{9\pi}{20} = 5.$$

TYCMJ 128. **by Mangho Ahuja**
Prove that

$$\tan\frac{\pi}{14}\left(\cos\frac{\pi}{14} + \cos\frac{3\pi}{14} + \cos\frac{5\pi}{14}\right) = \frac{1}{2}\ .$$

Inequalities: cos

SIAM 77-19. **by P. Barrucand**
Let

$$F_1(\theta) = \sum_{n=1}^{\infty}\frac{\cos^n\theta\cos n\theta - \cos^{2n}\theta}{n\,(1-2\cos^n\theta\cos n\theta + \cos^{2n}\theta)}\ ,$$

$$F_2(\theta) = \sum_{\substack{n=1\\ n\equiv 1 \pmod 2}}^{\infty}\frac{\cos^n\theta\cos n\theta - \cos^{2n}\theta}{n\,(1-2\cos^n\theta\cos n\theta + \cos^{2n}\theta)}\ .$$

It is conjectured that $F_1(\theta)$ and $F_2(\theta)$ are negative for

$$0 < \theta < \frac{\pi}{2}\ .$$

Inequalities: Huygens

CRUX 115. **by Viktors Linis**
Prove the following inequality of Huygens:

$$2\sin\alpha + \tan\alpha \ge 3\alpha, \quad 0 \le \alpha < \frac{\pi}{2}.$$

CRUX 167. **by Léo Sauvé**
Prove that

$$\alpha > \frac{3\sin\alpha}{2+\cos\alpha}$$

for $0 < \alpha < \pi/2$.

CRUX 303. **by Viktors Linis**
Prove that

$$2\sinh x + \tanh x \ge 3x, \quad x \ge 0.$$

Trigonometry

Inequalities: sin

AMM E2720. **by Ralph P. Boas**
Show that $\sin^2 x < \sin\left(x^2\right)$ for $0 < x \le (\pi/2)^{1/2}$.

CRUX 306. **by Irwin Kaufman**
Solve the following inequality:

$$\sin x \sin 3x > \frac{1}{4}.$$

PUTNAM 1978/A.5.
Let $0 < x_i < \pi$ for $i = 1, 2, \ldots, n$ and set

$$x = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Prove that

$$\prod_{i=1}^{n} \frac{\sin x_i}{x_i} \le \left(\frac{\sin x}{x}\right)^n.$$

Inequalities: sin and cos

CRUX 36. **by Léo Sauvé**
If m and n are positive integers, show that

$$\sin^{2m}\theta \cos^{2n}\theta \le \frac{m^m n^n}{(m+n)^{m+n}},$$

and determine the values of θ for equality to hold.

IMO 1977/4.
Four real constants a, b, A and B are given, and

$$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta.$$

Prove that if $f(\theta) \ge 0$ for all real θ, then

$$a^2 + b^2 \le 2 \quad \text{and} \quad \mathrm{A}^2 + \mathrm{B}^2 \le 1.$$

Inequalities: sin and tan

MM 1082. **by C. S. Gardner**
Prove that $\tan\sin x > \sin\tan x$ for $0 < x < \pi/2$.

Inequalities: tan

NAvW 521. **by M. E. Muldoon**
Prove that

$$\frac{\tan t}{t} < 2 - \left(1 - t^2\right)^{\frac{1}{2}}, \qquad 0 < t \le 1.$$

Inequalities: tan and cot

OSSMB G76.1-4.
Show that $\tan 3a \cot a$ cannot lie between 1/3 and 3.

Inequalities: tan and sec

AMM E2739. **by Marvin C. Papenfuss**
Prove that

$$x\sec^2 x - \tan x \le \frac{8\pi^2 x^3}{\left(\pi^2 - 4x^2\right)^2}, \qquad 0 \le x < \frac{\pi}{2}.$$

MM Q652. **by Murray S. Klamkin**
Show that

$$\sum_{i=1}^{n}(1 + \tan\alpha_i) \le \sqrt{2}\sum_{i=1}^{n}\sec\alpha_i$$

when $\sec\alpha_i > 0$. When does equality hold?

Infinite products

PARAB 425.
Show that $2\cos x + 1 = 4\cos^2 \frac{1}{2}x - 1$. Find

$$\lim_{n\to\infty}\left(2\cos\frac{x}{2} - 1\right)\left(2\cos\frac{x}{2^2} - 1\right)\cdots\left(2\cos\frac{x}{2^n} - 1\right).$$

OSSMB 75-17.
Prove that

$$\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\cdot\cos\frac{x}{16}\cdots = \frac{\sin x}{x}.$$

Infinite series

MM Q629. **by Norman Schaumberger**
Show that

$$\sum_{k=1}^{\infty}\tan^{-1}\frac{1}{2k^2} = \frac{\pi}{4}.$$

DELTA 5.2-3. **by Charles R. McConnell**
Show that

$$-\frac{1}{2}\log(1 - 2x\cos 3\theta + x^2) = x\cos 3\theta + \frac{x^2\cos 6\theta}{2} + \frac{x^3\cos 9\theta}{2} + \cdots.$$

For what real values of x and θ is this equation valid?

AMM 6241. **by Robert Baillie**
Prove that

$$\sum_{n=1}^{\infty}\frac{\sin(n)}{n} = \sum_{n=1}^{\infty}\left(\frac{\sin(n)}{n}\right)^2 = \frac{\pi - 1}{2}$$

and

$$\sum_{n=1}^{\infty}\frac{\sin^2(n)}{n^4} = \frac{(\pi-1)^2}{6}.$$

PME 363. **by Robert C. Gebhardt**
Does $\sum_{k=1}^{\infty}\frac{\sin k}{k}$ converge, and if so, to what?

CRUX 235. **by Viktors Linis**
Prove Gauss' *Theorema Elegantissimum:* If

$$f(x) = 1 + \frac{1}{2}\cdot\frac{1}{2}x^2 + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}x^4 + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{5}{6}x^6 + \cdots,$$

show that

$$\sin\phi f(\sin\phi)f'(\cos\phi) + \cos\phi f(\cos\phi)f'(\sin\phi) = \frac{2}{\pi\sin\phi\cos\phi}.$$

MM 1039. **by M. B. Gregory and J. M. Metzger**
Evaluate

$$\sum_{k=1}^{\infty}\frac{1}{k^2}\tan\frac{k\pi}{m}\tan\frac{k\pi}{n}.$$

TYCMJ 63. **by Norman Schaumberger**
For which values of k will $\sum_{n=1}^{\infty}\tan^k(1/n)$ converge?

SIAM 77-18. **by A. M. Liebetrau**

Show that

$$\sum_{j=1}^{\infty} \alpha_j^{-6} \left[\frac{\sin\alpha_j - \sinh\alpha_j}{\cos\alpha_j + \cosh\alpha_j} \right]^2 = \frac{1}{80},$$

where the α_j's are the positive solutions to the equation

$$(\cos\alpha)(\cosh\alpha) + 1 = 0.$$

Numerical evaluations

OSSMB 76-18.

Evaluate $\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ$.

SSM 3702. **by Tony To**

Prove:

$$(\cos 54^\circ + \cos 18^\circ) \tan 18^\circ = \sin 30^\circ.$$

CRUX 305. **by Bruce McColl**

How many distinct values does $\cos(\frac{1}{3}\sin^{-1}\alpha)$ have? What is the product of these values?

PENT 317. **by John A. Winterink**

Arc ABC of a circle has a measure of 150° and its center is at D. If $AB = 3$ and $BC = 2$, what is the value of $\cot(\frac{1}{2}\angle BDC)$?

OSSMB G79.3-2.

From the top of a hill, the angle of depression of a point D on the level plain below is 30°, and from the point three quarters of the way down the hill, the angle of depression of D is 15°. Find the tangent of the angle of inclination of the hill.

Recurrences

FQ B-308. **by Philip Mana**

(a) Let $c_n = \cos(n\theta)$ and find the integers a and b such that $c_n = ac_{n-1} + bc_{n-2}$ for $n = 2, 3, \ldots$.

(b) Let r be a real number such that $\cos(r\pi) = p/q$, with p and q relatively prime positive integers and q not in $\{1, 2, 4, 8\}$. Prove that r is not rational.

Series

MM 1036. **by Joseph Silverman**

If $a_0, a_1, \ldots, a_N$ are complex numbers such that

$$|a_N| > \sum_{k=0}^{N-1} |a_k|,$$

show that

$$\sum_{n=0}^{N} a_n \cos n\theta = 0$$

has at least $2N$ solutions for $0 \le \theta < 2\pi$.

Solution of equations: arctan

JRM 789. **by Hans Havermann**

Let $\theta_n = \sum_{i=1}^{n} \tan^{-1}\left(1/\sqrt{i}\right)$. Do there exist three positive integers (p, q, r) such that $\theta_p = \theta_q + 2\pi r$?

Solution of equations: sin and cos

CRUX 369. **by Hippolyte Charles**

Find all real solutions of the equation

$$\sin(\pi\cos x) = \cos(\pi\sin x).$$

MSJ 493.

Find all real values of x that satisfy the equation

$$\sin^{10} x + \cos^{10} x = 29/64.$$

NYSMTJ 69.
by 11th year Honors Class at Benjamin N. Cardozo H.S.

Find $\sin 2\theta$ if $\sin^6\theta + \cos^6\theta = 2/3$.

OSSMB G78.3-5.

(a) A statue standing on top of a 25 foot pillar subtends an angle θ whose tangent is .125 at a point 60 feet from the foot of the pillar. Find the height of the statue.

(b) Determine a and b such that

$$\frac{-3 + 4\cos^2\theta}{1 - 2\sin\theta} = a + b\sin\theta.$$

OSSMB G79.3-1.

Find all values of x that satisfy the equation

$$\sin x + 2\sin x\cos(a - x) = \sin a$$

where a is a real constant.

SSM 3715. **by Herta T. Freitag**

Find a positive integer n such that $\cos n^2$ equals $2\cos n \sin 4n$, where degree measure is being used.

MSJ 452. **by Steven R. Conrad**

Find the degree-measure of the least positive angle that satisfies

$$\sin 6x + \cos 4x = 0.$$

MATYC 120. **by Marc Glucksman**

Find the condition(s) for which the equation

$$a\sin\theta + b\cos\theta = c,$$

$0 \le \theta < 2\pi$, has exactly one root.

PME 385. **by John T. Hurt**

Find all α and β such that

$$\sin\alpha = \tan(\alpha - \beta) + \cos\alpha\tan\beta.$$

Solution of equations: tan

TYCMJ 125. **by Milton H. Hoehn**

Determine all values of $x \in (0, \pi)$ that satisfy

$$\tan x = \tan 2x \tan 3x \tan 4x.$$

PME 418. **by Robert C. Gebhardt**

Find all angles θ other than zero such that

$$\tan 11\theta = \tan 111\theta = \tan 1111\theta = \tan 11111\theta = \cdots .$$

Solution of equations: tan and sec

NYSMTJ 78. **by Norman Schaumberger**

Find all θ such that

$$(\tan\theta + \sec\theta)^{1/3} + (\tan\theta - \sec\theta)^{1/3} = 1.$$

Systems of equations

TYCMJ 115. **by Thomas E. Elsner**

Find all positive integer coefficients $A \neq B$ for which the system

$$\cos At + \cos Bt = 0$$
$$A \sin At + B \sin Bt = 0$$

has solutions.

Triangles

NYSMTJ 82. **by Madelaine Bates**

Show that the area of triangle ABC is numerically equal to its perimeter if and only if

$$a + b - c = 4(\cot C + \csc C).$$

NYSMTJ 97. **by Norman Schaumberger**

If A, B, and C are the angles of a triangle and K is its area, show that

$$a^2 + b^2 + c^2 = 4K(\cot A + \cot B + \cot C).$$

TYCMJ 143. **by K. R. S. Sastry**

Let AD, BE, and CF be the medians of triangle ABC. Prove that

$$\cot \angle DAB + \cot \angle EBC + \cot \angle FCA = 3(\cot A + \cot B + \cot C).$$

OSSMB G75.1-6.

Given the triangle ABC, where $a + b = 2c$, show that

$$\cot \frac{A}{2} + \cot \frac{B}{2} = 2 \cot \frac{C}{2}.$$

SSM 3786. **by Fred A. Miller**

If D is the midpoint of side BC of triangle ABC, the measure of angle BAD is θ, and the measure of angle CAD is ϕ, show that

$$\cot \theta - \cot \phi = \cot B - \cot C.$$

NYSMTJ 100. **by Bertram Kabak**

If A, B, and C are the angles of a triangle and

$$\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c},$$

show that the triangle is equilateral.

CRUX 423. **by Jack Garfunkel**

In a triangle ABC whose circumcircle has unit diameter, let m_a and t_a denote the lengths of the median and the internal angle bisector to side a, respectively. Prove that

$$t_a \leq \cos^2 \frac{A}{2} \cos \frac{B-C}{2} \leq m_a.$$

TYCMJ 49. **by Alan Wayne**

In triangle ABC, determine the maximum value of

$$\frac{\sin A + \sin B + \sin C}{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}}.$$

TYCMJ 118. **by Norman Gore**

Triangles ABC and DEF are inscribed in the same circle. Prove that

$$\sin A + \sin B + \sin C = \sin D + \sin E + \sin F$$

if and only if the perimeters of the triangles are equal.

TYCMJ 72. **by Clyde A. Bridger**

In the triangle ABC with sides $a > b > c$, prove or disprove that

$$\left(\frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b}\right)\left(\frac{c}{a-b} + \frac{a}{b-c} + \frac{b}{c-a}\right) = 1 - 8 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right).$$

OSSMB G77.1-4.

The bisectors of the interior angles of $\triangle ABC$ make angles of α, β, γ with the sides a, b, c respectively. Prove that

$$a \sin 2\alpha + b \sin 2\beta + c \sin 2\gamma = 0.$$

TYCMJ 109. **by Bertram Kabak**

Let K denote the area and R the circumradius of triangle ABC with angle $A \neq \pi/3$. Prove that

$$K = \frac{4R^2(\sin^2 A + \sin B \sin C) - b^2 - c^2}{2 \csc A - 4 \cot A}.$$

CRUX 126. **by Viktors Linis**

Show that, for any triangle ABC,

$$|OA|^2 \sin A + |OB|^2 \sin B + |OC|^2 \sin C = 2K,$$

where O is the center of the inscribed circle and K is the area of $\triangle ABC$.

CRUX 27. **by Léo Sauvé**

Given a triangle with angles A, B, and C, it is easy to verify that if $A = B = 45°$, then

$$\cos A \cos B + \sin A \sin B \sin C = 1.$$

Does the converse proposition hold?

NYSMTJ 80. **by Bertram Kabak**

If A, B, and C are the angles of a triangle, show that

$$\sin^2 C = \sin^2 A + \sin^2 B - 2 \sin A \sin B \cos C.$$

OSSMB G76.2-4.

Show that in any triangle ABC,

$$(b-c) \cos \frac{A}{2} = a \sin \frac{B-C}{2}.$$

TYCMJ 47. **by Bertram Kabak**

Prove that if $\sum_{i=1}^{3} A_i = \pi$, then

$$\sum_{i=1}^{3} \sin^2 A_i = 2 \sum_{i=1}^{3} \sin A_i \cdot \sin A_{i+1} \cdot \cos A_{i+2}$$

$(A_4 = A_1, A_5 = A_2)$.

SSM 3734. **by Ralph King and Joseph Stangle**

Given two sides a and b and their included angle C in triangle ABC, prove that

$$\tan A = \frac{a\sin C}{b - a\cos C}\ .$$

CRUX 268. **by Gali Salvatore**

Show that in $\triangle ABC$, with $a \geq b \geq c$, the sides are in arithmetic progression if and only if

$$2\cot\frac{B}{2} = 3(\tan\frac{C}{2} + \tan\frac{A}{2}).$$

CRUX 493.* **by R. C. Lyness**

(a) Let A, B, and C be the angles of a triangle. Prove that there are positive x, y, and z, each less than 1/2, simultaneously satisfying

$$y^2\cot\frac{B}{2} + 2yz + z^2\cot\frac{C}{2} = \sin A,$$
$$z^2\cot\frac{C}{2} + 2zx + x^2\cot\frac{A}{2} = \sin B,$$
$$x^2\cot\frac{A}{2} + 2xy + y^2\cot\frac{B}{2} = \sin C.$$

(b) In fact, 1/2 may be replaced by a smaller $k > 0.4$. What is the least value of k?

OSSMB G76.1-3.

A triangle ABC is such that $3AB = 2AC$. A point D on BC is such that $BD = 2DC$ and $AD = BC$. Show that

$$\tan\frac{\angle ADB}{2} = \sqrt{\frac{5}{19}}.$$

MM Q636. **by Richard L. Francis**

Does there exist a triangle such that the tangents of its angles are of the form x, $1 + x$, and $1 - x$?

OMG 18.2.5.

In any triangle ABC,

(a) prove

$$\frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2} = \frac{\tan B}{\tan C}\ .$$

(b) If $a : b : c = 4 : 5 : 6$, find $\tan A : \tan B$.

(c) If $\tan B = 2\tan C$, prove $a^2 + 3c^2 = 3b^2$.

SSM 3740. **by Fred A. Miller**

Let ABC be a triangle having a right angle at C. Construct a perpendicular to AB at A meeting line BC at E. Also construct a perpendicular to AB at B meeting line AC at D. Prove that

(a) the tangent of angle CED is equal to the cube of the tangent of angle BAC and

(b) the area of triangle ECD is equal to the area of triangle ABC.

PENT 293. **by Kenneth M. Wilke**

On a trigonometry test, one question asked for the largest angle of the triangle having sides 21, 41, and 50. L. A. Z. Thinker, a student, obtained the answer as follows:

Let C denote the desired angle, then

$$\sin C = \frac{50}{41} = 1.2195.$$

But $\sin 90° = 1$ and $.2195 = \sin 12°40'48''$. Therefore $C = 90° + 12°40'48'' = 102°40'48''$ which is correct.

Find another triangle having this property that is not similar to the given triangle.

PROBLEM CHRONOLOGY

Use this section to

- determine where a given problem was originally published
- determine where to find the solution
- find all references to a given problem from a specific journal.

We list every problem or solution that was published during the years 1975–1979 in a journal problem column that is covered by this index. For each proposed problem, we list the volume and page number where the proposal can be found. The list is sorted first by journal abbreviation, and then by problem number within that journal. If a journal has more than one problem column, the problems in each column are grouped together. The page reference is presented in the form:

vol(year)page

or

vol(year/issue)page

where	vol	is the volume number (if known),
	year	is the year of the volume,
	issue	is the issue number in which the problem appears, and
	page	is the page number where the problem appears.

The issue number will be included if the magazine numbers each issue beginning with page number 1 (instead of consecutively numbering throughout the year).

For each problem that was proposed during the years 1975–1979, we list references (in the problem columns) to all corrections, comments, and solutions to this problem. The page number reference is followed by a single character code that describes the nature of the reference. The codes are explained in the following table:

Code	Description
a	acknowledgment (out of order solvers list)
c	comment
r	reprint of a previously published problem
s	solution
v	version (correction to original problem proposal)
w	problem has been withdrawn
x	partial solution

Lists of solvers names are not usually referenced unless they appear out of order from their usual place of appearance in the journal, either immediately after the solution or at the end of the problem column in which the solution is published. This might happen for late solutions. The code "s" appears for a generalization as well as a normal solution. If a problem has multiple parts, sometimes solutions by different authors are given for each part. These are still considered solutions (code "s") as opposed to partial solutions (code "x"). Code "x" is reserved for the case where a complete solution is not known (at the time of publication), and a partial solution is being printed.

For a given problem, the references appear in chronological order. When making bibliographic references to these solutions, you should go back to the original source to get the complete list of page numbers. We have attempted to locate all references to problems published during the selected years, even if these references occurred later than 1979. All journals through January 1992 have been searched for references to problems that were originally published prior to 1980. We have also given the page numbers for all solutions or comments published during 1975–1979, even if the original problem was first published prior to 1975.

To find solutions to contest problems, consult the Citation Index (page 423).

JOURNALS COVERED BY THE CHRONOLOGY

Abbreviation	Name
AMM	The American Mathematical Monthly
CMB	Canadian Mathematical Bulletin
CRUX	Crux Mathematicorum
DELTA	Delta
FQ	The Fibonacci Quarterly
FUNCT	Function
ISMJ	Indiana State Mathematics Journal
JRM	Journal of Recreational Mathematics
MATYC	The MATYC Journal
MENEMUI	Menemui Matematik
MM	Mathematics Magazine
MSJ	The Mathematics Student Journal
NAvW	Nieuw Archief voor Wiskunde
NYSMTJ	The New York State Mathematics Teachers' Journal
OMG	Ontario Mathematics Gazette
OSSMB	Ontario Secondary School Mathematics Bulletin
PARAB	Parabola
PENT	The Pentagon
PME	The Pi Mu Epsilon Journal
SIAM	SIAM Review
SPECT	Mathematical Spectrum
SSM	School Science and Mathematics
TYCMJ	The Two-Year College Mathematics Journal

Notes: DELTA merged into MM.

AMM

Problem	Proposal	References
2797		88(1981)149v
3189		96(1989)260s
3834		85(1978)836c
3887		90(1983)486s 94(1987)1019c
3951		85(1978)836c
4003		85(1978)836c
4052		82(1975)1016s 85(1978)836c
4306		85(1978)836c
4444		85(1978)836c
4538		85(1978)836c
4555		85(1978)836c
4603		97(1990)937s
4638		85(1978)836c
4664		85(1978)836c
4744		85(1978)836c
5124		83(1976)662s 85(1978)836c
5297		88(1981)295s
5314		82(1975)672s 85(1978)836c
5385		83(1976)662s 85(1978)836c
5405		82(1975)1017s 85(1978)836c
5413		82(1975)85c 85(1978)836c 89(1982)279c
5415		85(1978)836c
5427		82(1975)673s 85(1978)836c
5437		83(1976)818s 85(1978)836c
5499		87(1980)65s
5540		90(1983)135s
5575		82(1975)674s 85(1978)836c
5589		83(1976)141s 85(1978)836c
5608		85(1978)500s, 836c
5643		82(1975)677s 85(1978)836c
5670		82(1975)677s 85(1978)836c
5687		82(1975)767c, 767s 83(1976)572v 85(1978)836c
5723		83(1976)62s, 64s 85(1978)836c
5735		97(1990)937s
5773		82(1975)943s 85(1978)836c
5790		89(1982)215s
5794		88(1981)214s
5861		82(1975)767x 88(1981)150s
5871		82(1975)530x 83(1976)573c 85(1978)829s
5872		85(1978)834c 90(1983)403s
5878		82(1975)678c
5880		82(1975)857s
5881		85(1978)834c
5884		85(1978)834c 87(1980)66s
5888		85(1978)834c 87(1980)583s
5889		85(1978)834c
5893		85(1978)835c
5895		82(1975)531s
5897		82(1975)532s
5910		85(1978)835c
5917		85(1978)835c
5927		85(1978)835c
5931		82(1975)86s
5932		82(1975)86s
5933		82(1975)87s, 768c
5934		82(1975)184s
5935		82(1975)88s
5936		82(1975)185s 83(1976)65s
5937		82(1975)308s
5938		82(1975)185s
5939		82(1975)533s
5940		82(1975)186s
5941		82(1975)309s
5942		82(1975)186s, 768s
5943		82(1975)309s
5944		82(1975)310s
5945		82(1975)410s
5946		82(1975)310s
5947		82(1975)411s
5948		82(1975)413s
5949		82(1975)536s
5950		82(1975)414s
5951		82(1975)679c
5952		82(1975)679s, 680s 83(1976)819c
5953		83(1976)574s
5954		85(1978)835c
5955		82(1975)415s
5956		85(1978)835c
5957		82(1975)770s
5958		82(1975)858s
5959		82(1975)859s
5960		82(1975)859s
5961		82(1975)861s
5962		82(1975)943s
5963		82(1975)861s
5964		82(1975)944c
5965		82(1975)944s
5966		82(1975)945s
5967		82(1975)1018s
5968		82(1975)1020s
5969		82(1975)1020s
5970		83(1976)65s
5971		83(1976)66s
5972		83(1976)67s
5973		83(1976)142c, 142x
5974		83(1976)143s
5975		83(1976)144s
5976		83(1976)145s
5977		83(1976)206s
5978		83(1976)206s
5979		83(1976)207s
5980		85(1978)835c 88(1981)540v
5981		83(1976)209s
5982		83(1976)209s
5983		83(1976)293c, 293x, 294c
5984		83(1976)294s
5985		83(1976)295s
5986		83(1976)295s
5987		83(1976)297s
5988		83(1976)386s
5989		83(1976)749c
5990		83(1976)387s
5991		85(1978)835c
5992		83(1976)388s
5993		83(1976)749s
5994		83(1976)389s
5995		83(1976)390s
5996		85(1978)283s
5997		83(1976)490s
5998		83(1976)491s
5999		83(1976)492s 84(1977)62s
6000		83(1976)492s
6001		83(1976)493c
6002		83(1976)494s
6003		83(1976)575s
6004		83(1976)575s
6005		85(1978)835c
6006	82(1975)84	83(1976)576s
6007	82(1975)84	83(1976)663s
6008	82(1975)84	83(1976)664s
6009	82(1975)84	83(1976)666s
6010	82(1975)84	83(1976)666s, 667s
6011	82(1975)85	83(1976)750s
6012	82(1975)183	83(1976)751s
6013	82(1975)183	83(1976)752s
6014	82(1975)183	83(1976)752s
6015	82(1975)183	83(1976)820s
6016	82(1975)184	83(1976)820c
6017	82(1975)184	83(1976)821s
6018	82(1975)307	83(1976)821s

6019	82(1975)307	84(1977)63s
6020	82(1975)307	84(1977)65c
6021	82(1975)307	84(1977)66s
6022	82(1975)308	84(1977)67s
6023	82(1975)308	84(1977)67s 90(1983)136s, 487c
6024	82(1975)409	85(1978)835c 87(1980)68s
6025	82(1975)409	84(1977)141s
6026	82(1975)409	84(1977)142s
6027	82(1975)409	84(1977)143s
6028	82(1975)410	85(1978)835c
6029	82(1975)410	85(1978)835c
6030	82(1975)528	84(1977)143s
6031	82(1975)528	84(1977)144s
6032	82(1975)529	84(1977)222s
6033	82(1975)529	84(1977)223s
6034	82(1975)529	84(1977)224s
6035	82(1975)529	84(1977)225s
6036	82(1975)671	84(1977)226s 85(1978)830s
6037	82(1975)671	84(1977)226s
6038	82(1975)671	84(1977)301s
6039	82(1975)671	84(1977)301s
6040	82(1975)672	84(1977)302s
6041	82(1975)672	84(1977)302s
6042	82(1975)766	84(1977)303s
6043	82(1975)766	84(1977)304s 92(1985)363s
6044	82(1975)766	84(1977)392s 87(1980)583c
6045	82(1975)766	84(1977)394s
6046	82(1975)766	84(1977)395s
6047	82(1975)766	84(1977)396s
6048	82(1975)856	84(1977)397c
6049	82(1975)856	84(1977)397s
6050	82(1975)856	85(1978)687s
6051	82(1975)857	84(1977)492x
6052	82(1975)857	84(1977)492s 87(1980)68s
6053	82(1975)857	84(1977)493s
6054	82(1975)941	84(1977)494s
6055	82(1975)942	85(1978)600s
6056	82(1975)942	84(1977)494s
6057	82(1975)942	84(1977)495s, 496s 94(1987)1020c
6058	82(1975)942	84(1977)576s
6059	82(1975)942	84(1977)577s
6060	82(1975)1016	85(1978)390x
6061	82(1975)1016	84(1977)577s
6062	82(1975)1016	84(1977)578x 88(1981)152s 89(1982)503c
6063	82(1975)1016	84(1977)579s
6064	82(1975)1016	84(1977)580s
6065	82(1975)1016	84(1977)580s
6066	83(1976)62	84(1977)660s
6067	83(1976)62	84(1977)661s
6068	83(1976)62	84(1977)661s
6069	83(1976)62	84(1977)662s
6070	83(1976)62	84(1977)662s
6071	83(1976)62	83(1976)572v 84(1977)663s
6072	83(1976)140	84(1977)744s
6073	83(1976)140	84(1977)745s
6074	83(1976)140	84(1977)746s
6075	83(1976)141	84(1977)747s
6076	83(1976)141	85(1978)835c 88(1981)152s
6077	83(1976)141	84(1977)747s
6078	83(1976)205	84(1977)829s, 830c
6079	83(1976)205	84(1977)748s
6080	83(1976)205	85(1978)503s, 503x
6081	83(1976)205	84(1977)830s
6082	83(1976)205	85(1978)503s 86(1979)597c
6083	83(1976)205	84(1977)830s
6084	83(1976)292	84(1977)832s
6085	83(1976)292	84(1977)833s
6086	83(1976)292	85(1978)54s
6087	83(1976)293	85(1978)284s
6088	83(1976)293	85(1978)55s
6089	83(1976)293	87(1980)495c
6090	83(1976)385	85(1978)122s
6091	83(1976)385	85(1978)55s
6092	83(1976)385	85(1978)123s
6093	83(1976)386	85(1978)56s
6094	83(1976)386	85(1978)57s
6095	83(1976)386	85(1978)59s
6096	83(1976)489	85(1978)124s
6097	83(1976)489	85(1978)286s
6098	83(1976)489	85(1978)125s
6099	83(1976)489	85(1978)204s
6100	83(1976)490	85(1978)205s
6101	83(1976)490	85(1978)126s
6102	83(1976)572	85(1978)504s
6103	83(1976)572	85(1978)205s
6104	83(1976)573	85(1978)206s
6105	83(1976)573	85(1978)207s
6106	83(1976)573	85(1978)208s
6107	83(1976)573	85(1978)287s
6108	83(1976)661	85(1978)289s
6109	83(1976)661	85(1978)291s
6110	83(1976)661	
6111	83(1976)661	85(1978)390s
6112	83(1976)661	85(1978)391s
6113	83(1976)661	85(1978)392s
6114	83(1976)748	85(1978)392s
6115	83(1976)748	85(1978)393s
6116	83(1976)748	85(1978)505s
6117	83(1976)748	85(1978)505s
6118	83(1976)748	85(1978)506s
6119	83(1976)748	
6120	83(1976)817	85(1978)601s
6121	83(1976)817	85(1978)602s
6122	83(1976)817	85(1978)603s
6123	83(1976)817	
6124	83(1976)818	
6125	83(1976)818	87(1980)495s 89(1982)503c 91(1984)60s
6126	84(1977)61	85(1978)604s
6127	84(1977)62	85(1978)604s
6128	84(1977)62	85(1978)688s 86(1979)597s
6129	84(1977)62	85(1978)689s
6130	84(1977)62	85(1978)689s
6131	84(1977)62	85(1978)690s
6132	84(1977)140	85(1978)690s
6133	84(1977)140	85(1978)771s
6134	84(1977)141	85(1978)771s
6135	84(1977)141	
6136	84(1977)141	85(1978)772s 87(1980)225c
6137	84(1977)141	85(1978)830s, 831c 87(1980)495c 88(1981)215c
6138	84(1977)221	85(1978)772s
6139	84(1977)221	85(1978)831s
6140	84(1977)221	88(1981)296s 89(1982)603c
6141	84(1977)221	
6142	84(1977)222	85(1978)773s
6143	84(1977)222	85(1978)774s
6144	84(1977)299	
6145	84(1977)300	85(1978)832s
6146	84(1977)300	85(1978)833s
6147	84(1977)300	86(1979)60s
6148	84(1977)300	86(1979)61s
6149	84(1977)301	86(1979)61s
6150	84(1977)391	86(1979)63s
6151	84(1977)391	86(1979)64s
6152	84(1977)391	86(1979)66s
6153	84(1977)392	86(1979)132s
6154	84(1977)392	86(1979)133s
6155	84(1977)392	86(1979)133s
6156	84(1977)491	86(1979)134s
6157	84(1977)491	
6158	84(1977)491	
6159	84(1977)491	86(1979)135s
6160	84(1977)491	86(1979)598s

Problem Chronology

6161	84(1977)491	86(1979)136s	
6162	84(1977)575	86(1979)227s	
6163	84(1977)575	86(1979)228s	
6164	84(1977)575	86(1979)229s	
6165	84(1977)575	86(1979)229s	
6166	84(1977)576	88(1981)296s	
6167	84(1977)576	86(1979)230s	
6168	84(1977)659	86(1979)312s	
6169	84(1977)659	88(1981)447s	
6170	84(1977)659	86(1979)231s	
6171	84(1977)660	86(1979)312s	
6172	84(1977)660		
6173	84(1977)660	86(1979)312s	
6174	84(1977)743	86(1979)313s	
6175	84(1977)743	86(1979)314s	
6176	84(1977)744	86(1979)314s	
6177	84(1977)744	86(1979)399s	
6178	84(1977)744	86(1979)400s	
6179	84(1977)744	87(1980)826s	
6180	84(1977)828	86(1979)401s	
6181	84(1977)828		
6182	84(1977)828	86(1979)510s	
6183	84(1977)829	85(1978)389v	86(1979)510s
6184	84(1977)829	87(1980)827s	
6185	84(1977)829	87(1980)759s	
6186	85(1978)53		
6187	85(1978)53	87(1980)583s	
6188	85(1978)53	88(1981)447s	
6189	85(1978)54		
6190	85(1978)54		
6191	85(1978)54	89(1982)134s	
6192	85(1978)121	86(1979)598c, 598s	
6193	85(1978)121	86(1979)710s	
6194	85(1978)122	86(1979)511s	
6195	85(1978)122	86(1979)710s	
6196	85(1978)122	86(1979)794s	
6197	85(1978)122		
6198	85(1978)203	86(1979)710s	
6199	85(1978)203	87(1980)226s	
6200	85(1978)203	87(1980)140s	
6201	85(1978)203	86(1979)869s	
6202	85(1978)203	86(1979)870s	
6203	85(1978)203	87(1980)68s	
6204	85(1978)282		
6205	85(1978)282	85(1978)828v	87(1980)227s
6206	85(1978)282	88(1981)215s	
6207	85(1978)282	88(1981)153s	
6208	85(1978)283	87(1980)228s	
6209	85(1978)283	87(1980)141s	
6210	85(1978)389	87(1980)228s	
6211	85(1978)389	87(1980)229x	
6212	85(1978)389		
6213	85(1978)389	89(1982)279s	
6214	85(1978)389		
6215	85(1978)390	87(1980)309s	
6216	85(1978)499		
6217	85(1978)499		
6218	85(1978)500	89(1982)134x	90(1983)408s
6219	85(1978)500	87(1980)141s	
6220	85(1978)500	87(1980)310s	
6221	85(1978)500	87(1980)310s	
6222	85(1978)599	87(1980)760s	
6223	85(1978)600	86(1979)795s	
6224	85(1978)600	87(1980)828x, 829x	89(1982)704s
6225	85(1978)600	87(1980)311s	
6226	85(1978)600	86(1979)796s	
6227	85(1978)600	87(1980)311s	
6228	85(1978)686	86(1979)870s	
6229	85(1978)686		
6230	85(1978)686	87(1980)142s	
6231	85(1978)686	87(1980)676s	
6232	85(1978)686	87(1980)312c	
6233	85(1978)686	87(1980)408s	
6234	85(1978)770	87(1980)829s, 830s	
6235	85(1978)770	87(1980)761s	
6236	85(1978)770	88(1981)69s	89(1982)65s
6237	85(1978)770	87(1980)496s	
6238	85(1978)770	87(1980)409s	
6239	85(1978)770	87(1980)410s	
6240	85(1978)828	87(1980)410s	
6241	85(1978)828	87(1980)496s, 497s	
6242	85(1978)828	87(1980)411s	
6243	85(1978)828	87(1980)498s	
6244	85(1978)828	87(1980)676s, 677c	
6245	85(1978)828	87(1980)584s	
6246	86(1979)59	87(1980)584s	
6247	86(1979)59	87(1980)761s	
6248	86(1979)59	87(1980)584s	
6249	86(1979)59	87(1980)831s	
6250	86(1979)60	87(1980)679s	
6251	86(1979)60	88(1981)154s	
6252	86(1979)131	87(1980)832s	
6253	86(1979)132	87(1980)762s	
6254	86(1979)132	87(1980)679s	
6255	86(1979)132	87(1980)679s	89(1982)704c
6256	86(1979)132	90(1983)487s	
6257	86(1979)132	88(1981)216s	
6258	86(1979)226		
6259	86(1979)226	88(1981)448s	
6260	86(1979)226	87(1980)680s	
6261	86(1979)226	88(1981)70s	
6262	86(1979)226	88(1981)71s	
6263	86(1979)226	88(1981)154s	
6264	86(1979)311	88(1981)449s	
6265	86(1979)311	87(1980)763s	
6266	86(1979)311	88(1981)216s	
6267	86(1979)398	88(1981)69s	
6268	86(1979)398	88(1981)217s	
6269	86(1979)399	88(1981)72s	
6270	86(1979)509		
6271	86(1979)509	88(1981)217s	
6272	86(1979)509	88(1981)353s	
6273	86(1979)596	88(1981)354s	
6274	86(1979)597	88(1981)219s	90(1983)60a
6275	86(1979)597	88(1981)355s	
6276	86(1979)709	88(1981)356s	
6277	86(1979)709	88(1981)449s	
6278	86(1979)709	88(1981)541s	
6279	86(1979)793	88(1981)542s	90(1983)488s
6280	86(1979)793	88(1981)623s	
6281	86(1979)793		
6282	86(1979)869	88(1981)357s	
6283	86(1979)869	88(1981)624s	
6284	86(1979)869	89(1982)136s	

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E435		83(1976)813s 85(1978)836c
E570		83(1976)285c 85(1978)836c
E585		83(1976)134s 85(1978)836c
E604		82(1975)400r
E966		83(1976)378r 84(1977)568s 85(1978)836c
E984		83(1976)567c 84(1977)739c 87(1980)303c
E1030		82(1975)1010s, 1011s 85(1978)836c
E1073		82(1975)72r 83(1976)135s 85(1978)836c
E1075		83(1976)54s 85(1978)836c
E1150		85(1978)836c
E1243		84(1977)58c, 567c 86(1979)593c, 914v
E1255		82(1975)661c 84(1977)652s 85(1978)836c
E1298		82(1975)661s 85(1978)836c
E1445		82(1975)73r

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E1822		83(1976)53r 84(1977)569s 85(1978)836c
E1847		85(1978)836c
E2003		89(1982)274x
E2125		83(1976)567s
E2289		85(1978)835c 87(1980)489s
E2331		82(1975)1012c, 1012s 85(1978)836c
E2344		82(1975)937s 85(1978)836c
E2349		83(1976)54s 85(1978)836c
E2372		84(1977)387c
E2384		83(1976)285s 85(1978)836c
E2392		83(1976)380s 84(1977)59c 85(1978)836c
E2401		83(1976)198s 85(1978)836c
E2428		82(1975)401c
E2432		85(1978)835c
E2434		82(1975)402c, 402s
E2438		85(1978)835c
E2440		82(1975)74s, 75s, 76c 84(1977)567c
E2446		82(1975)77s
E2447		82(1975)78s, 80c
E2448		82(1975)80s
E2450		82(1975)81s, 82c
E2451		82(1975)83s
E2452		82(1975)169c 83(1976)568s
E2453		82(1975)170s
E2454		82(1975)171s, 172s
E2455		82(1975)173s
E2456		82(1975)301s
E2457		82(1975)175s, 176s
E2458		82(1975)302s
E2459		82(1975)178s, 181c
E2460		82(1975)303s
E2461		82(1975)304s, 305c
E2462		85(1978)835c 92(1985)360s
E2463		82(1975)305s, 306s
E2464		82(1975)403s, 404s
E2465		82(1975)405s, 406s
E2466		82(1975)406s
E2467		82(1975)407s, 408s
E2468		83(1976)288c 84(1977)59c
E2469		82(1975)521s
E2470		82(1975)523s
E2471		82(1975)523s
E2472		82(1975)525s
E2473		82(1975)526s, 527s
E2474		82(1975)527s
E2475		82(1975)662s
E2476		82(1975)663x
E2477		82(1975)664s
E2478		82(1975)667s, 668s
E2479		82(1975)668s, 669c
E2480		82(1975)670s
E2481		82(1975)1013s
E2482		82(1975)757s, 757v
E2483		82(1975)758s, 759s
E2484		82(1975)761c, 761s
E2485		82(1975)762s
E2486		82(1975)852s
E2487		82(1975)764s
E2488		82(1975)765s 89(1982)757s
E2489		82(1975)853c, 853s
E2490		82(1975)854s
E2491		82(1975)854c, 854s
E2492		82(1975)855s
E2493		82(1975)855s
E2494		85(1978)835c
E2495		82(1975)168v, 938s
E2496		82(1975)939s
E2497		82(1975)939c, 939s
E2498		83(1976)382s
E2499		82(1975)1015s
E2500		82(1975)1015s
E2501		82(1975)940s, 940v
E2502		82(1975)941s
E2503		83(1976)58s, 59s
E2504		83(1976)289s
E2505		83(1976)59s
E2506		83(1976)60c, 60s
E2507		83(1976)61s
E2508		83(1976)200s
E2509		83(1976)136s
E2510	82(1975)73	83(1976)137s, 138s
E2511	82(1975)73	83(1976)291s
E2512	82(1975)73	83(1976)139s
E2513	82(1975)73	83(1976)140s
E2514	82(1975)73	83(1976)200c, 200s
E2515	82(1975)74	83(1976)201s
E2516	82(1975)168	83(1976)201s, 202s
E2517	82(1975)168	83(1976)204s
E2518	82(1975)169	83(1976)291s
E2519	82(1975)169	83(1976)382s
E2520	82(1975)169	83(1976)383s
E2521	82(1975)169	85(1978)835c
E2522	82(1975)300	83(1976)384s
E2523	82(1975)300	83(1976)384s, 385c
E2524	82(1975)300	83(1976)741s
E2525	82(1975)300	83(1976)483s
E2526	82(1975)300	83(1976)484s 88(1981)539c
E2527	82(1975)301	83(1976)485s
E2528	82(1975)400	83(1976)486s
E2529	82(1975)400	83(1976)487s
E2530	82(1975)400	85(1978)835c
E2531	82(1975)400	83(1976)488s
E2532	82(1975)400	83(1976)569s
E2533	82(1975)401	83(1976)570c, 570s
E2534	82(1975)520	83(1976)571s
E2535	82(1975)520	83(1976)657s
E2536	82(1975)521	83(1976)657s
E2537	82(1975)521	83(1976)658s
E2538	82(1975)521	83(1976)659s
E2539	82(1975)521	83(1976)742c
E2540	82(1975)659	83(1976)659s
E2541	82(1975)659	83(1976)660s
E2542	82(1975)660	83(1976)743s
E2543	82(1975)660	83(1976)744s
E2544	82(1975)660	83(1976)745s
E2545	82(1975)660	83(1976)747s
E2546	82(1975)756	83(1976)813s
E2547	82(1975)756	83(1976)814s
E2548	82(1975)756	83(1976)815s
E2549	82(1975)756	83(1976)815s
E2550	82(1975)756	83(1976)815s
E2551	82(1975)756	83(1976)816s, 817c
E2552	82(1975)851	84(1977)60s
E2553	82(1975)851	84(1977)60s
E2554	82(1975)851	84(1977)61s
E2555	82(1975)851	84(1977)135s
E2556	82(1975)852	84(1977)136s
E2557	82(1975)852	84(1977)137s
E2558	82(1975)936	84(1977)138c, 138s
E2559	82(1975)936	84(1977)140s
E2560	82(1975)936	84(1977)140s
E2561	82(1975)936	84(1977)217s
E2562	82(1975)937	84(1977)218s
E2563	82(1975)937	84(1977)218s
E2564	82(1975)1009	84(1977)219s, 654s
E2565	82(1975)1009	84(1977)220s
E2566	82(1975)1010	84(1977)220s, 221c
E2567	82(1975)1010	84(1977)570s
E2568	82(1975)1010	84(1977)295s
E2569	82(1975)1010	84(1977)296c
E2570	83(1976)53	84(1977)296s
E2571	83(1976)53	84(1977)297c

