A Census of Convex Lattice Polygons with at most one Interior Lattice Point

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ABSTRACT

We find all convex lattice polygons in the plane (up to equivalence) with at most one interior lattice point.

A lattice point in the plane is a point with integer coordinates. A lattice segment is a line segment whose endpoints are lattice points. A lattice polygon is a simple polygon whose vertices are lattice points.

In 1980, Arkinstall [1] proved that, up to lattice equivalence (defined below), there is just one convex lattice *hexagon* containing a single interior lattice point. In this article, we will extend this result by finding all convex lattice *polygons* with at most one interior lattice point. Having this characterization will help in proving inequalities about lattice polygons containing just one interior lattice point which in turn may help in the investigation of general convex bodies in the plane with lattice point constraints (see [3], [4], and [5]).

To understand when two lattice polygons are "equivalent", we must first review some definitions concerning standard transformations of the plane. An affine transformation is a linear transformation followed by a translation. A unimodular transformation is one that preserves area. To be unimodular, the matrix corresponding to a linear transformation must have determinant 1. If furthermore, the entries of the matrix are integers, then the transformation is known as an integral unimodular transformation and has the property that it preserves the number of lattice points in a set. An integral unimodular affine transformation (also known as an equiaffinity) in the plane can be expressed by the 3×3 matrix in the equation

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$$

where a, b, c, d, e, and f are integers and $ad-bc=1$. This includes an integral translation by the vector $< e, f >$.

Two lattice polygons are said to be *lattice equivalent* if one can be transformed into the other via an integral unimodular affine transformation.

A *shear* is an integral unimodular transformation that leaves all the points on a line fixed. It is also referred to as a shear about this line. In the plane, a shear about the x-axis

Reprinted from Ars Combinatoria 28(1989)83–96

is given by the equations

$$x' = x + ky$$
$$y' = y$$
$$k \in Z.$$

Such a shear is said to have magnitude k, and if $k = 1$, the shear is called a *unit shear*.

Let g denote the number of lattice points in the interior of the convex lattice polygon being considered.

Let us first review three results that Arkinstall proved in [1]. Recall that a *trapezium* is a quadrilateral with no two sides parallel.

The Lattice Trapezium Theorem.

A convex lattice trapezium must contain an interior lattice point.

The Lattice Pentagon Theorem.

A convex lattice pentagon must contain an interior lattice point.

The Central Hexagon Theorem. *A convex lattice hexagon containing precisely one interior lattice point is lattice equivalent to the centrally symmetric hexagon shown in figure 1.*

```
. o o
o . o
o o .
```

Figure 1
Unique convex lattice hexagon with g=1

The dots represent lattice points and the circles are the vertices of the polygon.

We will also need the following well-known theorem (for a proof, see Coxeter [2]):

Pick's Formula. *If a lattice polygon of area A contains b lattice points on its boundary and g lattice points in its interior, then $A = b/2 + g - 1$.*

We define the *lattice length* of a lattice segment to be one less than the number of lattice points on that segment. Thus, if the segment is parallel to an axis, the lattice length is the same as the Euclidean length.

We will also need the following lemma.

The x-axis Lemma. *Let AB be a side of a convex lattice polygon, K, in the plane. Let m be the lattice length of AB. Then there is an integral unimodular affine transformation that maps A into the origin, maps B into the point $(m, 0)$ on the positive x-axis, and maps all the other vertices of K into points above the x-axis.*

Proof. First translate the point A to the origin. This is an integer translation. Let B be transformed into the point (p, q) by this translation. We seek integers a, b, c, d such that $ad - bc = 1$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms B into a point on the x-axis. In other words, we want

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

for some positive integer x. Let $r = \gcd(p,q)$. Then we merely need pick $c = q/r$ and $d = -p/r$. This makes $cp + dq = 0$ and a and b need be chosen so that $ad - bc = 1$. Such positive integers can be found by a well-known theorem from number theory since q/r and p/r are relatively prime. $\square$

We shall begin by characterizing those lattice polygons with no interior lattice points. This characterization will be in terms of the following basic figures:

Notation. By TRIANG(p,h) we denote the triangle whose vertices are $(0,0)$, $(p,0)$, and $(0,h)$. By TRAP(p,q,h) we denote the trapezoid whose vertices are $(0,0)$, $(p,0)$, $(0,h)$, and (q,h).

