A Theorem about Collinear Lattice Points

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Abstract. Let S be a set of m^n+1 lattice points in E^n. Then either some two points of S span a hole (have a lattice point not in S between them), or some $m+1$ points of S are collinear.

A lattice point is a point in E^n with integer coordinates. The set of all lattice points in E^n is denoted by Z^n. In this note, we will look at some results that show when there must be m collinear lattice points in a collection of lattice points in Z^n.

Definition. Two lattice points, x, and y, are said to *span a hole* in a set S if there is some lattice point between x and y that is not in S. A set of lattice points, S, *contains a hole*, if some two points of S span a hole.

We now prove the following Ramsey-like theorem: (For other Ramsey-like theorems in E^n, see section 5.6 of [1] or section 21 of [2].)

Theorem 1. *Let S be a set of m^n+1 lattice points in E^n. Then either some two points of S span a hole, or some $m+1$ points of S are collinear.*

First note that the set S can be a rather complicated looking set. An example is shown in figure 1 consisting of 25 lattice points in the plane that form a set with no holes and no 6 lattice points in a row. Adding any 26th lattice point, however, (without adding any holes) will force some 6 lattice points to be collinear.

```
. . . . o o o o
. . o o o o o .
. . o o o o o .
. o o o o o . .
o o o o o . . .
. . . o . . . .
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Figure 1
25 lattice points forming a
non-trivial lattice-point set
with no holes and no 6 in a row

Proof. Consider the coordinates of the points modulo m. Since there are only m^n distinct ordered n-tuples of integers modulo m, some two of these must be congruent (mod m). Suppose the two points have coordinates $(x_1, x_2, \ldots, x_n)$ and $(x'_1, x'_2, \ldots, x'_n)$. Then $x_i \equiv x'_i$ (mod m) for $i = 1, 2, \ldots, n$. Now consider the points

$$(x_1 + \frac{x'_1 - x_1}{m}k, x_2 + \frac{x'_2 - x_2}{m}k, \ldots, x_n + \frac{x'_n - x_n}{m}k)$$

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as k varies from 0 to m. This is a set of $m+1$ collinear points. Furthermore, each point is a lattice point, since $m|(x'_i - x_i)$ for all i by the congruence condition. Finally, all the $m+1$ points belong to S since the first and last ones do and S contains no holes. □

Note that the above proof actually gives an effective (and even efficent) procedure for finding the $m+1$ collinear lattice points; it is not merely an existence proof.

We note that the quantity $m^n + 1$ is best possible in the above theorem, for we can always find m^n lattice points with no holes in which no $m+1$ are collinear. Namely, take the m^n lattice points inside and on the n-cube with m lattice points along each edge.

Theorem 1 can be rephrased in a number of ways.

Definition. A set, S, of lattice points is *2-convex*, if it does not contain a hole.

Proposition 1a. *Let S be a set of $m^n + 1$ lattice points in E^n that is 2-convex. Then S must contain some $m+1$ lattice points that are collinear.*

Definition. A set, S, of lattice points is *lattice-convex*, if any lattice point in the convex hull of S is also in S.

The concept of lattice-convexity differs from 2-convexity as can be seen by figure 2 which shows that 2-convexity does not imply lattice-convexity.

```
. . o
o . .
. o .
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Figure 2
A set that is 2-convex
but is not lattice convex

However, if x and y are two lattice points in a lattice-convex set S, then any lattice point between x and y must also be a member of S. Thus lattice-convexity implies 2-convexity and we may reformulate Theorem 1 as follows:

Proposition 1b. *Let S be a set of $m^n + 1$ lattice points in E^n that is lattice-convex. Then S must contain some $m+1$ lattice points that are collinear.*

We can view lattice points in E^n as vectors emenating from the origin. Such vectors are called lattice vectors.

Proposition 1c. *Let S be a set of $m^n + 1$ lattice vectors in E^n. Then either there is a lattice vector, not in S, that is a convex linear combination of two lattice vectors in S or else some $m+1$ vectors in S form an arithmetic progression.*

This formulation of Theorem 1 follows from the observation that if $m+1$ vectors form an arithmetic progression, then their endpoints are collinear.

We can also view Theorem 1 in the light of lattice points inside convex bodies.

Proposition 1d. *Let K be a convex body in E^n containing at least $m^n + 1$ lattice points. Then some $m+1$ of these lattice points must be collinear.*

This formulation of the theorem follows immediately from the observation that the set of lattice points inside a convex body forms a lattice-convex set.

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References

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2. J. Hammer, "Unsolved Problems Concerning Lattice Points", Pitman, London: 1977.