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E2572	83(1976)53	84(1977)298s	
E2573	83(1976)54	84(1977)298s, 299s	
E2574	83(1976)54	84(1977)387s	
E2575	83(1976)133	84(1977)388s	
E2576	83(1976)133	84(1977)388s	
E2577	83(1976)133	84(1977)389s	
E2578	83(1976)133	84(1977)390s	
E2579	83(1976)133	84(1977)487s	
E2580	83(1976)133	84(1977)488s	
E2581	83(1976)197	84(1977)488s	
E2582	83(1976)197	84(1977)489s	
E2583	83(1976)198	84(1977)571s	
E2584	83(1976)198	84(1977)489s, 490s	
E2585	83(1976)198	84(1977)490s	
E2586	83(1976)198	84(1977)572s	
E2587	83(1976)284	84(1977)572s	
E2588	83(1976)284	84(1977)573s	
E2589	83(1976)284	84(1977)574s	
E2590	83(1976)284	84(1977)654s	
E2591	83(1976)284	84(1977)655s	
E2592	83(1976)285	84(1977)656s	
E2593	83(1976)378	84(1977)739s	
E2594	83(1976)379	85(1978)835c	
E2595	83(1976)379	84(1977)657s	
E2596	83(1976)379	85(1978)835c	
E2597	83(1976)379	84(1977)657s, 658s	
E2598	83(1976)379	84(1977)659s	
E2599	83(1976)482	84(1977)740s	
E2600	83(1976)482	84(1977)741s	
E2601	83(1976)482	84(1977)741s, 742s	
E2602	83(1976)482	84(1977)742s	
E2603	83(1976)483	84(1977)743s	
E2604	83(1976)483	84(1977)821s	
E2605	83(1976)566	84(1977)822s	
E2606	83(1976)566	84(1977)823s	
E2607	83(1976)566	84(1977)824s	
E2608	83(1976)567	85(1978)835c	88(1981)148s
E2609	83(1976)567	84(1977)825s, 826c	
E2610	83(1976)567	85(1978)198s	
E2611	83(1976)656	85(1978)199s	
E2612	83(1976)656	85(1978)199s	
E2613	83(1976)656	84(1977)827s	
E2614	83(1976)657	85(1978)48s	
E2615	83(1976)657	85(1978)49s	
E2616	83(1976)657	85(1978)50s	
E2617	83(1976)740	85(1978)51s	
E2618	83(1976)740	85(1978)51s	
E2619	83(1976)740	85(1978)52c, 52s	
E2620	83(1976)740	85(1978)117s	
E2621	83(1976)741	85(1978)118s	
E2622	83(1976)741	85(1978)119s	
E2623	83(1976)812	85(1978)119s	
E2624	83(1976)812	85(1978)120s	
E2625	83(1976)812	85(1978)121s	
E2626	83(1976)812	85(1978)200s	
E2627	83(1976)812	85(1978)201s	
E2628	83(1976)813	85(1978)202s	
E2629	84(1977)57	85(1978)277s	
E2630	84(1977)57	85(1978)279s	
E2631	84(1977)57	85(1978)279s	
E2632	84(1977)57	85(1978)280s	
E2633	84(1977)58	85(1978)281s	
E2634	84(1977)58	85(1978)282s	
E2635	84(1977)134	85(1978)385s	
E2636	84(1977)134	85(1978)386s	
E2637	84(1977)134	85(1978)386s	
E2638	84(1977)135	85(1978)387s	
E2639	84(1977)135	85(1978)388s	
E2640	84(1977)135	85(1978)388s	
E2641	84(1977)216	85(1978)496s	
E2642	84(1977)216	85(1978)497s	
E2643	84(1977)217	85(1978)497s	
E2644	84(1977)217	85(1978)497s	
E2645	84(1977)217	85(1978)498s	
E2646	84(1977)217	85(1978)499s	
E2647	84(1977)294	85(1978)594s	
E2648	84(1977)294	85(1978)595c, 595s	
E2649	84(1977)294	85(1978)596s	86(1979)504a
E2650	84(1977)294	85(1978)597s	
E2651	84(1977)295	85(1978)598s	
E2652	84(1977)295	84(1977)567v	85(1978)765s
E2653	84(1977)386	85(1978)599s	
E2654	84(1977)386	85(1978)766s	
E2655	84(1977)386	85(1978)682s	
E2656	84(1977)386	85(1978)766s	
E2657	84(1977)386	85(1978)683s	
E2658	84(1977)387	86(1979)504s	
E2659	84(1977)486	85(1978)767s	
E2660	84(1977)487	85(1978)683s	
E2661	84(1977)487	85(1978)685s	
E2662	84(1977)487	85(1978)685s	
E2663	84(1977)487	85(1978)686s	
E2664	84(1977)487	85(1978)768s	
E2665	84(1977)567	85(1978)769s	
E2666	84(1977)567	85(1978)824s	
E2667	84(1977)567	85(1978)825s	
E2668	84(1977)568	85(1978)825s	
E2669	84(1977)568	85(1978)825s	
E2670	84(1977)568	85(1978)826s	
E2671	84(1977)651	85(1978)827c, 827s	
E2672	84(1977)651	86(1979)56s	
E2673	84(1977)652	86(1979)57s	
E2674	84(1977)652	86(1979)57s	
E2675	84(1977)652	86(1979)58s	
E2676	84(1977)652	86(1979)58s	
E2677	84(1977)738	86(1979)394s	
E2678	84(1977)738	86(1979)506s	
E2679	84(1977)738	86(1979)59s	
E2680	84(1977)738	86(1979)128s	
E2681	84(1977)738	86(1979)129s	
E2682	84(1977)738	86(1979)223s	
E2683	84(1977)820	86(1979)129s	
E2684	84(1977)820	86(1979)130s	
E2685	84(1977)820	86(1979)130s	
E2686	84(1977)820	86(1979)131s	
E2687	84(1977)820	86(1979)785s	
E2688	84(1977)820		
E2689	85(1978)47	86(1979)224s	
E2690	85(1978)48	86(1979)308s	
E2691	85(1978)48	86(1979)225s	
E2692	85(1978)48	86(1979)394s, 395s	
E2693	85(1978)48	86(1979)308s	
E2694	85(1978)48	86(1979)506s	
E2695	85(1978)116	86(1979)309s	
E2696	85(1978)116	86(1979)507s	
E2697	85(1978)116	86(1979)225s	
E2698	85(1978)116	86(1979)309s	
E2699	85(1978)117	86(1979)310s	
E2700	85(1978)117	86(1979)311s	
E2701	85(1978)197	86(1979)396s	
E2702	85(1978)197		
E2703	85(1978)198	86(1979)397s	
E2704	85(1978)198	86(1979)398s	
E2705	85(1978)198	86(1979)398s	
E2706	85(1978)198	86(1979)593s	
E2707	85(1978)276	86(1979)508s	
E2708	85(1978)276	86(1979)594s	
E2709	85(1978)276	86(1979)55v, 703s, 704s	
E2710	85(1978)276	86(1979)594s	
E2711	85(1978)277	86(1979)595s	
E2712	85(1978)277	86(1979)704s, 705s	
E2713	85(1978)384		
E2714	85(1978)384	86(1979)596s	
E2715	85(1978)384	86(1979)705s	87(1980)304s
E2716	85(1978)384	89(1982)594s	
E2717	85(1978)384		

E2718	85(1978)384	86(1979)509s
E2719	85(1978)495	86(1979)787s 91(1984)143a
E2720	85(1978)495	86(1979)706s, 707c, 707s
E2721	85(1978)496	86(1979)865s
E2722	85(1978)496	86(1979)708c
E2723	85(1978)496	86(1979)788s, 789s
E2724	85(1978)496	86(1979)708s
E2725	85(1978)593	86(1979)790s
E2726	85(1978)593	87(1980)61s
E2727	85(1978)594	86(1979)791s 90(1983)55s
E2728	85(1978)594	86(1979)792s
E2729	85(1978)594	87(1980)137s
E2730	85(1978)594	86(1979)866s
E2731	85(1978)681	86(1979)866s
E2732	85(1978)681	86(1979)867s
E2733	85(1978)682	86(1979)868s
E2734	85(1978)682	86(1979)869s
E2735	85(1978)682	87(1980)577s
E2736	85(1978)682	89(1982)131s
E2737	85(1978)764	87(1980)305s
E2738	85(1978)764	87(1980)61s, 62c
E2739	85(1978)765	87(1980)62s
E2740	85(1978)765	92(1985)591x
E2741	85(1978)765	87(1980)63s
E2742	85(1978)765	87(1980)63s
E2743	85(1978)823	87(1980)221s
E2744	85(1978)823	86(1979)503v 88(1981)705s
E2745	85(1978)824	87(1980)222s
E2746	85(1978)824	87(1980)64s
E2747	85(1978)824	86(1979)592v 87(1980)305s 90(1983)59a
E2748	85(1978)824	87(1980)138s
E2749	86(1979)55	87(1980)138s
E2750	86(1979)55	87(1980)138s
E2751	86(1979)56	88(1981)291s
E2752	86(1979)56	89(1982)757c, 757s
E2753	86(1979)56	87(1980)139s
E2754	86(1979)56	87(1980)139s
E2755	86(1979)127	87(1980)222s
E2756	86(1979)128	87(1980)223s
E2757	86(1979)128	
E2758	86(1979)128	87(1980)405s
E2759	86(1979)128	
E2760	86(1979)128	87(1980)223s
E2761	86(1979)222	87(1980)224s
E2762	86(1979)223	87(1980)405s 88(1981)350c, 350s 90(1983)59a
E2763	86(1979)223	90(1983)56s 91(1984)204c
E2764	86(1979)223	87(1980)306s
E2765	86(1979)223	87(1980)307s
E2766	86(1979)223	87(1980)406s
E2767	86(1979)307	87(1980)490s
E2768	86(1979)307	87(1980)406s
E2769	86(1979)307	87(1980)308s
E2770	86(1979)307	87(1980)491s
E2771	86(1979)308	87(1980)407s
E2772	86(1979)308	88(1981)350s
E2773	86(1979)393	87(1980)492s
E2774	86(1979)393	
E2775	86(1979)393	87(1980)578s, 579s
E2776	86(1979)393	87(1980)493s
E2777	86(1979)393	87(1980)494s
E2778	86(1979)393	87(1980)580s
E2779	86(1979)503	
E2780	86(1979)503	88(1981)764s 90(1983)59a
E2781	86(1979)503	87(1980)580s 90(1983)59a
E2782	86(1979)503	87(1980)581s 90(1983)59a
E2783	86(1979)504	87(1980)581s 90(1983)59a
E2784	86(1979)504	88(1981)209s
E2785	86(1979)592	87(1980)672s
E2786	86(1979)592	87(1980)672s
E2787	86(1979)592	87(1980)673s
E2788	86(1979)592	87(1980)673s
E2789	86(1979)592	87(1980)674s
E2790	86(1979)593	87(1980)755s
E2791	86(1979)702	87(1980)675s
E2792	86(1979)702	87(1980)756s 90(1983)59a
E2793	86(1979)703	88(1981)707s
E2794	86(1979)703	87(1980)756c
E2795	86(1979)703	87(1980)757s
E2796	86(1979)703	87(1980)824s
E2797	86(1979)785	88(1981)149s
E2798	86(1979)785	87(1980)824s
E2799	86(1979)785	87(1980)825s 89(1982)334c 91(1984)143a
E2800	86(1979)785	87(1980)825s
E2801	86(1979)785	
E2802	86(1979)785	88(1981)67s
E2803	86(1979)864	88(1981)149s
E2804	86(1979)864	
E2805	86(1979)864	88(1981)68s
E2806	86(1979)864	88(1981)210s 90(1983)59a
E2807	86(1979)865	88(1981)68s
E2808	86(1979)865	88(1981)211s

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S1	86(1979)54	87(1980)134s
S2	86(1979)55	87(1980)134c, 134s
S3	86(1979)55	87(1980)136s
S4	86(1979)127	87(1980)219s
S5	86(1979)127	87(1980)219s
S6	86(1979)222	87(1980)302s
S7	86(1979)222	87(1980)403s
S8	86(1979)222	87(1980)487s
S9	86(1979)306	87(1980)488s
S10	86(1979)306	87(1980)575s
S11	86(1979)392	87(1980)753s
S12	86(1979)392	87(1980)576s
S13	86(1979)392	87(1980)670s
S14	86(1979)503	90(1983)335s
S15	86(1979)503	87(1980)670s
S16	86(1979)591	87(1980)754s
S17	86(1979)591	87(1980)822s, 823s
S18	86(1979)592	88(1981)64s, 65s
S19	86(1979)702	88(1981)147s
S20	86(1979)702	88(1981)207s
S21	86(1979)784	88(1981)443x
S22	86(1979)863	88(1981)348s
S23	86(1979)863	88(1981)537s

CMB

Problem	Proposal	References
P169		25(1982)506c
P191		22(1979)520s, 521c
P207		23(1980)118s
P212		25(1982)506c
P217		19(1976)380s
P222		18(1975)616s
P223		18(1975)618s
P226		19(1976)250s
P227		18(1975)619s
P228		18(1975)619s
P229		19(1976)122s
P230		19(1976)123s, 123v
P231		19(1976)381s
P232		20(1977)148s
P233		19(1976)251s
P234		19(1976)124s
P235		23(1980)382s
P236		19(1976)124s
P237		20(1977)149s
P238		19(1976)252s
P239		20(1977)520s
P240		20(1977)518s
P241	18(1975)615	19(1976)382s

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P242	18(1975)615	20(1977)274s
P243	18(1975)615	22(1979)248s
P244	18(1975)616	20(1977)150s
P245	18(1975)616	20(1977)274s
P246	19(1976)121	22(1979)250s
P247	19(1976)121	20(1977)520s
P248	19(1976)121	20(1977)276s
P249	19(1976)122	22(1979)251s
P250	19(1976)249	22(1979)122s
P251	19(1976)249	20(1977)522s
P252	19(1976)249	22(1979)252s
P253	20(1977)517	22(1979)252s, 253c
P254	19(1976)380	20(1977)523s
P255	19(1976)379	22(1979)253s
P256	19(1976)379	20(1977)522s
P257	20(1977)147	20(1977)517v 22(1979)386s
P258	20(1977)147	20(1977)517v, 523s 22(1979)125a
P259	20(1977)147	22(1979)387s
P260	20(1977)147	22(1979)388s
P261	20(1977)148	20(1977)524s 22(1979)125a
P264	20(1977)273	22(1979)122s
P265	20(1977)273	20(1977)525s
P266	20(1977)273	20(1977)517v 22(1979)389s
P267	20(1977)518	22(1979)123s
P268	20(1977)518	25(1982)506c
P269	20(1977)518	22(1979)125s
P270	22(1979)121	23(1980)119v, 120s, 121s, 253a
P271	22(1979)121	23(1980)122s
P272	22(1979)121	23(1980)124s, 125s
P273	22(1979)247	23(1980)249s
P274	22(1979)248	23(1980)249s
P275	22(1979)248	23(1980)250s
P276	22(1979)248	23(1980)251s 25(1982)508a, 508c
P277	22(1979)385	25(1982)506c
P278	22(1979)386	23(1980)507x 24(1981)252s
P279	22(1979)386	23(1980)508s
P280	22(1979)386	23(1980)509s
P281	22(1979)519	24(1981)127s, 256a

CRUX

Problem	Proposal	References
1	1(1975)3	1(1975)12s
2	1(1975)3	1(1975)12s
3	1(1975)3	1(1975)14c, 14s
4	1(1975)3	1(1975)15s
5	1(1975)3	1(1975)15s
6	1(1975)3	1(1975)17s, 27a
7	1(1975)4	1(1975)18s
8	1(1975)4	1(1975)19s
9	1(1975)4	1(1975)19s
10	1(1975)4	1(1975)20c, 20s, 49c
11	1(1975)7	1(1975)27s
12	1(1975)7	1(1975)27s
13	1(1975)7	1(1975)27s
14	1(1975)7	1(1975)28s
15	1(1975)8	1(1975)28s
16	1(1975)8	1(1975)29s
17	1(1975)8	1(1975)29s, 30s
18	1(1975)8	1(1975)31s, 32s 2(1976)42c, 69c
19	1(1975)8	1(1975)32s
20	1(1975)8	1(1975)33s, 34s
21	1(1975)11	1(1975)40c, 40s, 58a
22	1(1975)11	1(1975)40s, 58a
23	1(1975)11	1(1975)41c, 41s
24	1(1975)11	1(1975)42s
25	1(1975)11	1(1975)42s, 43c, 43s, 58c
26	1(1975)12	1(1975)43s
27	1(1975)12	1(1975)44s
28	1(1975)12	1(1975)44s
29	1(1975)12	1(1975)45s, 46c
30	1(1975)12	1(1975)46s
31	1(1975)25	1(1975)58s
32	1(1975)25	1(1975)59c, 59s
33	1(1975)25	1(1975)60s
34	1(1975)25	1(1975)60c, 60s
35	1(1975)25	1(1975)61s
36	1(1975)25	1(1975)61s
37	1(1975)26	1(1975)62c, 62s
38	1(1975)26	1(1975)64c, 64s
39	1(1975)26	1(1975)64s, 65c, 65s 2(1976)7s
40	1(1975)26	1(1975)66s
41	1(1975)38	1(1975)72c, 72s
42	1(1975)38	1(1975)73s
43	1(1975)38	1(1975)73x, 85s
44	1(1975)38	1(1975)74s
45	1(1975)39	1(1975)74s 6(1980)213s
46	1(1975)39	1(1975)75c, 75s
47	1(1975)39	1(1975)76s
48	1(1975)39	1(1975)77s
49	1(1975)39	1(1975)77s
50	1(1975)39	1(1975)78s, 80c
51	1(1975)48	1(1975)86s 2(1976)7a
52	1(1975)48	1(1975)87s
53	1(1975)48	1(1975)88s 2(1976)7a
54	1(1975)48	1(1975)89s
55	1(1975)48	1(1975)89s 2(1976)7a
56	1(1975)48	1(1975)89s, 90c
57	1(1975)49	1(1975)56v, 91s
58	1(1975)49	2(1976)43s
59	1(1975)49	1(1975)91s, 92c, 92s
60	1(1975)49	1(1975)92s
61	1(1975)56	1(1975)98s 2(1976)26a
62	1(1975)56	1(1975)99s 2(1976)7a
63	1(1975)56	1(1975)99s
64	1(1975)57	1(1975)100s
65	1(1975)57	1(1975)100s 2(1976)7a, 69s
66	1(1975)57	1(1975)100s
67	1(1975)57	1(1975)101s
68	1(1975)57	1(1975)101s
69	1(1975)57	1(1975)102s
70	1(1975)57	1(1975)102s
71	1(1975)71	2(1976)8s
72	1(1975)71	2(1976)9s
73	1(1975)71	2(1976)9s
74	1(1975)71	2(1976)10s
75	1(1975)71	2(1976)10s, 11c, 11s, 172a
76	1(1975)71	2(1976)12s
77	1(1975)71	2(1976)12s
78	1(1975)72	2(1976)13s, 14c
79	1(1975)72	2(1976)15s, 16c, 16v
80	1(1975)72	2(1976)17c, 17s
81	1(1975)84	2(1976)26s, 27c
82	1(1975)84	2(1976)27s
83	1(1975)84	2(1976)28c, 28s 9(1983)278c
84	1(1975)84	2(1976)29s
85	1(1975)84	2(1976)29s, 30s
86	1(1975)84	2(1976)30s
87	1(1975)84	2(1976)32s
88	1(1975)85	2(1976)33c, 33s 5(1979)48c
89	1(1975)85	2(1976)33s, 34c
90	1(1975)85	2(1976)34s, 36c 8(1982)279s
91	1(1975)97	2(1976)44s, 69a
92	1(1975)97	2(1976)44s, 45c, 69a
93	1(1975)97	2(1976)45s, 46c, 111s 10(1984)293s
94	1(1975)97	2(1976)25v, 46s, 47s
95	1(1975)97	2(1976)47s, 48s
96	1(1975)97	2(1976)48s
97	1(1975)97	2(1976)48s, 49c, 69a
98	1(1975)97	2(1976)49c, 49s
99	1(1975)98	2(1976)50s, 51s, 52c, 69s
100	1(1975)98	2(1976)52s, 53c
101	2(1976)5	2(1976)72c, 72s
102	2(1976)5	2(1976)73s, 74s
103	2(1976)5	2(1976)74s, 75s

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104	2(1976)5	2(1976)76s
105	2(1976)5	2(1976)77s, 78c
106	2(1976)6	2(1976)78s, 79s
107	2(1976)6	2(1976)79s, 80c
108	2(1976)6	2(1976)80s, 81s
109	2(1976)6	2(1976)81s, 83c
110	2(1976)6	2(1976)84s, 85s, 87c 14(1988)16s
111	2(1976)25	2(1976)95c, 95s, 111a
112	2(1976)25	2(1976)96c, 96s
113	2(1976)25	2(1976)97s
114	2(1976)25	2(1976)98s
115	2(1976)25	2(1976)98c, 98s, 112c, 112s, 137s, 138c
116	2(1976)25	2(1976)100s
117	2(1976)26	2(1976)100s
118	2(1976)26	2(1976)101s
119	2(1976)26	2(1976)102s
120	2(1976)26	2(1976)103s, 104s, 139c
121	2(1976)41	2(1976)113s, 114s
122	2(1976)41	2(1976)114s, 115c, 115s, 116c
123	2(1976)41	2(1976)116s, 117s, 118c, 119c
124	2(1976)41	2(1976)119s
125	2(1976)41	2(1976)120s, 121c, 121s, 139c
126	2(1976)41	2(1976)123s, 172a
127	2(1976)41	2(1976)124c, 124s, 125c, 140c, 221c
128	2(1976)41	2(1976)125s, 126c, 141a
129	2(1976)42	2(1976)126s, 127c
130	2(1976)42	2(1976)128s 3(1977)44s
131	2(1976)67	2(1976)141c, 141s, 172a
132	2(1976)67	2(1976)142s, 143c, 172c 3(1977)11c
133	2(1976)67	2(1976)144c, 147c, 148c, 149c, 221c
134	2(1976)68	2(1976)151c, 151s, 152c, 173s, 174s, 222c, 222s 3(1977)12c, 44c
135	2(1976)68	2(1976)153s, 154c, 154s, 223c 3(1977)45c
136	2(1976)68	2(1976)155c, 155s
137	2(1976)68	2(1976)156s, 157c
138	2(1976)68	2(1976)157s, 158c
139	2(1976)68	2(1976)158s
140	2(1976)68	3(1977)13s, 46c
141	2(1976)93	2(1976)175s
142	2(1976)93	2(1976)175s, 176s, 177c 3(1977)106c
143	2(1976)93	2(1976)178s, 180c
144	2(1976)94	2(1976)181s
145	2(1976)94	2(1976)181c, 182c, 225c 3(1977)16c, 18c, 67s
146	2(1976)94	2(1976)182s
147	2(1976)94	2(1976)183c, 183s
148	2(1976)94	2(1976)183s, 184c
149	2(1976)94	2(1976)184s 3(1977)47c, 47s
150	2(1976)94	2(1976)185s, 186c
151	2(1976)109	2(1976)195s
152	2(1976)109	2(1976)196s
153	2(1976)110	2(1976)196s, 197c, 197s 3(1977)19c, 19s
154	2(1976)110	2(1976)159v, 197s, 198c, 226c 3(1977)20c, 108c, 191c, 191s
155	2(1976)110	2(1976)199s 3(1977)22c
156	2(1976)110	2(1976)199s
157	2(1976)110	2(1976)200s, 201c
158	2(1976)111	2(1976)201s
159	2(1976)111	2(1976)202c, 202s
160	2(1976)111	2(1976)203c, 203s 3(1977)23c
161	2(1976)135	2(1976)226s
162	2(1976)135	2(1976)226s, 227c
163	2(1976)135	2(1976)228s
164	2(1976)135	2(1976)230c, 230s
165	2(1976)135	2(1976)230s, 231c, 231s
166	2(1976)136	2(1976)231s, 232c
167	2(1976)136	3(1977)23s
168	2(1976)136	2(1976)233s
169	2(1976)136	2(1976)234c, 234s
170	2(1976)136	2(1976)170v 3(1977)25s, 26c
171	2(1976)170	3(1977)26s
172	2(1976)170	3(1977)28s, 29c
173	2(1976)171	3(1977)47c, 68s
174	2(1976)171	3(1977)48s
175	2(1976)171	3(1977)49s, 50c
176	2(1976)171	3(1977)30s, 69c
177	2(1976)171	3(1977)50s, 52c, 132c, 133s
178	2(1976)171	3(1977)53c, 53s
179	2(1976)171	3(1977)54s, 55c
180	2(1976)172	3(1977)56s
181	2(1976)193	3(1977)57c, 57s 4(1978)37a
182	2(1976)193	3(1977)58s
183	2(1976)193	3(1977)69s
184	2(1976)193	3(1977)70s
185	2(1976)194	3(1977)70s, 71s 4(1978)37a
186	2(1976)194	3(1977)71s
187	2(1976)194	3(1977)72c, 72s
188	2(1976)194	3(1977)73c, 73s
189	2(1976)194	3(1977)74c, 75c, 193c, 252c 4(1978)37a 15(1989)75s
190	2(1976)194	3(1977)76s
191	2(1976)219	3(1977)77s, 78s
192	2(1976)219	3(1977)79c, 79s, 80c
193	2(1976)219	3(1977)81s
194	2(1976)219	3(1977)82c, 82s
195	2(1976)220	3(1977)84s, 87s, 195c
196	2(1976)220	3(1977)108s
197	2(1976)220	3(1977)108s, 156c
198	2(1976)220	3(1977)111s
199	2(1976)220	3(1977)112s, 113s, 114c, 299c
200	2(1976)220	3(1977)133s, 228c
201	3(1977)9	3(1977)136c, 136s
202	3(1977)9	3(1977)137s, 138c
203	3(1977)9	3(1977)138s, 140c
204	3(1977)10	3(1977)140s, 141c
205	3(1977)10	3(1977)142s, 196a
206	3(1977)10	3(1977)143c, 143s
207	3(1977)10	3(1977)144c, 144s
208	3(1977)10	3(1977)157s, 158c
209	3(1977)10	3(1977)159s
210	3(1977)10	3(1977)160s, 163c, 163s, 196c, 197s 4(1978)13c, 16c, 193c
211	3(1977)42	3(1977)164c, 164s
212	3(1977)42	3(1977)165c, 165s
213	3(1977)42	3(1977)166s
214	3(1977)42	3(1977)166s
215	3(1977)42	3(1977)167s, 168c, 168s, 198a
216	3(1977)42	3(1977)170s, 198c
217	3(1977)43	3(1977)172s
218	3(1977)43	3(1977)172s
219	3(1977)43	3(1977)173s, 175c
220	3(1977)43	3(1977)175c, 175s
221	3(1977)65	3(1977)199s, 200s
222	3(1977)65	3(1977)200s, 201c, 201s
223	3(1977)65	3(1977)202s, 203c
224	3(1977)65	3(1977)203s, 203v
225	3(1977)65	3(1977)204s, 205s
226	3(1977)66	3(1977)205s, 206c
227	3(1977)66	3(1977)228s, 230c
228	3(1977)66	3(1977)230s
229	3(1977)66	3(1977)231s
230	3(1977)66	3(1977)233s, 234c, 235c
231	3(1977)104	3(1977)236s, 237c
232	3(1977)104	3(1977)238s, 240c, 240s 4(1978)17s
233	3(1977)104	3(1977)253s
234	3(1977)104	3(1977)154v, 257s, 258c, 299a
235	3(1977)105	3(1977)258s
236	3(1977)105	3(1977)260s

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237	3(1977)105	3(1977)261s
238	3(1977)105	3(1977)262s
239	3(1977)105	3(1977)263s
240	3(1977)105	3(1977)264s, 299c 4(1978)18a, 37c
241	3(1977)130	3(1977)265c, 265s, 299a
242	3(1977)130	3(1977)266s, 267s
243	3(1977)130	3(1977)268s
244	3(1977)130	4(1978)19s
245	3(1977)130	4(1978)21s
246	3(1977)131	4(1978)22c, 22s
247	3(1977)131	4(1978)24s, 38c
248	3(1977)131	3(1977)154v 4(1978)27c, 27s, 102c
249	3(1977)131	4(1978)28s, 29c
250	3(1977)132	4(1978)39s, 40c, 40s 5(1979)17x
251	3(1977)154	4(1978)42s, 43c
252	3(1977)154	4(1978)44s, 47c
253	3(1977)154	4(1978)49s, 50c
254	3(1977)155	4(1978)50s
255	3(1977)155	4(1978)52c, 52s
256	3(1977)155	4(1978)53x, 103s, 161a
257	3(1977)155	4(1978)54s
258	3(1977)155	4(1978)56s
259	3(1977)155	4(1978)57c, 57s
260	3(1977)155	4(1978)58s, 59c 9(1983)81c
261	3(1977)189	4(1978)67s, 68c, 69c
262	3(1977)189	4(1978)70s
263	3(1977)189	4(1978)71s
264	3(1977)189	4(1978)73s
265	3(1977)190	4(1978)74s, 75c, 104a
266	3(1977)190	4(1978)75x
267	3(1977)190	4(1978)76s, 77c, 104s
268	3(1977)190	4(1978)78s, 79c
269	3(1977)190	4(1978)79s, 80c, 81c, 82c 6(1980)45c
270	3(1977)190	4(1978)82s
271	3(1977)226	4(1978)85s
272	3(1977)226	4(1978)86s, 87c
273	3(1977)226	4(1978)87s
274	3(1977)226	4(1978)88s
275	3(1977)227	4(1978)105s
276	3(1977)227	4(1978)107s, 108c
277	3(1977)227	4(1978)109s
278	3(1977)227	4(1978)110c, 110s
279	3(1977)227	4(1978)110s
280	3(1977)227	4(1978)111s, 112c
281	3(1977)250	4(1978)113s
282	3(1977)250	4(1978)114s, 115c, 135a
283	3(1977)250	4(1978)195s
284	3(1977)250	4(1978)115s, 116s
285	3(1977)251	4(1978)116s, 117s, 118s
286	3(1977)251	4(1978)119s, 120c
287	3(1977)251	4(1978)135s, 136c, 138c
288	3(1977)251	4(1978)136s
289	3(1977)251	4(1978)139c, 139s, 140c
290	3(1977)251	4(1978)142s, 144c 11(1985)222c
291	3(1977)297	4(1978)147s, 148s
292	3(1977)297	4(1978)148s, 149c
293	3(1977)297	4(1978)150s
294	3(1977)297	4(1978)161s, 162c
295	3(1977)297	4(1978)162s, 163c, 163s
296	3(1977)297	4(1978)164s
297	3(1977)298	4(1978)165s, 167c
298	3(1977)298	4(1978)167s
299	3(1977)298	4(1978)170s
300	3(1977)298	4(1978)172s, 173c
301	4(1978)11	4(1978)174c, 174s
302	4(1978)11	4(1978)176s
303	4(1978)11	4(1978)177s
304	4(1978)11	4(1978)178c, 178s
305	4(1978)11	4(1978)180s, 227a
306	4(1978)12	4(1978)196s, 197c
307	4(1978)12	4(1978)198c, 198s
308	4(1978)12	4(1978)199s
309	4(1978)12	4(1978)200s
310	4(1978)12	4(1978)202s, 203s, 204c
311	4(1978)35	4(1978)204s, 205c
312	4(1978)35	4(1978)205s, 207c
313	4(1978)35	4(1978)207s, 208s
314	4(1978)35	4(1978)209s
315	4(1978)35	4(1978)227s
316	4(1978)36	4(1978)228s, 229s
317	4(1978)36	4(1978)230s
318	4(1978)36	4(1978)231s, 233s
319	4(1978)36	4(1978)235c, 235s
320	4(1978)36	4(1978)238s
321	4(1978)65	4(1978)252c, 252s 5(1979)18a
322	4(1978)65	4(1978)254s 5(1979)18a
323	4(1978)65	4(1978)255c, 255s 5(1979)18a
324	4(1978)66	4(1978)257s
325	4(1978)66	4(1978)258s 5(1979)18a, 49c
326	4(1978)66	5(1979)18s
327	4(1978)66	4(1978)260s 5(1979)18a
328	4(1978)66	4(1978)260s
329	4(1978)66	4(1978)262s
330	4(1978)67	4(1978)263s
331	4(1978)100	4(1978)265s
332	4(1978)100	4(1978)267c, 267s, 285a 5(1979)18a
333	4(1978)101	4(1978)269c, 269s
334	4(1978)101	4(1978)285s
335	4(1978)101	4(1978)287s
336	4(1978)101	4(1978)288c, 288s
337	4(1978)101	4(1978)289s
338	4(1978)101	4(1978)290s, 291s 5(1979)23a
339	4(1978)102	4(1978)292c
340	4(1978)102	4(1978)293s, 294c, 294s
341	4(1978)133	4(1978)296s, 297s
342	4(1978)133	6(1980)319x
343	4(1978)133	4(1978)298x
344	4(1978)133	5(1979)23s
345	4(1978)134	5(1979)25s
346	4(1978)134	5(1979)26s, 29c, 30c 10(1984)294c
347	4(1978)134	4(1978)191v 5(1979)50s
348	4(1978)134	5(1979)50s, 51s, 52s
349	4(1978)134	5(1979)53s
350	4(1978)135	5(1979)54s
351	4(1978)159	5(1979)54s
352	4(1978)159	5(1979)55c, 55s
353	4(1978)159	5(1979)56s
354	4(1978)159	5(1979)57s, 58c, 59s
355	4(1978)160	5(1979)78x, 79x, 80c, 168c
356	4(1978)160	5(1979)80s, 82c
357	4(1978)160	5(1979)83s
358	4(1978)161	5(1979)84s
359	4(1978)161	5(1979)85s
360	4(1978)161	5(1979)87s, 88c
361	4(1978)191	5(1979)88c, 88s
362	4(1978)191	5(1979)89s, 90c
363	4(1978)191	5(1979)111s
364	4(1978)192	5(1979)113s
365	4(1978)192	5(1979)115s
366	4(1978)192	5(1979)117c, 117s
367	4(1978)192	5(1979)118s
368	4(1978)192	5(1979)134s
369	4(1978)192	5(1979)135s
370	4(1978)193	5(1979)135s
371	4(1978)224	5(1979)136s, 137c
372	4(1978)224	5(1979)138s
373	4(1978)225	5(1979)139s
374	4(1978)225	5(1979)140s
375	4(1978)225	5(1979)142s
376	4(1978)225	5(1979)143s, 145c
377	4(1978)226	5(1979)146c, 146s

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378	4(1978)226	5(1979)147s, 148c, 148s
379	4(1978)226	5(1979)149s, 150c
380	4(1978)226	5(1979)171s
381	4(1978)250	5(1979)172s
382	4(1978)250	5(1979)172s, 173s
383	4(1978)250	5(1979)174s, 175c, 175s
384	4(1978)250	5(1979)176s, 178c
385	4(1978)250	5(1979)178s, 179c
386	4(1978)251	5(1979)179s, 180×
387	4(1978)251	6(1980)47×, 114s, 285a
388	4(1978)251	5(1979)201s, 202c
389	4(1978)251	5(1979)202s, 203c, 203s
390	4(1978)251	5(1979)205s, 206c
391	4(1978)282	5(1979)207s, 208c
392	4(1978)282	5(1979)208s, 209c
393	4(1978)283	5(1979)210s, 211c
394	4(1978)283	5(1979)229s
395	4(1978)283	5(1979)232s
396	4(1978)283	5(1979)233s, 234c
397	4(1978)283	5(1979)234s, 235s
398	4(1978)284	5(1979)235s
399	4(1978)284	5(1979)237s, 239c, 241c
400	4(1978)284	5(1979)243s, 294c
401	5(1979)14	5(1979)267c, 267s
402	5(1979)15	5(1979)267s, 268c
403	5(1979)15	5(1979)269s
404	5(1979)15	5(1979)270s
405	5(1979)15	5(1979)272s
406	5(1979)16	5(1979)273s
407	5(1979)16	5(1979)273s, 275c
408	5(1979)16	5(1979)295s 9(1983)114c
409	5(1979)16	5(1979)277s
410	5(1979)17	5(1979)296c, 298c
411	5(1979)46	5(1979)299s, 300s
412	5(1979)47	5(1979)300s, 301s 6(1980)214a
413	5(1979)47	5(1979)302s, 303c
414	5(1979)47	5(1979)304s, 305s, 306c
415	5(1979)47	5(1979)306s, 307c
416	5(1979)47	5(1979)307s
417	5(1979)47	5(1979)309s
418	5(1979)48	6(1980)17s, 18c
419	5(1979)48	6(1980)19s
420	5(1979)48	6(1980)21s
421	5(1979)76	6(1980)23s
422	5(1979)76	6(1980)24s, 25c
423	5(1979)76	6(1980)26s
424	5(1979)77	6(1980)27s, 28c
425	5(1979)77	6(1980)29s
426	5(1979)77	6(1980)30s
427	5(1979)77	6(1980)31s, 49c
428	5(1979)77	6(1980)50s
429	5(1979)77	6(1980)51s
430	5(1979)78	6(1980)52s, 53c
431	5(1979)107	6(1980)55s
432	5(1979)108	6(1980)57s
433	5(1979)108	6(1980)58s
434	5(1979)108	6(1980)59×
435	5(1979)108	6(1980)60s
436	5(1979)109	6(1980)61s, 62c
437	5(1979)109	6(1980)63c, 63s, 64c
438	5(1979)109	6(1980)79s
439	5(1979)109	6(1980)81s
440	5(1979)110	6(1980)83s
441	5(1979)131	6(1980)84s, 85c
442	5(1979)131	6(1980)86s
443	5(1979)132	6(1980)88×
444	5(1979)132	6(1980)90s
445	5(1979)132	6(1980)92s
446	5(1979)132	6(1980)94s
447	5(1979)132	6(1980)115s
448	5(1979)133	6(1980)117s
449	5(1979)133	6(1980)118s
450	5(1979)133	6(1980)120s, 214c
451	5(1979)166	6(1980)122s
452	5(1979)166	6(1980)123c, 123s
453	5(1979)166	6(1980)124s
454	5(1979)166	6(1980)125s
455	5(1979)167	6(1980)127s
456	5(1979)167	6(1980)128s
457	5(1979)167	6(1980)155s
458	5(1979)167	6(1980)158s
459	5(1979)167	6(1980)158s
460	5(1979)167	6(1980)160s
461	5(1979)199	6(1980)161s
462	5(1979)199	6(1980)162c, 162s
463	5(1979)199	6(1980)163s
464	5(1979)200	6(1980)185s, 186s
465	5(1979)200	6(1980)188s, 216a
466	5(1979)200	6(1980)189c, 189s, 252a
467	5(1979)200	6(1980)191s
468	5(1979)200	6(1980)192s
469	5(1979)200	6(1980)193s
470	5(1979)201	6(1980)194s
471	5(1979)228	6(1980)196s 7(1981)240a
472	5(1979)228	6(1980)196s
473	5(1979)229	6(1980)197c
474	5(1979)229	6(1980)198s
475	5(1979)229	6(1980)216s
476	5(1979)229	6(1980)217s
477	5(1979)229	6(1980)218s, 285a
478	5(1979)229	6(1980)219s 11(1985)189c, 190c 13(1987)151c
479	5(1979)229	6(1980)220s
480	5(1979)229	6(1980)222s
481	5(1979)264	6(1980)222s
482	5(1979)265	6(1980)223s
483	5(1979)265	6(1980)227s, 285a
484	5(1979)265	6(1980)253s, 285c
485	5(1979)265	6(1980)256s
486	5(1979)266	6(1980)258s
487	5(1979)266	6(1980)259s
488	5(1979)266	6(1980)260s, 261s, 262s
489	5(1979)266	6(1980)263s, 288a
490	5(1979)266	6(1980)288c
491	5(1979)291	6(1980)290s, 291c 7(1981)20c
492	5(1979)291	6(1980)291s 7(1981)50s, 117a, 277c 8(1982)79c
493	5(1979)291	6(1980)294× 7(1981)51c
494	5(1979)291	6(1980)297×
495	5(1979)291	7(1981)20s
496	5(1979)291	6(1980)323s
497	5(1979)293	6(1980)324s
498	5(1979)293	6(1980)325s
499	5(1979)293	6(1980)327s
500	5(1979)293	6(1980)328s, 329s

Problem	Proposal	References
PS1–1	5(1979)13	5(1979)44s 6(1980)310s
PS1–2	5(1979)13	5(1979)44s
PS1–3	5(1979)13	5(1979)45s
PS2–1	5(1979)13	5(1979)66s
PS2–2	5(1979)13	5(1979)67s
PS2–3	5(1979)13	5(1979)67s
PS3–1	5(1979)13	5(1979)106s
PS3–2	5(1979)14	5(1979)106s
PS3–3	5(1979)14	5(1979)106s
PS4–1	5(1979)44	5(1979)129s
PS4–2	5(1979)44	5(1979)130s
PS4–3	5(1979)44	5(1979)130s
PS5–1	5(1979)66	5(1979)162s
PS5–2	5(1979)66	5(1979)163s
PS5–3	5(1979)66	5(1979)164s
PS6–1	5(1979)105	5(1979)197s
PS6–2	5(1979)105	5(1979)198s
PS6–3	5(1979)105	5(1979)198s
PS7–1	5(1979)259	5(1979)289s

PS7–2	5(1979)259	5(1979)290s
PS7–3	5(1979)259	5(1979)290s
PS8–1	5(1979)288	6(1980)11s
PS8–2	5(1979)289	6(1980)12s
PS8–3	5(1979)289	6(1980)13s

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Problem	Proposal	References
4.2–1		5(1975)45s
4.2–2		5(1975)46s
4.2–3		5(1975)47s
4.2–4		5(1975)47s
5.1–1	5(1975)48	5(1975)94s
5.1–2	5(1975)48	5(1975)95s
5.1–3	5(1975)48	5(1975)95s
5.2–1	5(1975)96	6(1976)92s
5.2–2	5(1975)96	6(1976)93s
5.2–3	5(1975)96	6(1976)43s
6.1–1	6(1976)44	6(1976)92s
6.1–2	6(1976)44	6(1976)93s
6.1–3	6(1976)45	6(1976)93s
6.1–4	6(1976)45	6(1976)93s
6.2–1	6(1976)94	
6.2–2	6(1976)94	
6.2–3	6(1976)94	