Theorem 1 (Characterization of convex lattice polygons with $g = 0$). *If K is a convex lattice polygon with $g = 0$, then K is lattice equivalent to one of the following polygons:*

1. *TRIANG$(p,1)$ where p is any positive integer.*
2. *TRIANG$(2,2)$.*
3. *TRAP$(p,q,1)$ where p and q are any positive integers.*

Proof. Polygon K must have fewer than 5 sides because the Lattice Pentagon Theorem shows that if K had 5 or more sides, it would contain an interior lattice point. We thus need only consider two cases: triangles and quadrilaterals.

Case 1: The polygon is a triangle.

Using the x-axis Lemma, we can find an integral unimodular affine transformation that maps the largest side into the positive x-axis, with one vertex, A, at the origin, a second vertex, B, on the x-axis at $(p,0)$ where p is a positive integer. The third vertex, C, maps to some point above the x-axis. Let h denote the height of vertex C from the x-axis.

If $h = 1$, then C is at height 1 above the x-axis, so we can apply a shear about the x-axis to move point C to the y-axis. This shows that in this case, the polygon is equivalent to TRIANG$(p,1)$.

So assume that $h > 1$. Let r be the length of the line segment formed by the intersection of the line $y = 1$ with triangle ABC. (The relative interior of this segment lies wholly within $\triangle ABC$ because $h > 1$.) Clearly $p > r$.

By considering similar triangles we get $r/(h-1) = p/h$. Solving for r gives $r = p(h-1)/h$. Since we must have $r \leq 1$ (otherwise the segment will contain some lattice point in its interior), we find that $p(h-1)/h \leq 1$ from which we can conclude that either $p = 1$ or

$$h \leq 1 + \frac{1}{p-1}. \qquad (*)$$

If $p = 1$, then h can be arbitrary, and we find that the triangle is equivalent to TRIANG$(h,1)$. If $p > 1$, then from (*) and the fact that $h \geq 2$, we see that $p = 2$ and $h \leq 2$. So $h = 2$ and $p = 2$.

Thus $r = p(h-1)/h = 1$. Let the line $y = 1$ meet AC at E and BC at F. Point E must be a lattice point since EF is of length 1 and does not contain a lattice point in its interior. Since E is at height 1 above the x-axis, we can find a shear that will move E onto the y-axis. This shows that the triangle is equivalent to TRIANG$(2,2)$.

Case 2: The polygon is a quadrilateral.

Polygon K must be a trapezoid, for a non-trapezoidal quadrilateral must contain a lattice point by the Lattice Trapezium Theorem.

Using the x-axis Lemma, we can find an integral unimodular affine transformation that maps the larger base into the positive x-axis, with one vertex, A, at the origin, a second vertex, B, on the x-axis at $(p, 0)$ where p is a positive integer, and the other two vertices, C and D, being above the x-axis. Since unimodular affine transformations preserve parallelism, we have $CD \parallel AB$. Label the vertices so that D is to the left of C. Ratios on parallel line segments are preserved, so $AB \geq DC$. Let q be the length of DC. We have $p \geq q > 0$.

Consider triangle ABD. This triangle cannot be lattice equivalent to TRIANG$(2, 2)$, for if it were, then diagonal BD would contain a lattice point between B and D and this would yield a lattice point interior to the quadrilateral, a contradiction. Thus, by what has already been proven, $\triangle ABD$ must be lattice equivalent to TRIANG$(p, 1)$. Apply a shear about the x-axis to move point D to the y-axis. The resulting figure is of the form TRAP$(p, q, 1)$. $\square$

We now move on to characterizing all convex lattice polygons with $g = 1$.

Proposition 2 (Characterization of lattice triangles with $g = 1$). *If K is a lattice triangle with $g = 1$, then K is lattice equivalent to one of the following five triangles:*

```
                                    o . . .
o . . . .     o . . .     . o .     . . . .     . . o
. + . . .     . + . .     . + .     . + . .     o + .
o . . . o     o . . o     o . o     o . . o     . o .

   (a)          (b)        (c)        (d)        (e)
```

Figure 2
All lattice triangles with g=1

The interior lattice point is marked with a "+" sign.

Proof. Using the x-axis Lemma, we can apply an integral unimodular affine transformation to map the side of the triangle with largest lattice length onto the x-axis, with A at the origin and B at $(p, 0)$. The third point, C, maps into a point above the x-axis. Let h be the height of the triangle, that is, the distance from C to the x-axis. We find that h must be larger than 1 otherwise the triangle will have no interior lattice points.