FQ

Problem	Proposal	References	
B–141		13(1975)370c	
B–274		13(1975)95c, 95s	14(1976)94c
B–275		13(1975)95s	
B–276		13(1975)96s	
B–277		13(1975)96s	
B–278		13(1975)96s	
B–279		13(1975)96v, 286s	
B–280		13(1975)191s	
B–281		13(1975)191s	
B–282		13(1975)192s	
B–283		13(1975)192s	
B–284		13(1975)192c	
B–285		13(1975)192s	
B–286		13(1975)286s	
B–287		13(1975)286s	
B–288		13(1975)287s	
B–289		13(1975)287s	
B–290		13(1975)287s	
B–291		13(1975)288s	
B–292		13(1975)374s	
B–293		13(1975)374s	
B–294		13(1975)375s	
B–295		13(1975)375s	
B–296		13(1975)376s	
B–297		13(1975)377s	
B–298	13(1975)94	14(1976)94s	
B–299	13(1975)94	14(1976)94s	
B–300	13(1975)94	14(1976)94s	
B–301	13(1975)94	14(1976)95s	
B–302	13(1975)94	14(1976)95s	
B–303	13(1975)95	14(1976)96s	
B–304	13(1975)190	14(1976)188s	
B–305	13(1975)190	14(1976)189s	
B–306	13(1975)190	14(1976)189s	
B–307	13(1975)190	14(1976)190s	
B–308	13(1975)190	14(1976)191s	
B–309	13(1975)191	14(1976)191s	
B–310	13(1975)285	14(1976)287s	
B–311	13(1975)285	14(1976)287s	
B–312	13(1975)285	14(1976)287s	
B–313	13(1975)285	14(1976)288s	
B–314	13(1975)285	14(1976)288s	
B–316	13(1975)373	14(1976)470s	
B–317	13(1975)373	14(1976)471s	
B–318	13(1975)373	14(1976)471s	
B–319	13(1975)373	14(1976)472s	
B–320	13(1975)373	14(1976)472s	
B–321	13(1975)373	14(1976)472s	
B–322	14(1976)93	15(1977)94s	
B–323	14(1976)93	15(1977)94s	
B–324	14(1976)93	15(1977)95s	
B–325	14(1976)93	15(1977)95s	
B–326	14(1976)93	15(1977)95s	
B–327	14(1976)93	15(1977)95s	
B–328	14(1976)188	15(1977)190s	
B–329	14(1976)188	15(1977)190s	
B–330	14(1976)188	15(1977)191s	
B–331	14(1976)188	15(1977)191s	
B–332	14(1976)188	15(1977)191s	
B–333	14(1976)188	15(1977)192s	
B–334	14(1976)286	15(1977)286s	
B–335	14(1976)286	15(1977)286s	
B–336	14(1976)286	15(1977)286s	
B–337	14(1976)286	15(1977)286s	
B–338	14(1976)286	15(1977)287s	
B–339	14(1976)286	15(1977)288s	
B–340	14(1976)470	15(1977)376s	
B–341	14(1976)470	15(1977)376s	
B–342	14(1976)470	15(1977)376s	
B–343	14(1976)470	15(1977)376s	
B–344	14(1976)470	15(1977)377s	
B–345	14(1976)470	15(1977)377s	
B–346	15(1977)93	16(1978)89s	
B–347	15(1977)93	16(1978)89s	
B–348	15(1977)93	16(1978)90s	
B–349	15(1977)93	16(1978)90s	
B–350	15(1977)93	16(1978)91s	
B–351	15(1977)94	16(1978)91s	
B–352	15(1977)189	16(1978)185s	
B–353	15(1977)189	16(1978)185s	
B–354	15(1977)189	16(1978)185s	
B–355	15(1977)189	16(1978)186s	
B–356	15(1977)189	16(1978)186s	
B–357	15(1977)189	16(1978)186s	
B–358	15(1977)285	16(1978)474s	
B–359	15(1977)285	16(1978)474s	
B–360	15(1977)285	16(1978)474s	
B–361	15(1977)285	16(1978)475s	
B–362	15(1977)285	16(1978)475s	
B–363	15(1977)285	16(1978)476s	
B–364	15(1977)375	16(1978)563s	
B–365	15(1977)375	16(1978)563s	
B–366	15(1977)375	16(1978)563s	
B–367	15(1977)375	16(1978)564s	
B–368	15(1977)375	16(1978)564s	
B–369	15(1977)375	16(1978)565s	
B–370	16(1978)88	17(1979)91s	
B–371	16(1978)88	17(1979)91s	18(1980)85c
B–372	16(1978)88	17(1979)92s	
B–373	16(1978)88	17(1979)92s	
B–374	16(1978)88	17(1979)93s	
B–375	16(1978)89	17(1979)93s	
B–376	16(1978)184	17(1979)185s	
B–377	16(1978)184	17(1979)185s	
B–378	16(1978)184	17(1979)185s	
B–379	16(1978)184	17(1979)186s	
B–380	16(1978)184	17(1979)186s	
B–381	16(1978)184	17(1979)187s	
B–382	16(1978)473	17(1979)282c, 282s	
B–383	16(1978)473	17(1979)283c	
B–384	16(1978)473	17(1979)283s	
B–385	16(1978)473	17(1979)283s	
B–386	16(1978)473	17(1979)284c, 284s	
B–387	16(1978)473	17(1979)284c, 284s	
B–388	16(1978)562	17(1979)370s	

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B–389	16(1978)562	17(1979)371s
B–390	16(1978)562	17(1979)371s
B–391	16(1978)562	17(1979)372s
B–392	16(1978)562	17(1979)373s
B–393	16(1978)562	17(1979)373s
B–394	17(1979)90	18(1980)85s
B–395	17(1979)90	18(1980)86s
B–396	17(1979)90	18(1980)87s
B–397	17(1979)90	18(1980)87c, 87s
B–398	17(1979)90	18(1980)88s
B–399	17(1979)90	18(1980)89s, 89v
B–400	17(1979)184	18(1980)187s
B–401	17(1979)184	18(1980)187s
B–402	17(1979)184	18(1980)188s
B–403	17(1979)184	18(1980)188s
B–404	17(1979)184	18(1980)188s
B–405	17(1979)184	18(1980)189s
B–406	17(1979)281	18(1980)274s
B–407	17(1979)281	18(1980)274s
B–408	17(1979)281	18(1980)275c
B–409	17(1979)281	18(1980)275s
B–410	17(1979)282	18(1980)275s
B–411	17(1979)282	18(1980)276s
B–412	17(1979)369	18(1980)371s
B–413	17(1979)369	18(1980)371s
B–414	17(1979)369	18(1980)372s
B–415	17(1979)369	18(1980)372c
B–416	17(1979)370	18(1980)372c
B–417	17(1979)370	18(1980)373s

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H–91		29(1991)186v, 187s
H–123		16(1978)189c
H–125		27(1989)95c
H–152		26(1988)284s
H–179		14(1976)88v
H–188		13(1975)370c
H–206		13(1975)370c
H–211		16(1978)154s 26(1988)90s, 283c
H–213		16(1978)165s 26(1988)91s
H–215		26(1988)285s
H–216		13(1975)90s
H–217		13(1975)91s
H–218		13(1975)92s
H–219		13(1975)185s
H–220		13(1975)187s
H–221		13(1975)188s
H–223		13(1975)370s
H–225		16(1978)569v 17(1979)95s
H–226		13(1975)281s
H–227		13(1975)370s
H–229		13(1975)371s
H–230		14(1976)89s
H–231		14(1976)89s
H–232		14(1976)90s
H–233		14(1976)90s
H–234		14(1976)182s
H–235		14(1976)184s
H–236		14(1976)184s
H–237		14(1976)92v, 186s
H–238		14(1976)282s
H–239		13(1975)370v 14(1976)283s
H–240		14(1976)284s
H–241		13(1975)370c 14(1976)285s
H–243		14(1976)285s
H–244		14(1976)466s
H–245	13(1975)89	14(1976)468s
H–246	13(1975)89	14(1976)469s
H–247	13(1975)89	15(1977)89s
H–248	13(1975)89	15(1977)90s
H–249	13(1975)185	15(1977)91s
H–250	13(1975)185	15(1977)92s
H–251	13(1975)185	15(1977)185s
H–252	13(1975)281	15(1977)187s
H–253	13(1975)281	15(1977)188s
H–254	13(1975)281	17(1979)288r
H–255	13(1975)369	15(1977)281s
H–256	13(1975)369	15(1977)374s
H–257	13(1975)369	15(1977)283s
H–258	14(1976)88	15(1977)284s 16(1978)96a
H–259	14(1976)88	15(1977)284s 16(1978)96a
H–260	14(1976)88	17(1979)288r
H–261	14(1976)182	15(1977)371s
H–262	14(1976)182	15(1977)372s 16(1978)96a
H–263	14(1976)182	15(1977)373s 16(1978)96a
H–264	14(1976)282	16(1978)92s
H–265	14(1976)282	16(1978)94s, 189a
H–266	14(1976)282	16(1978)94s, 189a
H–267	14(1976)466	15(1977)192v 16(1978)190s
H–268	14(1976)466	16(1978)191s, 569a
H–269	15(1977)89	16(1978)478s
H–270	15(1977)89	16(1978)479s, 569a
H–271	15(1977)89	16(1978)480v 17(1979)288r
H–272	15(1977)185	16(1978)567s
H–273	15(1977)185	16(1978)568s
H–274	15(1977)281	17(1979)95s, 192a
H–275	15(1977)281	17(1979)191s
H–276	15(1977)371	17(1979)287s
H–277	15(1977)371	22(1984)91s
H–278	16(1978)92	17(1979)375s 18(1980)96a
H–279	16(1978)92	17(1979)376s 18(1980)96a
H–280	16(1978)92	17(1979)377s 18(1980)96a
H–281	16(1978)188	18(1980)91s, 192a 19(1981)191a
H–282	16(1978)188	18(1980)93s
H–283	16(1978)188	18(1980)94s, 192a 19(1981)191a
H–284	16(1978)188	18(1980)191s 19(1981)384c
H–285	16(1978)477	18(1980)281s
H–286	16(1978)477	18(1980)281s
H–287	16(1978)477	26(1988)283s
H–288	16(1978)477	18(1980)282s
H–289	16(1978)477	18(1980)283s
H–290	16(1978)566	18(1980)285s
H–291	16(1978)566	18(1980)286s
H–292	16(1978)566	18(1980)286s
H–293	16(1978)566	18(1980)287s
H–294	16(1978)567	18(1980)280v, 375s 20(1982)288a
H–295	17(1979)94	18(1980)281v, 376s
H–296	17(1979)94	18(1980)377x
H–297	17(1979)94	18(1980)378s
H–298	17(1979)94	18(1980)379s
H–299	17(1979)189	19(1981)94s
H–300	17(1979)189	
H–301	17(1979)190	19(1981)96s
H–302	17(1979)286	19(1981)190s
H–303	17(1979)286	19(1981)191s
H–304	17(1979)286	
H–305	17(1979)286	19(1981)191x
H–306	17(1979)287	26(1988)286s
H–307	17(1979)374	25(1987)285s 26(1988)90c
H–308	17(1979)374	19(1981)382s
H–309	17(1979)374	
H–310	17(1979)375	19(1981)383s, 384c

FUNCT

Problem	Proposal	References
1.1.1	1(1977/1)23	1(1977/3)25s
1.1.2	1(1977/1)29	1(1977/3)28s
1.1.3	1(1977/1)30	1(1977/4)8s
1.1.4	1(1977/1)30	1(1977/2)31s
1.1.5	1(1977/1)30	1(1977/3)28s
1.1.6	1(1977/1)30	1(1977/3)28s
1.1.7	1(1977/1)30	1(1977/4)9s
1.1.8	1(1977/1)30	1(1977/4)11s
1.1.9	1(1977/1)30	1(1977/4)9s
1.1.10	1(1977/1)30	1(1977/4)13s, 15c

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1.2.1	1(1977/2)23	1(1977/4)22s	
1.2.2	1(1977/2)29	1(1977/5)27s	
1.2.3	1(1977/2)29	1(1977/4)22s	
1.2.4	1(1977/2)30	1(1977/3)27s	
1.2.5	1(1977/2)30	1(1977/3)27s, 27v	
1.2.6	1(1977/2)31	2(1978/1)19s	4(1980/3)27c
1.2.7	1(1977/2)31	1(1977/5)27s	3(1979/1)28s
1.3.1	1(1977/3)6	1(1977/4)31s	
1.3.2	1(1977/3)29	2(1978/3)11s	
1.3.3	1(1977/3)29	1(1977/5)27s	
1.3.4	1(1977/3)29	1(1977/4)31s	
1.3.5	1(1977/3)30	1(1977/4)31s	1(1977/5)28s
1.3.6	1(1977/3)30	1(1977/5)29s	
1.3.7	1(1977/3)30	1(1977/5)29s	
1.4.1	1(1977/4)32	1(1977/5)29s	
1.4.2	1(1977/4)32	1(1977/5)30s	
1.4.3	1(1977/4)32	1(1977/5)31s	
1.4.4	1(1977/4)32	1(1977/5)31s	
1.4.5	1(1977/4)32	1(1977/5)32s	
1.5.1	1(1977/5)32	3(1979/1)28s	
1.5.2	1(1977/5)32	2(1978/2)7s	
1.5.3	1(1977/5)32	2(1978/1)28s	
1.5.4	1(1977/5)32	2(1978/3)29s	
2.1.1	2(1978/1)32	2(1978/5)28s	3(1979/1)21c
2.1.2	2(1978/1)32	2(1978/5)29s	
2.1.3	2(1978/1)32	2(1978/5)29s	
2.1.4	2(1978/1)32	2(1978/3)30s, 32c	
2.2.1	2(1978/2)7	2(1978/3)30s	
2.2.2	2(1978/2)7	2(1978/3)30s	
2.2.3	2(1978/2)27	2(1978/3)31s	
2.2.4	2(1978/2)27	3(1979/1)28s	3(1979/2)29s
2.3.1	2(1978/3)25	2(1978/5)31s	
2.3.2	2(1978/3)32	3(1979/1)30r	3(1979/3)27s
2.3.3	2(1978/3)32	2(1978/4)31s	
2.3.4	2(1978/3)32	2(1978/5)32s	
2.3.5	2(1978/3)32	2(1978/5)30s	
2.4.1	2(1978/4)32	3(1979/1)29s	
2.4.2	2(1978/4)32	3(1979/1)29s	
2.4.3	2(1978/4)32	3(1979/1)29s	
2.4.4	2(1978/4)32	3(1979/1)29s	3(1979/3)27c
2.5.1	2(1978/5)20	3(1979/3)29s	
2.5.2	2(1978/5)32	3(1979/2)29s	
2.5.3	2(1978/5)32	3(1979/2)30s	
2.5.4	2(1978/5)32	3(1979/3)29s	
3.1.1	3(1979/1)30	3(1979/4)27s	
3.1.2	3(1979/1)30	3(1979/4)28x	3(1979/5)26s
3.1.3	3(1979/1)30	3(1979/3)30s	
3.1.4	3(1979/1)30	3(1979/2)30s	
3.1.5	3(1979/1)30	3(1979/4)28s	
3.1.6	3(1979/1)31	3(1979/4)29s	
3.2.1	3(1979/2)31	3(1979/4)29s	
3.2.2	3(1979/2)31	3(1979/5)26s	
3.2.3	3(1979/2)31	3(1979/4)27c	3(1979/5)26s
3.2.4	3(1979/2)31	3(1979/5)27s	
3.2.5	3(1979/2)31	3(1979/4)30s	
3.2.6	3(1979/2)31	3(1979/4)30s	
3.2.7	3(1979/2)32	3(1979/5)27s	
3.2.8	3(1979/2)32	3(1979/5)28s	
3.3.1	3(1979/3)30	3(1979/5)28s	
3.3.2	3(1979/3)32	4(1980/2)27s	
3.3.3	3(1979/3)32	3(1979/5)28s	
3.3.4	3(1979/3)32	3(1979/5)29s	
3.3.5	3(1979/3)32	5(1981/1)27v	5(1981/3)16s
3.4.1	3(1979/4)32	4(1980/2)30s	4(1980/3)29c
3.4.2	3(1979/4)32	4(1980/1)28s	
3.4.3	3(1979/4)32	4(1980/1)28s	
3.5.1	3(1979/5)30	4(1980/1)30s	
3.5.2	3(1979/5)30	4(1980/3)28s	
3.5.3	3(1979/5)30	4(1980/3)30s	
3.5.4	3(1979/5)30	4(1980/3)30s	

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9.13		10(1975/1)7s	
9.14		10(1975/1)8s	
10.1	10(1975/1)6	10(1975/2)6s	
10.2	10(1975/1)6		
10.3	10(1975/1)6	10(1975/2)7s	
10.4	10(1975/1)6	10(1975/4)5s	
10.5	10(1975/1)6	10(1975/2)8s	
10.6	10(1975/2)5	10(1975/3)6s	
10.7	10(1975/2)5	10(1975/3)7s	
10.8	10(1975/2)5		
10.9	10(1975/2)5		
10.10	10(1975/2)6	10(1975/3)8s	
10.11	10(1975/3)4	10(1975/4)6s	
10.12	10(1975/3)4	10(1975/4)6s	11(1976/1)8a
10.13	10(1975/3)4		
10.14	10(1975/3)4		
10.15	10(1975/3)4	10(1975/4)7s	
10.16	10(1975/4)8	11(1976/1)8s	
10.17	10(1975/4)8	11(1976/1)8s	
11.1	11(1976/1)7		
11.2	11(1976/1)7	11(1976/2)9s	
11.3	11(1976/1)7	11(1976/2)10s	
11.4	11(1976/1)7	11(1976/2)11s	
11.5	11(1976/1)8	11(1976/2)11s	
11.6	11(1976/2)7	11(1976/3)6s	
11.7	11(1976/2)7	11(1976/3)6s	
11.8	11(1976/2)7	11(1976/3)7s	
11.9	11(1976/2)7	11(1976/3)8s	
11.10	11(1976/2)7		
11.11	11(1976/3)2	11(1976/4)7s	
11.12	11(1976/3)2		
11.13	11(1976/3)2	11(1976/4)7s	
11.14	11(1976/3)2		
11.15	11(1976/3)2	11(1976/4)8s	
11.16	11(1976/4)5		
11.17	11(1976/4)5		
11.18	11(1976/4)5		
11.19	11(1976/4)5		
11.20	11(1976/4)5		
12.1	12(1977/1)5	12(1977/2)6s	
12.2	12(1977/1)5	12(1977/2)7s	
12.3	12(1977/1)5	12(1977/2)7s	
12.4	12(1977/1)5	12(1977/2)7s	
12.5	12(1977/1)5	12(1977/2)8s	
12.6	12(1977/1)5	12(1977/2)9s	
12.7	12(1977/1)5	12(1977/2)9s	
12.8	12(1977/1)5	12(1977/2)10s	
12.9	12(1977/1)5	12(1977/2)10s	
12.10	12(1977/1)5	12(1977/2)10s	
12.11	12(1977/2)6	12(1977/3)5s	
12.12	12(1977/2)6	12(1977/3)5s	
12.13	12(1977/2)6	12(1977/3)6s	
12.14	12(1977/2)6	12(1977/3)6s	
12.15	12(1977/2)6	12(1977/3)6s	
12.16	12(1977/2)6		
12.17	12(1977/2)6	12(1977/3)7s	
12.18	12(1977/2)6	12(1977/3)7s	
12.19	12(1977/3)4		
12.20	12(1977/3)4	12(1977/4)6s	
12.21	12(1977/3)4	12(1977/4)7s	
12.22	12(1977/3)5		
12.23	12(1977/3)5		
12.24	12(1977/3)5		
12.25	12(1977/3)5		
12.26	12(1977/3)5	12(1977/4)7s	
12.27	12(1977/3)5	12(1977/4)7s	
12.28	12(1977/4)5		
12.29	12(1977/4)5	13(1978/1)9s	
12.30	12(1977/4)5	13(1978/1)10s	
12.31	12(1977/4)5		

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12.32	12(1977/4)6		
13.1	13(1978/1)9		
13.2	13(1978/1)9	13(1978/2)6s	
13.3	13(1978/1)9	13(1978/2)7s	
13.4	13(1978/1)9	13(1978/2)7s	
13.5	13(1978/1)9	13(1978/2)7s	
13.6	13(1978/1)9	13(1978/2)7s	
13.7	13(1978/1)9		
13.8	13(1978/1)9		
13.9	13(1978/2)5		
13.10	13(1978/2)5		
13.11	13(1978/2)5	13(1978/3)6s	
13.12	13(1978/2)5	13(1978/3)7s	
13.13	13(1978/2)5		
13.14	13(1978/2)5	13(1978/3)7s	
13.15	13(1978/2)5		
13.16	13(1978/2)5		
13.17	13(1978/2)5		
13.18	13(1978/2)5		
13.19	13(1978/3)6	13(1978/4)6s	
13.20	13(1978/3)6	13(1978/4)6s	
13.21	13(1978/3)6	13(1978/4)7s	
13.22	13(1978/3)6	13(1978/4)7s	
13.23	13(1978/3)6	13(1978/4)8s	
13.24	13(1978/4)5	14(1979/1)7s	
13.25	13(1978/4)5	14(1979/3)7s	14(1979/4)1c
13.26	13(1978/4)5	14(1979/3)7s	
13.27	13(1978/4)5	14(1979/3)8s	
13.28	13(1978/4)5	14(1979/3)8s	
14.1	14(1979/1)6	14(1979/2)6s	
14.2	14(1979/1)6	14(1979/2)7s	
14.3	14(1979/1)6	14(1979/2)7s	
14.4	14(1979/1)6	14(1979/3)6s	
14.5	14(1979/1)7	14(1979/2)8×	14(1979/3)6s
14.6	14(1979/2)6	14(1979/3)3s	
14.7	14(1979/2)6	14(1979/3)3s	
14.8	14(1979/2)6	14(1979/3)3s	
14.9		14(1979/3)4s	
14.10		14(1979/3)4s	
14.11	14(1979/2)6		
14.12	14(1979/2)6	14(1979/3)4s	
14.13	14(1979/2)6	14(1979/3)5s	
14.14	14(1979/2)6	14(1979/3)5s	
14.15	14(1979/3)2		
14.16	14(1979/3)2		
14.17	14(1979/3)3		
14.18	14(1979/3)3		
14.19	14(1979/3)3		
14.20	14(1979/4)4		
14.21	14(1979/4)4		
14.22	14(1979/4)4		
14.23	14(1979/4)4		
14.24	14(1979/4)4		

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J9.20		10(1975/1)6s	
J10.1	10(1975/1)6	10(1975/2)6s	
J10.2	10(1975/1)6	10(1975/2)6s	
J10.3	10(1975/1)6		
J10.4	10(1975/1)6	10(1975/4)2s	
J10.5	10(1975/1)6		
J10.6	10(1975/2)5	10(1975/3)4s	
J10.7	10(1975/2)5	10(1975/3)5s	
J10.8	10(1975/2)5	10(1975/3)5s	
J10.9	10(1975/2)5	10(1975/3)6s	
J10.10	10(1975/2)5	10(1975/3)6s	
J10.11	10(1975/3)3	10(1975/4)2s	
J10.12	10(1975/3)3	10(1975/4)3s	
J10.13	10(1975/3)3	10(1975/4)4s	
J10.14	10(1975/3)4		
J10.15	10(1975/3)4		
J10.16	10(1975/4)8		
J10.17	10(1975/4)8		
J11.1	11(1976/1)6	11(1976/2)8s	
J11.2	11(1976/1)6	11(1976/2)8s	
J11.3	11(1976/1)6		
J11.4	11(1976/1)6	11(1976/2)9s	
J11.5	11(1976/1)6	11(1976/2)9s	
J11.6	11(1976/2)7	11(1976/3)3s	
J11.7	11(1976/2)7	11(1976/3)3s	12(1977/2)6s
J11.8	11(1976/2)7	11(1976/3)4s	
J11.9	11(1976/2)7	11(1976/3)4c, 4v	
J11.10	11(1976/2)7	11(1976/3)5s, 6s	
J11.11	11(1976/3)2	11(1976/4)5s	
J11.12	11(1976/3)2	11(1976/4)6s	
J11–13	11(1976/3)2		
J11.13		11(1976/4)6s	
J11–14	11(1976/3)2		
J11.14		11(1976/4)6s, 7c	
J11.15	11(1976/3)2	11(1976/4)7s	
J11.16	11(1976/4)4	12(1977/1)5s	12(1977/2)6a
J11.17	11(1976/4)5		
J11.18	11(1976/4)5	12(1977/1)6s	
J11.19	11(1976/4)5		
J11.20	11(1976/4)5	12(1977/1)6s	

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43		8(1976)53×
50		8(1976)55s
58		9(1977)127r, 129× 10(1978)131c
71		9(1977)138s 20(1988)301s
75		8(1976)56s
81a	9(1977)130	10(1978)131s
89		8(1976)59s
92		8(1976)61s
96		8(1976)62s
98		9(1977)139c
112		8(1976)145c 9(1977)208c
117		9(1977)32s
120		9(1977)37s
121		9(1977)38s
126		9(1977)209r
162		9(1977)209c
163		9(1977)41s, 42s
164		8(1976)145c 9(1977)216s
166		8(1976)146s 10(1978)316c
167		9(1977)209c
170		8(1976)147s
175		9(1977)42s
177		8(1976)148s
180		9(1977)141s, 142s, 143s
184		9(1977)45s
185		9(1977)45s, 144s 10(1978)132c 13(1981)141c
198		8(1976)149s
201		9(1977)48c, 50c 10(1978)56c, 316c
202		9(1977)53s, 54s
210		9(1977)54s, 56s
211		9(1977)56s
212		9(1977)58a, 58s, 59c, 60c, 79c
213		9(1977)294r
214		9(1977)218s
216		9(1977)145s 10(1978)316c
217		9(1977)146s
218		8(1976)236c
227		9(1977)147s
228		9(1977)294r
229		9(1977)295r
230		9(1977)295r
232		8(1976)150s, 151s
242		9(1977)220s 10(1978)316c
246		9(1977)61s
247		9(1977)62s

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249		9(1977)63s 10(1978)316c
251		9(1977)64s
258		9(1977)65s
260		9(1977)222s 10(1978)316c
261		9(1977)66s
262		9(1977)66s
281		12(1980)291s
288		9(1977)223s
291		9(1977)224s
294		9(1977)299s
296		8(1976)66s
297		8(1976)66s
298		8(1976)66s 9(1977)151v
299		8(1976)66s
300		8(1976)66s
303		10(1978)58s
306		9(1977)300s 10(1978)316c
309		9(1977)303x 10(1978)160a
311		9(1977)304s
313		9(1977)305s
314		9(1977)306s
315		9(1977)307s
316		10(1978)59s
317		9(1977)308s 10(1978)240a
318		8(1976)67s
319		8(1976)151s
320		8(1976)67s, 68s
321		8(1976)69s
322		8(1976)70s
323		9(1977)308s
324		8(1976)70s, 237c
325		8(1976)70s
326		8(1976)71s
327		8(1976)71s
328		8(1976)71s
329		8(1976)71s
330		8(1976)151s
331		8(1976)151c, 151s
332		8(1976)152s
333		8(1976)152s
334		8(1976)152s
335		8(1976)153s
336		8(1976)153s
337		8(1976)153s
338		8(1976)153s
339		8(1976)153s
340		8(1976)154s
341		9(1977)310s
342		9(1977)310s
343		9(1977)311s
344		9(1977)312s
345		9(1977)312s 10(1978)160a
346		8(1976)155s
347		9(1977)314s
348		9(1977)314s
349		9(1977)315s
350		9(1977)316s
351		9(1977)317s 10(1978)316c
352		9(1977)318s
353		11(1979)47r 12(1980)57x
354		9(1977)319s
355		10(1978)62s
356		10(1978)62s
357		8(1976)155s
358		8(1976)155s
359		8(1976)156s
360		8(1976)156s
361		8(1976)156s
362		8(1976)156s
363		8(1976)156s
364	8(1976)44	9(1977)68s
365	8(1976)44	9(1977)68s
366	8(1976)44	9(1977)69s
367	8(1976)45	9(1977)69s
368	8(1976)45	9(1977)70c, 70s
369	8(1976)45	9(1977)71s
370	8(1976)46	9(1977)72s
371	8(1976)47	9(1977)72s
372	8(1976)47	11(1979)48c
373	8(1976)47	11(1979)49s
374	8(1976)47	9(1977)73s
375	8(1976)47	9(1977)74s
376	8(1976)48	9(1977)151c
377	8(1976)49	9(1977)75s
378	8(1976)49	9(1977)75s, 76s
379	8(1976)49	10(1978)65s
380	8(1976)50	9(1977)76s
381	8(1976)50	9(1977)77s, 78c
382	8(1976)136	10(1978)66s
383	8(1976)137	10(1978)67s
384	8(1976)137	10(1978)68s
385	8(1976)137	10(1978)288s
386	8(1976)137	10(1978)133s
387	8(1976)138	10(1978)134s
388	8(1976)138	11(1979)49r 12(1980)60s
389	8(1976)139	10(1978)135s
390	8(1976)140	10(1978)290s
391	8(1976)140	10(1978)137s
392	8(1976)141	10(1978)140s, 240a
393	8(1976)141	10(1978)291s
394	8(1976)141	10(1978)140s
395	8(1976)141	10(1978)142s, 320a
396	8(1976)141	10(1978)142s
397	8(1976)143	9(1977)226s
398	8(1976)143	9(1977)226s 10(1978)160a
399	8(1976)143	9(1977)227s 10(1978)160a
400	8(1976)143	9(1977)227s 10(1978)160a
401	8(1976)143	9(1977)227s
402	8(1976)143	9(1977)227s
403	8(1976)144	9(1977)228c 10(1978)240a
404	8(1976)144	9(1977)228s
405	8(1976)144	9(1977)228s
406	8(1976)144	9(1977)229s
407	8(1976)144	9(1977)229s
408	8(1976)144	9(1977)229s
409	8(1976)227	9(1977)229s
410	8(1976)227	9(1977)229s
411	8(1976)227	9(1977)229c
412	8(1976)227	9(1977)230s 10(1978)160a
413	8(1976)228	9(1977)230s
414	8(1976)228	9(1977)152v, 230s 10(1978)160a
415	8(1976)228	9(1977)230s
416	8(1976)228	9(1977)152v, 231s 10(1978)160a
417	8(1976)228	9(1977)231s
418	8(1976)228	9(1977)231s
419	8(1976)229	10(1978)214s
420	8(1976)229	10(1978)215s
421	8(1976)230	10(1978)72s
422	8(1976)230	9(1977)152v, 209r 10(1978)216s
423	8(1976)231	10(1978)217s 12(1980)94c
424	8(1976)231	10(1978)217s
425	8(1976)231	10(1978)218s
426	8(1976)231	10(1978)219s
427	8(1976)232	10(1978)292c
428	8(1976)308	9(1977)282s
429	8(1976)308	9(1977)282s 10(1978)240a
430	8(1976)308	9(1977)282s
431	8(1976)308	9(1977)282s
432	8(1976)309	9(1977)283s
433	8(1976)309	9(1977)283s
434	8(1976)309	9(1977)283s
435	8(1976)309	9(1977)283s 10(1978)160a
436	8(1976)309	9(1977)284s
437	8(1976)309	9(1977)284s 10(1978)208c
438	8(1976)310	9(1977)284s
439	8(1976)310	9(1977)284s 10(1978)160a

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440	8(1976)311	10(1978)293s
441	8(1976)311	10(1978)294s
442	8(1976)311	10(1978)295s
443	8(1976)312	10(1978)296s 12(1980)115c
444	8(1976)312	11(1979)50c 12(1980)60s
445	8(1976)312	10(1978)297s
446	8(1976)313	10(1978)298s
447	8(1976)313	10(1978)298s
448	8(1976)313	11(1979)51s
449	9(1977)21	10(1978)42s
450	9(1977)21	10(1978)42s
451	9(1977)21	10(1978)42s
452	9(1977)21	10(1978)42s
453	9(1977)21	10(1978)43s, 240a
454	9(1977)22	10(1978)43s
455	9(1977)22	10(1978)43s
456	9(1977)22	10(1978)43s
457	9(1977)22	10(1978)43s
458	9(1977)22	10(1978)44s
459	9(1977)22	10(1978)44s
460	9(1977)23	10(1978)44s
461	9(1977)23	10(1978)44s
462	9(1977)24	11(1979)52s
463	9(1977)24	10(1978)72s
464	9(1977)25	10(1978)73s
465	9(1977)25	11(1979)54s
466	9(1977)25	10(1978)74s
467	9(1977)25	10(1978)74s, 160a
468	9(1977)26	10(1978)143r
469	9(1977)26	10(1978)75s, 240a
470	9(1977)26	10(1978)144s
471	9(1977)27	11(1979)56s
472	9(1977)27	10(1978)76s
473	9(1977)28	10(1978)77s
474	9(1977)28	10(1978)146s
475	9(1977)28	10(1978)146c
476	9(1977)29	10(1978)77s, 160a
477	9(1977)30	10(1978)47s
478	9(1977)31	10(1978)49r
479	9(1977)31	10(1978)49r
480	9(1977)31	10(1978)50s
481	9(1977)125	10(1978)116s, 240a, 320a
482	9(1977)125	10(1978)116s, 240a, 320a
483	9(1977)125	10(1978)117s, 240a, 320a
484	9(1977)125	10(1978)117s, 240a, 320a
485	9(1977)126	10(1978)117s, 240a
486	9(1977)126	10(1978)117s, 240a
487	9(1977)126	10(1978)118s, 240a, 320a
488	9(1977)126	10(1978)118s, 240a, 320a
489	9(1977)126	10(1978)118s, 240a, 320a
490	9(1977)126	10(1978)118s, 240a, 320a
491	9(1977)126	10(1978)118s, 240a
492	9(1977)126	10(1978)118s, 240a
493	9(1977)130	10(1978)147s
494	9(1977)131	10(1978)220s
495	9(1977)132	10(1978)221c 11(1979)145c
496	9(1977)132	10(1978)148s, 240a
497	9(1977)132	10(1978)148s 12(1980)94c
498	9(1977)132	10(1978)149s
499	9(1977)132	10(1978)150s
500	9(1977)133	10(1978)152s
501	9(1977)133	10(1978)153s
502	9(1977)133	11(1979)146r 12(1980)159s
503	9(1977)134	10(1978)154s, 240a, 320a
504	9(1977)134	10(1978)155s, 240a, 320a
505	9(1977)134	10(1978)156s 12(1980)94c
506	9(1977)134	
507	9(1977)135	11(1979)146c
508	9(1977)136	10(1978)120r
509	9(1977)136	10(1978)121s
510	9(1977)136	10(1978)122r
511	9(1977)137	10(1978)122s, 320a
512	9(1977)137	10(1978)123s, 320a
513	9(1977)137	10(1978)125s, 316v
514	9(1977)206	10(1978)206s, 320a
515	9(1977)206	10(1978)207s, 320a
516	9(1977)206	10(1978)207s, 320a
517	9(1977)206	10(1978)207s, 320a
518	9(1977)206	10(1978)207s, 320a
519	9(1977)207	10(1978)207s, 320a
520	9(1977)207	10(1978)207s, 320a
521	9(1977)207	10(1978)207s, 320a
522	9(1977)207	10(1978)208s, 320a
523	9(1977)207	10(1978)208s, 320a
524	9(1977)207	10(1978)208s, 320a
525	9(1977)207	10(1978)208s, 320a
526	9(1977)207	10(1978)208s, 320a
527	9(1977)210	10(1978)221s, 320a
528	9(1977)210	10(1978)223s, 240a 11(1979)58s
529	9(1977)211	10(1978)223s
530	9(1977)212	10(1978)225s 12(1980)94c
531	9(1977)212	10(1978)225s
532	9(1977)212	10(1978)226s
533	9(1977)212	10(1978)226r 11(1979)218s
534	9(1977)212	10(1978)227s
535	9(1977)213	10(1978)229s
536	9(1977)213	10(1978)230s
537	9(1977)213	10(1978)230s, 240a
538	9(1977)214	10(1978)232s, 320a
539	9(1977)214	10(1978)232r
540	9(1977)214	10(1978)233s
541	9(1977)215	10(1978)235c, 235s, 240a
542	9(1977)280	10(1978)276s, 320a
543	9(1977)280	10(1978)276s
544	9(1977)280	10(1978)276s
545	9(1977)280	10(1978)276s
546	9(1977)280	10(1978)277s
547	9(1977)281	10(1978)277s
548	9(1977)281	10(1978)277s
549	9(1977)281	10(1978)277s
550	9(1977)281	10(1978)277s
551	9(1977)281	10(1978)278s
552	9(1977)281	10(1978)278s
553	9(1977)281	10(1978)278s
554	9(1977)295	10(1978)300s
555	9(1977)296	10(1978)302s
556	9(1977)296	10(1978)303s
557	9(1977)296	11(1979)59s
558	9(1977)296	10(1978)304s
559	9(1977)297	10(1978)306s
560	9(1977)297	10(1978)306s
561	9(1977)297	10(1978)307s
562	9(1977)297	10(1978)308s
563	9(1977)297	10(1978)309s
564	9(1977)298	10(1978)310s
565	9(1977)298	10(1978)311s
566	9(1977)298	10(1978)311c, 311s
567	9(1977)298	10(1978)314s
568	9(1977)298	11(1979)147s
569	9(1977)286	10(1978)279x
570	9(1977)286	10(1978)280s 12(1980)299s
571	9(1977)287	10(1978)281s
572	9(1977)287	10(1978)281r
573	9(1977)287	10(1978)282c 11(1979)132s
574	10(1978)40	11(1979)31s
575	10(1978)40	11(1979)31s
576	10(1978)40	11(1979)31s
577	10(1978)40	11(1979)31s
578	10(1978)41	11(1979)31s
579	10(1978)41	11(1979)32s
580	10(1978)41	11(1979)32s
581	10(1978)41	11(1979)32s
582	10(1978)41	11(1979)32s
583	10(1978)41	11(1979)33s
584	10(1978)41	11(1979)33s
585	10(1978)42	11(1979)33s

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586	10(1978)45	11(1979)43s	
587	10(1978)46	11(1979)137r	
588	10(1978)46	11(1979)45s	
589	10(1978)46	11(1979)46s	
590	10(1978)46	11(1979)137s	
591	10(1978)51	10(1978)316v	11(1979)64r 12(1980)62s
592	10(1978)52	11(1979)64s	
593	10(1978)52	11(1979)65s	
594	10(1978)52	11(1979)65s	
595	10(1978)52	11(1979)66s	
596	10(1978)52	11(1979)66s	
597	10(1978)53	11(1979)67s	
598	10(1978)53	11(1979)69s	
599	10(1978)53	11(1979)70s	
600	10(1978)54	11(1979)70r	12(1980)63s
601	10(1978)54	11(1979)71s	12(1980)64s
602	10(1978)54	11(1979)75s	
603	10(1978)55	11(1979)76s	
604	10(1978)55	11(1979)76s	
605	10(1978)114	11(1979)124s	12(1980)51a
606	10(1978)114	11(1979)124s	12(1980)51a
607	10(1978)114	11(1979)125s	12(1980)51a
608	10(1978)114	11(1979)125s	12(1980)51a
609	10(1978)115	11(1979)125s	12(1980)51a
610	10(1978)115	11(1979)125s	12(1980)51a
611	10(1978)115	11(1979)125s	12(1980)51a
612	10(1978)115	11(1979)125s	12(1980)51a
613	10(1978)115	11(1979)126s	12(1980)51a
614	10(1978)115	11(1979)126s	
615	10(1978)116	11(1979)126s	
616	10(1978)116	11(1979)126s	
617	10(1978)116	11(1979)126s	
618	10(1978)119	11(1979)139s	
619	10(1978)119	11(1979)140c	
620	10(1978)120	11(1979)141s	
621	10(1978)120	11(1979)143s	
622	10(1978)120	11(1979)144r	12(1980)160x
623	10(1978)127	11(1979)149s	
624	10(1978)128	11(1979)150s	
625	10(1978)128	11(1979)304x	
626	10(1978)128	11(1979)151s	
627	10(1978)128	11(1979)152s	
628	10(1978)129	11(1979)220s	
629	10(1978)129	11(1979)152s	12(1980)80a
630	10(1978)129	11(1979)154s	
631	10(1978)129	11(1979)156s	
632	10(1978)130	11(1979)157s	
633	10(1978)204	11(1979)209s	12(1980)50a
634	10(1978)204	11(1979)209s	12(1980)50a
635	10(1978)204	11(1979)209s	12(1980)50a
636	10(1978)204	11(1979)210s	12(1980)50a
637	10(1978)204	11(1979)210s	12(1980)50a
638	10(1978)204	11(1979)210s	12(1980)50a
639	10(1978)204	11(1979)210s	12(1980)50a
640	10(1978)204	11(1979)210s	12(1980)50a
641	10(1978)206	11(1979)210s	12(1980)50a
642	10(1978)206	11(1979)211s	12(1980)50a
643	10(1978)206	11(1979)211s	12(1980)50a
644	10(1978)206	11(1979)211s	
645	10(1978)210	11(1979)223s	
646	10(1978)210	11(1979)224s	
647	10(1978)211	11(1979)226s	
648	10(1978)211	11(1979)226s	
649	10(1978)211	11(1979)227s	
650	10(1978)211	11(1979)227s	
651	10(1978)211	11(1979)229s	
652	10(1978)212	11(1979)230c	
653	10(1978)212	11(1979)230s	
654	10(1978)212	11(1979)231c	12(1980)222c
655	10(1978)212	11(1979)232x	
656	10(1978)213	11(1979)233s	
657	10(1978)213	11(1979)233s	
658	10(1978)213	11(1979)234s	
659	10(1978)213	11(1979)235s	
660	10(1978)274	11(1979)296s	12(1980)50a, 217a
661	10(1978)274	11(1979)296s	12(1980)50a, 217a
662	10(1978)274	11(1979)296s	12(1980)50a, 217a
663	10(1978)274	11(1979)297s	12(1980)50a, 135a, 217a
664	10(1978)275	11(1979)297s	12(1980)50a, 217a
665		12(1980)50a, 217a	
665–1	10(1978)275	11(1979)297s	
665–2	10(1978)275	11(1979)297s	
665–3	10(1978)275	11(1979)297s	
666	10(1978)275	11(1979)298s	12(1980)50a, 217a
667	10(1978)275	11(1979)298s	12(1980)50a, 217a
668	10(1978)275	11(1979)298s	12(1980)50a, 217a
669	10(1978)275	11(1979)298s	12(1980)50a, 217a
670	10(1978)276	11(1979)298s	12(1980)50a, 217a
671	10(1978)283	11(1979)306s	
672	10(1978)283	11(1979)307s	
673	10(1978)284	11(1979)308s	12(1980)80a
674	10(1978)284	11(1979)309s, 320c	12(1980)80a, 240a
675	10(1978)284	11(1979)309s	
676	10(1978)284	11(1979)310s	12(1980)80a
677	10(1978)284	11(1979)310r	12(1980)300s
678	10(1978)284	11(1979)311s	12(1980)80a
679	10(1978)285	11(1979)312s	
680	10(1978)285	11(1979)312r	
681	10(1978)286	11(1979)313s	12(1980)80a
682	10(1978)286	11(1979)314s	12(1980)80a
683	10(1978)287	11(1979)315s	
684	10(1978)287	11(1979)316c	
685	10(1978)287	11(1979)317s	
686	11(1979)28	12(1980)47s, 134a, 217a	
687	11(1979)28	12(1980)47s, 134a	
688	11(1979)28	12(1980)47s, 134a	
689	11(1979)28	12(1980)48s, 134a	
690	11(1979)29	12(1980)48s, 134a	
691	11(1979)29	12(1980)48s, 134a	
692	11(1979)29	12(1980)48s, 134a	
693	11(1979)29	12(1980)48s, 134a	
694	11(1979)29	12(1980)48s, 134a	
695	11(1979)30	12(1980)49s, 134a	
696	11(1979)30	12(1980)49s, 134a	
697	11(1979)30	12(1980)49s, 134a	
698	11(1979)30	12(1980)49s, 50s, 134a, 217a	
699	11(1979)35	12(1980)65s	
700	11(1979)35	12(1980)66c	
701	11(1979)35	12(1980)67c	
702	11(1979)36	12(1980)67s	
703	11(1979)36	12(1980)68s	
704	11(1979)36	12(1980)69s	
705	11(1979)37	12(1980)70s, 240a	
706	11(1979)37	12(1980)70s	
707	11(1979)37	12(1980)71s	
708	11(1979)37	12(1980)72s	
709	11(1979)38	12(1980)73x	
710	11(1979)38	12(1980)74s	
711	11(1979)38	12(1980)75s, 240a	
712	11(1979)38	12(1980)75s	
713	11(1979)38	12(1980)141s	
714	11(1979)39		
715	11(1979)39	12(1980)77s, 240a, 291a	
716	11(1979)122	12(1980)132s, 217a, 291a	
717	11(1979)122	12(1980)132s, 217a, 291a	
718	11(1979)122	12(1980)132s, 217a, 291a	
719	11(1979)122	12(1980)133s, 217a, 291a	
720	11(1979)123	12(1980)133s, 217a, 291a	
721	11(1979)123	12(1980)133s, 291a	
722	11(1979)123	12(1980)133s, 217a, 291a	
723	11(1979)123	12(1980)133s, 217a, 291a	
724	11(1979)123	12(1980)133s, 217a, 291a	
725	11(1979)124	12(1980)134s, 217a, 291a	

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726	11(1979)124	12(1980)134s, 217a, 291a
727	11(1979)124	12(1980)134s
728	11(1979)127	12(1980)145s
729	11(1979)128	12(1980)145s
730	11(1979)128	12(1980)146s
731	11(1979)128	12(1980)147s
732	11(1979)128	12(1980)149s
733	11(1979)129	12(1980)150s
734	11(1979)129	12(1980)151x
735	11(1979)129	12(1980)152s
736	11(1979)129	12(1980)153s
737	11(1979)130	12(1980)154s
738	11(1979)130	12(1980)155s
739	11(1979)130	12(1980)156s
740	11(1979)131	12(1980)157c
741	11(1979)131	12(1980)158s 13(1981)160a
742	11(1979)207	12(1980)214s, 291a
743	11(1979)207	12(1980)214s, 291a
744	11(1979)207	12(1980)214s, 291a
745	11(1979)207	12(1980)215s, 291a
746	11(1979)207	12(1980)215s, 291a
747	11(1979)208	12(1980)215s, 291a
748	11(1979)208	12(1980)215s, 291a
749	11(1979)208	12(1980)215s, 291a
750	11(1979)208	12(1980)215s, 291a
751	11(1979)208	12(1980)216s, 291a
752	11(1979)208	12(1980)216s, 291a
753	11(1979)209	12(1980)216s, 291a
754	11(1979)209	12(1980)216s, 291a
755	11(1979)213	12(1980)222s
756	11(1979)214	12(1980)223s, 320a
757	11(1979)214	12(1980)224c, 224x
758	11(1979)214	12(1980)227c
759	11(1979)214	12(1980)227s
760	11(1979)214	12(1980)228s, 320a
761	11(1979)215	12(1980)228s, 320a
762	11(1979)215	12(1980)229s, 320a
763	11(1979)215	
764	11(1979)215	12(1980)230x, 320a
765	11(1979)215	12(1980)230s
766	11(1979)215	12(1980)231s
767	11(1979)216	12(1980)232s
768	11(1979)216	12(1980)232s
769	11(1979)216	12(1980)233s, 235s
770a	11(1979)216	12(1980)235s
770b	11(1979)294	12(1980)289s 13(1981)55a
771	11(1979)294	12(1980)289s 13(1981)55a, 136a
772	11(1979)294	12(1980)289s 13(1981)55a
773	11(1979)295	12(1980)289s
774	11(1979)295	12(1980)289s 13(1981)55a
775	11(1979)295	12(1980)289s 13(1981)55a, 136a
776	11(1979)295	12(1980)290s 13(1981)55a, 136a
777	11(1979)295	12(1980)290s 13(1981)55a
778	11(1979)295	12(1980)290s 13(1981)55a, 136a
779	11(1979)295	12(1980)290s 13(1981)55a, 136a
780	11(1979)296	12(1980)290s 13(1981)55a
781	11(1979)296	
782	11(1979)299	12(1980)302s
783	11(1979)300	12(1980)302x
784	11(1979)300	12(1980)304s 13(1981)80a, 160a
785	11(1979)300	12(1980)304s 13(1981)80a, 160a
786	11(1979)300	12(1980)305s
787	11(1979)301	12(1980)306s
788	11(1979)301	12(1980)307s
789	11(1979)301	13(1981)300s, 320a
790	11(1979)302	12(1980)310s 13(1981)80a, 160a
791	11(1979)302	12(1980)311s
792	11(1979)302	12(1980)311s
793	11(1979)302	12(1980)312s 13(1981)80a, 160a
794	11(1979)302	12(1980)314s 13(1981)80a, 160a
795	11(1979)303	12(1980)315s 13(1981)80a, 160a
796	11(1979)303	12(1980)315s
797	11(1979)303	12(1980)316s 13(1981)80a
798	11(1979)303	12(1980)317s 13(1981)80a, 160a

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C1	8(1976)233	9(1977)233s
C2	8(1976)234	9(1977)235s, 237s
C3	8(1976)234	9(1977)238s
C4	8(1976)234	9(1977)239s
C5	8(1976)305	10(1978)47r
C6	8(1976)306	9(1977)289s 10(1978)160a
C7	8(1976)306	9(1977)290s
C8	8(1976)306	9(1977)291c, 291s
C9	8(1976)306	9(1977)292s

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56		9(1975/1)50s
57		9(1975/1)50s
58		9(1975/1)51s
59		9(1975/1)51s
60		9(1975/1)52s
61		9(1975/2)51s
62		9(1975/2)52s
63		9(1975/2)52s
64		9(1975/2)53s
65		9(1975/3)45s
66		9(1975/3)47s
67		9(1975/3)47s
68		9(1975/3)49s
69		9(1975/3)50s 11(1977)142s
70	9(1975/1)49	10(1976)43s, 201c 14(1980)155s
71	9(1975/1)49	10(1976)44s
72	9(1975/1)49	10(1976)45s
73	9(1975/1)49	10(1976)45s
74	9(1975/2)51	10(1976)122s
75	9(1975/2)51	10(1976)123s
76	9(1975/2)51	10(1976)124s
77	9(1975/2)51	10(1976)124s
78	9(1975/3)45	10(1976)201s
79	9(1975/3)45	10(1976)201s
80	9(1975/3)45	10(1976)202s
81	9(1975/3)45	10(1976)203s
82	10(1976)43	11(1977)63s, 145c
83	10(1976)43	11(1977)64s, 65s
84	10(1976)43	11(1977)67s
85	10(1976)43	11(1977)67s
86	10(1976)122	11(1977)143s
87	10(1976)122	11(1977)144s
88	10(1976)122	11(1977)144s
89	10(1976)122	11(1977)144s
90	10(1976)122	11(1977)145s
91	10(1976)200	11(1977)222s
92	10(1976)200	11(1977)222s
93	10(1976)200	11(1977)223s
94	10(1976)200	11(1977)224s
95	10(1976)200	11(1977)224s
96	11(1977)63	12(1978)78s
97	11(1977)63	12(1978)79s
98	11(1977)63	12(1978)79s
99	11(1977)63	12(1978)80s
100	11(1977)63	12(1978)80s
101	11(1977)142	12(1978)174s
102	11(1977)142	12(1978)175s
103	11(1977)142	12(1978)175s
104	11(1977)142	12(1978)176s
105	11(1977)221	12(1978)254s
106	11(1977)221	12(1978)255s
107	11(1977)221	12(1978)256s, 256v
108	11(1977)221	12(1978)256s
109	11(1977)222	13(1979)65s
110	12(1978)78	13(1979)67s 14(1980)155c
111	12(1978)78	13(1979)68s
112	12(1978)78	13(1979)69s

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113	12(1978)78	13(1979)69s
114	12(1978)78	13(1979)70s
115	12(1978)173	13(1979)136s
116	12(1978)173	13(1979)137s
117	12(1978)173	13(1979)137s
118	12(1978)173	13(1979)138s
119	12(1978)173	13(1979)138s
120	12(1978)253	13(1979)215s
121	12(1978)253	13(1979)216s
122	12(1978)253	13(1979)217s
123	12(1978)253	13(1979)218s
124	12(1978)254	13(1979)219s
125	13(1979)64	14(1980)73s
126	13(1979)64	14(1980)74s
127	13(1979)64	14(1980)75s
128	13(1979)65	14(1980)75s
129	13(1979)65	14(1980)76s
130	13(1979)135	14(1980)156s
131	13(1979)135	14(1980)157s
132	13(1979)135	14(1980)157s
133	13(1979)135	14(1980)233s
134	13(1979)136	14(1980)234s
135	13(1979)214	14(1980)235s
136	13(1979)214	14(1980)236s
137	13(1979)214	14(1980)237s
138	13(1979)214	15(1981)72s
139	13(1979)215	15(1981)73s

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Problem	Proposal	References	
0.2.1		1(1979/1)54s	
0.2.2		1(1979/1)56s	
0.3.1		1(1979/1)57s	
0.3.2		1(1979/1)58s	
1.1.1	1(1979/1)52		
1.1.2	1(1979/1)53	1(1979/3)58s	
1.1.3	1(1979/1)53	1(1979/2)47s	1(1979/3)58a
1.2.1	1(1979/2)46		
1.2.2	1(1979/2)46	1(1979/3)59s	
1.3.1	1(1979/3)56		
1.3.2	1(1979/3)56		
1.3.3	1(1979/3)57		

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Problem	Proposal	References	
643		56(1983)112s	
879		48(1975)300a	
880		48(1975)53s, 300a	
881		48(1975)54s	
882		48(1975)54s, 301c, 302c	
883		48(1975)55s	
884		48(1975)56s	
885		48(1975)57s	
886		48(1975)58c, 301c	
888		48(1975)300a	
889		48(1975)300a	
893		48(1975)301a	
894		48(1975)117s, 301a	
895		48(1975)118s, 301a	
896		48(1975)119c, 301a	
897		48(1975)120s	
898		48(1975)120s	
899		48(1975)121s	
900		48(1975)121s	
901		48(1975)182s	
902		48(1975)183s	
903		48(1975)184s	
905		48(1975)184s	
906		48(1975)185s, 301a	
907		48(1975)186s	
908		48(1975)241s	
909		48(1975)241s	
910		48(1975)242s	
911		48(1975)244s	
912		48(1975)245s	
913		48(1975)246s	
914		48(1975)247s	49(1976)254c, 254s
915		48(1975)295s, 296s	
916		48(1975)297s	
917		48(1975)297s	
918		48(1975)298s	
919		48(1975)299s	
920		48(1975)300s	
921		48(1975)300s	
922	48(1975)51	49(1976)44s	
923	48(1975)51	49(1976)45s	
924	48(1975)51	49(1976)46s	
925	48(1975)51	49(1976)46s	
926	48(1975)51	49(1976)46s	
927	48(1975)51	49(1976)47s	
928	48(1975)52	49(1976)48s	
929	48(1975)115	49(1976)97s	
930	48(1975)115	49(1976)97s	
931	48(1975)115	49(1976)98s	
932	48(1975)115	49(1976)99s	
933	48(1975)115	49(1976)100s	
934	48(1975)116	49(1976)100s, 254c, 254s	
935	48(1975)116	49(1976)255s	
936	48(1975)116	49(1976)101s	59(1986)179c
937	48(1975)180	49(1976)150s	
938	48(1975)180	49(1976)151s	
939	48(1975)180	49(1976)152s	
940	48(1975)180	49(1976)152s	
941	48(1975)181	49(1976)153s	
942	48(1975)181	49(1976)153s	
943	48(1975)181	49(1976)212s	
944	48(1975)181	49(1976)214s	
945	48(1975)238	49(1976)215s	
946	48(1975)238	49(1976)215s	
947	48(1975)238	49(1976)216s	
948	48(1975)238	49(1976)217s	
949	48(1975)238	49(1976)218s	
950	48(1975)239	49(1976)256s	
951	48(1975)239	49(1976)256s	
952	48(1975)239	49(1976)257s	
953	48(1975)239	50(1977)100s	
954	48(1975)293	49(1976)257s	
955	48(1975)293	50(1977)47s	
956	48(1975)293	49(1976)258s	
957	48(1975)293	50(1977)103s, 104c	
958	48(1975)293	50(1977)49s	
959	48(1975)294	50(1977)50s, 212c	
960	48(1975)294	50(1977)52s	
961	48(1975)294	50(1977)52s	
962	48(1975)294	50(1977)165s	
963	49(1976)43	50(1977)53s	
964	49(1976)43	50(1977)104s	
965	49(1976)43	50(1977)166s	
966	49(1976)43	50(1977)166s	59(1986)52c
967	49(1976)43	50(1977)167s	
968	49(1976)44	50(1977)168s	
969	49(1976)44	50(1977)169s	
970	49(1976)95	50(1977)213s	
971	49(1976)95	50(1977)214s	
972	49(1976)95	50(1977)215s	
973	49(1976)95	49(1976)211v	50(1977)215s
974	49(1976)95	50(1977)216s	
975	49(1976)96	50(1977)266s	
976	49(1976)96	50(1977)267s	
977	49(1976)96	50(1977)268s	
978	49(1976)149	50(1977)268s	
979	49(1976)149	50(1977)269s	
980	49(1976)149	50(1977)270s	

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981	49(1976)149	50(1977)271s
982	49(1976)149	50(1977)271s
983	49(1976)149	51(1978)70s
984	49(1976)150	51(1978)195c, 195s
985	49(1976)150	49(1976)211v 51(1978)70s
986	49(1976)150	51(1978)196s
987	49(1976)150	51(1978)71s
988	49(1976)211	51(1978)71s
989	49(1976)211	51(1978)72s
990	49(1976)211	51(1978)128s
991	49(1976)211	51(1978)129s
992	49(1976)211	51(1978)129s
993	49(1976)212	51(1978)130s
994	49(1976)212	51(1978)130s
995	49(1976)212	51(1978)130s
996	49(1976)252	51(1978)196s
997	49(1976)252	51(1978)198s, 199c, 199s
998	49(1976)252	51(1978)199s
999	49(1976)252	51(1978)200s
1000	49(1976)253	51(1978)201s
1001	49(1976)253	51(1978)246s
1002	49(1976)253	51(1978)247s
1003	50(1977)46	51(1978)247c, 247s
1004	50(1977)46	51(1978)248s
1005	50(1977)46	51(1978)249s
1006	50(1977)46	51(1978)306s
1007	50(1977)46	
1008	50(1977)99	56(1983)113s
1009	50(1977)99	51(1978)307s
1010	50(1977)99	51(1978)307s
1011	50(1977)99	51(1978)308s
1012	50(1977)99	52(1979)48s
1013	50(1977)163	52(1979)48c, 48s
1014	50(1977)163	50(1977)221v 52(1979)318s 58(1985)244c
1015	50(1977)164	
1016	50(1977)164	52(1979)49s
1017	50(1977)164	52(1979)49s
1018	50(1977)164	52(1979)50s
1019	50(1977)164	52(1979)50s
1020	50(1977)164	52(1979)51s
1021	50(1977)211	52(1979)51s
1022	50(1977)211	52(1979)52s
1023	50(1977)211	52(1979)53s
1024	50(1977)211	52(1979)53s
1025	50(1977)265	52(1979)53s
1026	50(1977)265	52(1979)55s
1027	50(1977)265	52(1979)114s
1028	50(1977)265	52(1979)180s
1029	51(1978)69	52(1979)180s, 182c
1030	51(1978)69	52(1979)115s
1031	51(1978)69	52(1979)116s
1032	51(1978)69	52(1979)117s
1033	51(1978)127	52(1979)182s
1034	51(1978)127	52(1979)183c, 183s
1035	51(1978)127	52(1979)259s
1036	51(1978)127	52(1979)260s
1037	51(1978)128	52(1979)319s 58(1985)244c
1038	51(1978)128	52(1979)319s
1039	51(1978)193	52(1979)260s, 261s
1040	51(1978)193	52(1979)261s, 262s
1041	51(1978)193	52(1979)262s 58(1985)244c
1042	51(1978)193	52(1979)263s
1043	51(1978)193	52(1979)320c, 320s
1044	51(1978)194	52(1979)263s
1045	51(1978)194	52(1979)264s
1046	51(1978)194	52(1979)264s
1047	51(1978)194	52(1979)265s
1048	51(1978)245	52(1979)321s 58(1985)244c
1049	51(1978)245	52(1979)322s
1050	51(1978)245	52(1979)322s
1051	51(1978)245	53(1980)50s
1052	51(1978)245	53(1980)50s
1053	51(1978)245	53(1980)51s
1054	51(1978)305	53(1980)52s
1055	51(1978)305	53(1980)53s
1056	51(1978)305	53(1980)54s
1057	51(1978)305	53(1980)113s, 114s
1058	52(1979)46	53(1980)114s
1059	52(1979)46	53(1980)115s
1060	52(1979)46	53(1980)116s
1061	52(1979)46	53(1980)116s
1062	52(1979)46	53(1980)117s
1063	52(1979)47	53(1980)181s
1064	52(1979)47	53(1980)181s, 183s
1065	52(1979)47	53(1980)184s
1066	52(1979)113	53(1980)184s
1067	52(1979)113	53(1980)185s
1068	52(1979)113	53(1980)186c, 186x
1069	52(1979)113	53(1980)245s
1070	52(1979)113	53(1980)245s
1071	52(1979)114	53(1980)247x 54(1981)141s
1072	52(1979)179	53(1980)247s
1073	52(1979)179	
1074	52(1979)258	53(1980)248s
1075	52(1979)258	53(1980)249s
1076	52(1979)258	53(1980)249s
1077	52(1979)258	53(1980)250s
1078	52(1979)258	53(1980)251s
1079	52(1979)258	53(1980)301s
1080	52(1979)316	53(1980)302s
1081	52(1979)316	53(1980)302s
1082	52(1979)316	53(1980)302s
1083	52(1979)316	53(1980)303s
1084	52(1979)317	54(1981)85s
1085	52(1979)317	53(1980)303s
1086	52(1979)317	53(1980)304s
1087	52(1979)317	53(1980)304s
1088	52(1979)317	54(1981)36x

Problem	Proposal	References
Q608	48(1975)52	48(1975)58s
Q609	48(1975)52	48(1975)58s
Q610	48(1975)52	48(1975)58s
Q611	48(1975)52	48(1975)58s
Q612	48(1975)52	48(1975)58s
Q613	48(1975)52	48(1975)58s
Q614	48(1975)116	48(1975)122s
Q615	48(1975)116	48(1975)122s
Q616	48(1975)116	48(1975)122s
Q617	48(1975)116	48(1975)122s
Q618	48(1975)117	48(1975)122s
Q619	48(1975)117	48(1975)122s
Q620	48(1975)181	48(1975)186s
Q621	48(1975)182	48(1975)186s
Q622	48(1975)182	48(1975)186s
Q623	48(1975)182	48(1975)186s
Q624	48(1975)182	48(1975)186s
Q625	48(1975)240	48(1975)248s
Q626	48(1975)240	48(1975)248s
Q627	48(1975)240	48(1975)248s
Q628	48(1975)295	48(1975)302s
Q629	48(1975)295	48(1975)302s
Q630	48(1975)295	48(1975)303s
Q631	49(1976)44	49(1976)48s
Q632	49(1976)44	49(1976)48s
Q633	49(1976)96	49(1976)101s
Q634	49(1976)96	49(1976)101s
Q635	49(1976)150	49(1976)154s
Q636	49(1976)150	49(1976)154s
Q637	49(1976)150	49(1976)154s
Q638	49(1976)212	49(1976)218s
Q639	49(1976)212	49(1976)218s
Q640	49(1976)253	49(1976)258s
Q641	49(1976)253	49(1976)258s
Q642	49(1976)253	49(1976)258s

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Q643	50(1977)47	50(1977)53s
Q644	50(1977)47	50(1977)53s
Q645	50(1977)164	50(1977)169s
Q646	50(1977)164	50(1977)169s
Q647	50(1977)164	50(1977)169s
Q648	50(1977)164	50(1977)169s
Q649	50(1977)266	50(1977)271s
Q650	50(1977)266	50(1977)271s
Q651	51(1978)128	51(1978)130s
Q652	51(1978)128	51(1978)130s
Q653	51(1978)194	51(1978)201s
Q654	51(1978)194	51(1978)201s
Q655	51(1978)246	51(1978)249s
Q656	52(1979)47	52(1979)55s
Q657	52(1979)47	52(1979)55s
Q658	52(1979)114	52(1979)117s
Q659	52(1979)114	52(1979)117s
Q660	52(1979)179	52(1979)184s
Q661	52(1979)179	52(1979)184s
Q662	52(1979)259	52(1979)265s
Q663	52(1979)317	52(1979)323s
Q664	52(1979)317	52(1979)323s

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406		22(1975/1)6s	
407		22(1975/1)6s	
408		22(1975/1)7s	
409		22(1975/1)7s	
410		22(1975/1)7s	
411		22(1975/2)6s	
412		22(1975/2)6s	
413		22(1975/2)7s	
414		22(1975/2)7s	
415		22(1975/1)5v	22(1975/2)7c
		22(1975/3)6s	
416	22(1975/1)5	22(1975/3)6s	
417	22(1975/1)5	22(1975/3)6s	
418	22(1975/1)5	22(1975/3)7s	
419	22(1975/1)5	22(1975/3)7s	
420	22(1975/1)5	22(1975/3)7s	
421	22(1975/2)5	22(1975/4)5s	
422	22(1975/2)5	22(1975/4)5s	
423	22(1975/2)5	22(1975/4)6s	
424	22(1975/2)5	22(1975/4)6s	
425	22(1975/2)5	22(1975/4)7s	
426	22(1975/3)5	23(1976/1)6s	
427	22(1975/3)5	23(1976/1)7s	
428	22(1975/3)5	23(1976/1)7s	
429	22(1975/3)5	23(1976/1)7s	
430	22(1975/3)5	23(1976/1)7s	
431	23(1976/1)8	23(1976/3)8s	
432	23(1976/1)8	23(1976/3)8s	
433	23(1976/2)8	23(1976/4)8s	
434	23(1976/2)8	23(1976/4)8s	
435	23(1976/3)8	24(1977/1)4s	
436	23(1976/3)8	24(1977/1)4s	
437	23(1976/4)8	24(1977/2)5s	
438	23(1976/4)8	24(1977/2)6s	
439	24(1977/1)4	24(1977/3)5s	
440	24(1977/1)4	24(1977/3)5c, 5s	
441	24(1977/2)5	24(1977/4)2s	
442	24(1977/2)5	24(1977/4)2s	
443	24(1977/3)5	25(1978/1)4s	
444	24(1977/3)5	25(1978/1)4s	
445	24(1977/4)2	25(1978/2)4s	
446	24(1977/4)2	24(1977/4)2c	25(1978/2)4s
447	25(1978/1)4	25(1978/5)4s	
448	25(1978/1)4	25(1978/5)4s	
449	25(1978/2)4	25(1978/6)4s	
450	25(1978/2)4	25(1978/6)4s	
451	25(1978/3)4	25(1978/7)2s	
452	25(1978/3)4	25(1978/7)2s	
453	25(1978/4)4	25(1978/8)2s	
454	25(1978/4)4	25(1978/8)2s	
455	25(1978/5)4	26(1979/1)2s	
456	25(1978/5)4	26(1979/1)3s	
457	25(1978/6)4	26(1979/2)3s	
458	25(1978/6)4	26(1979/2)3s	
459	25(1978/7)2	26(1979/3)3s	
460	25(1978/7)2	26(1979/3)3s	
461	25(1978/8)2	26(1979/3)3s	
462	25(1978/8)2	26(1979/4)3s	
463	26(1979/1)2	26(1979/6)2s	
464	26(1979/1)2	26(1979/6)2s	27(1980/1)5c
465	26(1979/1)2	26(1979/6)2c, 2s	
466	26(1979/1)2	26(1979/6)2s, 3c	27(1980/1)5c
467	26(1979/1)2	26(1979/6)3s	27(1980/1)5c
468	26(1979/2)2	26(1979/7)2c, 2s	
469	26(1979/2)2	26(1979/7)2c, 2s	
470	26(1979/2)2	26(1979/7)2s, 2v	
471	26(1979/2)2	26(1979/7)2s	
472	26(1979/2)2	26(1979/7)3c, 3s	27(1980/1)5c
473	26(1979/3)2	26(1979/8)2s	
474	26(1979/3)2	26(1979/8)2c, 2s	
475	26(1979/3)2	26(1979/8)2c, 2s	
476	26(1979/3)2	26(1979/8)2s, 3c	27(1980/1)5c
477	26(1979/3)2	26(1979/8)3c, 3s	27(1980/1)5c
478	26(1979/4)2	27(1980/1)5c, 5s	
479	26(1979/4)2	27(1980/1)5s	
480	26(1979/4)2	27(1980/1)5c, 5s	
481	26(1979/4)2	27(1980/1)6s	
482	26(1979/4)2	27(1980/1)5c, 6s	
483	26(1979/5)2	27(1980/1)5c	27(1980/2)3s
484	26(1979/5)2	27(1980/2)3s	
485	26(1979/5)2	27(1980/2)3s	
486	26(1979/5)2	27(1980/2)4s	
487	26(1979/5)2	27(1980/2)4s	
488	26(1979/6)2	27(1980/1)5c	27(1980/3)2s
489	26(1979/6)2	27(1980/1)5c	27(1980/3)2s
490	26(1979/6)2	27(1980/3)3s	
491	26(1979/6)2	27(1980/1)5c	27(1980/3)3s
492	26(1979/6)2	27(1980/1)5c	27(1980/3)3s
493	26(1979/7)2	27(1980/1)5c	27(1980/3)4s
494	26(1979/7)2	27(1980/3)4s	
495	26(1979/7)2	27(1980/3)4s	
496	26(1979/7)2	27(1980/3)4s	
497	26(1979/7)2	27(1980/3)4s	
498	26(1979/8)2	27(1980/1)5c	27(1980/4)3s
499	26(1979/8)2	27(1980/4)3s	
500	26(1979/8)2	27(1980/1)5c	27(1980/4)3s, 4s
501	26(1979/8)2	27(1980/1)5c	27(1980/4)4s
502	26(1979/8)2	27(1980/4)4s	

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Problem	Proposal	References
372		23(1975)83s
373		23(1975)84s, 85s, 86c
374		23(1975)86s
375		23(1975)88s, 89c
376		23(1975)89s
377		23(1975)90s
378		23(1975)246s
379		23(1975)92s
380		23(1975)248s
381		23(1975)94s
382		23(1975)178s, 179s
383		23(1975)180s
384		23(1975)181s, 182c
385		24(1976)81s
386		24(1976)82s, 83s, 84c
387		23(1975)183s, 184s
388		23(1975)190s
389		23(1975)191s

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390		23(1975)193s
391	23(1975)79	24(1976)190s
392	23(1975)79	23(1975)249s
393	23(1975)80	23(1975)250s
394	23(1975)80	23(1975)251s
395	23(1975)80	24(1976)192s
396	23(1975)81	23(1975)252s
397	23(1975)81	23(1975)252s
398	23(1975)81	24(1976)194s
399	23(1975)82	23(1975)254s, 255s
400	23(1975)82	23(1975)257s
401	23(1975)173	24(1976)84s
402	23(1975)173	24(1976)87s
403	23(1975)174	24(1976)88c, 88s
404	23(1975)174	24(1976)89s, 90s
405	23(1975)174	24(1976)93s
406	23(1975)175	24(1976)95s, 96s
407	23(1975)175	24(1976)98s
408	23(1975)176	24(1976)100s
409	23(1975)176	24(1976)101s
410	23(1975)176	24(1976)103s
411	23(1975)176	24(1976)104s
412	23(1975)176	24(1976)106s
413	23(1975)176	24(1976)107c, 107s, 189c
414	23(1975)242	24(1976)195s, 196s
415	23(1975)242	24(1976)198s, 201c
416	23(1975)242	24(1976)202s, 203c, 204s
417	23(1975)243	25(1977)89s
418	23(1975)243	24(1976)205s
419	23(1975)243	24(1976)206s, 207c
420	23(1975)244	24(1976)210s
421	23(1975)244	24(1976)211s
422	23(1975)244	24(1976)212s, 273c, 273s
423	23(1975)245	24(1976)213s
424	24(1976)77	24(1976)275s, 276c
425	24(1976)77	24(1976)276s
426	24(1976)78	24(1976)277s, 278s, 279c
427	24(1976)78	24(1976)279s
428	24(1976)78	25(1977)190s
429	24(1976)78	25(1977)192s
430	24(1976)79	24(1976)280s
431	24(1976)79	24(1976)281s
432	24(1976)79	24(1976)282s
433	24(1976)80	24(1976)283s
434	24(1976)80	24(1976)284s
435	24(1976)80	24(1976)285s
436	24(1976)184	25(1977)90s, 92c
437	24(1976)184	25(1977)426s
438	24(1976)185	25(1977)428s
439	24(1976)185	25(1977)429s
440	24(1976)185	25(1977)93s
441	24(1976)185	25(1977)93s, 94s
442	24(1976)186	25(1977)431s
443	24(1976)186	25(1977)95s
444	24(1976)187	25(1977)95s
445	24(1976)187	25(1977)97s
446	24(1976)187	25(1977)98s, 99s
447	24(1976)187	25(1977)100s, 101s
448	24(1976)270	25(1977)193s
449	24(1976)270	25(1977)194s
450	24(1976)270	25(1977)196s
451	24(1976)271	25(1977)197s
452	24(1976)271	25(1977)198s
453	24(1976)271	25(1977)199s
454	24(1976)272	25(1977)200s
455	24(1976)272	25(1977)201c, 201s
456	24(1976)272	25(1977)202s
457	24(1976)272	25(1977)204s
458	25(1977)86	26(1978)352s
459	25(1977)86	25(1977)434c, 434s
460	25(1977)87	25(1977)436s
461	25(1977)87	25(1977)438s
462	25(1977)87	25(1977)439s
463	25(1977)87	25(1977)441s
464	25(1977)88	25(1977)442c, 442s
465	25(1977)88	25(1977)443s
466	25(1977)88	25(1977)444s
467	25(1977)88	25(1977)445s
468	25(1977)186	26(1978)235s, 236c
469	25(1977)186	26(1978)237s
470	25(1977)187	26(1978)238s
471	25(1977)187	26(1978)241s
472	25(1977)187	26(1978)242s, 243c, 243s
473	25(1977)187	26(1978)244s
474	25(1977)187	26(1978)245s
475	25(1977)188	26(1978)246s, 248c
476	25(1977)188	26(1978)248c, 250s
477	25(1977)189	26(1978)251s
478	25(1977)423	26(1978)354s
479	25(1977)423	26(1978)356s
480	25(1977)424	26(1978)357s
481	25(1977)424	26(1978)358s
482	25(1977)424	26(1978)359s, 360c
483	25(1977)424	26(1978)361s, 362s
484	25(1977)425	26(1978)363s
485	25(1977)425	26(1978)363s
486	25(1977)425	26(1978)364s
487	25(1977)425	26(1978)365s
488	26(1978)231	26(1978)465s
489	26(1978)231	27(1979)271s
490	26(1978)232	26(1978)466s
491	26(1978)232	26(1978)468s
492	26(1978)232	26(1978)469s, 470s
493	26(1978)232	26(1978)470s
494	26(1978)232	26(1978)471s
495	26(1978)233	27(1979)274s
496	26(1978)233	26(1978)472s
497	26(1978)233	26(1978)474s
498	26(1978)233	26(1978)474s
499	26(1978)234	26(1978)475s
500	26(1978)234	26(1978)476s
501	26(1978)348	27(1979)136s
502	26(1978)348	27(1979)137s
503	26(1978)349	27(1979)137s
504	26(1978)349	27(1979)138s
505	26(1978)349	27(1979)140s
506	26(1978)350	27(1979)142s, 143s
507	26(1978)350	27(1979)143s
508	26(1978)350	27(1979)145s, 146s
509	26(1978)350	27(1979)147s
510	26(1978)351	27(1979)148s
511	26(1978)351	27(1979)150s
512	26(1978)462	27(1979)275s
513	26(1978)462	28(1980)119s, 120c
514	26(1978)463	27(1979)277s
515	26(1978)463	28(1980)120s
516	26(1978)463	27(1979)278s
517	26(1978)463	27(1979)279s, 280c
518	26(1978)463	27(1979)132v, 280s
519	26(1978)464	27(1979)282s
520	26(1978)464	27(1979)282s
521	26(1978)464	27(1979)283s
522	26(1978)464	27(1979)283s, 284s
523	27(1979)132	27(1979)412s
524	27(1979)132	27(1979)414s
525	27(1979)133	27(1979)415s
526	27(1979)133	27(1979)415s
527	27(1979)133	27(1979)417s
528	27(1979)133	28(1980)205s, 206c
529	27(1979)134	27(1979)418s, 419s, 420s
530	27(1979)134	28(1980)207s
531	27(1979)134	27(1979)421s
532	27(1979)134	28(1980)207s
533	27(1979)135	27(1979)422s, 423s
534	27(1979)267	28(1980)122s
535	27(1979)267	28(1980)123s, 124s

536	27(1979)267	28(1980)125s
537	27(1979)268	28(1980)127s, 129s
538	27(1979)268	
539	27(1979)268	
540	27(1979)268	28(1980)130c, 130s
541	27(1979)268	28(1980)131c, 131s, 205a
542	27(1979)269	28(1980)132s
543	27(1979)269	28(1980)133s
544	27(1979)269	28(1980)134s, 135c
545	27(1979)270	28(1980)136s, 137c
546	27(1979)408	28(1980)209s
547	27(1979)408	28(1980)211s
548	27(1979)408	28(1980)213s, 214c
549	27(1979)409	28(1980)214s
550	27(1979)409	28(1980)215s, 216c
551	27(1979)409	29(1981)106s
552	27(1979)409	29(1981)107s
553	27(1979)410	28(1980)216s
554	27(1979)410	29(1981)108s
555	27(1979)410	28(1980)218s
556	27(1979)410	28(1980)219s
557	27(1979)411	28(1980)220c, 220s
558	27(1979)411	28(1980)221s

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27		25(1975)171s
28		25(1975)21s
29		25(1975)21s
30		25(1975)22s
31		25(1975)22c, 55r, 125s
32		25(1975)22c, 56s
33		26(1976)151r 27(1977)54s
34		25(1975)56s, 126s
35		25(1975)126s
36		25(1975)57s
37	25(1975)20	25(1975)127s
38	25(1975)20	25(1975)127c, 127s
39	25(1975)20	25(1975)171s
40	25(1975)20	25(1975)172s 26(1976)150c 27(1977)100s
41	25(1975)55	25(1975)172s
42	25(1975)55	25(1975)172s
43	25(1975)55	25(1975)173s 26(1976)150c
44	25(1975)56	26(1976)19s
45	25(1975)124	26(1976)19s
46	25(1975)124	26(1976)97s
47	25(1975)124	26(1976)98s
48	25(1975)170	26(1976)99s, 150c
49	25(1975)170	26(1976)100s, 101c
50	25(1975)170	26(1976)151c, 151s
51	25(1975)170	26(1976)152x
52	26(1976)18	26(1976)152c 27(1977)51s
53	26(1976)18	26(1976)150c 27(1977)52s
54	26(1976)18	26(1976)152c 27(1977)52x 28(1978)157s
55	26(1976)96	27(1977)53s
56	26(1976)96	27(1977)53s
57	26(1976)97	27(1977)54c, 101s
58	26(1976)97	27(1977)54c, 98c, 137s 28(1978)53c
59	26(1976)151	27(1977)101s
60	26(1976)151	27(1977)102s 28(1978)53s
61	26(1976)151	27(1977)102s
62	27(1977)54	27(1977)137s
63	27(1977)54	27(1977)137s
64	27(1977)54	27(1977)136r 28(1978)78s, 82s
65	27(1977)54	27(1977)138s
66	27(1977)98	28(1978)54s
67	27(1977)99	28(1978)52r, 152s
68	27(1977)99	28(1978)55s
69	27(1977)99	28(1978)56c, 56s
70	27(1977)99	28(1978)56s
71	27(1977)136	28(1978)83s
72	27(1977)136	28(1978)83s
73	27(1977)136	28(1978)84s
74	28(1978)52	28(1978)152s
75	28(1978)52	28(1978)153s
76	28(1978)52	28(1978)154s
77	28(1978)53	28(1978)155s
78	28(1978)77	29(1979)57s
79	28(1978)77	29(1979)58s, 59s
80	28(1978)78	29(1979)60s
81	28(1978)78	29(1979)60s
82	28(1978)78	29(1979)61s
83	28(1978)150	29(1979)84s
84	28(1978)151	29(1979)85s
85	28(1978)151	29(1979)85s, 86s
86	28(1978)151	29(1979)88s
87	29(1979)56	29(1979)146s
88	29(1979)57	29(1979)147s
89	29(1979)57	29(1979)147s
90	29(1979)57	29(1979)148s
91	29(1979)57	29(1979)150s
92	29(1979)83	30(1980)55s
93	29(1979)83	30(1980)55s
94	29(1979)83	30(1980)56s
95	29(1979)84	30(1980)57s
96	29(1979)145	30(1980)170s
97	29(1979)145	30(1980)170s
98	29(1979)145	30(1980)171s
99	29(1979)145	30(1980)172s
100	29(1979)145	30(1980)173s

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OBG1	27(1977)99	27(1977)103s
OBG2	27(1977)136	27(1977)138s
OBG3	28(1978)53	28(1978)57s
OBG4	28(1978)78	28(1978)85s
OBG5	28(1978)78	28(1978)85s
OBG6	28(1978)151	28(1978)157s
OBG7	29(1979)57	29(1979)61s
OBG8	29(1979)84	29(1979)88s
OBG9	29(1979)146	29(1979)150s

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Problem	Proposal	References
14.1.1	14(1975/1)42	
14.1.2	14(1975/1)42	
14.1.3	14(1975/1)42	
14.2.1	14(1975/2)30	
14.2.2	14(1975/2)30	
14.2.3	14(1975/2)30	
14.3.1	14(1975/3)44	
14.3.2	14(1975/3)44	
14.3.3	14(1975/3)44	
15.1.1	15(1976/1)51	
15.1.2	15(1976/1)52	
15.1.3	15(1976/1)52	
15.2.1	15(1976/2)66	15(1976/3)61s
15.2.2	15(1976/2)66	15(1976/3)61s
15.2.3	15(1976/2)66	15(1976/3)61s
15.3.1	15(1976/3)59	
15.3.2	15(1976/3)59	
15.3.3	15(1976/3)60	
15.3.4	15(1976/3)60	
15.3.5	15(1976/3)60	
15.3.6	15(1976/3)60	
15.3.7	15(1976/3)60	
15.3.8	15(1976/3)60	
15.3.9	15(1976/3)60	
15.3.10	15(1976/3)60	
16.1.1	16(1977/1)64	
16.1.2	16(1977/1)64	

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16.1.3	16(1977/1)64	
16.1.4	16(1977/1)64	
16.1.5	16(1977/1)64	
16.1.6	16(1977/1)64	
16.1.7	16(1977/1)64	
16.1.8	16(1977/1)64	
16.1.9	16(1977/1)64	
16.1.10	16(1977/1)65	
16.2.1	16(1977/2)51	
16.2.2	16(1977/2)51	
16.2.3	16(1977/2)52	
16.2.4	16(1977/2)52	
16.2.5	16(1977/2)52	
16.2.6	16(1977/2)52	
16.2.7	16(1977/2)53	
17.1.1	17(1978/1)59	17(1978/3)59s
17.1.2	17(1978/1)59	17(1978/3)59s
17.1.3	17(1978/1)59	17(1978/3)60s
17.1.4	17(1978/1)59	17(1978/3)60s
17.1.5	17(1978/1)59	17(1978/3)60s
17.1.6	17(1978/1)59	17(1978/3)60s
17.1.7	17(1978/1)59	17(1978/3)60s
17.1.8	17(1978/1)59	17(1978/3)60s
17.1.9	17(1978/1)59	17(1978/3)61s
17.2.1	17(1978/2)58	
17.2.2	17(1978/2)58	
17.2.3	17(1978/2)58	
17.2.4	17(1978/2)58	
17.2.5	17(1978/2)58	
17.2.6	17(1978/2)58	
17.2.7	17(1978/2)58	
17.2.8	17(1978/2)58	
17.2.9	17(1978/2)58	
17.3.1	17(1978/3)58	
17.3.2	17(1978/3)58	
17.3.3	17(1978/3)58	
17.3.4	17(1978/3)58	
17.3.5	17(1978/3)58	
17.3.6	17(1978/3)59	
17.3.7	17(1978/3)59	
17.3.8	17(1978/3)59	
17.3.9	17(1978/3)59	
18.1.1	18(1979/1)56	18(1979/1)60s
18.1.2	18(1979/1)56	18(1979/1)60s
18.1.3	18(1979/1)56	18(1979/1)60s
18.1.4	18(1979/1)56	18(1979/1)60s
18.1.5	18(1979/1)56	18(1979/1)60s
18.1.6	18(1979/1)56	18(1979/1)60s
18.1.7	18(1979/1)56	18(1979/1)60s
18.1.8	18(1979/1)57	18(1979/1)60s
18.1.9	18(1979/1)57	18(1979/1)61s
18.2.1	18(1979/2)61	18(1979/2)66s
18.2.2	18(1979/2)62	18(1979/2)66s
18.2.3	18(1979/2)62	18(1979/2)66s
18.2.4	18(1979/2)62	18(1979/2)66s
18.2.5	18(1979/2)62	18(1979/2)66s
18.2.6	18(1979/2)62	18(1979/2)66s
18.2.7	18(1979/2)62	18(1979/2)67s
18.2.8	18(1979/2)63	18(1979/2)67s
18.2.9	18(1979/2)63	18(1979/2)67s
18.3.1	18(1979/3)65	18(1979/3)67s
18.3.2	18(1979/3)65	18(1979/3)67s
18.3.3	18(1979/3)65	18(1979/3)67s
18.3.4	18(1979/3)65	18(1979/3)67s
18.3.5	18(1979/3)65	18(1979/3)67s
18.3.6	18(1979/3)65	18(1979/3)68s
18.3.7	18(1979/3)65	18(1979/3)68s
18.3.8	18(1979/3)65	18(1979/3)68s
18.3.9	18(1979/3)65	18(1979/3)68s