By Pick's Formula, $2A = ph = b$. Since the base has the longest lattice length, $b \leq 3p$. Therefore $ph \leq 3p$ or $h \leq 3$. Thus h is either 2 or 3.

Let r be the length of the segment intercepted by the triangle on the line $y = 1$. Since $h > 1$, this segment goes through the interior of the triangle, and we must have $r \leq 2$ otherwise this segment would contain 2 lattice points in its interior. By similar triangles, we see that $(h - 1)/r = h/p$.

But $r \leq 2$ implies that

$$p = \frac{hr}{h-1} \leq \frac{2h}{h-1} \leq 4.$$

Case 1: $h = 2$:

In this case, $p = 2r \leq 4$, so $p = 1, 2, 3$, or 4. Point C lies along $y = 2$ and there are at most two locations for C to get inequivalent triangles, namely, $C = (j, 2), j = 0, 1$.

Case 1a: $h = 2$, $p = 4$:

Choice $C = (0, 2)$ produces TRIANG$(4, 2)$ with $g = 1$. Choice $C = (1, 2)$ is ruled out because $\triangle ABC$ has $g = 2$. This triangle is shown in figure 2a.

Case 1b: $h = 2$, $p = 3$:

Both $C = (0, 2)$ and $C = (1, 2)$ yield triangles with $g = 1$. However, we need only include one of these, namely, TRIANG$(3, 2)$, because the other one is equivalent to this one after a reflection about $x = 3/2$ followed by a unit shear about the x-axis. This triangle is shown in figure 2b.

Case 1c: $h = 2$, $p = 2$:

Choice $C = (0, 2)$ is ruled out because the resulting triangle has no interior lattice points. Choice $C = (1, 2)$ yields a valid triangle and is included in our classification. This triangle is shown in figure 2c.

Case 1d: $h = 2$, $p = 1$:

Both $C = (0, 2)$ and $C = (1, 2)$ are ruled out because the resulting triangles have no interior lattice points.

Case 2: $h = 3$:

In this case, $p = 3r/2 \leq 3$, so $p = 1, 2$, or 3. Point C lies along $y = 3$ and there at most three locations for C to get inequivalent triangles, namely, $C = (j, 3), j = 0, 1, 2$.

Case 2a: $h = 3$, $p = 3$:

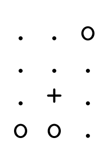
Both $C = (1, 3)$ and $C = (2, 3)$ give rise to triangles with more than 1 interior lattice point. $C = (0, 3)$ gives rise to a valid triangle, namely, TRIANG$(3, 3)$. This triangle is shown in figure 2d.

Case 2b: $h = 3$, $p = 2$:

Choice $C = (1, 3)$ yields a triangle with 2 interior lattice points. Choices $C = (0, 3)$ and $C = (2, 3)$ are valid and equivalent, but both have a side of lattice length 3 and are equivalent to TRIANG$(3, 2)$ discovered in case 1b.

Case 2c: $h = 3$, $p = 1$:

Choices $C = (0, 3)$ and $C = (1, 3)$ yield triangles with no interior lattice points and are ruled out. Choice $C = (2, 3)$ generates a valid triangle. This case is shown in the figure below.



A unit shear about the y-axis shows this triangle to be equivalent to the one in figure 2e. □

Note that the five lattice triangles listed are all inequivalent, because their sides have different lattice lengths. The triples of lattice lengths that can occur are: 4–2–2, 3–2–1, 2–1–1, 3–3–3, and 1–1–1.

Proposition 3 (Characterization of convex lattice quadrilaterals with $g = 1$). *If K is a convex lattice quadrilateral with $g = 1$, then K is lattice equivalent to one of the following six quadrilaterals:*

```
. . o      . o .      o . o      o . o      o . . .      . o .
o + .      o + o      . + o      . + .      . + o .      . + o
o o .      . o .      o . .      o . o      o . . o      o . o

 (a)        (b)        (c)        (d)         (e)         (f)
```

Figure 3
All convex lattice quadrilaterals with g=1

Proof. We divide the collection of such quadrilaterals into two sets – one in which the interior lattice point lies on a diagonal of the quadrilateral, and one in which the interior lattice point does not lie on a diagonal of the quadrilateral.

Case 1: The interior lattice point lies on a diagonal of the quadrilateral.

Call the quadrilateral $ABCD$, and assume that E is a lattice point on diagonal AC. Both triangles DAC and ABC have no interior lattice points, so by the previous theorem on the Characterization of lattice triangles with $g = 0$, we know the possible affine shapes for these triangles. The two triangles must be attached to each other along an edge of lattice length 2. Each triangle must be equivalent to either TRIANG$(p, 1)$ or TRIANG$(2, 2)$. We consider two cases; one in which both triangles are equivalent to TRIANG$(p, 1)$ and the other in which at least one triangle is equivalent to TRIANG$(2, 2)$. In this latter case, we may as well assume that it is $\triangle DAC$ that is equivalent to TRIANG$(2, 2)$.

Case 1a: $\triangle ACD$ is equivalent to TRIANG$(p, 1)$.

In this case, we must have $p = 2$ and we can transform $\triangle DAC$ into a right triangle with D going to $(0, 1)$, A going to $(0, 0)$, and C going to $(2, 0)$.

In this case, E goes to $(1, 0)$. Point B must be somewhere below the x-axis. Since $\triangle ABC$ has no interior lattice points, from Pick's Formula we see that its area must be $b/2+g-1 = 1$ since $g = 0$ and since $\triangle ABC$ is presumed to be equivalent to TRIANG$(2, 1)$ so that $b = 4$. With an area of 1 and a base (AC) of length 2, the altitude must have length 1. Thus B must lie along $y = -1$. Convexity considerations limit B to the 5 lattice points from $(0, -1)$ to $(4, -1)$. Choices $B = (0, -1)$ and $B = (4, -1)$ are ruled out because the resulting figure degenerates into a triangle. Choices $B = (1, -1)$, $B = (2, -1)$, and $B = (3, -1)$ are valid and are shown below.

```
o . .      o . .      o . . .
o + o      o + o      o + o .
. o .      . . o      . . . o
```

Unit shears show that the first two of these figures are equivalent to the ones shown in figures 3a and 3b respectively. The third figure need not be included in the characterization because the resulting quadrilateral is equivalent to the one shown in figure 3a. (To see this, apply a unit shear about the x-axis and the appropriate reflections.)

Case 1b: $\triangle ACD$ is equivalent to TRIANG$(2,2)$.

In this case, we can transform $\triangle DAC$ into a right triangle with D going to $(0,2)$, A going to $(0,0)$, and C going to $(2,0)$. If $\triangle ABC$ is equivalent to TRIANG$(2,1)$, then as in the previous argument, TRIANG$(2,1)$ must have area 1 and thus B must lie on the line $y=-1$. Convexity considerations leave just 4 locations for B. The first and last are ruled out because the resulting figure degenerates into a triangle. The other two cases, $B=(1,-1)$ and $B=(2,-1)$, are valid and are both equivalent to figure 3c. (Although not obvious at first, these two figures are equivalent after performing a unit shear about the x-axis followed by a reflection about the line $x=1$.) If $\triangle ABC$ is equivalent to TRIANG$(2,2)$, then a similar argument shows that B must line on $y=-2$, so B must vary from $(0,-2)$ to $(4,-2)$. Only $B=(2,-2)$ results in a valid quadrilateral, and this resulting parallelogram is equivalent (by a shear of magnitude 2 about the y-axis) to the square shown in figure 3d.

Case 2: The interior lattice point does not lie on a diagonal of the quadrilateral.

Let the quadrilateral be $ABCD$ and draw in diagonal AC. Assume that the interior lattice point is called E and that E lies inside triangle ABC. Since $\triangle ABC$ has exactly one interior lattice point, it must be lattice equivalent to one of the five triangles shown in figure 2. Furthermore, side AC must not contain any lattice points in its interior. This immediately rules out figures 2a and 2d because each side of those triangles contain an interior lattice point.

Triangle DAC is placed against triangle ABC with one side coinciding with side AC. This side must not contain any interior lattice points, so this narrows the side down to precisely one side of each of the remaining triangles in figure 2. (All three sides of the triangle in figure 2e are equivalent, so just pick any one side.)