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Problem	Proposal	References
74–11		11(1975/1)23s
74–12		11(1975/1)24c
74–13		11(1975/1)18s
74–14		11(1975/1)20s, 24a
74–15		11(1975/1)20s, 24a
74–16		11(1975/1)21s 11(1975/3)23c
74–17		11(1975/1)21s, 24a
74–18		11(1975/1)22s, 24a
75–1	11(1975/1)16	11(1975/2)18s, 19s
75–2	11(1975/1)16	11(1975/2)20s
75.2–7		12(1976/1)16s
75.2–8		12(1976/1)16s
75.2–9		12(1976/1)17s
75.2–10		12(1976/1)18s
75.2–11		12(1976/1)18s
75.2–12		12(1976/1)19s
75–3	11(1975/1)16	11(1975/2)20s, 21s
75.3–13		12(1976/1)19s
75.3–14		12(1976/1)20s
75.3–15		12(1976/1)21s
75.3–16		12(1976/1)22s
75.3–17		12(1976/1)22s
75.3–18		12(1976/1)23s
75–4	11(1975/1)16	11(1975/2)22s
75–5	11(1975/1)16	11(1975/2)23s
75–6	11(1975/1)16	11(1975/2)24s
75–7	11(1975/2)18	
75–8	11(1975/2)18	
75–9	11(1975/2)18	
75–10	11(1975/2)18	
75–11	11(1975/2)18	
75–12	11(1975/2)18	
75–13	11(1975/3)22	
75–14	11(1975/3)22	
75–15	11(1975/3)22	
75–16	11(1975/3)22	
75–17	11(1975/3)23	
75–18	11(1975/3)23	
76–1	12(1976/1)15	12(1976/2)20s
76–2	12(1976/1)15	12(1976/2)20s
76–3	12(1976/1)15	12(1976/2)21s
76–4	12(1976/1)15	12(1976/2)22s
76–5	12(1976/1)15	12(1976/2)23s
76–6	12(1976/1)15	12(1976/2)24s
76–7	12(1976/2)19	12(1976/3)21s
76–8	12(1976/2)19	12(1976/3)21s
76–9	12(1976/2)19	12(1976/3)22s
76–10	12(1976/2)19	12(1976/3)23s
76–11	12(1976/2)19	12(1976/3)24s
76–12	12(1976/2)19	12(1976/3)24s
76–13	12(1976/3)20	13(1977/1)20s
76–14	12(1976/3)20	13(1977/1)21s
76–15	12(1976/3)20	13(1977/1)21s
76–16	12(1976/3)20	13(1977/1)22s
76–17	12(1976/3)20	13(1977/1)22s
76–18	12(1976/3)20	13(1977/1)24s
77–1	13(1977/1)19	13(1977/2)21s
77–2	13(1977/1)19	13(1977/2)21s, 22s
77–3	13(1977/1)19	13(1977/2)22s
77–4	13(1977/1)19	13(1977/2)23s
77–5	13(1977/1)19	13(1977/2)19v
77–6	13(1977/1)19	13(1977/2)24s
77–7	13(1977/2)19	
77–8	13(1977/2)19	
77–9	13(1977/2)19	
77–10	13(1977/2)19	
77–11	13(1977/2)20	
77–12	13(1977/2)20	
77–13	13(1977/3)19	14(1978/1)16s
77–14	13(1977/3)19	14(1978/1)16s

77–15	13(1977/3)19	14(1978/1)17s	
77–16	13(1977/3)19	14(1978/1)17s	
77–17	13(1977/3)19	14(1978/1)18s	
77–18	13(1977/3)19	14(1978/1)19s	
78–1	14(1978/1)15	14(1978/2)23s	
78–2	14(1978/1)15	14(1978/2)24s	
78–3	14(1978/1)15	14(1978/2)22r	14(1978/3)18s
78–4	14(1978/1)15	14(1978/2)22r	14(1978/3)18s
78–5	14(1978/1)15	14(1978/2)22r	14(1978/3)19s
78–6	14(1978/1)15	14(1978/2)24s	
78–7	14(1978/2)22	14(1978/3)19s	
78–8	14(1978/2)22	14(1978/3)19s	
78–9	14(1978/2)23	14(1978/3)20s	
78–10	14(1978/3)17	15(1979/1)21s	
78–11	14(1978/3)17	15(1979/1)21s	
78–12	14(1978/3)17	15(1979/1)22s	
78–13	14(1978/3)17	15(1979/1)22s	
78–14	14(1978/3)17	15(1979/1)23s	
78–15	14(1978/3)17	15(1979/1)23s	
79–1	15(1979/1)20	15(1979/2)18s	
79–2	15(1979/1)20	15(1979/2)18s	
79–3	15(1979/1)20	15(1979/2)18s	
79–4	15(1979/1)20	15(1979/2)19s	
79–5	15(1979/1)20	15(1979/2)20s	
79–6	15(1979/1)20	15(1979/2)21s	
79–7	15(1979/2)17	16(1980/1)11s	
79–8	15(1979/2)17	16(1980/1)12s	
79–9	15(1979/2)17	16(1980/1)13s	
79–10	15(1979/2)17	16(1980/1)14s	
79–11	15(1979/2)17	16(1980/1)14s	
79–12	15(1979/2)17	16(1980/1)14s	
79–13	15(1979/3)23	16(1980/1)15s	
79–14	15(1979/3)23	16(1980/1)15s	
79–15	15(1979/3)23	16(1980/1)16s	
79–16	15(1979/3)23	16(1980/1)17c	
79–17	15(1979/3)23	16(1980/1)17s	
79–18	15(1979/3)23	16(1980/1)17s	

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G75.1–1	11(1975/1)7	11(1975/1)11s
G75.1–2	11(1975/1)7	11(1975/1)11s
G75.1–3	11(1975/1)7	11(1975/1)11s, 12s
G75.1–4	11(1975/1)7	11(1975/1)13s
G75.1–5	11(1975/1)7	11(1975/1)14s
G75.1–6	11(1975/1)7	11(1975/1)14s
G75.2–1	11(1975/2)6	11(1975/2)11s
G75.2–2	11(1975/2)6	11(1975/2)11s
G75.2–3	11(1975/2)6	11(1975/2)12s
G75.2–4	11(1975/2)6	11(1975/2)13s
G75.2–5	11(1975/2)6	11(1975/2)14s
G75.2–6	11(1975/2)6	11(1975/2)15s
G75.3–1	11(1975/3)12	11(1975/3)18s
G75.3–2	11(1975/3)12	11(1975/3)18s
G75.3–3	11(1975/3)12	11(1975/3)19s
G75.3–4	11(1975/3)12	11(1975/3)20s
G75.3–5	11(1975/3)12	11(1975/3)20s
G75.3–6	11(1975/3)12	11(1975/3)21s
G76.1–1	12(1976/1)6	12(1976/1)10s
G76.1–2	12(1976/1)6	12(1976/1)11s
G76.1–3	12(1976/1)6	12(1976/1)11s
G76.1–4	12(1976/1)6	12(1976/1)12s
G76.1–5	12(1976/1)6	12(1976/1)12s
G76.1–6	12(1976/1)6	12(1976/1)13s
G76.2–1	12(1976/2)7	12(1976/2)11s
G76.2–2	12(1976/2)7	12(1976/2)11s
G76.2–3	12(1976/2)7	12(1976/2)12s
G76.2–4	12(1976/2)7	12(1976/2)13s
G76.2–5	12(1976/2)7	12(1976/2)13s
G76.2–6	12(1976/2)7	12(1976/2)14s
G76.2–7	12(1976/2)7	12(1976/2)14s
G76.3–1	12(1976/3)7	12(1976/3)13s
G76.3–2	12(1976/3)7	12(1976/3)13s
G76.3–3	12(1976/3)7	12(1976/3)14s
G76.3–4	12(1976/3)7	12(1976/3)15s
G76.3–5	12(1976/3)7	12(1976/3)15s
G76.3–6	12(1976/3)7	12(1976/3)16s
G77.1–1	13(1977/1)5	13(1977/1)13s
G77.1–2	13(1977/1)5	13(1977/1)13s
G77.1–3	13(1977/1)5	13(1977/1)13s
G77.1–4	13(1977/1)5	13(1977/1)14s
G77.1–5	13(1977/1)5	13(1977/1)15s
G77.1–6	13(1977/1)5	13(1977/1)16s
G77.2–1	13(1977/2)9	13(1977/2)10s
G77.2–2	13(1977/2)9	13(1977/2)10s
G77.2–3	13(1977/2)9	13(1977/2)14s
G77.2–4	13(1977/2)9	13(1977/2)15s
G77.2–5	13(1977/2)9	13(1977/2)16s
G77.2–6	13(1977/2)9	13(1977/2)17s
G77.3–5		14(1978/1)2v
G78.1–1	14(1978/1)2	14(1978/1)3s
G78.1–2	14(1978/1)2	14(1978/1)4s
G78.1–3	14(1978/1)2	14(1978/1)4s
G78.1–4	14(1978/1)2	14(1978/1)5s
G78.1–5	14(1978/1)2	14(1978/1)5s
G78.1–6	14(1978/1)2	14(1978/1)7s
G78.2–1	14(1978/2)13	14(1978/2)14s
G78.2–2	14(1978/2)13	14(1978/2)14s
G78.2–3	14(1978/2)13	14(1978/2)14s
G78.2–4	14(1978/2)13	14(1978/2)15s
G78.2–5	14(1978/2)13	14(1978/2)16s
G78.3–1	14(1978/3)7	14(1978/3)7s
G78.3–2	14(1978/3)7	14(1978/3)8s
G78.3–3	14(1978/3)7	14(1978/3)8s
G78.3–4	14(1978/3)7	14(1978/3)9s
G78.3–5	14(1978/3)7	14(1978/3)10s
G78.3–6	14(1978/3)7	14(1978/3)10s
G79.1–1	15(1979/1)6	15(1979/1)7s
G79.1–2	15(1979/1)6	15(1979/1)8s
G79.1–3	15(1979/1)6	15(1979/1)9s
G79.1–4	15(1979/1)6	15(1979/1)10s
G79.1–5	15(1979/1)6	15(1979/1)11s
G79.1–6	15(1979/1)6	15(1979/1)11s
G79.2–1	15(1979/2)9	15(1979/2)10s
G79.2–2	15(1979/2)9	15(1979/2)10s
G79.2–3	15(1979/2)9	15(1979/2)10s
G79.2–4	15(1979/2)9	15(1979/2)10s
G79.2–5	15(1979/2)9	15(1979/2)10s
G79.2–6	15(1979/2)9	15(1979/2)11s
G79.2–7	15(1979/2)9	15(1979/2)11s
G79.2–8	15(1979/2)9	15(1979/2)11s
G79.3–1	15(1979/3)12	15(1979/3)12s
G79.3–2	15(1979/3)12	15(1979/3)13s
G79.3–3	15(1979/3)12	15(1979/3)13s
G79.3–4	15(1979/3)12	15(1979/3)13s
G79.3–5	15(1979/3)12	15(1979/3)14s
G79.3–6	15(1979/3)12	15(1979/3)14s

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Problem	Proposal	References	
251		11(1975/1)20s	11(1975/2)34a
252		11(1975/1)20s	11(1975/2)34a
253		11(1975/1)21s	11(1975/2)34a
254		11(1975/1)21s	
255		11(1975/1)22s	11(1975/2)34a
256		11(1975/1)23s	11(1975/2)34a
257		11(1975/1)23s	11(1975/2)34a
258		11(1975/1)24s	11(1975/2)34a
259		11(1975/1)24s	11(1975/2)34a
260		11(1975/1)25s	
261	11(1975/1)18	11(1975/2)27s	
262	11(1975/1)18	11(1975/2)27s	
263	11(1975/1)18	11(1975/2)28s	
264	11(1975/1)18	11(1975/2)28s	
265	11(1975/1)19	11(1975/2)29s	
266	11(1975/1)19	11(1975/2)30s	

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267	11(1975/1)19	11(1975/2)31s	
268	11(1975/1)19	11(1975/2)31s	
269	11(1975/1)19	11(1975/2)32s	
270	11(1975/1)20	11(1975/2)32s	
271	11(1975/1)20	11(1975/2)9c, 33s	
272	11(1975/1)20	11(1975/2)34s	
273	11(1975/2)25	11(1975/3)20s	12(1976/1)32a
274	11(1975/2)25	11(1975/3)21s	
275	11(1975/2)25	11(1975/3)22s	12(1976/1)32a
276	11(1975/2)26	11(1975/3)22s	12(1976/1)32a
277	11(1975/2)26	11(1975/3)22s	12(1976/1)32a
278	11(1975/2)26	11(1975/3)23s	12(1976/1)32a
279	11(1975/2)26	11(1975/3)23s	12(1976/1)32a
280	11(1975/2)26	11(1975/3)23s	
281	11(1975/2)26	11(1975/3)24s	12(1976/1)32a
282	11(1975/2)26	11(1975/3)24s	12(1976/1)32a
283	11(1975/2)26	11(1975/3)25s	
284	11(1975/2)27	11(1975/3)25s	
285	11(1975/3)18	12(1976/1)25s	12(1976/3)32a
286	11(1975/3)18	12(1976/1)25s	12(1976/3)32a
287	11(1975/3)18	12(1976/1)25s	12(1976/3)32a
288	11(1975/3)19	12(1976/1)26s	12(1976/3)32a
289	11(1975/3)19	12(1976/1)27s	12(1976/3)32a
290	11(1975/3)19	12(1976/1)27s	12(1976/3)32a
291	11(1975/3)19	12(1976/1)28s	12(1976/3)32a
292	11(1975/3)19	12(1976/1)29s	12(1976/3)32a
293	11(1975/3)19	12(1976/1)30s	12(1976/3)32a
294	11(1975/3)20	12(1976/1)30s	12(1976/3)32a
295	11(1975/3)20	12(1976/1)31s	12(1976/3)32a
296	11(1975/3)20	12(1976/1)32s	12(1976/3)32a
297	12(1976/1)22	12(1976/2)29s	12(1976/3)32a
298	12(1976/1)22	12(1976/2)29s	
299	12(1976/1)22	12(1976/2)30s	
300	12(1976/1)22	12(1976/2)30s	
301	12(1976/1)23	12(1976/2)31s	
302	12(1976/1)23	12(1976/2)31s	
303	12(1976/1)23	12(1976/2)32s	
304	12(1976/1)23	12(1976/2)33s	
305	12(1976/1)23	12(1976/2)34s	12(1976/3)32a
306	12(1976/1)24	12(1976/2)34s	12(1976/3)32a
307	12(1976/1)24	12(1976/2)35s	12(1976/3)32a
308	12(1976/1)24	12(1976/2)36s	12(1976/3)32a
309	12(1976/2)26	12(1976/3)26s	
310	12(1976/2)26	12(1976/3)26s	
311	12(1976/2)26	12(1976/3)26s	
312	12(1976/2)26	12(1976/3)27s	
313	12(1976/2)27	12(1976/3)27s	
314	12(1976/2)27	12(1976/3)28s	
315	12(1976/2)27	12(1976/3)29s	
316	12(1976/2)27	12(1976/3)29s	
317	12(1976/2)27	12(1976/3)30s	13(1977/1)36a
318	12(1976/2)28	12(1976/3)30s	13(1977/1)36a
319	12(1976/2)28	12(1976/3)31s	
320	12(1976/2)28	12(1976/3)32s	
321	12(1976/3)23	13(1977/1)27s	
322	12(1976/3)23	13(1977/1)28s	
323	12(1976/3)23	13(1977/1)28s	
324	12(1976/3)23	13(1977/1)28s	
325	12(1976/3)24	13(1977/1)29s	
326	12(1976/3)24	13(1977/1)29s	
327	12(1976/3)24	13(1977/1)30s	
328	12(1976/3)24	13(1977/1)31s	
329	12(1976/3)24	13(1977/1)31s	13(1977/2)36a
330	12(1976/3)25	13(1977/1)32s	
331	12(1976/3)25	13(1977/1)34s	
332	12(1976/3)25	13(1977/1)35s	
333	13(1977/1)24	13(1977/3)27s	
334	13(1977/1)24	13(1977/3)28s	
335	13(1977/1)25	13(1977/3)29s	
336	13(1977/1)25	13(1977/3)30s	
337	13(1977/1)25	13(1977/3)31s	14(1978/1)36a
338	13(1977/1)25	13(1977/3)31s	14(1978/1)36a
339	13(1977/1)26	13(1977/3)32s	14(1978/1)36a
340	13(1977/1)26	13(1977/3)33s	14(1978/1)36a
341	13(1977/1)26	13(1977/3)34s	14(1978/1)36a
342	13(1977/1)26	13(1977/3)35s	14(1978/1)36a
343	13(1977/1)27	13(1977/3)35s	14(1978/1)36a
344	13(1977/1)27	13(1977/3)36s	14(1978/1)26s, 36a
345	13(1977/2)34	14(1978/1)30s	
346	13(1977/2)34	14(1978/1)30s	
347	13(1977/2)34	14(1978/1)30s	
348	13(1977/2)35	14(1978/1)31s	
349	13(1977/2)35	14(1978/1)31s	
350	13(1977/2)35	14(1978/1)32s	
351	13(1977/2)35	14(1978/1)32s	
352	13(1977/2)35	14(1978/1)33s	
353	13(1977/2)35	14(1978/1)33s	
354	13(1977/2)35	14(1978/1)34s	
355	13(1977/2)36	14(1978/1)35s	
356	13(1977/2)36	14(1978/1)35s	
357	13(1977/3)25	14(1978/2)31s, 32c	
358	13(1977/3)25	14(1978/2)32s	
359	13(1977/3)25	14(1978/2)32s	
360	13(1977/3)25	14(1978/2)33s	
361	13(1977/3)26	14(1978/2)34s	
362	13(1977/3)26	14(1978/2)34s	
363	13(1977/3)26	14(1978/2)36s	
364	13(1977/3)26	14(1978/2)36s	
365	13(1977/3)26	14(1978/2)38s	
366	13(1977/3)27	14(1978/2)38s	
367	13(1977/3)27	14(1978/2)39s	
368	13(1977/3)27	14(1978/2)40s	
369	14(1978/1)28	14(1978/3)29s	
370	14(1978/1)28	14(1978/3)30s	
371	14(1978/1)28	14(1978/3)30s	
372	14(1978/1)28	14(1978/3)31s	
373	14(1978/1)28	14(1978/3)31s	
374	14(1978/1)28	14(1978/3)32s	
375	14(1978/1)29	14(1978/3)33s	
376	14(1978/1)29	14(1978/3)33s	
377	14(1978/1)29	14(1978/3)34s	
378	14(1978/1)29	14(1978/3)34s	
379	14(1978/1)29	14(1978/3)35s	
380	14(1978/1)29	14(1978/3)36s	
381	14(1978/2)30	15(1979/1)28s	
382	14(1978/2)30	15(1979/1)29s	15(1979/2)44a
383	14(1978/2)30	15(1979/1)29s	15(1979/2)44a
384	14(1978/2)30	15(1979/1)30s	15(1979/2)44a
385	14(1978/2)30	15(1979/1)31s	15(1979/2)44a
386	14(1978/2)30	15(1979/1)32s	15(1979/2)44a
387	14(1978/2)30	15(1979/1)32s	
388	14(1978/2)31	15(1979/1)33s	15(1979/2)44a
389	14(1978/2)31	15(1979/1)34s	15(1979/2)44a
390	14(1978/2)31	15(1979/1)34s	15(1979/2)44a
391	14(1978/2)31	15(1979/1)35s	
392	14(1978/2)31	15(1979/1)35s	
393	14(1978/3)28	15(1979/2)37s, 38s	15(1979/3)39a
394	14(1978/3)28	15(1979/2)38s	15(1979/3)39a
395	14(1978/3)28	15(1979/2)38s, 39s	
396	14(1978/3)28	15(1979/2)39s	15(1979/3)39a
397	14(1978/3)28	15(1979/2)40s	15(1979/3)39a
398	14(1978/3)29	15(1979/2)40s, 41s	15(1979/3)39a
399	14(1978/3)29	15(1979/2)41s	15(1979/3)39a
400	14(1978/3)29	15(1979/2)42s	15(1979/3)40s
401	14(1978/3)29	15(1979/2)42s	
402	14(1978/3)29	15(1979/2)43s	15(1979/3)39a
403	14(1978/3)29	15(1979/2)43s	
404	14(1978/3)29	15(1979/2)44s	
405	15(1979/1)26	15(1979/3)32s	
406	15(1979/1)26	15(1979/3)33s	
407	15(1979/1)26	15(1979/3)33s	
408	15(1979/1)26	15(1979/3)34s	
409	15(1979/1)26	15(1979/3)34s	

410	15(1979/1)27	15(1979/3)35s
411	15(1979/1)27	15(1979/3)36s
412	15(1979/1)27	15(1979/3)36s
413	15(1979/1)27	15(1979/3)37s
414	15(1979/1)28	15(1979/3)38s
415	15(1979/1)28	15(1979/3)38s
416	15(1979/1)28	15(1979/3)39s
417	15(1979/2)36	
418	15(1979/2)36	
419	15(1979/2)36	
420	15(1979/2)36	
421	15(1979/2)36	
422	15(1979/2)36	
423	15(1979/2)36	
424	15(1979/2)37	
425	15(1979/2)37	
426	15(1979/2)37	
427	15(1979/2)37	
428	15(1979/2)37	
429	15(1979/3)31	
430	15(1979/3)31	
431	15(1979/3)31	
432	15(1979/3)31	
433	15(1979/3)31	
434	15(1979/3)31	
435	15(1979/3)31	
436	15(1979/3)31	
437	15(1979/3)31	
438	15(1979/3)31	
439	15(1979/3)32	
440	15(1979/3)32	

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Q415		16(1980/1)22s

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Problem	Proposal	References
262		34(1975)105s
263		34(1975)106s, 107c
264		34(1975)108s
265		34(1975)109s, 110c
266		34(1975)110s, 111s
267		35(1975)34s
268		35(1975)35s, 36s
269		35(1975)36s
270		35(1975)37s
271		35(1975)38s
272	34(1975)103	35(1976)98s
273	34(1975)103	35(1976)99s
274	34(1975)103	35(1976)99c, 99s
275	34(1975)104	35(1976)100s
276	34(1975)104	35(1976)101s, 102c
277	35(1975)33	36(1976)32s, 33c
278	35(1975)33	36(1976)33s
279	35(1975)33	36(1976)34s
280	35(1975)33	36(1976)35s
281	35(1975)34	36(1976)35s
282	35(1976)97	36(1977)94s
283	35(1976)97	36(1977)95s
284	35(1976)97	36(1977)96s
285	35(1976)97	36(1977)97s
286	35(1976)98	36(1977)98s
287	36(1976)31	37(1977)27s
288	36(1976)31	37(1977)28s
289	36(1976)31	37(1977)29s, 30c
290	36(1976)31	36(1977)93c 37(1977)32s
291	36(1976)32	36(1977)93c 37(1977)33s, 34s
292	36(1977)93	37(1978)83s, 84c, 84s
293	36(1977)93	37(1978)85c, 85s
294	36(1977)93	37(1978)86s
295	36(1977)94	37(1978)87s
296	36(1977)94	37(1978)88s
297	37(1977)26	37(1978)82v 38(1978)28c 38(1979)80s
298	37(1977)26	38(1978)28s
299	37(1977)26	38(1978)30c, 30s
300	37(1977)26	38(1978)31s
301	37(1977)27	38(1978)32s
302	37(1978)82	38(1979)80s
303	37(1978)82	38(1979)81s
304	37(1978)82	38(1979)81s
305	37(1978)82	38(1979)82s
306	37(1978)83	38(1979)83s
307	38(1978)26	39(1979)32s
308	38(1978)27	39(1979)34s
309	38(1978)27	39(1979)35s
310	38(1978)27	39(1979)36s
311	38(1978)27	39(1979)38s
312	38(1979)78	39(1980)102s
313	38(1979)79	39(1980)103s
314	38(1979)79	39(1980)105s 40(1980)46a 40(1981)114c
315	38(1979)79	39(1980)105s, 107s 40(1980)46a 40(1981)114c
316	38(1979)79	39(1980)108s
317	39(1979)30	40(1980)38s 40(1981)114c
318	39(1979)30	40(1980)40s
319	39(1979)30	40(1980)41s
320	39(1979)31	40(1980)43s
321	39(1979)31	40(1980)44s

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Problem	Proposal	References
120		7(1983)609r
136		7(1983)609r
144		7(1983)610r
190		7(1983)610r
213		7(1984)673c, 673s, 673v
239		7(1983)610r 8(1984)44c
278		7(1983)610r
292		6(1975)107s, 108s
294		6(1975)193c
297		7(1980)206c
298		6(1975)192s
304		6(1975)122a
313		6(1975)109s
314		6(1975)109s
315		6(1975)110s
316		6(1975)111s
317		6(1975)112s
318		6(1975)113s
319		6(1975)115s, 116s, 117c, 118c
320		6(1975)118s
321		6(1975)119s
322		6(1975)119s
323		6(1975)120s
324		6(1975)121s
325		6(1975)122s
326		6(1975)180s, 181s 6(1976)244c
327		6(1975)182s
328		6(1975)183s
329		6(1975)184s
330		6(1975)185s
331		6(1975)186s, 187c, 187s
332		6(1975)188s
333		6(1975)188s
334		6(1975)189s
335		6(1975)190s
336		6(1975)191s
337		6(1975)191s
338	6(1975)104	6(1976)228s, 324a
339	6(1975)104	6(1976)230s
340	6(1975)104	6(1976)231s
341	6(1975)105	6(1976)232s, 309c

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342	6(1975)105	6(1976)233s
343	6(1975)105	6(1976)236s
344	6(1975)105	6(1976)237s
345	6(1975)106	6(1976)239s
346	6(1975)106	6(1976)240s
347	6(1975)106	6(1976)242s
348	6(1975)106	6(1976)242s
349	6(1975)106	6(1976)243s
350	6(1975)177	6(1976)310s
351	6(1975)178	6(1976)311s
352	6(1975)178	6(1976)312s
353	6(1975)178	6(1976)313s
354	6(1975)178	6(1976)314s
355	6(1975)178	6(1976)315c, 315s 6(1977)381c
356	6(1975)179	6(1976)316s
357	6(1975)179	6(1976)317s
358	6(1975)179	6(1976)317s
359	6(1975)179	6(1976)320s, 320v
360	6(1975)179	6(1976)321s 6(1977)381c
361	6(1975)180	6(1976)323s, 323v, 324c
362	6(1976)226	6(1977)368s, 369s
363	6(1976)227	6(1977)370s, 436s
364	6(1976)227	6(1976)309s 6(1977)371s 6(1978)501c
365	6(1976)227	6(1977)372s, 373c, 374c
366	6(1976)227	6(1977)374s, 435s, 436c
367	6(1976)227	6(1977)375s
368	6(1976)227	6(1977)376s
369	6(1976)227	6(1977)377s
370	6(1976)227	6(1977)378s
371	6(1976)227	6(1977)378s, 379s
372	6(1976)228	6(1977)379s
373	6(1976)228	6(1977)380s
374	6(1976)306	6(1977)421s
375	6(1976)306	6(1977)422s
376	6(1976)306	6(1977)423s, 424s
377	6(1976)306	6(1977)425s
378	6(1976)306	6(1977)426s
379	6(1976)308	6(1977)427s
380	6(1976)308	6(1977)427s
381	6(1976)308	6(1977)428s 6(1978)559c
382	6(1976)308	6(1977)429s
383	6(1976)308	6(1977)431s, 432s
384	6(1976)308	6(1977)434s
385	6(1976)309	6(1977)435s
386	6(1977)364	6(1978)485s, 486c
387	6(1977)365	6(1978)486s, 559c
388	6(1977)365	6(1978)488s
389	6(1977)366	6(1978)488c, 559c
390	6(1977)366	6(1978)489s
391	6(1977)366	6(1978)490s
392	6(1977)366	6(1978)491s, 559a
393	6(1977)366	6(1978)492s
394	6(1977)366	6(1978)493s, 559a
395	6(1977)367	6(1978)495s, 559a
396	6(1977)367	6(1978)496s, 559a
397	6(1977)367	6(1978)497s
398	6(1977)367	6(1978)499s, 500s
399	6(1977)417	6(1978)542s, 543s
400	6(1977)417	6(1978)544s, 545s
401	6(1977)417	6(1978)546s 6(1979)619c
402	6(1977)418	6(1978)550s
403	6(1977)418	7(1983)611r 8(1984)45s
404	6(1977)419	6(1978)551s
405	6(1977)419	6(1978)542v, 551c 7(1979)76s
406	6(1977)419	6(1978)552s, 553c, 554c
407	6(1977)419	6(1978)554s 8(1985)182c
408	6(1977)419	6(1978)555s
409	6(1977)419	6(1978)557s
410	6(1977)420	6(1978)557s
411	6(1977)421	6(1978)558s, 559s
412	6(1978)481	6(1979)620s
413	6(1978)481	6(1979)621s
414	6(1978)482	6(1979)623s
415	6(1978)482	6(1979)624s
416	6(1978)482	6(1979)625s
417	6(1978)483	6(1979)626s
418	6(1978)483	6(1979)627s
419	6(1978)483	6(1979)628c 7(1983)611r 8(1984)46s
420	6(1978)483	6(1979)628s, 629s
421	6(1978)483	6(1979)631s
422	6(1978)484	6(1979)632s
423	6(1978)484	6(1979)615v 7(1980)134s 7(1981)266c 7(1983)611r
424	6(1978)484	6(1979)633s
425	6(1978)539	7(1979)61s
426	6(1978)539	7(1979)62s
427	6(1978)539	7(1979)63s
428	6(1978)540	7(1979)64s
429	6(1978)540	7(1979)65s
430	6(1978)540	7(1979)67s
431	6(1978)540	7(1979)68s
432	6(1978)540	7(1979)69s
433	6(1978)540	7(1979)70s
434	6(1978)541	7(1979)73s
435	6(1978)541	7(1979)73s
436	6(1978)542	7(1979)74s
437	6(1978)542	7(1979)75s
438	6(1979)615	7(1980)135c, 190s 7(1981)267s
439	6(1979)616	7(1980)136s
440	6(1979)616	7(1980)137s
441	6(1979)616	7(1980)137s
442	6(1979)616	7(1980)139s
443	6(1979)617	7(1980)139s
444	6(1979)617	7(1980)140s
445	6(1979)617	7(1980)141s
446	6(1979)618	7(1980)143s
447	6(1979)618	7(1980)145s
448	6(1979)619	7(1980)146s
449	7(1979)57	7(1980)191s
450	7(1979)57	7(1980)191s
451	7(1979)58	7(1980)192s
452	7(1979)58	7(1980)193s, 194s
453	7(1979)58	7(1980)195s
454	7(1979)58	7(1980)195s
455	7(1979)58	7(1980)196s
456	7(1979)58	7(1980)197c 7(1981)262v 7(1983)612r
457	7(1979)58	7(1980)197s, 198c
458	7(1979)59	7(1980)199s
459	7(1979)59	7(1980)200s
460	7(1979)60	7(1980)201s
461	7(1979)60	7(1980)203s

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63–9		27(1985)447c 28(1986)234c
71–19		25(1983)403s
73–2		18(1976)492c, 492s
74–3		17(1975)171s
74–4		17(1975)172s
74–5		17(1975)174s, 175c, 175s
74–8		17(1975)687s
74–9		17(1975)690s, 691c
74–10		17(1975)691s, 693c
74–12		23(1981)102s
74–13		17(1975)693s
74–14		17(1975)694s
74–16		17(1975)695s
74–17		18(1976)119s
74–18		18(1976)120s
74–19		18(1976)121s
74–20		18(1976)123s
74–21		18(1976)126s

74–22		18(1976)130s
75–1	17(1975)167	18(1976)299s, 300c, 763a
		19(1977)148a
75–2	17(1975)167	18(1976)300s, 301s
75–3	17(1975)168	18(1976)302s, 303c, 763a
		19(1977)148a
75–4	17(1975)168	18(1976)303s
75–5	17(1975)169	18(1976)764c, 764s
75–6	17(1975)169	
75–7	17(1975)169	18(1976)305s
75–8	17(1975)565	18(1976)493s 19(1977)737a
75–9	17(1975)565	18(1976)494s
75–10	17(1975)566	18(1976)496s 19(1977)565a
75–11	17(1975)566	18(1976)497s
75–12	17(1975)566	18(1976)497c, 498c, 498s, 500s
		19(1977)334c
75–13	17(1975)567	
75–14	17(1975)567	18(1976)501x
75–15	17(1975)567	18(1976)503c, 503s 19(1977)148a
75–16	17(1975)685	18(1976)766s
75–17	17(1975)685	18(1976)767s, 768c
75–18	17(1975)686	18(1976)769s, 770c 19(1977)148a
75–19	17(1975)686	19(1977)738s
75–20	17(1975)686	18(1976)306v, 770s, 772c
75–21	17(1975)687	18(1976)773c, 773s
76–1	18(1976)117	19(1977)149s, 149x, 335c, 744c
		20(1978)184c
76–2	18(1976)117	19(1977)150s
76–3	18(1976)117	
76–4	18(1976)118	19(1977)153s
76–5	18(1976)118	19(1977)155s
76–6	18(1976)118	19(1977)155s
76–7	18(1976)294	
76–8	18(1976)295	19(1977)329s, 330s, 331c
76–9	18(1976)295	19(1977)331s
76–10	18(1976)296	19(1977)148v, 332x, 737a
		20(1978)183a
76–11	18(1976)296	19(1977)334s
76–12	18(1976)296	
76–13	18(1976)489	19(1977)565s, 737a 20(1978)183a
76–14	18(1976)489	19(1977)567v, 567x
76–15	18(1976)490	19(1977)568s
76–16	18(1976)490	23(1981)104s
76–17	18(1976)491	19(1977)740s 20(1978)856c
76–18	18(1976)762	19(1977)742s
76–19	18(1976)762	20(1978)184s, 863a
76–20	18(1976)762	19(1977)742s
76–21	18(1976)763	
76–22	18(1976)763	19(1977)743s 20(1978)183a
77–1	19(1977)146	20(1978)186s, 856c, 863v
77–2	19(1977)146	20(1978)187s, 863v
77–3	19(1977)147	20(1978)189c, 189s
77–4	19(1977)147	20(1978)190s
77–5	19(1977)148	
77–6	19(1977)328	20(1978)396x 22(1980)102v, 373v
77–7	19(1977)328	20(1978)398s
77–8	19(1977)329	20(1978)595c, 595s
77–9	19(1977)329	20(1978)400c, 400s 21(1979)140a
77–10	19(1977)329	20(1978)400s
77–11	19(1977)563	20(1978)597c, 597s
77–12	19(1977)563	20(1978)598c, 598s, 599c
		21(1979)140a, 258a
77–13	19(1977)564	20(1978)599s
77–14	19(1977)564	20(1978)857x
77–15	19(1977)564	20(1978)601c, 601s, 604a
		21(1979)140a
77–16	19(1977)736	20(1978)858s 21(1979)140a
77–17	19(1977)736	20(1978)859c, 859s
77–18	19(1977)736	20(1978)860s, 862c
77–19	19(1977)737	21(1979)140c, 141s
77–20	19(1977)737	20(1978)862s
78–1	20(1978)181	
78–2	20(1978)182	21(1979)143s, 144c
78–3	20(1978)182	21(1979)145s, 146c, 258a
78–4	20(1978)183	
78–5	20(1978)183	21(1979)146s
78–6	20(1978)394	21(1979)258s, 259c, 259s, 260c
78–7	20(1978)394	21(1979)560s
78–8	20(1978)394	21(1979)261s, 263c
78–9	20(1978)395	
78–10	20(1978)593	21(1979)397s 22(1980)102a
78–11	20(1978)593	21(1979)398s
78–12	20(1978)594	21(1979)398s
78–13	20(1978)594	21(1979)562x
78–14	20(1978)594	21(1979)400s
78–15	20(1978)594	21(1979)401s 22(1980)102a
78–16	20(1978)855	21(1979)564s
78–17	20(1978)855	21(1979)565s
78–18	20(1978)855	21(1979)567s, 568s
78–19	20(1978)855	21(1979)568s
78–20	20(1978)856	21(1979)569s
79–1	21(1979)139	
79–2	21(1979)139	22(1980)99s 23(1981)105c
79–3	21(1979)139	22(1980)100s
79–4	21(1979)139	
79–5	21(1979)140	22(1980)101s 23(1981)113a
79–6	21(1979)256	21(1979)256c
79–7	21(1979)256	22(1980)230s 23(1981)113a
79–8	21(1979)257	22(1980)231s
79–9	21(1979)257	22(1980)232s
79–10	21(1979)257	21(1979)257c 22(1980)234s
79–11	21(1979)395	22(1980)364s
79–12	21(1979)395	22(1980)366s
79–13	21(1979)396	22(1980)369s 23(1981)113a
79–14	21(1979)396	22(1980)369s
79–15	21(1979)396	22(1980)373s
79–16	21(1979)559	22(1980)504x
79–17	21(1979)559	
79–18	21(1979)559	22(1980)508s, 509s
79–19	21(1979)559	22(1980)509s
79–20	21(1979)560	22(1980)503s, 504c

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6.3		7(1975)69c
6.5		7(1975)68s
6.6		7(1975)68s
6.7		7(1975)69s
6.8		7(1975)69s
7.1	7(1975)31	7(1975)102s 8(1976)34c
7.2	7(1975)31	7(1975)103s
7.3	7(1975)31	7(1975)103s
7.4	7(1975)67	8(1976)34s
7.5	7(1975)67	8(1976)34s
7.6	7(1975)67	8(1976)34s
7.7	7(1975)102	8(1976)64s
7.8	7(1975)102	8(1976)65s
7.9	7(1975)102	8(1976)65s
8.1	8(1976)33	8(1976)92s
8.2	8(1976)33	8(1976)92s
8.3	8(1976)33	8(1976)93s
8.4	8(1976)64	9(1977)33s
8.5	8(1976)64	9(1977)33s
8.6	8(1976)64	9(1977)34s
8.7	8(1976)92	9(1977)64s
8.8	8(1976)92	9(1977)65s
8.9	8(1976)92	9(1977)65s
9.1	9(1977)32	9(1977)98s
9.2	9(1977)32	9(1977)98s
9.3	9(1977)32	9(1977)98s
9.4	9(1977)64	10(1978)32s
9.5	9(1977)64	10(1978)32s
9.6	9(1977)64	10(1978)33s
9.7	9(1977)97	10(1978)64s
9.8	9(1977)97	10(1978)64s

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9.9	9(1977)97	10(1978)65s
10.1	10(1978)31	10(1978)97s
10.2	10(1978)31	10(1978)97s
10.3	10(1978)31	10(1978)97v, 99s 11(1979)29c, 64s
10.4	10(1978)63	11(1979)28s
10.5	10(1978)63	11(1979)29s
10.6	10(1978)63	11(1979)29s
10.7	10(1978)97	11(1979)61s
10.8	10(1978)97	11(1979)61s
10.9	10(1978)97	11(1979)62s
11.1	11(1979)28	11(1979)100s
11.2	11(1979)28	11(1979)101s
11.3	11(1979)28	11(1979)101s
11.4	11(1979)61	12(1980)27s
11.5	11(1979)61	12(1980)27s
11.6	11(1979)61	12(1980)27s
11.7	11(1979)100	12(1980)61s
11.8	11(1979)100	12(1980)62s
11.9	11(1979)100	12(1980)62s

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680A		75(1975)563a
680B		78(1978)621s
2496		75(1975)743a
2547		79(1979)80s
2617		77(1977)621s
2928		79(1979)445c
3489		78(1978)714c, 714s
3513		75(1975)199a
3515		75(1975)199a
3517		75(1975)199a
3526		75(1975)199a
3529		75(1975)199a
3530		75(1975)199a
3531		75(1975)199a, 293c
3535		75(1975)199a
3537		75(1975)199a
3541		75(1975)473a
3543		75(1975)199a
3544		75(1975)199s
3545		75(1975)200s
3546		75(1975)201s
3547		75(1975)201s
3548		75(1975)202s
3549		75(1975)202s
3550		75(1975)294s, 381a, 386a, 473a
3551		75(1975)294c, 294s, 381a
3552		75(1975)296s, 381a
3553		75(1975)296s, 381a, 473a, 563c
3554		75(1975)382s
3555		75(1975)382s, 473a, 743a
3556		75(1975)383s, 473a
3557		75(1975)383s, 473a, 563a
3558		75(1975)383s, 473a
3559		75(1975)384s, 473a
3560		75(1975)385s, 473a
3561		75(1975)385s, 473a
3562		75(1975)474s, 563a
3563		75(1975)474s
3564		75(1975)475s
3565		75(1975)475s, 563a
3566		75(1975)476s, 563a
3567		75(1975)476s, 563a
3568	75(1975)204	75(1975)564s
3569	75(1975)204	75(1975)564s
3570	75(1975)204	75(1975)565s, 743a
3571	75(1975)204	75(1975)565s 76(1976)170a
3572	75(1975)204	75(1975)566c, 566s, 743a
3573	75(1975)204	75(1975)567s, 743a
3574	75(1975)297	75(1975)653s
3575	75(1975)297	75(1975)654s 76(1976)261a, 622c
3576	75(1975)297	75(1975)655s, 743a 76(1976)261a, 623c
3577	75(1975)297	75(1975)655s, 743a 76(1976)261a
3578	75(1975)298	75(1975)656s
3579	75(1975)298	75(1975)656c 76(1976)624s
3580	75(1975)386	75(1975)743s 76(1976)439a
3581	75(1975)386	75(1975)744s
3582	75(1975)386	75(1975)744s
3583	75(1975)387	75(1975)568v, 744s
3584	75(1975)387	75(1975)568v, 745s 76(1976)261a
3585	75(1975)387	75(1975)746s
3586	75(1975)477	76(1976)82s, 170a, 442a
3587	75(1975)477	76(1976)83s, 442a, 528c
3588	75(1975)477	76(1976)84s, 442a
3589	75(1975)477	76(1976)84c, 170a, 442a
3590	75(1975)477	76(1976)84s, 442a
3591	75(1975)478	76(1976)85s, 170a, 442a
3592	75(1975)568	76(1976)170s, 261a, 534a
3593	75(1975)568	76(1976)171s, 261a, 265a, 439a
3594	75(1975)568	76(1976)172s, 261a, 528c, 534a
3595	75(1975)568	76(1976)172s, 261a, 445a, 528c
3596	75(1975)568	76(1976)173s, 261a, 265a, 439a, 442a
3597	75(1975)657	76(1976)174s, 262s
3598	75(1975)657	76(1976)262s, 439a, 445a, 534a
3599	75(1975)657	76(1976)263s, 529c, 534a
3600	75(1975)657	76(1976)264s, 445a
3601	75(1975)657	76(1976)264s
3602	75(1975)658	76(1976)264s
3603	75(1975)658	76(1976)265s, 445a, 534a
3606	75(1975)747	76(1976)439s, 534a
3607	75(1975)747	76(1976)440s, 533a, 534a
3608	75(1975)747	76(1976)441s, 534a, 716c
3609	75(1975)748	76(1976)441s, 534a, 623c
3610	75(1975)748	76(1976)441s, 534a
3611	75(1975)748	76(1976)442s, 534a
3612	76(1976)85	76(1976)442s, 533a, 534a
3613	76(1976)85	76(1976)443s, 533a, 534a, 627a, 715c
3614	76(1976)86	76(1976)444s, 534a
3615	76(1976)86	76(1976)444s, 534a
3616	76(1976)86	76(1976)444c, 534a
3617	76(1976)86	76(1976)444c, 529s
3618	76(1976)174	76(1976)531s
3619	76(1976)175	76(1976)532s, 627a 77(1977)358a
3620	76(1976)175	76(1976)533s
3621	76(1976)175	76(1976)625s 77(1977)354c
3622	76(1976)175	76(1976)626s 77(1977)358a
3623	76(1976)175	76(1976)626s
3624	76(1976)266	76(1976)627s 77(1977)358a
3625	76(1976)266	76(1976)717s 77(1977)82a, 174a
3626	76(1976)266	76(1976)717s 77(1977)82a
3627	76(1976)266	76(1976)718x
3628	76(1976)266	76(1976)718s 77(1977)82a
3629	76(1976)266	77(1977)78s
3630	76(1976)445	77(1977)79s
3631	76(1976)445	77(1977)79s, 358a
3632	76(1976)445	77(1977)80s, 532c
3633	76(1976)445	77(1977)82s
3634	76(1976)446	77(1977)170s
3635	76(1976)446	77(1977)170s
3636	76(1976)446	77(1977)171s, 268a
3637	76(1976)446	77(1977)172s
3638	76(1976)446	77(1977)173s, 268a
3639	76(1976)446	77(1977)265s, 358a
3640	76(1976)446	77(1977)265s
3641	76(1976)446	77(1977)266s
3642	76(1976)527	77(1977)266s
3643	76(1976)527	77(1977)267s
3644	76(1976)527	77(1977)355s
3645	76(1976)527	77(1977)355s, 449a
3646	76(1976)528	77(1977)356s

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3647	76(1976)528	77(1977)357s	
3648	76(1976)621	77(1977)357s, 449a	
3649	76(1976)621	77(1977)444s	
3650	76(1976)621	77(1977)445s	
3651	76(1976)622	77(1977)446s	
3652	76(1976)622	77(1977)447s	
3653	76(1976)622	77(1977)447s, 536a	
3654	76(1976)714	77(1977)533s	78(1978)358a
3655	76(1976)714	77(1977)533s	
3656	76(1976)714	77(1977)534s	
3657	76(1976)715	77(1977)535s	
3658	76(1976)715	77(1977)535s	
3659	76(1976)715	77(1977)536s	
3660	77(1977)77	77(1977)622s	
3661	77(1977)77	77(1977)623s	
3662	77(1977)77	77(1977)624s	
3663	77(1977)78	77(1977)625s	
3664	77(1977)78	77(1977)626s	
3665	77(1977)78	77(1977)626s	
3666	77(1977)169	77(1977)714s	
3667	77(1977)169	77(1977)715s	
3668	77(1977)169	77(1977)715s	
3669	77(1977)169	77(1977)716s	
3670	77(1977)170	77(1977)716s	
3671	77(1977)170	77(1977)717s	
3672	77(1977)263	78(1978)82c, 82s	
3673	77(1977)263	78(1978)83s	
3674	77(1977)263	78(1978)84s	
3675	77(1977)263	78(1978)84s	
3676	77(1977)264	78(1978)85s	
3677	77(1977)264	78(1978)86s	
3678	77(1977)353	78(1978)171s	
3679	77(1977)353	78(1978)172s	
3680	77(1977)353	78(1978)172s	79(1979)712c
3681	77(1977)353	78(1978)174s	
3682	77(1977)353	78(1978)174s, 533c	
3683	77(1977)354	78(1978)176s	
3684	77(1977)443	78(1978)354s	
3685	77(1977)443	78(1978)355s	
3686	77(1977)443	78(1978)356s	
3687	77(1977)443	78(1978)356s	
3688	77(1977)444	78(1978)356s, 357s	
3689	77(1977)444	78(1978)357s, 449a	
3690	77(1977)530	78(1978)444s	
3691	77(1977)530	78(1978)445s, 537a	
3692	77(1977)531	78(1978)446s	
3693	77(1977)531	78(1978)447s, 537a	
3694	77(1977)531	78(1978)447s	
3695	77(1977)531	78(1978)448s	
3696	77(1977)620	78(1978)534s	
3697	77(1977)620	78(1978)534s	
3698	77(1977)620	78(1978)535s	
3699	77(1977)621	78(1978)536s	
3700	77(1977)621	78(1978)536s	
3701	77(1977)621	78(1978)622s	
3702	77(1977)713	78(1978)623s, 624s	
3703	77(1977)713	78(1978)624s	
3704	77(1977)713	78(1978)626s	
3705	77(1977)714	78(1978)626s	
3706	77(1977)714	78(1978)714s	
3707	77(1977)714	78(1978)715s	
3708	78(1978)81	78(1978)716s	
3709	78(1978)81	78(1978)717s	
3710	78(1978)81	78(1978)717s	
3711	78(1978)82	79(1979)81c, 81s	
3712	78(1978)82	79(1979)82s	
3713	78(1978)82	79(1979)83s	
3714	78(1978)170	79(1979)84s	
3715	78(1978)170	79(1979)86s	
3716	78(1978)170	79(1979)173s	
3717	78(1978)170	79(1979)173s, 174s	
3718	78(1978)170	79(1979)174s	
3719	78(1978)171	79(1979)175s	
3720	78(1978)353	79(1979)175s	
3721	78(1978)353	79(1979)260s	
3722	78(1978)353	79(1979)261s	
3723	78(1978)354	79(1979)261s	
3724	78(1978)354	79(1979)262s	
3725	78(1978)354	79(1979)264s	
3726	78(1978)443	79(1979)356s	
3727	78(1978)443	79(1979)356s	81(1981)439c
3728	78(1978)443	79(1979)357s, 358s	
3729	78(1978)443	79(1979)358s, 528c	
3730	78(1978)444	79(1979)358s	
3731	78(1978)444	79(1979)360s	
3732	78(1978)532	79(1979)446s, 529c	
3733	78(1978)533	79(1979)447s	
3734	78(1978)533	79(1979)448s	
3735	78(1978)533	79(1979)449s	
3736	78(1978)533	79(1979)449s, 712c	
3737	78(1978)533	79(1979)450s	
3738	78(1978)620	79(1979)529s	
3739	78(1978)620	79(1979)529s	80(1980)80a
3740	78(1978)620	79(1979)530s, 531s	
3741	78(1978)621	79(1979)532s, 717a	
3742	78(1978)621	79(1979)532s	
3743	78(1978)621	79(1979)533s	
3744	78(1978)712	79(1979)713s	
3745	78(1978)712	79(1979)713s	
3746	78(1978)712	79(1979)714s	
3747	78(1978)713	79(1979)716s	
3748	78(1978)713	79(1979)716s	
3749	78(1978)713	79(1979)717s	
3750	79(1979)79	80(1980)77s	
3751	79(1979)79	80(1980)78s	
3752	79(1979)79	80(1980)78s	
3753	79(1979)80	80(1980)78s	
3754	79(1979)80	80(1980)79s	
3755	79(1979)80	80(1980)79s	
3756	79(1979)172	80(1980)174s	
3757	79(1979)172	80(1980)174s	
3758	79(1979)172	80(1980)175s	
3759	79(1979)172	80(1980)176s	
3760	79(1979)173	80(1980)176s	
3761	79(1979)173	80(1980)176s	
3762	79(1979)259	80(1980)264s	
3763	79(1979)259	80(1980)264s	
3764	79(1979)259	80(1980)265s	
3765	79(1979)259	80(1980)265s	
3766	79(1979)259	80(1980)266s	
3767	79(1979)260	80(1980)267s	
3768	79(1979)355	80(1980)350s	
3769	79(1979)355	80(1980)351s	
3770	79(1979)355	80(1980)352s	
3771	79(1979)355	80(1980)353s	
3772	79(1979)356	80(1980)353s	
3773	79(1979)356	80(1980)355s	
3774	79(1979)444	80(1980)442s	
3775	79(1979)444	80(1980)443s, 444s	
3776	79(1979)444	80(1980)444s	
3777	79(1979)444	80(1980)446s	
3778	79(1979)445	80(1980)446s	
3779	79(1979)445	80(1980)447s	
3780	79(1979)527	80(1980)526s	
3781	79(1979)527	80(1980)527s	
3782	79(1979)528	80(1980)528s, 710s	
3783	79(1979)528	80(1980)528s	
3784	79(1979)528	80(1980)529s	
3785	79(1979)528	80(1980)529s	
3786	79(1979)711	80(1980)710s	
3787	79(1979)711	80(1980)711s	
3788	79(1979)711	80(1980)712s	
3789	79(1979)711	80(1980)712s, 714s	
3790	79(1979)712	80(1980)715s	
3791	79(1979)712	80(1980)715s	

Problem Chronology

TYCMJ

Problem	Proposal	References
17		6(1975/1)33s
18		6(1975/1)33s
19		6(1975/1)34s
20		6(1975/1)34s
21		6(1975/2)32s
22		6(1975/2)32s
23		6(1975/2)33s
24		6(1975/2)34s
25		6(1975/3)35s
26		6(1975/3)35s
27		6(1975/3)36s
28		6(1975/3)37s
29		6(1975/4)25s
30		6(1975/4)26s
31		6(1975/4)26s
32		6(1975/4)27s
33	6(1975/1)32	7(1976/1)29s
34	6(1975/1)32	7(1976/1)30s
35	6(1975/1)32	7(1976/1)31s
36	6(1975/1)32	7(1976/1)31s
37	6(1975/2)31	7(1976/2)50s, 50v
38	6(1975/2)31	7(1976/2)51s
39	6(1975/2)31	7(1976/2)52s
40	6(1975/2)31	7(1976/2)53s
41	6(1975/2)31	7(1976/2)53s
42	6(1975/3)34	7(1976/3)48s
43	6(1975/3)34	7(1976/3)49s
44	6(1975/3)34	7(1976/3)49s
45	6(1975/3)35	7(1976/3)50s
46	6(1975/3)35	7(1976/4)34s
47	6(1975/4)24	7(1976/4)35s
48	6(1975/4)24	7(1976/4)36s
49	6(1975/4)24	7(1976/4)37c, 37s
50	6(1975/4)24	8(1977)43s
51	6(1975/4)25	8(1977)44s
52	6(1975/4)25	7(1976/2)49v 8(1977)44v, 45s
53	6(1975/4)25	8(1977)46s
54	7(1976/1)28	8(1977)96s
55	7(1976/1)28	8(1977)97s
56	7(1976/1)28	8(1977)98s, 99s
57	7(1976/1)28	8(1977)100s
58	7(1976/1)28	8(1977)178s
59	7(1976/1)29	8(1977)179s
60	7(1976/1)29	8(1977)180s
61	7(1976/2)49	8(1977)181s, 181v
62	7(1976/2)49	8(1977)241s
63	7(1976/2)49	8(1977)241s
64	7(1976/2)49	8(1977)242s
65	7(1976/2)50	8(1977)243s
66	7(1976/2)50	8(1977)243s
67	7(1976/3)47	8(1977)293s
68	7(1976/3)47	8(1977)293s
69	7(1976/3)47	8(1977)294s
70	7(1976/3)47	8(1977)295s
71	7(1976/3)47	9(1978)41c, 41s
72	7(1976/3)48	9(1978)42s
73	7(1976/4)33	9(1978)43s, 43v
74	7(1976/4)33	9(1978)44c, 44s
75	7(1976/4)33	9(1978)45s
76	7(1976/4)33	9(1978)96s
77	7(1976/4)33	9(1978)97s
78	7(1976/4)33	9(1978)98s
79	8(1977)42	9(1978)99s
80	8(1977)42	9(1978)100c, 100s
81	8(1977)42	9(1978)177s
82	8(1977)42	9(1978)177s, 178s
83	8(1977)43	9(1978)178s
84	8(1977)95	9(1978)180s
85	8(1977)95	9(1978)181s
86	8(1977)95	9(1978)237s
87	8(1977)95	9(1978)238s
88	8(1977)96	9(1978)239s
89	8(1977)96	9(1978)240s
90	8(1977)177	9(1978)242s
91	8(1977)177	9(1978)298s
92	8(1977)177	9(1978)298s
93	8(1977)177	9(1978)299s
94	8(1977)177	9(1978)300s, 300v
95	8(1977)178	9(1978)301s
96	8(1977)240	10(1979)53s
97	8(1977)240	10(1979)54s
98	8(1977)240	10(1979)55s
99	8(1977)240	10(1979)56s
100	8(1977)240	10(1979)57s
101	8(1977)292	10(1979)128c, 128s
102	8(1977)292	10(1979)129s
103	8(1977)292	10(1979)130s
104	8(1977)292	10(1979)131s
105	8(1977)293	10(1979)211s
106	9(1978)40	10(1979)213s
107	9(1978)40	10(1979)214s
108	9(1978)40	10(1979)215s
109	9(1978)40	10(1979)216s
110	9(1978)41	10(1979)217s
111	9(1978)95	10(1979)294s
112	9(1978)95	10(1979)295s
113	9(1978)95	10(1979)296s
114	9(1978)95	10(1979)297s
115	9(1978)95	10(1979)298s
116	9(1978)176	10(1979)360s
117	9(1978)176	10(1979)361s
118	9(1978)176	10(1979)363s
119	9(1978)176	10(1979)363s
120	9(1978)177	10(1979)366s
121	9(1978)236	11(1980)62s
122	9(1978)236	11(1980)63s
123	9(1978)236	11(1980)64s
124	9(1978)236	11(1980)65s
125	9(1978)236	11(1980)132s
126	9(1978)297	11(1980)133s
127	9(1978)297	11(1980)134s
128	9(1978)297	11(1980)135s
129	9(1978)297	11(1980)137s
130	9(1978)297	10(1979)210v 11(1980)138c, 138s 13(1982)71a
131	10(1979)52	11(1980)209s
132	10(1979)53	11(1980)210s
133	10(1979)53	11(1980)211s
134	10(1979)53	11(1980)212s
135	10(1979)53	11(1980)213s
136	10(1979)127	11(1980)276s
137	10(1979)127	11(1980)277s
138	10(1979)127	11(1980)278s
139	10(1979)127	11(1980)278s
140	10(1979)128	11(1980)279s
141	10(1979)210	11(1980)337s
142	10(1979)211	11(1980)337s
143	10(1979)211	11(1980)338s
144	10(1979)211	11(1980)339s
145	10(1979)211	11(1980)340s
146	10(1979)293	12(1981)64s
147	10(1979)293	12(1981)64s
148	10(1979)294	12(1981)65s
149	10(1979)294	12(1981)66s
150	10(1979)294	12(1981)67s
151	10(1979)359	12(1981)155s
152	10(1979)359	12(1981)157s
153	10(1979)359	12(1981)158s, 159s
154	10(1979)360	12(1981)160s
155	10(1979)360	12(1981)162s, 213a

AUTHOR INDEX

Use this section to

- locate problems proposed by a given author
- find all published works (problems, solutions, or comments) by a given author that appear in a journal problem column covered by this index
- determine if an author proposed a problem without submitting a solution
- find the names of prominent problem proposers or problem solvers

In this section, we list the name of every person who has published a problem, solution or comment during the years 1975–1979 (in one of the columns covered by this index). We also list the name of every person who has published a solution or comment to a problem published in one of those years even if this solution appeared later than 1979. All journal issues through January 1992 have been scanned for solutions to problems covered by this index.

We have attempted to group together variant names under the longest name given, for example, works by M. Klamkin, M. S. Klamkin, and Murray Klamkin will appear under Murray S. Klamkin. We have also attempted to consolidate names when a nickname or variant spelling is used. Thus problems by Joe Konhauser would be listed under Joseph Konhauser. Similarly, shortened names such as Tom/Thomas, Chris/Christopher, Mike/Michael, etc. will appear using the longer name. Each reference to a problem, solution, or comment published by this person follows the author's name. It is given in the form

JNL prob vol(year/issue)page code

where		
	JNL	is the abbreviation of the journal name
	prob	is the problem number
	vol	is the volume number of the issue (if known)
	year	is the year of publication
	/issue	is given if the periodical numbers its pages beginning with page 1 in every issue
	page	is the page number where the reference begins
	code	is a single character code specifying the type of reference as listed below:

Code	Description
c	comment
p	problem proposal
s	solution
x	partial solution

To save space, we have omitted duplicate information from the reference list. For example, the journal name is listed just once per author. The journal name and problem number are given in boldface. Thus, a boldface problem number with no immediately preceding journal name refers to the last journal name listed. Similarly, if no volume or year information is listed for a reference, scan backwards for the last listed volume and year information for the journal in question. Multiple page number references in the same volume of a journal are separated by commas. References to different volumes in a given journal are separated by semicolons. An asterisk after a problem number indicates that the author submitted the problem without submitting a solution. For a given journal, the references are listed chronologically.

Thus, for example, an author entry of

AMATYC C–3 4(1982/1)54p, **A–3** 57s; 4(1983/2) **B–2** 67c. **TYCMJ 128** 11(1980)135s.

means that the author proposed problem C–3 in the AMATYC Review on page 54 of issue number 1 of volume 4 (published in 1982) and had his solution to problem A–3 published on page 57 of that same issue. In issue 2 of the 1983 volume (vol. 4), he had a comment to problem B–2 published on page 67. In the Two-Year College Mathematics Journal (TYCMJ), he had a solution to problem 128 published on page 135 of volume 11 published in 1980.

Only references to published material are indexed. If a person is listed in a solver's list or editor's comment indicating that he has solved a certain problem, his name will not appear in this author index unless his solution or comment was actually printed in the journal.

Works published under a pseudonym are indexed under that pseudonymic name.

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PSEUDONYMS

This section lists some of the pseudonyms appearing within the problem columns. An "editorial name" refers to a house name used by a problem column editor. A "consortium" refers to a group of people writing under a pseudonym. Obvious consortiums, such as the University of South Alabama Problem Group are not listed.

Pseudonym	Real Name
Elizabeth Andy	Clayton W. Dodge
Alfred Brousseau	Brother U. Alfred
Francis Cald	Les Marvin
Lewis Carroll	Charles L. Dodgson
Edmund Charles	Les Marvin
Hippolyte Charles	Léo Sauvé
Samuel Chort	Leroy F. Meyers
Jesse Croach Jr.	Les Marvin
Blanche Descarte	W. Tutte
Aloysisus Dorp	Stanley Rabinowitz
Harry Dweighter	Jacob E. Goodman
Shalosh B. Ekhad	Doron Zeilberger
D. O. Fantus	PME editorial name
Pauvre Fish	PME editorial name
Travis Fletcher	Les Marvin
P. J. Flores	Les Marvin
Fubine	Cipriano Ferraris
Garland Hopkins	Les Marvin
Zazou Katz	Leon Bankoff
Zelda Katz	Leon Bankoff
Nosmo King	Les Marvin
Nosmo King	PME editorial name
Susan Laird	Les Marvin
Henry Larson	Harry Nelson
Cyril P. Lewis	Les Marvin
Ray Lipman	Les Marvin
O. P. Lossers	consortium
O. P. Lossers Jr.	consortium
Magister Ludorum	Les Marvin
U. I. Lydna	Andy Liu
Benedict Marukian	Les Marvin
Diophantus McLeod	Les Marvin
D. C. Morley	Les Marvin
Paul Morphy IV	Les Marvin
L. P. Mullooly	Les Marvin
K. S. Murray	Murray S. Klamkin
Alfred E. Neuman	Les Marvin
Sherry Nolan	Harry Nelson
David R. Orndoff	Les Marvin
Peter Ørno	consortium
Edith Orr	Léo Sauvé
Archimedes O'Toole	Les Marvin
Alvin Owen	Les Marvin
Emil Prochaska	Les Marvin
Vance Revennaugh	Les Marvin
Victor Reyes	Les Marvin
Azriel Rosenfeld	Les Marvin
Gali Salvatore	Léo Sauvé
Vincent J. Seally	Les Marvin
Barbara Seville	Leon Bankoff
K. M. Seymour	Murray S. Klamkin
David L. Silverman	Les Marvin
Fred Walbrook	Les Marvin
Robert Walsh	Les Marvin
Mark Wetzel	Les Marvin
C. F. White	Les Marvin
N. R. White	Les Marvin
Marshall Willheit	Les Marvin
N. Withheld	CRUX editorial name
Harold Wyatt	Les Marvin

TITLE INDEX

Use this section to

- locate a problem from a keyword that appears in the title of the problem
- peruse the titles of problems that appear in this index.

Many journal problem columns assign a title to a published problem or solution. In this section, we present a keyword in context (KWIC) permuted index to these titles. Each title appears multiple times, once for each major word in the title. The entire title is given to provide contextual information to aid you in locating the desired item. A bullet (•) appears between the end of the title and the beginning of the title in the cyclic permutation.

For example: the title "Axiomatic Characterization of Distance" could appear in the title index three times, listed once for "Axiomatic", once for "Characterization" and once for "Distance". The three references would appear as follows:

MM 1126	Axiomatic Characterization of Distance
MM 1126	Characterization of Distance • Axiomatic
MM 1126	Distance • Axiomatic Characterization of

The title entries are alphabetized. Mathematical expressions beginning with a roman letter appear next to other words beginning with that letter. Other mathematical expressions appear at the beginning of the list.

Preceding each title is a reference to the problem that title is associated with. This reference consists of the journal abbreviation followed by the problem number. In the example above, **"MM 1126"** refers to problem 1126 from Mathematics Magazine. The list of journal abbreviations can be found on page 17.

All titles for problems that appeared in the years 1975–1979 in one of the problem columns covered by this index are listed (if the title appeared with a contribution that was published prior to 1992). All titles accompanying solutions and comments published in the years 1975–1979 are also given.

The journals that regularly assign titles to their problems are:

abbreviation	**journal**
AMM	The American Mathematical Monthly
FQ	The Fibonacci Quarterly
JRM	Journal of Recreational Mathematics
MATYC	The MATYC Journal
MM	Mathematics Magazine
SIAM	SIAM Review
TYCMJ	The Two-Year College Mathematics Journal

Consult page 17 for a more complete list of journal abbreviations.

Once you have located a problem that you are interested in, you can click the problem number to look up references to that problem in the Chronology section of this index (page 282).

Entries beginning with uninteresting words have been suppressed from the listing. The words suppressed are:

a	at	from	into	of	the	v
am	be	i	iv	on	there	was
an	but	ii	is	or	this	which
and	by	iii	it	than	those	with
as	for	in	its	that	to	

See also:

- the keyword index — to look for problems containing a given word in the statement of the problem
- the subject index — to look for problems related to a given topic

Title Index

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JOURNAL ISSUE CHECKLIST

Use this section to

- determine which journals are being indexed in this book
- determine which problem columns in those journals have been indexed
- find the name of the problem column editor during the years 1975–1979
- find the name of the publisher of the journal
- obtain the page numbers and issue numbers where the problem column appeared
- find out which problems were proposed in each issue

This issue checklist lists all the mathematical journals that contained problem columns (1975–1979) that have been indexed in this volume.

Each entry begins with a bullet and the abbreviation of the journal that is used in this index. For example, the abbreviation "AMM" is used to represent "The American Mathematical Monthly". Abbreviations were chosen to be about 2 to 6 characters in length and to be mnemonic for the journal in question. These abbreviations are used when forming the name of a problem. For example, "AMM 6048" refers to problem 6048 from the American Mathematical Monthly.

To the right of the abbreviation for the journal is the ISSN number for the journal. This is the International Standard Serial Number, assigned to most journals in the world. The ISSN given is the ISSN that the journal had during the years covered by this index (i.e. 1975–1979) and could be different from the current ISSN if the journal has changed name or publisher since 1979. For the current ISSN number (if different) consult the Journal Information section of Volume 1, which begins on page 435 of that volume. In that section, you will find all the information about current issues of this journal (i.e. the current publisher and current subscription information). In this checklist, you will only find information about the journal at the time the problems covered by this index were published.

On the line following the abbreviation is the full name of the journal. After that is the name of the publisher (for the period covered by this index). See the Journal Information section for the name of the current publisher (if different).

If the problem column has many editors or associate editors, they are all listed next. This list represents the names of the editors during the period covered by this index. For the name of the current problem column editor, see the Journal Information section (page 435, Volume 1). If there was only one editor for the problem column (the usual case), then his name is listed to the right of the name of the problem column.

Next follows detailed information about each issue of the journal that contains a problem column. If a journal contains more than one problem column, then each problem column is listed. If the problem column does not have a formal name, it is listed merely as “problem column”. To the right of the name of the problem column is the name of the problem column editor. Following the name of the problem column is a listing of each issue of the journal published in the years 1975–1979. For each issue, we give the page numbers of the problem column from that issue. We also give the date, volume number, and issue number for that issue. Thus, this checklist lets you confirm which journals and problem columns have been indexed. We also give the problem numbers of the new problems proposed in each issue. In a few cases, we have chosen to give the checklist for a journal’s problem column, but have not indexed the problems therein. In that case, the notation “problem column not indexed” is given. This is usually because the majority of the problems in that column are research-level problems or are otherwise dissimilar from the majority of the problems covered by this index. In general, physics problems, chess problems, puzzles, and non-mathematical problems have not been indexed. Also, some journals publish a lot of problems and/or short notes that are not part of any formal problem column. In that case, the problems have not been included in this index. In general, a problem column is a regular feature that has consecutively numbered problems spanning across many issues of the journal and contains solutions submitted by readers. If you are not sure if a particular column in a journal has been included in this index, consult this checklist.

If the name of the problem editor changed during the year, then the list of editors is given along with their terms of editorship.

If a journal normally has a problem column, but no problems or solutions were published in the years 1975-1979, then that journal will not be listed in this checklist.

Following the list of issues and problem columns are notes about standard columns that run in the specified journal but which have not been indexed. A statement of the form “column not indexed” means that we have no intention of indexing this column. If the journal has changed name since 1979, this is also noted. Additional notes of interest may also be given.

Problems from articles in these journals are not normally indexed. An article that reports about the problems given in a national or international mathematical competition or olympiad might have these problems indexed if the competition was held in 1975–1979 (not that the article about it appeared during these years). The Citation Index (page 423) lists those articles that reference problems covered by this index.

Although only problems published in the years 1975-1979 have been indexed, solutions to the problems proposed in these years (or earlier) are indexed if these solutions were published in 1975 or later. We have attempted to check all issues through August 1992 to find solutions to problems originally published in 1979 or earlier. In those cases, the solution has been indexed, in the sense that the names of the original featured solvers are listed in the Author Index (page 316) and the dates and page numbers where the solutions can be found are given in the Problem Chronology (page 282).

• AMM

ISSN 0002-9890

The American Mathematical Monthly

Publisher:. . . . . Taylor & Francis, Ltd.

Overall Problem Column Editors:
1975 - 1978. . . . . Emory P. Starke
1979. . . . . A. P. Hillman

Associate and Collaborating Editors: Joshua Barlaz, Eric S. Langford, Leonard Carlitz, Gulbank D. Chakerian, Haskell Cohen, S. Ashby Foote, Israel N. Herstein, Murray S. Klamkin, Daniel J. Kleitman, Roger C. Lyndon, Marvin Marcus, Christoph Neugebauer, W. C. Waterhouse, Albert Wilansky, and University of Maine Problems Group: Earl M. L. Beard, George S. Cunningham, Clayton W. Dodge, Oskar Feichtinger, William R. Geiger, Ramesh Gupta, Philip M. Locke, John C. Mairhuber, Curtis S. Morse, Grattan P. Murphy, Edward S. Northam and William L. Soule, Jr.

∆ Problems dedicated to Emory P. Starke

Date	Year	Vol	Issue	Pages	Proposals
Jan	1979	86	1	54-55	S1-S3
Feb	1979	86	2	127	S4-S5
Mar	1979	86	3	222	S6-S8
Apr	1979	86	4	306	S9-S10
May	1979	86	5	392	S11-S13
Jun/Jul	1979	86	6	503	S14-S15
Aug/Sep	1979	86	7	591-592	S16-S18
Oct	1979	86	8	702	S19-S20
Nov	1979	86	9	784	S21
Dec	1979	86	10	863	S22-S23

E. P. Starke Problem Column Editor:
1979. . . . . A. P. Hillman

∆ Elementary Problems

Date	Year	Vol	Issue	Pages	Proposals
Jan	1975	82	1	72-83	E2510-2515
Feb	1975	82	2	168-182	E2516-2521
Mar	1975	82	3	299-307	E2522-2527
Apr	1975	82	4	399-409	E2528-2533
May	1975	82	5	520-528	E2534-2539
Jun/Jul	1975	82	6	659-671	E2540-2545
Aug/Sep	1975	82	7	755-765	E2546-2551
Oct	1975	82	8	851-856	E2552-2557
Nov	1975	82	9	936-941	E2558-2563
Dec	1975	82	10	1009-1015	E2564-2569
Jan	1976	83	1	53-61	E2570-2574
Feb	1976	83	2	132-140	E2575-2580
Mar	1976	83	3	197-204	E2581-2586
Apr	1976	83	4	284-292	E2587-2592
May	1976	83	5	378-385	E2593-2598
Jun/Jul	1976	83	6	482-489	E2599-2604
Aug/Sep	1976	83	7	566-572	E2605-2610
Oct	1976	83	8	656-661	E2611-2616
Nov	1976	83	9	740-747	E2617-2622
Dec	1976	83	10	812-817	E2623-2628
Jan	1977	84	1	57-61	E2629-2634
Feb	1977	84	2	134-140	E2635-2640
Mar	1977	84	3	216-221	E2641-2646
Apr	1977	84	4	294-299	E2647-2652
May	1977	84	5	386-391	E2653-2658
Jun/Jul	1977	84	6	486-490	E2659-2664
Aug/Sep	1977	84	7	567-574	E2665-2670
Oct	1977	84	8	651-659	E2671-2676
Nov	1977	84	9	738-743	E2677-2682
Dec	1977	84	10	820-828	E2683-2688
Jan	1978	85	1	47-53	E2689-2694
Feb	1978	85	2	116-121	E2695-2700
Mar	1978	85	3	197-202	E2701-2706
Apr	1978	85	4	276-282	E2707-2712
May	1978	85	5	383-388	E2713-2718
Jun/Jul	1978	85	6	495-499	E2719-2724
Aug/Sep	1978	85	7	593-599	E2725-2730
Oct	1978	85	8	681-686	E2731-2736
Nov	1978	85	9	764-769	E2737-2742
Dec	1978	85	10	823-827	E2743-2748
Jan	1979	86	1	55-59	E2749-2754
Feb	1979	86	2	127-131	E2755-2760
Mar	1979	86	3	222-225	E2761-2766
Apr	1979	86	4	307-311	E2767-2772
May	1979	86	5	393-398	E2773-2778
Jun/Jul	1979	86	6	503-509	E2779-2784
Aug/Sep	1979	86	7	592-596	E2785-2790
Oct	1979	86	8	702-709	E2791-2796
Nov	1979	86	9	784-793	E2797-2802
Dec	1979	86	10	864-869	E2803-2808

Elementary Problem Column Editors:
Jan - Feb 1975. . . . . U. of Maine Problems Group
Feb 1975 - Jul 1978. . . . . . U. of Waterloo Problems Group
after Jul 1978. . . . . J. L. Brenner

∆ Advanced Problems

Date	Year	Vol	Issue	Pages	Proposals
Jan	1975	82	1	84-89	6006-6011
Feb	1975	82	2	183-187	6012-6017
Mar	1975	82	3	307-310	6018-6023
Apr	1975	82	4	409-416	6024-6029
May	1975	82	5	528-538	6030-6035
Jun/Jul	1975	82	6	671-681	6036-6041
Aug/Sep	1975	82	7	766-770	6042-6047
Oct	1975	82	8	856-862	6048-6053
Nov	1975	82	9	941-945	6054-6059
Dec	1975	82	10	1016-1021	6060-6065
Jan	1976	83	1	62-67	6066-6071
Feb	1976	83	2	140-145	6072-6077
Mar	1976	83	3	205-210	6078-6083
Apr	1976	83	4	292-297	6084-6089
May	1976	83	5	385-390	6090-6095
Jun/Jul	1976	83	6	489-494	6096-6101
Aug/Sep	1976	83	7	572-576	6102-6107
Oct	1976	83	8	661-667	6108-6113
Nov	1976	83	9	748-753	6114-6119
Dec	1976	83	10	817-821	6120-6125
Jan	1977	84	1	61-67	6126-6131

Feb	1977	84	2	140-144	6132-6137
Mar	1977	84	3	221-226	6138-6143
Apr	1977	84	4	299-304	6144-6149
May	1977	84	5	391-397	6150-6155
Jun/Jul	1977	84	6	491-496	6156-6161
Aug/Sep	1977	84	7	575-580	6162-6167
Oct	1977	84	8	659-663	6168-6173
Nov	1977	84	9	743-748	6174-6179
Dec	1977	84	10	828-834	6180-6185
Jan	1978	85	1	53-59	6186-6191
Feb	1978	85	2	121-126	6192-6197
Mar	1978	85	3	203-210	6198-6203
Apr	1978	85	4	282-291	6204-6209
May	1978	85	5	389-393	6210-6215
Jun/Jul	1978	85	6	499-506	6216-6221
Aug/Sep	1978	85	7	599-604	6222-6227
Oct	1978	85	8	686-690	6228-6233
Nov	1978	85	9	770-774	6234-6239
Dec	1978	85	10	828-834	6240-6245
Jan	1979	86	1	59-66	6246-6251
Feb	1979	86	2	131-136	6252-6257
Mar	1979	86	3	226-232	6258-6263
Apr	1979	86	4	311-315	6264-6266
May	1979	86	5	398-401	6267-6269
Jun/Jul	1979	86	6	509-511	6270-6272
Aug/Sep	1979	86	7	596-598	6273-6275
Oct	1979	86	8	709-711	6276-6278
Nov	1979	86	9	793-796	6279-6281
Dec	1979	86	10	869-871	6282-6284

Advanced Problem Column Editors:
1975 - Jul 1978. . . . . J. Barlaz
after Jul 1978. . . . . Roger C. Lyndon

Notes:
•Research problems not indexed
•Unsolved Problem Column not indexed

• CMB ISSN 0008-4395

Canadian Mathematical Bulletin
Publisher: Canadian Mathematical Society

Δ Problems and Solutions

Date	Year	Vol	Issue	Pages	Proposals
Mar	1975	18	1	none	none
Jun	1975	18	2	none	none
Sep	1975	18	3	none	none
Oct	1975	18	4	615-620	P241-245
Mar	1976	19	1	121-125	P246-249
Jun	1976	19	2	249-253	P250-252
Sep	1976	19	3	379-382	P255-258
Dec	1976	19	4	none	none
Mar	1977	20	1	147-150	P257-261
Jun	1977	20	2	273-276	P264-266
Sep	1977	20	3	none	none
Dec	1977	20	4	517-525	P253,267-269
Mar	1978	21	1	none	none
Jun	1978	21	2	none	none
Sep	1978	21	3	none	none
Dec	1978	21	4	none	none
Mar	1979	22	1	121-125	P270-272
Jun	1979	22	2	247-253	P273-276
Sep	1979	22	3	385-389	P277-280
Dec	1979	22	4	519-522	P281

Problem Column Editors:
vol 18 - vol 21. . . . . E.C. Milner
vol 22. . . . . E.J. Barbeau

Notes:
•Problem numbers jumped from 252 to 255 in Volume 19, numbers 2 and 3.
•Problem numbers 257 and 258 were repeated in Volume 20, number 1.
•Between Volume 20, number 1 and 20, number 2, problem numbers P262 and P263 were skipped.

• CRUX ISSN 0705-0348

Crux Mathematicorum
Publisher: Algonquin College

Δ Problems - Problèmes Léo Sauvé

Date	Year	Vol	Issue	Pages	Proposals
Mar	1975	1	1	3-4	1-10
Apr	1975	1	2	7-8	11-20
May	1975	1	3	11-22	21-30
Jun	1975	1	4	25-36	31-40
Jul	1975	1	5	38-46	41-50
Aug	1975	1	6	48-49	51-60
Sep	1975	1	7	56-67	61-70
Oct	1975	1	8	71-81	71-80
Nov	1975	1	9	84-93	81-90
Dec	1975	1	10	97-102	91-100
Jan	1976	2	1	5-17	101-110
Feb	1976	2	2	25-36	111-120
Mar	1976	2	3	41-53	121-130
Apr	1976	2	4	67-88	131-140
May	1976	2	5	93-106	141-150
Jun/Jul	1976	2	6	109-128	151-160
Aug/Sep	1976	2	7	135-159	161-170
Oct	1976	2	8	170-186	171-180
Nov	1976	2	9	193-204	181-190
Dec	1976	2	10	219-234	191-200
Jan	1977	3	1	9-30	201-210
Feb	1977	3	2	42-58	211-220
Mar	1977	3	3	65-87	221-230
Apr	1977	3	4	104-114	231-240
May	1977	3	5	130-146	241-250
Jun/Jul	1977	3	6	154-176	251-260
Aug/Sep	1977	3	7	189-206	261-270
Oct	1977	3	8	226-240	271-280
Nov	1977	3	9	250-269	281-290
Dec	1977	3	10	297-299	291-300

Jan	1978	4	1	11-30	301-310
Feb	1978	4	2	35-60	311-320
Mar	1978	4	3	65-89	321-330
Apr	1978	4	4	100-120	331-340
May	1978	4	5	133-150	341-350
Jun/Jul	1978	4	6	159-180	351-360
Aug/Sep	1978	4	7	191-210	361-370
Oct	1978	4	8	224-240	371-380
Nov	1978	4	9	250-270	381-390
Dec	1978	4	10	282-300	391-400
Jan	1979	5	1	14-30	401-410
Feb	1979	5	2	46-60	411-420
Mar	1979	5	3	76-90	421-430
Apr	1979	5	4	107-120	431-440
May	1979	5	5	131-150	441-450
Jun/Jul	1979	5	6	166-180	451-460
Aug/Sep	1979	5	7	199-212	461-470
Oct	1979	5	8	228-244	471-480
Nov	1979	5	9	264-278	481-490
Dec	1979	5	10	291-310	491-500

∆ Olympiad Corner — Murray S. Klamkin

Date	Year	Vol	Issue	Pages	Proposals
Jan	1979	5	1	12-14	PS1-1 to 3-1
Feb	1979	5	2	44-46	PS4-1 to 4-3
Mar	1979	5	3	62-69	PS5-1 to 5-3
Apr	1979	5	4	102-107	PS6-1 to 6-3
May	1979	5	5	128-131	none
Jun/Jul	1979	5	6	160-165	none
Aug/Sep	1979	5	7	193-199	none
Oct	1979	5	8	220-228	none
Nov	1979	5	9	259-264	PS7-1 to 7-3
Dec	1979	5	10	288-291	PS8-1 to 8-3

Columns are numbered through 60.

Notes:
- Puzzle Corner not indexed.
- Only the Practice Sets from the Olympiad Corner are indexed. These are given the prefix "PS". The Olympiad Corner has many problems that are not indexed.
- The Olympiad Corner contains problems and solutions from many competitions. See the contest list in the citation index for these references.

• DELTA

ISSN 0011-801X

Delta
Publisher: Waukesha Mathematical Society

∆ Problems and Solutions — R. S. Luthar

Date	Year	Vol	Issue	Pages	Proposals
Spr	1975	5	1	45-48	5.1.1-3
Fall	1975	5	2	94-96	5.2.1-3
Spr	1976	6	1	43-45	6.1.1-4
Fall	1976	6	2	92-94	6.2.1-3

• FQ

ISSN 0015-0517

The Fibonacci Quarterly
Publisher: The Fibonacci Association

∆ Elementary Problems — A. P. Hillman

Date	Year	Vol	Issue	Pages	Proposals
Feb	1975	13	1	94-96	B298-303
Apr	1975	13	2	190-192	B304-309
Oct	1975	13	3	285-288	B310-314
Dec	1975	13	4	373-377	B316-321
Feb	1976	14	1	93-96	B322-327
Apr	1976	14	2	188-192	B328-333
Oct	1976	14	3	286-288	B334-339
Nov	1976	14	4	none	none
Dec	1976	14	5	470-473	B340-345
Feb	1977	15	1	93-96	B346-351
Apr	1977	15	2	189-192	B352-357
Oct	1977	15	3	285-288	B358-363
Dec	1977	15	4	375-377	B364-369
Feb	1978	16	1	88-91	B370-375
Apr	1978	16	2	184-187	B376-381
Jun	1978	16	3	none	none
Aug	1978	16	4	none	none
Oct	1978	16	5	473-476	B382-387
Dec	1978	16	6	562-565	B388-393
Feb	1979	17	1	90-93	B394-399
Apr	1979	17	2	184-188	B400-405
Oct	1979	17	3	281-285	B406-411
Dec	1979	17	4	369-373	B412-417

Notes:
- There is no problem numbered B–315.

∆ Advanced Problems — Raymond E. Whitney

Date	Year	Vol	Issue	Pages	Proposals
Feb	1975	13	1	89-93	H245-248
Apr	1975	13	2	185-189	H249-251
Oct	1975	13	3	281-284	H252-254
Dec	1975	13	4	369-372	H255-257
Feb	1976	14	1	88-92	H258-260
Apr	1976	14	2	182-187	H261-263
Oct	1976	14	3	282-285	H264-266
Nov	1976	14	4	none	none
Dec	1976	14	5	466-469	H267-268
Feb	1977	15	1	89-92	H269-271
Apr	1977	15	2	185-188	H272-273
Oct	1977	15	3	281-284	H274-275
Dec	1977	15	4	371-374	H276-277
Feb	1978	16	1	92-96	H278-280
Apr	1978	16	2	188-192	H281-284
Jun	1978	16	3	none	none
Aug	1978	16	4	none	none
Oct	1978	16	5	477-480	H285-289
Dec	1978	16	6	566-569	H290-294
Feb	1979	17	1	94-96	H295-298
Apr	1979	17	2	189-192	H299-301
Oct	1979	17	3	286-288	H302-306
Dec	1979	17	4	374-377	H307-310

• FUNCT

ISSN 0313-6825

Function

Publisher: Monash University

ΔProblem Section

Date	Year	Vol	Issue	Pages	Proposals
Feb	1977	1	1	23,29-31	1.1.1-10
Apr	1977	1	2	23,29-31	1.2.1-7
Jun	1977	1	3	6,25,27-30	1.3.1-7
Aug	1977	1	4	8-9,11-16, 22,31-32	1.4.1-5
Oct	1977	1	5	27-32	1.5.1-4
Feb	1978	2	1	19-22,28-29,32	2.1.1-4
Apr	1978	2	2	7,27	2.2.1-4
Jun	1978	2	3	11,25,29-32	2.3.1-5
Aug	1978	2	4	31-32	2.4.1-4
Oct	1978	2	5	20,28-32	2.5.1-4
Feb	1979	3	1	28-31	3.1.1-6
Apr	1979	3	2	29-32	3.2.1-8
Jun	1979	3	3	27-32	3.3.1-5
Aug	1979	3	4	27-32	3.4.1-3
Oct	1979	3	5	26-30	3.5.1-4

• ISMJ

Indiana School Mathematics Journal

Δ Problems - Junior Section

Date	Year	Vol	Issue	Pages	Proposals
Aug	1974	10	1	5-8	J10.1-5
Dec	1974	10	2	4-8	J10.6-10
Feb	1975	10	3	3-8	J10.11-15
Apr	1975	10	4	2-8	J10.16-17
Sep	1975	11	1	6-11	J11.1-5
Dec	1975	11	2	6-12	J11.6-10
Feb	1976	11	3	2-8	J11.11-15
May	1976	11	4	4-8	J11.16-20

Δ Problems - Open Section

Date	Year	Vol	Issue	Pages	Proposals
Aug	1974	10	1	5-8	10.1-5
Dec	1974	10	2	4-8	10.6-10
Feb	1975	10	3	3-8	10.11-15
Apr	1975	10	4	2-8	10.16-17
Sep	1975	11	1	6-11	11.1-5
Dec	1975	11	2	6-12	11.6-10
Feb	1976	11	3	2-8	11.11-15
May	1976	11	4	4-8	11.16-20

Δ Problems

Date	Year	Vol	Issue	Pages	Proposals
Sep	1976	12	1	4-7	12.1-10
Nov	1976	12	2	5-12	12.11-18
Feb	1977	12	3	4-8	12.19-27
Apr	1977	12	4	5-8	12.28-32
Sep	1977	13	1	8-11	13.1-8
Dec	1977	13	2	4-8	13.9-18
Feb	1978	13	3	5-8	13.19-23
May	1978	13	4	5-8	13.24-28
Sep	1978	14	1	6-7	14.1-5
Dec	1978	14	2	5-8	14.6-14
Feb	1979	14	3	2-8	14.15-19
Apr	1979	14	4	1-4	14.20-24

• JRM

ISSN 0022-412X

Journal of Recreational Mathematics

Publisher: Baywood Publishing Company, Inc.