In each case, transform the triangle so that the side, AC, of lattice length 1 maps to the x-axis with A at the origin. We find that figures 2b, 2c, and 2e are lattice equivalent to figures b′, c′, and e′ below.

```
. . . o
. . . .
. . . .        . . o
. . . .        . . .        . . o
. . . .        . . .        . . .
. + . .        . + .        . + .
o o . .        o o .        o o .

 (b′)           (c′)         (e′)
```

As before, point D must lie on the line $y=-1$. In each case, convexity considerations limit point D to one possible location, namely the point $(0,-1)$. Two of these three cases (figures b′ and c′ above) give rise to valid quadrilaterals. These are equivalent to those shown in figures 3e and 3f respectively. Triangle e′ yields a valid quadrilateral, but this one need not be counted since its interior lattice point happens to be contained on diagonal BD. □

Note that the six quadrilaterals appearing in our characterization are all in fact inequivalent. This is because the lattice lengths of their sides differ or their interior lattice point is situated differently. The sequence of lattice lengths are: 1–1–1–1, 1–1–1–1, 2–2–1–1, 2–2–2–2, 3–2–1–1, and 2–1–1–1. The first two are inequivalent because the interior lattice point lies on both diagonals of one, but on only one diagonal of the other.

Proposition 4 (Characterization of convex lattice pentagons with $g = 1$). *If K is a convex lattice pentagon with $g = 1$, then K is lattice equivalent to one of the following three pentagons:*

```
. o .        . o .        o o .
o + o        o + o        . + o
o o .        o . o        o . o

 (a)          (b)          (c)
```

Figure 4
All convex lattice pentagons with g=1

Proof. First we begin with a claim.

Claim. *The interior lattice point must lie on a diagonal.*

Let the pentagon be $ABCDE$. Suppose the interior lattice point, F, does not lie on a diagonal. We claim that F lies inside quadrilateral $ACDE$. For suppose that interior lattice point F lies inside triangle ABC. Then $AFCDE$ would be a convex lattice pentagon, so would contain an interior lattice point, by the Lattice Pentagon Theorem. This is a contradiction because it would show that the original pentagon contained two interior lattice points. Thus we may assume that F lies inside quadrilateral $ACDE$, or in other words, that F lies outside of $\triangle ABC$.

By looking at figure 3, it can be seen that all convex quadrilaterals with precisely one interior lattice point, F, which does not lie on any diagonal of the quadrilateral has the property that F is collinear with a vertex of the quadrilateral and some lattice point on the interior of a side of the quadrilateral (figures 3e and 3f only). Thus one side of quadrilateral $ACDE$ contains a lattice point, Q, and QF passes through a vertex of the quadrilateral. Point Q can not lie on DE because we would arrive at a contradiction (via the Lattice Pentagon Theorem) whether QF passes through A or C. Thus, we may assume without loss of generality that Q lies on CD and that AQ passes through F.

Now by the same reasoning as before, point F must lie inside quadrilateral $ABCE$. Since F does not lie on any diagonal of this quadrilateral, we can conclude that there must be a lattice point, R, on one edge of this quadrilateral (cases 3e and 3f again). It cannot be on CE, so R must lie on AE, AB, or BC.

(i) If R lies on side AE, then, since F lies outside of $\triangle ADE$, $RFQDE$ is a convex lattice pentagon which would imply the presence of another lattice point interior to the original pentagon – a contradiction.

(ii) If R lies on side AB, then, since F lies outside of $\triangle ABC$, $RFQCB$ is a convex lattice pentagon which would imply the presence of another lattice point interior to the original pentagon – a contradiction.

(iii) If R lies on side BC, then, since F lies outside of $\triangle BCD$, $RBFDQ$ is a convex lattice pentagon which would imply the presence of another lattice point interior to the original pentagon – a contradiction.

We have arrived at a contradiction in each case and so we must conclude that F lies on a diagonal of the pentagon and the claim is established.

Thus, the interior lattice point lies on a diagonal. Let the diagonal containing a lattice point be AD. This diagonal divides the pentagon into a quadrilateral and a triangle. Call the quadrilateral $ABCD$ so that the pentagon is named $ABCDE$. Quadrilateral $ABCD$ has one side of lattice length 2 and contains no interior lattice points. Thus it must be equivalent to TRAP$(p, 2, 1)$ for some integer p. Map this quadrilateral into one with A going to $(0, 1)$, B going to $(0, 0)$, C going to $(p, 0)$ and D going to $(2, 1)$ and E lying above AD.