Δ Problems and Conjectures

Year	Vol	Issue	Pages	Proposals
1975-76	8	1	46-73	370-381
1975-76	8	2	136-142,145-158	382-396
1975-76	8	3	229-232	419-427
1975-76	8	4	311-314	440-448
1976-77	9	1	24-29,32-79	462-476
1976-77	9	2	127-135,138-150	81a, 493-507
1976-77	9	3	208-232	527-541
1976-77	9	4	294-320	554-568
1977-78	10	1	51-80	591-604
1977-78	10	2	127-160	623-632
1977-78	10	3	210-240	645-659
1977-78	10	4	283-320	671-685
1978-79	11	1	34-39,47-80	699-715
1978-79	11	2	127-131,145-160	728-741
1978-79	11	3	212-235,238-240	755-770
1978-79	11	4	299-320	782-798

Problem Column Editor:
vol 8/1 - vol 10/1. . . . . David L. Silverman
vol 10/2 - vol 11/4. . . . . Friend H. Kierstead Jr.

Associate Editors:
vol 10/1 - vol 10/2. . . . . Harvey J. Hinden
vol 10/4 - vol 11/4. . . . . . . John Brinn and Romae Cormier

∆ Computer Challenge Corner

Year	Vol	Issue	Pages	Proposals
1975-76	8	3	233-234	1-4
1975-76	8	4	305-307	5-9
1976-77	9	1	30-31	477-480
1976-77	9	2	136-137	508-513
1976-77	9	3	233-240	none
1976-77	9	4	286-293	569-573
1977-78	10	1	45-50	586-590
1977-78	10	2	119-126	618-622
1977-78	10	3	none	none
1977-78	10	4	279-282	none
1978-79	11	1	43-46	none
1978-79	11	2	132-144	none

Problem Column Editor:
vol 8 - vol 9. . . . . David L. Silverman
vol 10 - vol 11. . . . . Friend H. Kierstead Jr.

∆ Alphametics

Year	Vol	Issue	Pages	Proposals
1975-76	8	1	44-45	364-369
1975-76	8	2	143-144	397-408
1975-76	8	3	227-228	409-418
1975-76	8	4	308-310	428-439
1976-77	9	1	21-23	449-461
1976-77	9	2	125-126	481-492
1976-77	9	3	206-207	514-526
1976-77	9	4	280-285	542-553
1977-78	10	1	40-44	574-585
1977-78	10	2	114-118	605-617
1977-78	10	3	204-209	633-644
1977-78	10	4	274-278	660-670
1978-79	11	1	28-33	686-698
1978-79	11	2	122-126	716-727
1978-79	11	3	207-211	742-754
1978-79	11	4	294-298	770-780

Problem Column Editor:
vol 8. . . . . David L. Silverman

vol 9 - vol 11. . . . . Steven Kahan

Notes:
- Hinden was first spelled with an “in”, then changed to an “en”.
- Problem number 81+ was changed to 81a for consistency.
- There were two problems numbered 770. The suffixes “a” and “b” were attached to distinguish them.

• MATYC

ISSN 0300-7650

The MATYC Journal

∆ Problem Department. . . . . Martin J. Brown

Date	Year	Vol	Issue	Pages	Proposals
Winter	1975	9	1	49-53	70-73
Spring	1975	9	2	51-53	74-77
Fall	1975	9	3	45-50	78-81
Winter	1976	10	1	43-46	82-85
Spring	1976	10	2	122-124	86-90
Fall	1976	10	3	200-203	91-95
Winter	1977	11	1	63-68	96-100
Spring	1977	11	2	142-145	101-104
Fall	1977	11	3	221-225	105-109
Winter	1978	12	1	78-80	110-114
Spring	1978	12	2	173-176	115-119
Fall	1978	12	3	253-256	120-124
Winter	1979	13	1	64-70	125-129
Spring	1979	13	2	135-139	130-134
Fall	1979	13	3	214-219	135-139

• MENEMUI

ISSN 0126-9003

Menemui Matematik

∆ Problems and Solutions

Date	Year	Vol	Issue	Pages	Proposals
	1979	1	1	52-59	1.1.1-3
	1979	1	2	46-49	1.2.1-2
	1979	1	3	56-60	1.3.1-3

• MM ISSN 0025-570X

Mathematics Magazine

Publisher: . . . . Taylor & Francis, Ltd.

∆ Problems and Solutions Dan Eustice

Date	Year	Vol	Issue	Pages	Proposals
Jan	1975	48	1	50-58	922-928
Mar	1975	48	2	115-122	929-936
May	1975	48	3	180-186	937-944
Sep	1975	48	4	238-247	945-953
Nov	1975	48	5	293-302	954-962
Jan	1976	49	1	43-48	963-969
Mar	1976	49	2	95-101	970-977
May	1976	49	3	149-154	978-987
Sep	1976	49	4	211-218	988-995
Nov	1976	49	5	252-258	996-1002
Jan	1977	50	1	46-53	1003-1007
Mar	1977	50	2	99-104	1008-1012
May	1977	50	3	163-169	1013-1020
Sep	1977	50	4	211-216	1021-1024
Nov	1977	50	5	265-271	1025-1028
Jan	1978	51	1	69-72	1029-1032
Mar	1978	51	2	127-132	1033-1038
May	1978	51	3	193-201	1039-1047
Sep	1978	51	4	245-249	1048-1053
Nov	1978	51	5	305-308	1054-1057
Jan	1979	52	1	46-55	1058-1065
Mar	1979	52	2	113-118	1066-1071
May	1979	52	3	179-184	1072-1073
Sep	1979	52	4	258-265	1074-1079
Nov	1979	52	5	316-323	1080-1088

∆ Quickies

Date	Year	Vol	Issue	Pages	Proposals
Jan	1975	48	1	52,58	Q608-613
Mar	1975	48	2	116-117,122	Q614-619
May	1975	48	3	181-182,186	Q620-624
Sep	1975	48	4	240,248	Q625-627
Nov	1975	48	5	295,302-303	Q628-630
Jan	1976	49	1	44,48	Q631-632
Mar	1976	49	2	96,101	Q633-634
May	1976	49	3	150,154	Q635-637
Sep	1976	49	4	212,218	Q638-639
Nov	1976	49	5	253,258	Q640-642
Jan	1977	50	1	47,53	Q643-644
Mar	1977	50	2	none	none
May	1977	50	3	164,169	Q645-648
Sep	1977	50	4	none	none
Nov	1977	50	5	266,271	Q649-650
Jan	1978	51	1	none	none
Mar	1978	51	2	128,132	Q651-652
May	1978	51	3	194,201	Q653-654
Sep	1978	51	4	246,249	Q655
Nov	1978	51	5	none	none
Jan	1979	52	1	47,55	Q656-657
Mar	1979	52	2	114,118	Q658-659
May	1979	52	3	179,184	Q660-661
Sep	1979	52	4	259,265	Q662
Nov	1979	52	5	317,323	Q663-664

*Associate Editor: J. S. Frame. Assistant Editors: Don Bonar, William McWorter Jr., and L. F. Meyers.
*Starting with the Sep., 1975 issue, (Volume 48, number 4), Leroy F. Meyers becomes Associate Editor.

• MSJ ISSN 0095-7089

The Mathematics Student (Reston)

Publisher: NCTM

∆ Problem Section

Date	Year	Vol	Issue	Pages	Proposals
Oct	1974	22	1	5-7	416-420
Dec	1974	22	2	5-7	421-425
Feb	1975	22	3	5-7	426-430
Apr	1975	22	4	5-7	none
Oct	1975	23	1	6-8	431-432
Dec	1975	23	2	8	433-434
Feb	1976	23	3	8	435-436
Apr	1976	23	4	8	437-438
Oct	1976	24	1	4	439-440
Dec	1976	24	2	5-6	441-442
Feb	1977	24	3	5	443-444
Apr	1977	24	4	2-3	445-446
Oct	1977	25	1	4	447-448
Nov	1977	25	2	4	449-450
Dec	1977	25	3	4	451-452
Jan	1978	25	4	4	453-454
Feb	1978	25	5	4	455-456
Mar	1978	25	6	4	457-458
Apr	1978	25	7	2	459-460
May	1978	25	8	2	461-462

∆ Competition Corner

Oct	1978	26	1	2-3	463-467
Nov	1978	26	2	2-3	468-472
Dec	1978	26	3	2-4	473-477
Jan	1979	26	4	2-3	478-482
Feb	1979	26	5	2	483-487
Mar	1979	26	6	2-3	488-492
Apr	1979	26	7	2-3	493-497
May	1979	26	8	2-3	498-502

Problem Column Editor:
vol 22 – vol 25. . . . . Steven R. Conrad
vol 26. . . . . George Berzsenyi

• NAvW

ISSN 0028-9825

Nieuw Archief voor Wiskunde (3rd series)

Publisher: Dutch Mathematical Society

Δ Problem Section — J. H. Van Lint

Date	Year	Vol	Issue	Pages	Proposals
Mar	1975	23	1	79-94	391-400
Jul	1975	23	2	173-194	401-413
Nov	1975	23	3	242-257	414-423
Mar	1976	24	1	77-107	424-435
Jul	1976	24	2	184-214	436-447
Nov	1976	24	3	270-286	448-457
Mar	1977	25	1	86-101	458-467
Jul	1977	25	2	186-204	468-477
Nov	1977	25	3	423-446	478-487
Mar	1978	26	1	231-253	488-500
Jul	1978	26	2	348-366	501-511
Nov	1978	26	3	462-478	512-522
Mar	1979	27	1	132-152	523-533
Jul	1979	27	2	267-285	534-545
Nov	1979	27	3	407-424	546-558

• NYSMTJ

ISSN 0545-6584

New York State Mathematics Teachers' Journal

Publisher: Association of Mathematics Teachers of New York State

Problem Column Editor:
vol 25 - vol 27/2. . . . . David E. Bock
vol 27/3 - vol 29. . . . . Sidney Penner

Δ Problems and Solutions

Date	Year	Vol	Issue	Pages	Proposals
Jan	1975	25	1	20-22	37-40
Apr	1975	25	2	55-57	41-44
Jun	1975	25	3	124-127	45-47
Oct	1975	25	4	170-173	48-51
	1976	26	1		
Jan	1976	26	2	18-19	52-54
	1976	26	3		
May	1976	26	4	96-101	55-58
	1976	26	5		
Sep	1976	26	6	150-152	59-61
Win	1977	27	1	50-54	62-65
Spr	1977	27	2	98-103	66-70
Fall	1977	27	3	136-138	71-73
Win	1977/78	28	1	52-58	74-77
Spr/Sum	1978	28	2	77-85	78-82
Fall	1978	28	3	150-158	83-86
Win	1978/79	29	1	56-62	87-91
Spr	1979	29	2	83-89	92-95

• OMG

ISSN 0030-3011

Ontario Mathematics Gazette

Publisher: Ontario Association for Mathematics Education

Δ Problems — Arn Harris

Date	Year	Vol	Issue	Pages	Proposals
Mar	1976	14	3	44	14.3.1-3
Sep	1976	15	1	51-52	15.1.1-3
Dec	1976	15	2	66	15.2.1-3
Mar	1977	15	3	59-61	15.3.1-10
Sep	1977	16	1	64-65	16.1.1-10
Dec	1977	16	2	51-53	16.2.1-7
Mar	1978	16	3	none	none
Sep	1978	17	1	58-59	17.1.1-9
Dec	1978	17	2	58	17.2.1-9
Mar	1979	17	3	58-61	17.3.1-9
Sep	1979	18	1	55-57,60-61	18.1.1-9
Dec	1979	18	2	61-63,66-67	18.2.1-9
Mar	1980	18	3	65,67-68	18.3.1-9

*Assistant Editor: (for problems) Tom Griffiths
*McKay and McKnight wrote the problems in Volume 17, numbers 1 and 2, and Volume 18, numbers 1, 2, and 3.
*Starting with Volume 18, number 1, the editor changes to R.S. Smith. (Assistant editor: Walker Schofield)

• OSSMB

ISSN 0380-6235

Ontario Secondary School Mathematics Bulletin

Publisher: University of Waterloo

Δ Problems Section — K.D. Fryer

Date	Year	Vol	Issue	Pages	Proposals
May	1975	11	1	16-24	75.1-6
Sep	1976	12	2	19-24	76.7-12
May	1978	14	1	15-19	78.1-6
Sep	1978	14	2	22-25	78.3-5,78.7-9
Dec	1978	14	3	17-21	78.10-15
May	1979	15	1	20-23	79.1-6
Sep	1979	15	2	17-21	79.7-12
Dec	1979	15	3	23	79.13-18

*Problems editor: R.A. Honsberger
*Starting with Volume 14, number 1, the Problems editor changes to: Professor E.M. Moskal

Notes:
- We were unable to locate v.11 nos. 2 and 3, v.12 nos. 1 and 3, and all of v.13. As a result, those issues are not indexed in this volume.

• PARAB

Parabola

Publisher: University of New South Wales

∆ Problem Section — R.K. James

Date	Year	Vol	Issue	Pages	Proposals
Feb/Mar	1975	11	1	18-25	261-272
May/Jun	1975	11	2	25-34	273-284
Aug/Sep	1975	11	3	18-26	285-296
Feb/Mar	1976	12	1	22-33	297-308
May/Jun	1976	12	2	26-36	309-320
Aug/Sep	1976	12	3	23-32	321-332
Feb/Mar	1977	13	1	24-36	333-344
May/Jun	1977	13	2	34-36	345-356
Aug/Sep	1977	13	3	25-36	357-368
Term 1	1978	14	1	28-36	369-380
Term 2	1978	14	2	30-40	381-392
Term 3	1978	14	3	28-36	393-404
1st Term	1979	15	1	26-36	405-416
2nd Term	1979	15	2	36-44	417-428
3rd Term	1979	15	3	31-40	429-440

Notes:
- Problem editor: Mr. C.D. Cox
- Volume 14, numbers 1-3, the editor changes to M. Hirschhorn.
- Volume 15, numbers 1-3, the editor is J.H. Loxton.

• PENT

ISSN 0031-4870

The Pentagon

Publisher: Kappa Mu Epsilon

∆ The Problem Corner — Kenneth M. Wilke

Date	Year	Vol	Issue	Pages	Proposals
Spring	1975	34	2	103-111	272-276
Fall	1975	35	1	33-39	277-281
Spring	1976	35	2	97-103	282-286
Fall	1976	36	1	31-35	287-290
Spring	1977	36	2	93-98	292-296
Fall	1977	37	1	26-34	297-301
Spring	1978	37	2	82-88	302-306
Fall	1978	38	1	26-32	307-311
Spring	1979	38	2	78-83	312-316
Fall	1979	39	1	30-39	317-321

Notes:
- Volume numbers for Spring/Fall 1979 have been corrected from those printed in the journal.
- There are two problems numbered 289. (Fall 1976)

• PME

ISSN 0031-952X

Pi Mu Epsilon Journal

Publisher: Pi Mu Epsilon Fraternity

∆ Problem Department — Leon Bankoff

Date	Year	Vol	Issue	Pages	Proposals
Spr	1975	6	2	104-122	338-349
Fall	1975	6	3	177-193	350-361
Spr	1976	6	4	226-244	362-373
Fall	1976	6	5	306-324	374-385
Spr	1977	6	6	364-381	386-398
Fall	1977	6	7	417-437	399-411
Spr	1978	6	8	481-501	412-424
Fall	1978	6	9	539-559	425-437
Spr	1979	6	10	615-633	438-448
Fall	1979	7	1	57-76	449-461

• SIAM

ISSN 0036-1445

SIAM Review

Publisher: Society for Industrial and Applied Mathematics

∆ Problems and Solutions — Murray S. Klamkin

Date	Year	Vol	Issue	Pages	Proposals
Jan	1975	17	1	167-175	75-1 to 75-7
Apr	1975	17	2	none	none
Jul	1975	17	3	565-567	75-8 to 75a-15
Oct	1975	17	4	685-695	75-16 to 75-21
Jan	1976	18	1	117-130	76-1 to 76-6
Apr	1976	18	2	294-306	76-7 to 76-12
Jul	1976	18	3	489-503	76-13 to 76-17
Oct	1976	18	4	762-773	76-18 to 76-22
Jan	1977	19	1	146-155	77-1 to 77-5
Apr	1977	19	2	328-335	77-6 to 77-10
Jul	1977	19	3	563-568	77-11 to 77-15
Oct	1977	19	4	736-744	77-16 to 77-20
Jan	1978	20	1	181-190	78-1 to 78-5
Apr	1978	20	2	394-400	78-6 to 78-9
Jul	1978	20	3	593-604	78-10 to 78-15
Oct	1978	20	4	855-863	78-16 to 78-20
Jan	1979	21	1	139-146	79-1 to 79-5
Apr	1979	21	2	256-263	79-6 to 79-10
Jul	1979	21	3	395-401	79-11 to 79-15
Oct	1979	21	4	559-569	79-16 to 79-20

• SPECT

ISSN 0025-5653
Mathematical Spectrum
Publisher: Applied Probability Trust

∆ Problems and Solutions — David W. Sharpe

Year	Vol	Issue	Pages	Proposals
1974/75	7	1	31	7.1-3
1974/75	7	2	67-70	7.4-6
1974/75	7	3	102-103	7.7-9
1975/76	8	1	33-34	8.1-3
1975/76	8	2	64-65	8.4-6
1975/76	8	3	91-95	8.7-9
1976/77	9	1	32-34	9.1-3
1976/77	9	2	64-65	9.4-6
1976/77	9	3	97-99	9.7-9
1977/78	10	1	31-34	10.1-3
1977/78	10	2	63-65	10.4-6
1977/78	10	3	97-99	10.7-9
1978/79	11	1	28-29	11.1-3
1978/79	11	2	61-65	11.4-6
1978/79	11	3	100-101	11.7-9

• SSM

ISSN 0036-6803
School Science and Mathematics
Publisher: Wiley Blackwell Publishers

Problem Column Editors:
Jan 1975 to Jun 1976 — Margaret F. Willerding
Oct 1976 to Dec 1979 — N. J. Kuenzi and Bob Prielipp

∆ Problem Department

Date	Year	Vol	Issue	Pages	Proposals
Jan	1975	75	1	none	none
Feb	1975	75	2	199-204	3568-3573
Mar	1975	75	3	293-298	3574-3579
Apr	1975	75	4	381-387	3580-3585
May/Jun	1975	75	5	473-478	3586-3591
Oct	1975	75	6	563-568	3592-3596
Nov	1975	75	7	653-658	3597-3603
Dec	1975	75	8	743-748	3606-3611
Jan	1976	76	1	82-86	3612-3617
Feb	1976	76	2	170-175	3618-3623
Mar	1976	76	3	261-266	3624-3629
Apr	1976	76	4	none	none
May/Jun	1976	76	5	439-446	3630-3641
Oct	1976	76	6	527-534	3642-3647
Nov	1976	76	7	621-627	3648-3653
Dec	1976	76	8	714-718	3654-3659
Jan	1977	77	1	77-82	3660-3665
Feb	1977	77	2	169-174	3666-3671
Mar	1977	77	3	263-268	3672-3677
Apr	1977	77	4	353-358	3678-3683
May/Jun	1977	77	5	443-449	3684-3689
Oct	1977	77	6	530-536	3690-3695
Nov	1977	77	7	620-627	3696-3701
Dec	1977	77	8	712-717	3702-3707
Jan	1978	78	1	81-87	3708-3713
Feb	1978	78	2	170-177	3714-3719
Mar	1978	78	3	none	none
Apr	1978	78	4	353-358	3720-3725
May/Jun	1978	78	5	443-449	3726-3731
Oct	1978	78	6	532-537	3732-3737
Nov	1978	78	7	620-627	3738-3743
Dec	1978	78	8	712-718	3744-3749
Jan	1979	79	1	79-87	3750-3755
Feb	1979	79	2	172-176	3756-3761
Mar	1979	79	3	259-264	3762-3767
Apr	1979	79	4	355-361	3768-3773
May/Jun	1979	79	5	444-450	3774-3779
Oct	1979	79	6	527-534	3780-3785
Nov	1979	79	7	none	none
Dec	1979	79	8	711-717	3786-3791

Notes:
•There were no problems numbered 3604 or 3605.

• TYCMJ

ISSN 0049-4925

The Two Year College Mathematics Journal

Publisher:. . . . . Taylor & Francis, Ltd.
Problem column editor: Erwin Just
Associate editor: Samuel A. Greenspan
Assistant editor: Stanley Friedlander

Δ Problems and Solutions Erwin Just

Date	Year	Vol	Issue	Pages	Proposals
Feb	1975	6	1	32-34	33-36
May	1975	6	2	31-35	37-41
Sep	1975	6	3	34-37	42-46
Dec	1975	6	4	24-28	47-53
Feb	1976	7	1	28-32	54-60
May	1976	7	2	49-53	61-66
Sep	1976	7	3	47-50	67-72
Dec	1976	7	4	33-37	73-78
Jan	1977	8	1	42-46	79-83
Mar	1977	8	2	95-100	84-89
Jun	1977	8	3	177-181	90-95
Sep	1977	8	4	240-243	96-100
Nov	1977	8	5	292-295	101-105
Jan	1978	9	1	40-45	106-110
Mar	1978	9	2	95-100	111-115
Jun	1978	9	3	176-181	116-120
Sep	1978	9	4	236-242	121-125
Nov	1978	9	5	297-302	126-130
Jan	1979	10	1	52-57	131-135
Mar	1979	10	2	127-131	136-140
Jun	1979	10	3	210-217	141-145
Sep	1979	10	4	293-299	146-150
Nov	1979	10	5	359-367	151-155

Notes:
•Associate Editor: Samuel A. Greenspan
•Added Stanley Friedlander as the Assistant Editor starting with the Jan., 1978 issue, (Volume 9, number 1).

UNSOLVED PROBLEMS

Use this section to

- locate problems that are still unsolved
- determine the names of proposers who have submitted unsolved problems.

This section lists those problems that were proposed during the years 1975–1979 in one of the journals covered by this index but whose complete solution has not been published as of May 1992.

An index of the proposers of these unsolved problems follows the statements of the problems (see page 422).

A problem is not listed as unsolved if
- the journal ceased publication before the solution could be printed
- the problem column indicates that it is a practice problem whose solution they do not intend to publish
- the problem was withdrawn
- the solution to the problem has appeared in an article listed in the citation index.

If the original problem consisted of several parts, only those parts that remain unsolved are listed.

Readers making progress on these problems should correspond with the problem column editor of the problem column in which the problem appeared. Do **not** send comments to the editors or publisher of this index. The names and addresses of the current problem column editors can be found in the Current Journal Information section of this index. If the solution to one of these problems appears as a journal article that you think we might miss when we prepare the citation index for our next volume, then MathPro Press would be pleased to receive a reference to the paper containing a solution or partial solution.

If the Problem Chronology in this index shows a problem to be unsolved but it is not listed in this section, then you should also consult the Citation Index (beginning on page 423) to see if an article has been published that contains the solution to that problem.

Unsolved Problems

AMM 6016. **by C. J. Moreno**

Let $D(n) = \prod p$, where the product runs over those primes p such that $p-1$ divides $2n$. Find an asymptotic formula for

$$\sum_{n \le x} D(n).$$

AMM 6020. **by C. W. Anderson and Dean Hickerson**

A pair of distinct numbers (k, m) is called a friendly pair (k is a friend of m) if $\Sigma(k) = \Sigma(m)$, where $\Sigma(n) = \sigma(n)/n$, where $\sigma(n)$ is the sum of the divisors of n. Show that the density of solitary numbers (numbers without friends) is zero.

AMM 6028. **by F. D. Hammer**

Is there a polynomial in two variables with integral coefficients which is a bijection from $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}$? If so, how many such polynomials are there?

AMM 6029. **by P. P. Carreras**

Let $E[t]$ be a linear space provided with a separated locally convex topology t. Show that $E[t]$ is bornological if and only if every absolutely convex bornivorous and algebraically closed subset of $E[t]$ is a t-neighborhood of the origin.

AMM 6048. **by H. M. Edgar**

A positive integer n is said to be harmonic if the ratio

$$\frac{n\tau(n)}{\sigma(n)}$$

is again integral.

(a) Are there any harmonic numbers other than 1 that are perfect squares?

(b) Do there exist infinitely many harmonic numbers?

AMM 6051. **by Jochem Zowe**

Let X be a real vector space, Y an ordered vector space and p a sublinear map of X into Y, i.e., $p(\lambda x) = \lambda p(x)$ and $p(x + x') \le p(x) + p(x')$ for all $x, x' \in X$ and all real nonnegative λ. Does there always exist a linear map T of X into Y such that $Tx \le p(x)$ for all $x \in X$?

AMM 6060. **by Daniel Sokolowsky**

For fixed $k \ge 2$, A_i, B_i $(i = 1, 2, \dots, k)$ are $2k$ subsets of a finite set S. What is the largest possible value of $n = |S|$ such that the following three conditions can hold simultaneously for $i = 1, 2, \dots, k$?

(*i*) $A_i \cap B_i = \varnothing$

(*ii*) $|A_i \cup B_i| = n - 1$

(*iii*) For each $x \in S$, $\{x\}$ is the intersection of an appropriate subcollection of the $2k$ sets A_i, B_i $(i = 1, 2, \dots, k)$.

AMM 6089. **by E. Ehrhart**

Let K be a convex body in $\mathbb{R}^n$ of Jordan content

$$V(K) > \frac{(n+1)^n}{n!}$$

with $n > 2$ and with centroid at the origin. Does $K \cup (-K)$ contain a convex body C, symmetric in the origin, for which $V(C) > 2^n$?

AMM 6110. **by David M. Battany**

Let p and q be primes; not both even. Let m, n and v be integers; $m, n \ge 2$; $v \ge 0$. For each value of v, prove that there exists at most one pair of powers (p^m, q^n) such that $p^m - q^n = 2^v$.

AMM 6119. **by M. J. Pelling**

AMM 6216. **by M. J. Pelling**

Are there any algebraic number fields A with the property that $A = A_1 + A_2$ (qua abelian groups), where A_1, A_2 are proper subfields of A?

AMM 6123. **by E. G. Kundert**

Let s be any integer larger than 1 and let ε_i be the following function defined on the integers:

$$\varepsilon_i = \begin{cases} 0 & \text{if } i \equiv 0, 6 \\ 1 & \text{if } i \equiv 2, 4, 7, 11 \\ -1 & \text{if } i \equiv 1, 5, 8, 10 \qquad \pmod{12} \\ 2 & \text{if } i \equiv 9 \\ -2 & \text{if } i \equiv 3 \end{cases}$$

Show that the following identity holds:

$$\sum_{1 \le i, j \le s} \varepsilon_i \varepsilon_j \binom{j+1}{s-i} \binom{s+1}{j+1} 3^{\lfloor i/2 \rfloor + \lfloor j/2 \rfloor - \lfloor (s-2)/2 \rfloor} = -3\varepsilon_s.$$

AMM 6124. **by Thomas E. Elsner**

Let Y be a compactification of a completely regular space X. Is there a base B for Y such that the smallest algebra of sets containing B has no element in $Y - X$?

AMM 6135. **by Paul Erdős**

Denote by $P(n)$ the greatest prime factor of n and put

$$A(x, y) = \prod_{1 \le i \le y - x} (x + i).$$

An integer n is called exceptional if for some $x \le n \le y$, $(P(A(x, y)))^2$ divides $A(x, y)$.

Prove that the density of exceptional numbers is 0 and estimate the number $E(x)$ not exceeding x as well as you can.

AMM 6141. **by Dennis Johnson and Herbert Taylor**

Can the Borromean Rings be drawn without crossing on a surface of genus 2?

AMM 6144. **by Carl Pomerance**

If n is a natural number, denote by $A(n)$ the arithmetic mean of the divisors of n.

(a) Prove that the asymptotic density of the set of n, for which $A(n)$ is an integer, is 1.

(b) Show that for any N there is an integer m such that $A(n) = m$ has at least N solutions.

(c) If it exists, find the asymptotic density of the set of integers m for which $A(n) = m$ has a solution.

AMM 6157. **by C. C. Chen and D. E. Daykin**

(a) Find integers Δ, p with the following property: Whenever the lines of the complete graph K_p are colored so that every vertex is on at most Δ lines of each color, there is a triangle whose lines have different colors.

(b) Find integers δ, p, n with the following property: Whenever the lines of a complete graph K_p are colored with n colors so that every vertex is on at least δ lines of each color, there is a triangle whose lines have different colors.

AMM 6158. **by M. J. Pelling**

Prove that if R is a bounded convex region of the plane of area 1 then there is a $d > 0$ independent of R such that R is equivalent under an area preserving affine transformation to a region of diameter at most d. What is the best possible value of d?

AMM 6172. **by Doug Hensley**

Give an example, if possible, of two planar lattices of unit determinant that do not possess a common bounded measurable fundamental domain. Do any two distinct lattices possess a common fundamental domain?

AMM 6181. **by J. M. Arnaudies**

Let n be an integer with $n \geq 3$, and let $A_0, A_1, \ldots, A_n$ be n single-valued real functions defined and continuous on a given topological Hausdorff space T. Suppose that for all $t \in T$, the 2-form

$$A_0x^n + A_1x^{n-1}y + \ldots + A_ny^n$$

(where the A_i take their values for t) defines n real distinct lines in the 2-dimensional real projective space.

Characterize spaces T such that, for any choice of the A_i, there exists a system of continuous functions (P_1, Q_1, P_2, $Q_2, \ldots, P_n$, Q_n), real-valued, defined on T, satisfying the formal equality,

$$A_0x^n + A_1x^{n-1}y + \cdots + A_ny^n$$
$$= (P_1x + Q_1y)(P_2x + Q_2y)\cdots(P_nx + Q_ny).$$

AMM 6186. **by Ronald Evans**

Let $r, k \in \mathbb{N}$, where r is fixed. Fix $\beta > 1$. Let

$$F_r(k) = \sum (j_1j_2\cdots j_r)^{\beta-1},$$

where the sum is over all vectors $(j_1, j_2, \ldots, j_r) \in \mathbb{N}^r$ for which $j_1 + j_2 + \cdots + j_r = k$. Prove that

$$F_r(k) \sim \frac{\Gamma^r(\beta)}{\Gamma(r\beta)}k^{\beta r-1} \qquad \text{as} \qquad k \to \infty.$$

AMM 6189. **by Edward T. H. Wang**

Prove or disprove that for each natural number $n \geq 2$, one can arrange the numbers $1, 2, \ldots, n$ in a sequence such that the sum of any two adjacent terms is a prime.

AMM 6190. **by D. E. Daykin and D. J. Kleitman**

Let n be a square free integer that is not prime. Let F be a set of divisors of n such that neither the product of two elements of F nor n^2 divided by such a product is in F. What is the maximal proportion of the divisors of n that may lie in F?

AMM 6197. **by Manuel Scarowsky**

Let p be a prime; a and b positive integers; and let (x_0, y_0) be a solution of $ax + by = p$ in positive integers with x_0 minimal, if such exists (otherwise take $x_0 = 0$). Find an estimate for $\sum_{a,b} x_0$.

AMM 6204. **by F. David Hammer**

(a) If all proper subgroups of an infinite abelian group are free (as abelian groups), then the group is free.

(b) Find a weaker hypothesis for (a).

(c) Delete abelian in (a).

AMM 6211. **by Alvin J. Paullay and Sidney Penner**

Suppose that each square of an $n \times n$ chessboard is colored either black or white. A square, formed by the horizontal and vertical lines of the board, will be called chromatic if its four distinct corner squares are all of the same color.

Find the smallest n such that, with any such coloring, every $n \times n$ board must contain a chromatic square.

AMM 6212. **by A. A. Mullin**

Prove that $\lfloor \pi^n \rfloor$ is prime for only finitely many positive integers n.

AMM 6214. **by Leonard Carlitz**

Let k and t be fixed integers, $k \geq 2$, $t \geq 0$ and let $A_k(kn + t)$ denote the number of permutations of

$$Z_{kn+t} = \{1, 2, 3, \ldots, kn + t\}$$

such that

$$a_{kj+1} < a_{kj+2} < \cdots < a_{kj+k},$$
$$a_{kj+k} > a_{kj+k+1} \quad (j = 0, 1, \ldots, n-1)$$
$$a_{kn+1} < a_{kn+2} < \cdots < a_{kn+t}.$$

It has recently been proved as a corollary of a general result that $A_4(2n+1) = 2^{-n}A_2(2n+1)$. Prove this identity by a direct combinatorial argument.

AMM 6217. **by M. J. Pelling**

Let B be a subset of the nonnegative integers having positive density. Is it always true that there is an infinite subset X of B and an infinite sequence $k_1 < k_2 < \cdots$ of integers such that all the translates $X + k_i \subseteq B$?

AMM 6229. **by David W. Erbach**

Suppose that the plane is tiled with regular hexagons in the customary manner. Color each black or white independently with probability $1/2$. What is the expected size of a connected monochromatic component? What is the probability that there is an infinite component?

AMM 6232. **by Allan Wm. Johnson, Jr.**

Prove or disprove: Given any integer $G > 13$, there exist distinct integers $x_i > 0$ such that

$$G^3 = \sum_{i=1}^{5} x_i^3.$$

AMM 6258. **by John S. Lew**

Let $X = (x_{jk})$ be an $m \times n$ matrix, where $1 < m < n$ and the x_{jk} are algebraically independent indeterminates over the field $\mathbb{C}$ of complex numbers. Let X' be the transpose of X. Prove that $\det(XX')$ is an irreducible polynomial over $\mathbb{C}$.

AMM 6270. **by Kenneth S. Williams**

Let p be a prime congruent to 1 modulo 8. Let ε_{2p} denote the fundamental unit of the real quadratic field $Q\left(\sqrt{2p}\,\right)$ and let $h(-2p)$ denote the class number of the imaginary quadratic field $Q\left(\sqrt{-2p}\,\right)$. Prove that if the norm of ε_{2p} is -1, then

$$h(-2p) \equiv \begin{cases} 0 \pmod 8, & \text{if } p \equiv 1 \pmod{16} \\ 4 \pmod 8, & \text{if } p \equiv 9 \pmod{16}. \end{cases}$$

AMM 6281. **by Clark Kimberling**

If $A = (1, a_1, a_2, \ldots)$ is a sequence of 1's and 2's, let $B = (1, b_1, b_2, \ldots)$ where b_n is the length of the nth maximal string of identical symbols in A. If $B = A$, then A must be $(1, 2, 2, 1, 1, 2, 1, 2, 2, 1, \ldots)$. By a run is meant a finite subsequence of consecutive terms of A. Its complement is obtained by interchanging all 1's and 2's.

Prove or disprove:

(a) The complement of every run is also a run;

(b) Every run occurs infinitely many times.

AMM E2521. **by John A. Cross**

An instructor has a file of p questions of equal diagnostic value in testing students on a certain topic. He gives q-question tests repeatedly ($q < p$). How many test forms can he compose if any n-size subset, $1 \leq n < q$, of the p questions may appear on at most two tests, and no subset of size $m > n$ may appear on more than one test? Determine an algorithm for composing the set of possible tests, for any allowable p, q, n.

AMM E2530. **by F. Loupekine**

(a) Show that it is possible to partition the natural numbers into three classes so that if (x, y, z) is a primitive Pythagorean triple, then x, y, z are in different classes.

(b) Can such a partition be made if the above is to hold for all Pythagorean triples, not just primitive ones?

AMM E2539. **by A. Vince**

Let F_n denote the nth Fibonacci number. Prove or disprove: If $m^2 \mid F_n$, then $m \mid n$.

AMM E2569. **by Harry Dweighter**

The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?

AMM E2571. **by Sidney Kravitz**

Find all numbers n for which $\sigma(n) = 2n - 2$.

AMM E2594. **by David P. Robbins**

Suppose that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are vectors corresponding to the edges of an oriented regular polygon. Since their sum is 0, an object undergoing displacements by each of these vectors in some order traces out a closed polygon. If this order is chosen at random, what is the probability that the polygon does not intersect itself?

AMM E2596. **by Mark A. Spikell**

Suppose one is supplied with a collection of Cuisenaire rods having dimension $1 \times 1 \times a$, where the length a belongs to a finite set A of positive integers and the number of rods of length $a \in A$ may be supposed to be unlimited. For which s can one build a $1 \times s \times s$ square from one's collection?

AMM E2688. **by David Jackson**

Let $\{f_i\}$ and $\{g_i\}$ $(i = 0, 1, 2, \ldots)$ be the solutions of the recurrence equation

$$u_{m+1} = -u_m - m(m+1)xu_{m-1}$$

satisfying the initial conditions $f_0 = 0$, $f_1 = 1$ and $g_0 = 1$, $g_1 = -1$, respectively. Show that the coefficient of x^{n-1} in the Maclaurin expansion of $-f_n/g_n$ is t_{2n-1} where

$$\tan x = \sum_{n \geq 1} t_{2n-1} \frac{x^{2n-1}}{(2n-1)!} .$$

AMM E2702. **by David Jackson**

Let $a = (a_1, a_2, \ldots, a_{2m})$ be a non-decreasing sequence of positive integers. Let S denote the set of sequences obtained from a by permuting its terms. Let A, B, C be the subsets of S consisting of those sequences $s = (s_1, s_2, \ldots, s_{2m})$ that satisfy

$$s_1 < s_2 \geq s_3 < s_4 \geq \cdots \geq s_{2m-1} < s_{2m},$$

$$\prod_{i=1}^{2m}(s_i - a_i) > 0, \qquad \prod_{i=1}^{2m}(s_i - a_i) < 0,$$

respectively. Show that $|A|$ is equal to the absolute value of $|B| - |C|$.

AMM E2713. **by Saul Singer**

A stack of x rings is given, decreasing in size from the bottom up. In addition, y empty stacks are provided ($y \geq 2$). Let $N(x, y)$ be the minimum number of moves necessary to transfer the rings to one of the empty stacks subject to the following two rules:

(1) Move just one ring at a time,

(2) at no time can a larger ring be placed atop a smaller one.

It is conjectured that

$$N(x, y) = \sum_{k=1}^{m} 2^{k-1} \binom{k+y-3}{y-2} + 2^m \left[x - \binom{m+y-2}{y-1} \right],$$

where m is the largest integer such that the expression in the brackets is nonnegative.

AMM E2717. **by E. Ehrhart**

Find the number of symmetric 4×4 matrices whose entries are all the integers from 1 to 10 and whose row-sums are all equal.

AMM E2722. **by Clark Kimberling**

A ball is drawn from an urn containing one red ball and one green ball. If it is red it is returned to the urn with one additional red ball and one additional green ball, but if it is green no balls are put into the urn. After the first drawing, subsequent drawings take place following the same rules. Find the probability that the urn contains at least one green ball at all times.

AMM E2740. **by Victor Pambuccian**

Show that if P is a convex polyhedron, one can find a square all of whose vertices are on four different faces of P.

AMM E2757. **by Harry D. Ruderman**

Let a, b, c be three lines in $\mathbb{R}^3$. Find points A, B, C on a, b, c, respectively, such that $AB+BC+CA$ is a minimum.

AMM E2759. **by Hugh L. Montgomery**

Suppose that $a^{-1} \leq f''(x) \leq 2a^{-1}$ for $0 \leq x \leq a$, where $a \geq 8$. Prove that there exists a lattice point (m, n) such that $0 \leq m \leq a$ and $|f(m) - n| \leq 2a^{-1/2}$.

AMM E2774. **by James Propp**

Prove or disprove that, given a convex two-dimensional figure S, six translates of S can fit inside a homothetic figure three times as large as S in linear dimensions.

AMM E2779. by H. Schwerdtfeger

(a) Let $A = \left(a^{(1)}a^{(2)}\dots a^{(n)}\right)$ be a non-singular matrix, over a field F, whose columns $a^{(j)}$ represent points in the n-dimensional affine space S_n. Let π be the hyperplane passing through the points $a^{(1)},\dots,a^{(n)}$. Let $b\in S_n$, $b\neq 0$, and B be the matrix $(b\,b\dots b)$. Show that the determinant $|A-B|=0$ if and only if $b\in\pi$.

(b) Generalize statement (a) to a more general matrix of rank one, namely $B=(\gamma_1 b\gamma_2 b\dots\gamma_n b)$, $\gamma_1\gamma_2\dots\gamma_n\neq 0$, $\gamma_j\in F$.

(c) If A is singular and Σ is the subspace of S_n generated by the columns of A, show that there is no b in Σ such that $|A-B|\neq 0$, with $B=(b\,b\dots b)$.

AMM E2794. by Robert A. Leslie

Let m, n, r, and c be positive integers with $rm=cn$. How many $m\times n$ matrices are there with each entry either 0 or 1 and where every row sum is r and every column sum is c?

AMM E2804. by Harry D. Ruderman

Let k be a positive integer and S_k be the set of integers j expressible in the form

$$j=k|ab|+a+b,$$

where a, b, run through the nonzero integers. Find the cardinality of the set of positive integers not in S_k.

AMM S21. by Paul Erdős

Let

$$A(n,k)=(n+1)(n+2)\cdots(n+k),$$

$$B(n,k)=\operatorname{lcm}\left[n+1,n+2,\dots,n+k\right],$$

and

$$\alpha(n,k)=\frac{A(n,k)}{B(n,k)}\ .$$

Do m, n, and k exist with $m>n+k-1$ and $\alpha(m,k)=\alpha(n,k)$?

CMB P268. by P. Erdős and E. C. Milner

A graph $G=(V,E)$ is said to be realized if there is a family of sets $\{A_x : x\in V\}$ associated with the vertices of G such that $A_x\subset\{0,1,2\dots\}$ and such that $\{x,y\}$ is an edge of G if and only if $A_x\cap A_y=\varnothing$. Is it true that any bipartite graph on $2^{\aleph_0}$ vertices is realizable?

CMB P277. by Allan M. Krall and D. J. Allwright

Let $R(z)$ be a rational function of the complex variable z, and let Γ be the locus of $R(ix)$ for x real. Prove that Γ partitions the plane into finitely many regions.

CRUX 133. submitted by Kenneth S. Williams

Let f be the operation that takes a positive integer n to $n/2$ (if n even) and to $3n+1$ (if n odd). Prove or disprove that any positive integer can be reduced to 1 by successively applying f to it.

CRUX 154. by Kenneth S. Williams

Let p_n denote the nth prime. Prove or disprove that the following method finds p_{n+1} given $p_1,p_2,\dots,p_n$.

In a row list the integers from 1 to p_n-1. Corresponding to each r $(1\le r\le p_n-1)$ in this list, say $r=p_1^{a_1}\dots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1}\dots p_n^{a_{n-1}}$ in a second row. Let l be the smallest odd integer not appearing in the second row. The claim is that $l=p_{n+1}$.

CRUX 250. by Gilbert W. Kessler

For integers m and n, if $|3^m-2^n|\neq 1$, is there always a prime between 3^m and 2^n?

CRUX 266. by Daniel Rokhsar

Let d_n be the first digit in the decimal representation of $n!$. Find expressions for d_n and $\sum_{i=0}^{n} d_i$.

CRUX 339. by Steven R. Conrad

Is $\binom{37}{2}=666$ the only binomial coefficient $\binom{n}{r}$ whose decimal representation consists of a single digit repeated k times with $k\ge 3$?

CRUX 342. by James Gary Propp

For fixed even n with $n>2$, the set of all positive integers is partitioned into the (disjoint) subsets $S_1,S_2,\dots,S_n$ as follows: for each positive integer m, we have $m\in S_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the n subsets.