If $p = 1$, then the quadrilateral is shown below.

```
. . .
o + o
o o .
```

Triangle EAD must be equivalent to either TRIANG$(2, 1)$ or TRIANG$(2, 2)$. If $\triangle EAD$ is equivalent to TRIANG$(2, 1)$, then triangle EAD has area 1 and no interior lattice points, so E must lie along $y = 2$. Only two possibilities arise, $E = (1, 2)$ and $E = (2, 2)$. The first one is shown in figure 4a. The second one is equivalent to this one by a reflection about $x = 1$ followed by a unit shear about the x-axis.

If $\triangle EAD$ is equivalent to TRIANG$(2, 2)$, then triangle EAD has area 2 and no interior lattice points, so E must lie along $y = 3$. Only 5 choices make the resulting figure convex, $(0, 3)$ through $(4, 3)$. Two of these are ruled out because the resulting polygon is not a pentagon. Two of these are ruled out because $\triangle ADE$ is not equivalent to TRIANG$(2, 2)$. We are left with the one case, $E = (2, 3)$ which results in the pentagon shown in the figure below. This pentagon is equivalent to figure 4c as can be seen by applying a unit shear about the line $x = 1$.

```
. . o
. . .
o + o
o o .
```

If $p = 2$, then the quadrilateral is shown below.

```
. . .
o + o
o . o
```

Triangle EAD must be equivalent to either TRIANG$(2,1)$ or TRIANG$(2,2)$. If $\triangle EAD$ is equivalent to TRIANG$(2,1)$, then triangle EAD has area 1 and no interior lattice points, so E must lie along $y = 2$. The only valid spot yields $E = (1,2)$ and the resulting pentagon is shown in figure 4b.

If $\triangle EAD$ is equivalent to TRIANG$(2,2)$, then triangle EAD has area 2 and no interior lattice points, so E must lie along $y = 3$. Only 3 possible choices for point E exist, and each is easily ruled out.

If $p = 3$, then the quadrilateral is shown below.

```
. . . .
. . . .
o + o .
o . . o
```

There is no way to locate a triangle equivalent to the triangle TRIANG$(2,1)$ or TRIANG$(2,2)$ above AD and wind up with a convex pentagon. The same remark holds true for $p > 3$. Thus we have found all the pentagons with one interior point lying on a diagonal. □

Note that the three pentagons described in the characterization (figure 4) are all in fact inequivalent. This is because the lattice lengths of their sides differ. The sequences of lattice lengths are: 1–1–1–1–1, 2–1–1–1–1, and 2–2–1–1–1.

Finally, note that the characterization of lattice hexagons with $g = 1$ is given by the Central Hexagon Theorem, first proved by Arkinstall.

This covers all cases because there are no convex n-gons with $n > 6$ that contain precisely one interior lattice point. For suppose that $A_1A_2A_3A_4A_5A_6A_7\ldots$ is a convex n-gon with $n \geq 7$. Then $A_1A_2A_3A_4A_5$ is a convex lattice pentagon and must contain an interior lattice point, P, by the Lattice Pentagon Theorem. But then $A_1PA_5A_6A_7$ is also a convex lattice pentagon and it must contain an interior lattice point Q. Thus the original n-gon contains at least two interior lattice points.

We can summarize these results with the following theorem.

Theorem 5 (Characterization of convex lattice polygons with $g = 1$). *If K is a convex lattice polygon with $g = 1$, then K is lattice equivalent to precisely one of the following 15 polygons:*

```
                              o . . .
o . . . .   o . . .   . o .   . . . .   . . o
. + . . .   . + . .   . + .   . + . .   o + .
o . . . o   o . . o   o . o   o . . o   . o .

. . o       . o .     o . o   o . o     o . . .
o + .       o + o     . + .   . + .     . + o .
o o .       . o .     o o .   o . o     o . . o

. o .       . o .     o o .   . o .     . o o
o + .       o + o     . + o   o + o     o + o
o . o       o . o     o . o   o o .     o o .
```

Figure 5
All convex lattice polygons with g=1

Acknowledgement. I would like to thank the referee for finding an error in an earlier draft of this paper and for making several useful suggestions.

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Errata. Added after publication on 1/29/05.

Proposition 3 is incorrect. The following quadrilateral was missed.

```
o o . .
. + . .
o . . o
```

Figure 3(g)

So there are in fact 16 convex lattice polygons with $g = 1$. A revision and corrected proof of Proposition 3 is available from the author, `stan@MathProPress.com`.