Prove that $m\in S_n$ for all sufficiently large m.

CRUX 343. by Steven R. Conrad

It is known that the greatest integer function satisfies the functional equation

$$f(nx)=\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)$$

for all real x and positive intrgers n. Are there other functions which satisfy this equation? Find as many as possible.

CRUX 355. by James Gary Propp

Given a finite sequence $A=(a_n)$ of positive integers, we define the family of sequences

$$A_0=A;\quad A_i=(b_r),\quad i=1,2,3,\dots,$$

where b_r is the number of times that the rth lowest term of A_{i-1} occurs in A_{i-1}.

For example, if $A=A_0=(2,4,2,2,4,5)$, then $A_1=(3,2,1)$, $A_2=(1,1,1)$, $A_3=(3)$, and $A_4=(1)=A_5=A_6=\dots$.

The degree of a sequence A is the smallest i such that $A_i=(1)$.

Let $A(d)$ be the length of the shortest sequence of degree d. Find a formula, recurrence relation, or asymptotic approximation for $A(d)$.

Given sequences A and B, define C as the concatenation of A and B. Find sharp upper and lower bounds on the degree of C in terms of the degrees of A and B.

CRUX 410. by James Gary Propp

Are there only finitely many powers of 2 that have no zeros in their decimal expansions?

CRUX 434. by Harold N. Shapiro

It is known that all the solutions in positive integers x,y,m,n of the equation

$$(m!)^x=(n!)^y$$

are given by $m=n=1$; and $m=n,\ x=y$.

Prove this result without using Bertrand's Postulate or equivalent results from number theory.

CRUX 443. by Allan Wm. Johnson Jr.

Does there exist a set of more than seven consecutive squares with the property that each has its decimal digits summing to a square?

CRUX 473. **by A. Liu**

The set of all positlve integers is partitioned into the disjoint subsets $T_1, T_2, T_3, \ldots$ as follows: for each positive integer m, we have $m \in T_k$ if and only if k is the largest integer such that m can be written as the sum of k distinct elements from one of the subsets. Prove that each T_k is finite.

CRUX 490. **by Michael W. Ecker**

Are there infinitely many palindromic primes?

CRUX 493. **by R. C. Lyness**

Let A, B, C be the angles of a triangle. It is known that there are positive x, y, z, each less than $\frac{1}{2}$, simultaneously satisfying

$$y^2 \cot\frac{B}{2} + 2yz + z^2 \cot\frac{C}{2} = \sin A,$$

$$z^2 \cot\frac{C}{2} + 2zx + x^2 \cot\frac{A}{2} = \sin B,$$

$$x^2 \cot\frac{A}{2} + 2xy + y^2 \cot\frac{B}{2} = \sin C.$$

In fact, $\frac{1}{2}$ may be replaced by a smaller $k > 0.4$. What is the least value of k?

CRUX 494. **by Rufus Isaacs**

Let r_j, $j = 1, \ldots, k$, be the roots of a polynomial with integral coefficients and leading coefficient 1.

Prove or disprove: for any positive integer n,

$$n \Bigm| \sum_j \Bigl(\sum_{d|n} r_j^d \mu(n/d)\Bigr),$$

where μ is the Möbius function.

FQ B-408. **by Lawrence Somer**

Let $d \in \{2, 3, \ldots\}$ and $G_n = F_{dn}/F_n$. Let p be an odd prime and $z = z(p)$ be the least positive integer n with $F_n \equiv 0 \pmod{p}$. For $d = 2$ and $z(p)$ an even integer $2k$, it is known that

$$F_{n+1}G_{n+k} \equiv F_n G_{n+k+1} \pmod{p}.$$

Establish a generalization for $d \geq 2$.

FQ B-416. **by Gene Jakubowski and V. E. Hoggatt, Jr.**

Let F_n be the Fibonacci sequence (defined for all integers n). Prove that every positive integer m has at least one representation of the form

$$m = \sum_{j=-N}^{N} \alpha_j F_j,$$

with each α_j in $\{0, 1\}$ and $\alpha_j = 0$ when j is an integral multiple of 3.

FQ H-254. **by R. Whitney**

Evaluate

$$\sum_{k=0}^{n} F_{\binom{n}{k}}.$$

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FQ H-260. **by H. Edgar**

Are there infinitely many subscripts, n, for which F_n or L_n are prime?

FQ H-271. **by R. Whitney**

Define the binary dual, D, as follows:

$$D = \left\{ t \Bigm| t = \prod_{i=0}^{n} (a_i + 2i); \quad a_i \in \{0, 1\}; \quad n \geq 0 \right\}$$

Let $\bar{D}$ denote the complement of D, with respect to the set of positive integers. Form a sequence, $\{S_n\}_{n=1}^{\infty}$, by arranging $\bar{D}$ in increasing order. Find a formula for S_n.

FQ H-296. **by C. Kimberling**

Suppose x and y are positive real numbers with $y > 1$. Find the least positive integer n for which

$$\left\lfloor \frac{x}{n+y} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor.$$

FQ H-300. **by James L. Murphy**

Given two relatively prime positive integers A and B, form a multiplicative Fibonacci sequence $\{A_i\}$ with $A_1 = A$, $A_2 = B$, and $A_{i+2} = A \times A_{i+1}$. Now form the sequence of partial sums $\{S_n\}$ where

$$S_n = \sum_{i=1}^{n} A_i.$$

$\{S_n\}$ is a subsequence of the arithmetic sequence $\{Y_n\}$ where $T_n = A + nB$, and by Dirichlet's theorem we know that infinitely many of the T_n are prime. The question is: Does such a sparse subsequence $\{S_n\}$ of the arithmetic sequence $A + nB$ also contain infinitely many primes?

FQ H-304. **by V. E. Hoggatt, Jr.**

(a) Show that there is a unique partition of the positive integers, $\mathbb{N}$, into two sets, A_1 and A_2, such that

$$A_1 \cup A_2 = \mathbb{N}, \quad A_1 \cap A_2 = \varnothing,$$

and no two distinct elements from the same set add up to a Lucas number.

(b) Show that every positive integer, M, which is not a Lucas number is the sum of two distinct elements of the same set.

FQ H-305. **by Martin Schechter**

For fixed positive integers, m, n, define a Fibonacci-like sequence as follows:

$$S_1 = 1,\ S_2 = m,\ S_k = \begin{cases} mS_{k-1} + S_{k-2} & \text{if } k \text{ is even} \\ nS_{k-1} + S_{k-2} & \text{if } k \text{ is odd} \end{cases}$$

Show that the sequence obtained when $[m = 1, n = 4]$ and when $[m = 1, n = 8]$, respectively, have only the element 1 in common.

FQ H-309. **by David Singmaster**

Let f be a permutation of $\{1, 2, \ldots, m-1\}$ such that the terms $i + f(i)$ are all distinct $\pmod{m}$. Characterize and/or enumerate such f.

MENEMUI 1.1.1. **by T. N. T. Goodman**

For $n = 1, 2, 3, \ldots$, show that

$$\sum_{j=1}^{n} \int_0^{\pi} \{\cos\theta(u - \pi)\sec\theta\pi - 1\} \csc\frac{u}{2}\, du = 2n\log n.$$

where

$$\theta = \frac{1}{2} - \frac{2j-1}{2n}.$$

MENEMUI 1.3.3. **by S. L. Lee**

If f is continuously differentiable up to derivatives of 4th order and $f(-1) = f(1) = 0$, find the least constant A such that

$$\left| \int_{-\sqrt{3}}^{\sqrt{3}} f(x)\,dx \right| \le A.$$

MM 1007. **by Thomas E. Elsner**

It is known that given a nonnegative integer n, there is a positive integer k, such that k occurs in exactly n distinct Pythagorean triples (x, y, z), $x < y < z$, $x^2 + y^2 = z^2$. For each n, determine $m_n = \min\{k : k$ occurs in exactly n Pythagorean triples$\}$.

MM 1015. **by Allan W. Johnson, Jr.**

Show that for $n \ge 5$ there are $2n+1$ distinct, positive, odd, square-free integers whose reciprocals add to one.

MM 1021. **by Peter Ørno**

Prove or disprove that a countably infinite set of positive real numbers with a finite nonzero cluster point can be arranged in a sequence, $\{a_n\}$, so that $\{(a_n)^{1/n}\}$ is convergent.

MM 1068. **by James Propp**

Given a simple closed curve S, let the "navel" of S denote the envelope of the family of lines that bisect the area within S.

If S is arbitrary (or bounds a convex set), find a sharp upper bound for the ratio of the area within the navel of S to the area within S.

MM 1073. **by James Propp**

Let A and B be the unique nondecreasing sequences of odd integers and even integers, respectively, such that for all $n \ge 1$, the number of integers i satisfying $A_i = 2n - 1$ is A_n and the number of integers i satisfying $B_i = 2n$ is B_n. That is, $A = (1, 3, 3, 3, 5, 5, 5, 7, 7, 7, 9, 9, 9, 9, 9, \ldots)$ and $B = (2, 2, 4, 4, 6, 6, 6, 6, 8, 8, 8, 8, \ldots)$. Is the difference $|A_n - B_n|$ bounded?

MM 1088. **by Alan Wayne**

For each positive integer m, how many triangles with integer sides are there that have an area equal to m times the perimeter?

PME 389. **by Paul Erdős**

Find a sequence of positive integers $1 \le a_1 < a_2 < \cdots$ that omits infinitely many integers from every arithmetic progression (in fact it has density 0) but which contains all but a finite number of terms of every geometric progression. Prove also that there is a set S of real numbers which omits infinitely many terms of any arithmetic progression but contains every geometric progression (disregarding a finite number of terms).

SIAM 75-6. **by P. C. T. de Boer and G. S. S. Ludford**

Show that there exists a continuous solution of

$$y'' = \left(2y^\alpha - x\right) y, \qquad \alpha > 0,$$

for $-\infty < x < \infty$ such that

$$y \sim (x/2)^{1/\alpha} \left[1 + (1-\alpha)/\alpha^3 x^3 + \cdots\right]$$

as $x \to +\infty$; and that, for some $k(\alpha)$, $y \sim k\mathrm{Ai}(-x)$ as $x \to -\infty$.

SIAM 75-13. **by M. Golberg**

Let P denote an $n \times n$ primitive stochastic matrix and let R denote a diagonal matrix with diagonal $(r_1, r_2, \ldots, r_n)$, where $0 \le r_i \le 1$. Determine

$$\lim_{N\to\infty} \frac{1}{N} \left\{ \sum_{k=1}^{N} \frac{(\mathrm{P}+\mathrm{R})^k}{\left(1 + \sum_{i=1}^{n} \frac{r_i}{n}\right)^k} \right\}.$$

SIAM 75-14. **by M. W. Green, A. J. Korsak, and M. C. Pease**

It has been found in practice that the following very simple (but very effective) procedure always converges for any n starting trial roots:

$$x_i' = \frac{x_i - P\left(x_i\right)}{\prod_{j\ne i}\left(x_i - x_j\right)}\ , \qquad i = 1, 2, \ldots, n,$$

where $P(x)$ is an arbitrary (complex coefficient) monic polynomial in x of degree n. In fact, even when $P(x)$ has multiple roots, the above procedure still converges, but only linearly (as opposed to quadratically in the distinct root case). Show that this procedure is globally convergent outside of a set of measure zero in the starting space and describe this set for $n > 2$.

SIAM 76-3. **by S. A. Rice**

Determine the inverse Laplace transforms, or at least asymptotic formulas for large time t, of the following three functions:

$$\frac{I_v(x)}{I_v(y)}\ ,$$

$$\frac{I_v(x) I_v(z) K_v(y)}{I_v(y)}\ ,$$

$$I_v(z) K_v(x).$$

Here $I_v(x)$ and $K_v(x)$ are modified Bessel's functions of the first and second kind, respectively, and $v = \sqrt{as}$, where s is the Laplace transform parameter, a is a constant, and $x \ne y \ne z$.

SIAM 76-7. **by R. D. Spinetto**

Suppose a company wants to locate k service centers that will service n communities and suppose that the company wants to locate these k centers in k of the communities so that the total population distance traveled by the people in the $n-k$ communities without service centers to those communities with service centers is minimized. This problem can be set up as a 0-1 integer programming problem as follows. Let

$$x_{jj} = \begin{cases} 1 & \text{if community } j \text{ gets a service center,} \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$x_{ij} = \begin{cases} 1 & \text{if community } i \text{ is to be serviced by a center} \\ & \text{in community } j, \\ 0 & \text{otherwise.} \end{cases}$$

Let p_i be the population of community i and let d_{ij} be the distance from community i to community j. The problem then is to minimize

$$\sum_{i=1}^{n} \sum_{j=1}^{n} p_i d_{ij} x_{ij},$$

subject to constraints

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for } i = 1, 2, 3, \ldots, n;$$

$$x_{ij} - x_{jj} \le 0 \qquad \text{for } i = 1,2,3,\dots,n, \text{ and for } j = 1,2,3,\dots,n;$$

$$\sum_{j=1}^{n} x_{jj} = k,$$

and with the added condition that each of the variables x_{ii} and x_{ij} takes on only the values of 0 or 1.

If one ignores this last 0-1 condition and solves the problem as though it were a linear programming problem, then one will find that very often (but not always) an optimal extreme point solution to this linear programming problem will in fact be a 0-1 extreme point. Perhaps this is due to the fact that most of the extreme points of the polyhedron determined by the constraints shown above are in fact 0-1 extreme points, but it cannot be proven. This, in turn, suggests the following problems:

(a) What are the smallest n and k for which there exists a linear programming problem of the above form which will have only non-0-1 optimal extreme point solutions?

(b) Can the non-0-1 extreme points of polyhedrons determined by the constraints shown above be characterized in any set theoretic way that would be useful in developing more efficient algorithms for solving this facility location problem?

SIAM 76-10. by L. Wijnberg and M. L. Glasser

If $\alpha > 1$, $v \ge 0$, and

$$S_v(x) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \binom{m+n}{m} (2\alpha)^m J_{v+m+2n+1}(x),$$

it is known that

$$S_v(x) = \frac{1}{2}\left\{ \frac{e^{\alpha x}\left[(1+\alpha^2)^{1/2} - \alpha\right]^v}{(1+\alpha^2)^{1/2} - G_v(\alpha,x)} \right\},$$

where

$$G_v(\alpha,x) = \sum_{k=0}^{\infty} \alpha^{-k-1} J_v^{(k)}(x).$$

Can a similar result be found for $0 < \alpha < 1$?

Also, is there a closed form for $G_v(\alpha,x)$?

SIAM 76-12. by A. S. Perelson and C. Delisi

The following system of nonlinear differential equations

$$\frac{dx_n}{dt} = 2k\sum_{m=1}^{n-1} x_{n-m}y_m - 2x_n(kS + k'n) + k'\sum_{m=n}^{\infty}(2x_m + y_m), \qquad n = 1,2,\dots,$$

$$\frac{dy_n}{dt} = 4k\sum_{m=1}^{n} z_{n-m}x_m + k\sum_{m=1}^{n-1} y_{n-m}y_m - y_n\left[k(S+L) + (2n-1)k'\right] + 2k'\left[\sum_{m=n+1}^{\infty} x_m + \sum_{m=n+1}^{\infty} y_n + \sum_{m=n}^{\infty} z_m\right],$$
$$n = 1,2,\dots,$$

$$\frac{dz_n}{dt} = 2k\sum_{m=1}^{n} z_{n-m}y_m - 2z_n(kL + k'n) + k'\sum_{m=n+1}^{\infty}(2z_m + y_m), \qquad n = 0,1,2,\dots,$$

where

$$S = \sum_{m=1}^{\infty} y_m + 2\sum_{m=0}^{\infty} z_m$$

and

$$L = \sum_{m=1}^{\infty} (y_m + 2x_m),$$

subject to the initial conditions $x_1(0) = a$, $x_n(0) = 0$ $(n = 2,3,\dots)$, $y_n(0) = 0 = z_n(0)$ $(n = 1,2,\dots)$, $z_0 = b$, with k and k' being nonnegative constants, can be solved by a combinatorial method.

The problem we pose is to generate the combinatoric solution via direct methods applied to equations one through three.

SIAM 76-14. by L. Carlitz

The following formulas appear in an earlier paper:

$$\sum_{i=0}^{m}\sum_{j=0}^{n} (-1)^{i+j} \frac{\binom{m}{i}^2\binom{n}{j}^2}{\binom{m+n}{i+j}} = \delta_{mn},$$

$$\sum_{r=0}^{\min(i,j,k)} \frac{\binom{i}{r}\binom{j}{r}\binom{k}{r}}{\binom{i+j+k}{r}} = \frac{(j+k)!(k+i)!(i+j)!}{i!j!k!(i+j+k)!}.$$

Simpler proofs of these would be desirable.

SIAM 76-21. by P. Barrucand

Define the polynomials $\{p_n(x,m,\gamma)\}$ by the generating function

$$\sum p_n(x,m,\gamma)t^n = \frac{\exp(xt)}{[\Gamma(1+\gamma+t)]^m},$$

m positive integer, $\gamma > -1$.

Prove that for every n, all the zeros of $p_n(x)$ are real and give an asymptotic formula for the lesser-in-modulus (i.e., the greater) negative zeros.

SIAM 77-5. by M. L. Glasser

Let

$$S(r) = \sum_{k=1}^{\infty} (-1)^{k+1} \sinh y \operatorname{csch} ky \left(y = \cosh^{-1} r\right).$$

Prove whether or not $S(r)$ is monotone between $S(1) = \log 2$ and $S(\infty) = 1$.

SIAM 77-14. by G. K. Kristiansen

Let $P = \{p_{rs}\}$ be a symmetric matrix having

(1) $p_{rs} = 0$ for $|r-s| > 1$ and $p_{rs} > 0$ otherwise,
(2) spectral radius 1, and
(3) $p_{s-1,s} + p_{s+1,s} \le 1$ for all s.

Denote by e^T the $1 \times n$ matrix with all entries 1, and let

$$I = \{\delta_{rs}\}$$

be the $n \times n$ unit matrix. Let c be a nonnegative $n \times 1$ matrix with $e^T c = 1$. Prove or disprove that the matrix

$$F = \left(I - ce^T\right) P$$

has spectral radius at most equal to 1. If a counterexample is found, try to minimize the order n.

SIAM 78-1. **by J. S. Lew**

Let (x, y) be an arbitrary point of the Euclidean unit disc D, let $a(p; x, y)$ denote the average l^p distance to a random disc point (u, v), and let $b(p; r)$ denote the rotational average of this function $a(p; x, y)$:

$$D = \left\{(x, y) : x^2 + y^2 \le 1\right\},$$

$$a(p; x, y) = \int\int_D \left\{|x-u|^p + |y-v|^p\right\}^{1/p} du\, dv/\pi,$$

$$b(p; r) = \int_0^{2\pi} a(p; r\cos\theta, r\sin\theta)\, d\theta/(2\pi).$$

To measure the deviation from this average, we introduce the ratio of these quantities and we consider its extrema on the disc:

$$c(p; x, y) = a(p; x, y)/\left[b\left(p; \sqrt{x^2+y^2}\,\right)\right],$$

$$\lambda(p) = \inf\left\{c(p; x, y) : (x, y) \in D\right\},$$

$$\mu(p) = \sup\left\{c(p; x, y) : (x, y) \in D\right\}.$$

Conjecture: $\lambda(p) \uparrow 1$ and $\mu(p) \downarrow 1$ as either $p \uparrow 2$ or $p \downarrow 2$.

SIAM 78-4. **by C. L. Mallows**

Find the symmetric cumulative distribution function $G(x)$ satisfying $dG(0) = \alpha$, $0 < \alpha < 1$ that minimizes the integral

$$I_f = \int_{-\infty}^{\infty} \frac{\left(f'(x)\right)^2}{f(x)}\, dx,$$

where $f(x)$ is the convolution

$$f(x) = \int_{-\infty}^{\infty} \phi(x-u)\, dG(u),$$

with $\phi(u)$ the standard Gaussian density

$$\phi(u) = (2\pi)^{-1/2} \exp\left[-\frac{1}{2}u^2\right].$$

It is believed that G is a step function, so that

$$f(x) = \sum p_j \phi\left(x - g_j\right),$$

with $g_{-j} = -g_j$, $p_{-j} = p_j > 0$, $p_0 = \alpha$.

SIAM 78-9. **by W. Aiello and T. V. Narayana**

Suppose we assign positive integer weights to the vote of each member of a board of directors that consists of n members so that the following conditions apply:

(1) Different subsets of the board always have different total weights so that there are no ties in voting (tie-avoiding).

(2) Any subset of size k will always have more weight than any subset of size $k-1$ $(k = 1, \ldots, n)$ so that any majority carries the vote, abstentions allowed (nondistorting).

A solution is given in Table 1 below for $n = 1, \ldots, 7$ that can be extended very easily from any n to $n+1$. It is conjectured that this is a minimal dominance solution. Here, an increasing sequence $(y_1, \ldots, y_n)$ is said to dominate another increasing sequence $(x_1, \ldots, x_n)$ if $y_i \ge x_i$ $(i = 1, \ldots, n)$. So a solution $(x_1, \ldots, x_n)$ is minimal dominant if no other solution $(y_1, \ldots, y_n)$ exists such that $x_i \ge y_i$ $(i = 1, \ldots, n)$. The underlined values along the diagonal of vector elements are the I_n values, where:

$$I_1 = I_2 = 1 \qquad \text{and} \qquad I_{2n+1} = 2I_n,$$

$$I_{2n+2} = 2I_{2n+1} - I_n.$$

SIAM 78-13. **by T. D. Rogers**

Given n points distributed uniformly in the unit circle, with $n > 2$, associate with each such point the region in the circle whose points are closer to it than the remaining $n-1$ a priori given points. If $A_1 \le A_2 \le \cdots \le A_n$ is the ordered enumeration of the areas of these regions, what are the expected values of the A_i's?

SIAM 79-1. **by I. Lux**

Let V be an arbitrary three-dimensional spatial region. Let $P = (\mathrm{r}, \omega)$, a six-dimensional phase space point, where $\mathrm{r} \in V$ and ω is a directional unit vector. Define a function $M_\lambda(P)$ through the following integral equation

$$M_\lambda(P) = 1 - e^{-D} + \frac{\lambda}{4\pi}\int_0^D e^{-\lambda x}\, dx \int M_\lambda\left(P'\right) d\omega'$$

where $P' = \left(\mathrm{r} + x\omega, \omega'\right)$, λ is an arbitrary but positive parameter, D is the distance between the point r and the boundary of V along the direction ω and the integral over $d\omega'$ is a double integral over the surface of a unit sphere. Prove or disprove that

$$\left.\frac{d}{d\lambda} M_\lambda(P)\right]_{\lambda=1} \ge 0.$$

SIAM 79-4. **by K. L. McAvaney**

For positive integer n, maximize the number of $n \times n$ matrices each containing all of $1, 2, \ldots, n^2$ such that any two entries appear simultaneously in at most one row of all the matrices.

SIAM 79-6. **by L. B. Klebanov**

Let $f(x)$, $g(x)$ be two probability densities on $\mathbb{R}^1$ with $g(x) > 0$. Suppose that the condition

$$\int_{-\infty}^{\infty} (u-c) \prod_{j=1}^{n} f\left(x_j - u\right) g(u)\, du = 0$$

holds for all $x_1, x_2, \ldots, x_n$ such that $\sum_{j=1}^n x_j = 0$ where $n \ge 3$ and c is some constant. Prove that

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}.$$

SIAM 79-16. **by D. Singmaster**

Determine the resistances $R(n, i)$ between two nodes a distance i apart in an n-cubical network if all of the edges are of unit resistance.

SIAM 79-17. **by W. R. Utz**

Determine an algorithm, better than complete enumeration, for the following problem: Given a nonnegative integer matrix, permute the entries in each column independently so as to minimize the largest row sum.

AUTHOR INDEX TO UNSOLVED PROBLEMS

CITATION INDEX

Use this section to

- find articles in journals that reference a specific problem that you are interested in
- locate problems that reference or generalize a specific problem
- find articles about specific mathematical competitions or problems from such competitions
- find information about and reviews of problem books

We have scanned the 1975–1979 issues of many journals looking for articles that refer to problems in their list of references. We have also examined the many problems and solutions in the problem columns covered by this index looking for references to other problems. In this way, you can find those articles and problems that refer to a problem you are interested in.

The citation index lists those problems that have been referred to during the years 1975–1979. They are sorted by journal abbreviation followed by problem number. Since articles may reference problems from journals that ceased publication prior to 1975 or they may reference problems from journals not indexed elsewhere in this book, the list of referenced journal abbreviations is given on the following page and is larger than the list given on the inside back cover of this book.

Citations are of two types:

ABBREV number	refers to a problem from a contest or journal problem column
[REF year]	refers to a book or journal article listed in the bibliography.

A reference enclosed in square brackets is a bibliographic reference and refers to a book or journal article. See the bibliography (beginning on page 431) for the complete reference. The reference consists of a reference word (typically the last name of the author of the article) followed by the year of publication, as in "[Johnson 1984]". The date may be followed by an additional letter, a, b, c, etc., if more than one work assigned a given reference word appeared in the same year, for example: "[Trigg 1983b]".

A reference not enclosed in square brackets is a reference to a contest problem or a problem from a journal problem column. It consists of the abbreviation for the contest or journal name followed by the problem number. For example: "AMM E1071" refers to problem E1071 from the American Mathematical Monthly. Contest problems give the year of the contest preceding the problem number and separated from it by a slash. Thus, "USA 1982/3" refers to problem 3 from the 1982 USA Mathematical Olympiad.

A given problem may be cited by more than one reference. In that case, the references are separated by commas.

There are five sections to this citation index, occurring in the following order:

• References to Problems	gives citations to specific problems from a journal or contest
• Contest References	gives citations to specific contests
• Biographical Notes	gives citations to biographical notes (including obituaries)
• Problem Indexes	gives citations to indexes of problem collections
• Reviews of Problem Books	gives citations for reviews of problem books

The journals that were scanned for citations are:
AMM, CRUX, FQ, FUNCT, JRM, MATYC, MM, NAvW, PARAB, PENT, PME, SIAM, SPECT, SSM, and TYCMJ.

ABBREVIATIONS USED IN THE CITATION INDEX

Abbreviation	Journal or Contest Name
AHSME	American High School Mathematics Examination
AHSPE	Alberta High School Prize Examination
AMC	Australian Mathematics Competition
AMM	The American Mathematical Monthly
AMP	Arch. Math. Phys.
ATRML	Atlantic Region Mathematics Competition
BIBLIOGRAPHIES	A Bibliography of Mathematical Competitions
BRITAIN	British Mathematical Olympiad
CANADA	Canadian Mathematics Olympiad
CARLETON	Carleton University Mathematics Competition for high school students
CMB	Canadian Mathematical Bulletin
COLLOQ	Colloquium Mathematicum
CRUX	Crux Mathematicorum
CZECH	Czechoslovakian Mathematics Olympiad
DC	The Descartes Competition
EC	The Euclid Contest
EDUC	The Educational Times
ELEM	Elemente der Mathematik
EOTVOS	Eötvös Mathematical Competition (Hungary)
FQ	The Fibonacci Quarterly
FUNCT	Function
FUND	Fundamenta Mathematica
GAZ	The Mathematical Gazette
GDIARY	Gentleman's Diary
GMNYMF	The Greater Metropolitan New York Math Fair
HUNGARY	Kürschak Mathematical Competition
IM	L'Intermédiaire des Mathématiciens
IMO	International Mathematical Olympiad
JDMV	Jahresbericht der Deutschen Mathematiker-Vereinigung
JIMS	Journal of the Indian Mathematical Society
JMC	Junior Mathematics Contest
JRM	Journal of Recreational Mathematics
KVANT	Kvant
MATYC	The MATYC Journal
MJHSSMC	MATYC Journal High School Student Mathematics Contest
MM	Mathematics Magazine

Abbreviation	Journal or Contest Name
MS	Mathematics Student (Reston)
MSJ	The Mathematics Student Journal
NAM	Nouvelles Annales de Mathématique
NAvW	Nieuw Archief voor Wiskunde
NCIML	Nassau County Interscholastic Mathematics League Contest
NSTYCML	National Student Two-Year College Mathematics League
NSW	New South Wales Mathematical Olympiad
NSWSMC	New South Wales School Mathematics Competition
NTvW	Nieuw Tijdschrift voor Wiskunde
NYCIML	New York City Interscholastic Mathematics League
NYCSIML	New York City Senior Interscholastic Mathematics League
NYSML	New York State Mathematics League
NYSMTJ	The New York State Mathematics Teachers' Journal
OSSMB	Ontario Secondary School Mathematics Bulletin
PARAB	Parabola
PENT	The Pentagon
PME	The Pi Mu Epsilon Journal
PMMC	Peking Municipality Mathematical Competitions
PRAXIS	Praxis der Mathematik
PUTNAM	William Lowell Putnam Mathematical Competition (USA)
SCAND	Mathematica Scandinavica
SIAM	SIAM Review
SKFMC	Special K Freshman Mathematics Contest
SPECT	Mathematical Spectrum
SPHINX	Sphinx
SSM	School Science and Mathematics
STANFORD	Stanford University Competitive Examination in Mathematics
TECH	Technology Review
THESIS	Mathesis
TIMES	Mathematical Questions and Solutions from the Educational Times
TYCMJ	The Two-Year College Mathematics Journal
USA	USA Mathematical Olympiad
WO	Wiskundige Opgaven

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NSWSMC	[Mack 1979]
NSWSMC 1975	[News 1975a], [News 1975b]
NSWSMC 1976	[News 1976f], [News 1976g]
NSWSMC 1977	[News 1977h], [News 1977i]
NSWSMC 1978	[News 1978g], [News 1978h]
NSWSMC 1979	[News 1979g]
NYCIML	[News 1976b]
OLYMPIADS	[Turner 1978]
PUTNAM 1974	[Hillman 1975], [OSSMB 1975b]
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SKFMC 1974	[OSSMB 1975a]
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Martin Gardner	[Renz 1979]
Michael Greening	[PARAB 1978]
R. Robinson Rowe	[Nelson 1978], [Sauvé 1978]
K. R. S. Sastry	[Contributors 1976]
David Silverman	[Nelson 1978]

Problem Indexes

Journal	Indexed by
SSM	[Grinstein 1975]

Reviews of Problem Books

Problem Book	Reviewed by
[Barbeau 1977b]	[Reviews 1977]
[Berloquin 1976]	[McGinty 1978]
[Brousseau 1972]	[Trigg 1979b]
[Fixx 1978]	[Vignette 1979]
[Fujimura 1978]	[Kennedy 1979a]
[Greitzer 1978]	[Trigg 1978c]
[Honsberger 1978]	[Trigg 1979e]
[Kahan 1978]	[Wilke 1978]
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Glossary

[See also the Notation section beginning on page 3.]

abundant number — A positive integer, n, such that $\sigma(n) > 2n$.

alphametic — A cryptarithm in which the letters, which represent distinct digits, form related words or meaningful phrases.

amicable numbers — Two numbers are said to be amicable if each is equal to the sum of the proper divisors of the other.

ball — A sphere together with its interior.

bijection — A one-to-one function.

Caliban puzzle — A logic puzzle in which one is asked to infer one or more facts from a set of given facts.

Catalan number — A member of the sequence 1, 1, 2, 5, 14, 42, 132, . . ., where the nth term C_n equals $\binom{2n}{n}/(n+1)$.

ceiling function — $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

centroid — The center of mass of a figure. The centroid of a triangle is the intersection of the medians.

cevian — A line segment extending from a vertex of a triangle to the opposite side.

Chebyshev polynomials — $T_n(x) = \cos(n \arccos x)$ and $U_n(x) = \sin[(n+1)\arccos x]/\sin(\arccos x)$.

circumcenter — The circumcenter of a triangle is the center of the circumscribed circle.

circumcircle — The circle circumscribed about a figure.

coprime — Integers m and n are coprime if $\gcd(m, n) = 1$.

cryptarithm — A number puzzle in which an indicated arithmetical operation has some or all of its digits replaced by letters or symbols and where the restoration of the original digits is required. Each letter represents a unique digit.

cyclic polygon — A polygon whose vertices lie on a circle.

deficient number — A positive integer, n, such that $\sigma(n) < 2n$.

digimetic — A cryptarithm in which digits represent other digits.

disc — A circle together with its interior.

Diophantine equation — An equation that is to be solved in integers.

dodecahedral number — A number of the form $n(27n^2 - 27n + 6)/6$.

domino — Two congruent squares joined along an edge.

escribed circle — An escribed circle of a triangle is a circle tangent to one side of the triangle and to the extensions of the other sides.

excenter — The center of an excircle.

excircle — An escribed circle of a triangle.

exradius — An exradius of a triangle is the radius of an escribed circle.

Farey sequence — The sequence obtained by arranging in numerical order all the proper fractions having denominators not greater than a given integer.

Fermat number — A number of the form $2^{2^n} + 1$.

Fibonacci number — A member of the sequence 0, 1, 1, 2, 3, 5 . . . where each number is the sum of the previous two numbers.

floor function — $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

focal radius — A line segment from the focus of an ellipse to a point on the perimeter of the ellipse.

geoboard — A flat board into which nails have been driven in a regular rectangular pattern. These nails represent the lattice points in the plane.

Gergonne point — In a triangle, the lines from the vertices to the points of contact of the opposite sides with the inscribed circle meet in a point called the Gergonne point.

gnomon magic square — A 3×3 array in which the elements in each 2×2 corner have the same sum.

golden ratio — $(1 + \sqrt{5})/2$.

golden rectangle — A rectangle whose sides are in the golden ratio.

harmonic mean — The harmonic mean of two numbers a and b is $\frac{2ab}{a+b}$.

hexagonal number — A number of the form $n(2n - 1)$.

hexomino — A six-square polyomino.

Heronian triangle — A triangle with integer sides and integer area.

homeomorphism — A one-to-one continuous transformation that preserves open and closed sets.

homomorphism — A function that preserves the operators associated with the specified structure.

incenter — The incenter of a triangle is the center of its inscribed circle.

incircle — The circle inscribed in a given figure.

isogonal conjugate — Isogonal lines of a triangle are cevians that are symmetric with respect to the angle bisector. Two points are isogonal conjugates if the corresponding lines to the vertices are isogonal.

isotomic conjugate — Two points on the side of a triangle are isotomic if they are equidistant from the midpoint of that side. Two points inside a triangle are isotomic conjugates if the corresponding cevians through these points meet the opposite sides in isotomic points.

L-tetromino — A tetromino in the shape of the letter L.

lattice point — A point with integer coordinates.

Legendre polynomials — $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$.

Lucas number — A member of the sequence 2, 1, 3, 4, 7 . . . where each number is the sum of the previous two numbers.

magic square — A square array of n numbers such that the sum of the n numbers in any row, column, or main diagonal is a constant (known as the magic sum).

Glossary

Malfatti circles	Three equal circles that are mutually tangent and each tangent to two sides of a given triangle.
medial triangle	The triangle whose vertices are the midpoints of the sides of a given triangle.
Mersenne number	A number of the form $2^n - 1$.
Mersenne prime	A Mersenne number that is prime.
monic polynomial	A polynomial in which the coefficient of the term of highest degree is 1.
monochromatic triangle	A triangle whose vertices are all colored the same.
Nagel point	In a triangle, the lines from the vertices to the points of contact of the opposite sides with the excircles to those sides meet in a point called the Nagel point.
nine point center	In a triangle, the circumcenter of the medial triangle is called the nine point center.
nonagonal number	A number of the form $n(7n - 5)/2$.
orthic triangle	The triangle whose vertices are the feet of the altitudes of a given triangle.
orthocenter	The point of intersection of the altitudes of a triangle.
palindrome	A positive integer whose digits read the same forward and backwards.
palindromic	A positive integer is said to be palindromic with respect to a base b if its representation in base b reads the same forward and backwards.
pandiagonal magic square	A magic square in which all the broken diagonals as well as the main diagonals add up to the magic constant.
pandigital	A decimal integer is called pandigital if it contains each of the digits from 0 to 9.
Pascal's triangle	A triangular array of binomial coefficients.
pedal triangle	The pedal triangle of a point P with respect to a triangle ABC is the triangle whose vertices are the feet of the perpendiculars dropped from P to the sides of $\triangle ABC$.
Pell number	The nth term in the sequence 0, 1, 2, 5, $12, \ldots$ defined by the recurrence: $P_0 = 0$, $P_1 = 1$, and $P_n = 2P_{n-1} + P_{n-2}$.
pentagonal number	A number of the form $n(3n - 1)/2$.
pentomino	A five-square polyomino. (The name *pentomino* is a registered trademark of Solomon W. Golomb.)
perfect number	A positive integer, n, such that $\sigma(n) = 2n$.
polyomino	A planar figure consisting of congruent squares joined edge-to-edge.
primitive Pythagorean triangle	A right triangle whose sides are relatively prime integers.
pronic number	A number of the form $n(n + 1)$.
Pythagorean triangle	A right triangle whose sides are integers.
Pythagorean triple	An ordered set of three positive integers (a, b, c) such that $a^2 + b^2 = c^2$.
repdigit	An integer all of whose digits are the same.
repunit	An integer consisting only of 1's.
rusty compass	A pair of compasses that are fixed open in a given position.
skeleton division	A long division in which most or all of the digits have been replaced by asterisks to form a cryptarithm.
square number	A number of the form n^2.
Stirling numbers	$\left\{ {n \atop k} \right\}$ are Stirling numbers of the second kind. $\left[{n \atop k} \right]$ are Stirling numbers of the first kind. $x^{\overline{n}} = \sum_k \left[{n \atop k} \right] x^k$ and $x^n = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}}$.
symmedian	Reflection of a median of a triangle about the corresponding angle bisector.
symmetric function	Function of n variables whose value given n arguments does not depend on the order of the arguments.
tetrahedral number	A number of the form $n(n^2 + 3n + 2)/6$.
tetration	Multiplication is iterated addition, exponentiation is iterated multiplication, and *tetration* is iterated exponentiation.
tetromino	A four-square polyomino.
trapezium	A quadrilateral in which no sides are parallel.
trapezoid	A quadrilateral in which two sides are parallel.
triangular number	A number of the form $n(n + 1)/2$.
tromino	A three-square polyomino.
unimodal	A finite sequence is unimodal if it first increases and then decreases.
unimodular	A square matrix is unimodular if its determinant is 1.
unitary divisor	A divisor d of c is called unitary if $\gcd(d, c/d) = 1$.
unit fraction	A fraction $1/d$ with d an integer.
X-pentomino	A pentomino in the shape of the letter X.
zeta function	$\zeta(s)$ stands for the Riemann Zeta Function: $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$.

KEYWORD INDEX

Use this section to

- find problems that contain a specific word or phrase
- locate problems related to a given topic.

This section lists most words and two-word phrases that appear in the problems covered by this book, and also lists all words used to classify these problems. Look up a word or phrase and then you will find the references to those problems or classifications that contain this word or phrase.

The problem references look like

JNL number

or

CONTEST year/number

where

JNL	is the journal abbreviation
CONTEST	is the abbreviation for the contest
year	is the year of the contest

and

number	is the problem number.

If several consecutive problem references are from the same journal, the journal (or contest) abbreviation is listed just once, followed by the list of problem numbers.

The classification references look like

►SUB/subject 2/subject 3 [x]

where

SUB	is the abbreviation for the primary subject in the classification containing the keyword
subject 2	is the second subject in the classification containing the keyword, if such precedes the keyword in that classification
subject 3	is the third subject in the classification containing the keyword, if such precedes the keyword in that classification

and

[x]	is the number of problems having this particular classification, when that number is greater than one. (The case [1] is suppressed.)

To find the text for problems referenced by classification, look up the appropriate subsection in the Subject Index (beginning on page 15) pertaining to that classification. Here are the primary subject abbreviations:

AL = Algebra	GT = Game Theory	RM = Recreational Mathematics
AN = Analysis	HA = Higher Algebra	ST = Set Theory
AM = Applied Mathematics	LA = Linear Algebra	SG = Solid Geometry
C = Combinatorics	NT = Number Theory	T = Topology
G = Geometry	P = Probability	TR = Trigonometry

Each problem appearing in this book is listed just once in the subject index under its primary classification. The keyword index can help you locate a problem from a given topic when that topic is a secondary classification for the problem. Pick a keyword that is either likely to occur in the problem or accurately describes a particular aspect of that problem, and then look it up in this keyword index.

Uninteresting words such as “the”, “like”, “of”, “that”, “each”, “is”, etc. have been suppressed from the listing. Important mathematical words that occur more than 50 times are listed in the keyword index but the references are suppressed. For example, it would serve no purpose to list all 693 problems that contain the word “triangle”. You should consult a narrower term such as “scalene triangle”.

To save space, two-word phrases are usually listed only under the first word. Thus “regular pentagon” is listed under “regular” but not under “pentagon”. Thus, you should look up the narrowest term first associated with a topic you are interested in. If that does not occur, then try a broader term.

See also:

- the Subject Index — to find problems concerning a given topic
- the Title Index — to search for keywords in the title of a problem.

Keyword Index

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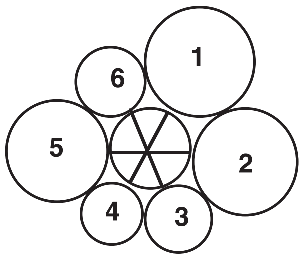
About the Cover:



The seven coins shown on the cover illustrate an interesting result known as the Seven Circles Theorem. For more details and additional references, see reference [Rabinowitz 1987]. Inside the six outer circles are figures associated with problems indexed in this book:

1. A remarkable property of the least common multiple of binomial coefficients submitted by Peter L. Montgomery as problem AMM E2686.
2. A magic pentagram containing distinct integers between 1 and 12 associated with with problem JRM 385 submitted by Vance Revennaugh.
3. A well-known congruence involving binomial coefficients which appeared as problem PARAB 355.
4. An alphametic puzzle by Sidney Kravitz published as problem JRM 717.
5. A challenging geometry problem by F. David Hammer published as problem MATYC 121.
6. An unusual variant of the alternating harmonic series submitted by Harry D. Ruderman as problem AMM 6105.

About the Authors:

Stanley Rabinowitz (right) received his Ph.D. in mathematics from the Polytechnic University (of New York) working under the direction of Erwin Lutwak in the areas of convexity, combinatorics, and number theory. Professionally, he is a software engineer and computer consultant, but math problem solving has been his hobby most of his life. He has had over 300 problems published and is a regular contributor, both as solver and proposer, to the problem columns of over a dozen journals from around the world. He served as editor for a problem column in *The Fibonacci Quarterly*.

Mark Bowron is an over-the-road mathematrucker who spends most of his time driving an 18-wheeler throughout the lower 48 states and Canada for Marten Transport, Ltd. (based in Mondovi, Wisconsin). He is surprised more mathematicians do not drive truck for a living: after all, one has ample time to solve problems on the job, the pay is not bad, and (perhaps best of all), there are no exams to grade.

About the Publisher:

MathPro Press

MathPro Press was founded in 1989 by Stanley Rabinowitz for the purpose of publishing indexes to problems from the mathematical literature. It also specializes in publishing mathematical problem books, compendiums of mathematical results, and books about ways of using computers to help solve mathematics problems.

The MathPro logo illustrates a result discovered by Dr. Rabinowitz using the Cabri-géomètre computer program and submitted as problem MM 1364 [Rabinowitz 1991]. The text is set in Computer Modern Bold.

INDEX TO MATHEMATICAL PROBLEMS 1975–1979