# Conic Sections and Limits

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The standard form of an ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where the center is $(h, k)$ and $a$ and $b$ are the lengths of the semi-major and semi-minor axes. Also, $a^2 = b^2 + c^2$ where $c$ is the distance from the center to a focus.

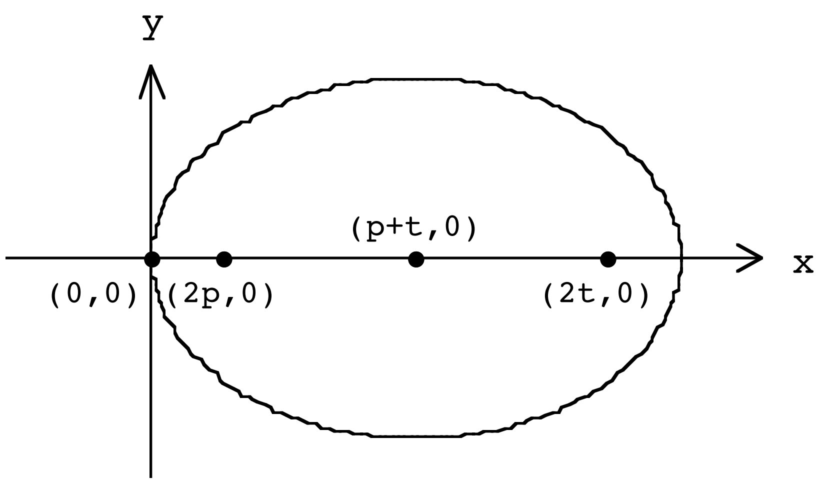


Figure 1

Consider an ellipse (see Fig. 1) which passes through the origin and has its foci at the points $(2p, 0)$ and $(2t, 0)$. Its center is at $(p+t, 0)$, $p > 0$.

$$h = p + t \quad k = 0 \quad a = p + t \quad c = t - p$$

$$b = \sqrt{a^2 - c^2} = \sqrt{(t+p)^2 - (t-p)^2} = \sqrt{4pt}.$$

Hence the equation of the ellipse is

$$\frac{[x - (p+t)]^2}{(p+t)^2} + \frac{y^2}{4pt} = 1.$$

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It is now necessary to show that if $p$ remains constant and $t \to \infty$, the curve approaches the shape of a parabola. Solving for $y^2$,

$$\begin{aligned} y^2 &= 4pt\left[1 - \frac{(x-[p+t])^2}{(p+t)^2}\right] \\ &= 4pt - \frac{4pt[x-(p+t)]^2}{(p+t)^2} \\ &= \frac{4pt(p+t)^2 - 4pt[x-(p+t)]^2}{(p+t)^2} \end{aligned}$$

and after simplifying,

$$y^2 = 4ptx(2t+2p-x)/(p+t)^2.$$

Letting $t = 1/u$,

$$y^2 = \frac{4px(2/u+2p-x)}{u(p+1/u)^2} = \frac{4px(2+2pu-ux)}{p^2u^2+2pu+1}.$$

From this form it can be seen that as $t \to \infty$ or $u \to 0$, the equation of the curve approaches $y^2 = 8px$ which is the equation of a parabola. Q.E.D.

Another fact, not so obvious, is that a parabola is also a limiting case of a hyperbola as one focus tends to infinity.

The proof is similar. The standard form of a hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$.

Consider a hyperbola which passes through the origin and has its foci at the points $(2p, 0)$ and $(2t, 0)$. See Figure 2. (Note that $c = t - p$ since $p < 0$ and so $b^2 = -4pt$.)

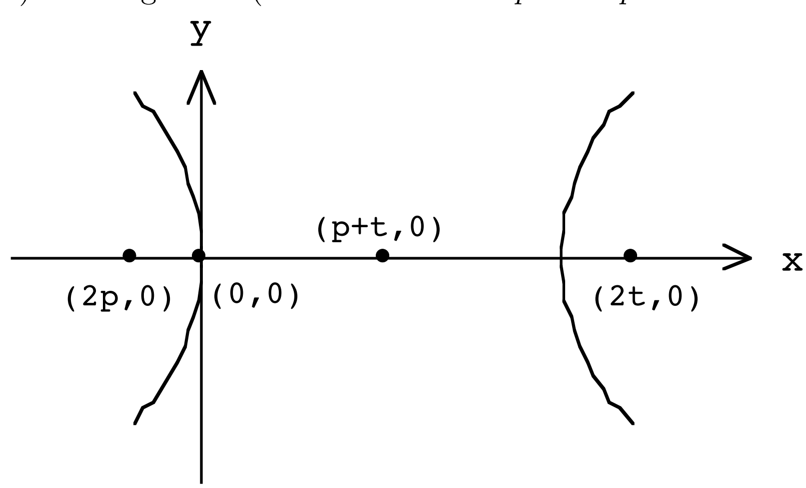


Figure 2

Its equation is

$$\frac{[x-(p+t)]^2}{(p+t)^2} - \frac{y^2}{-4pt} = 1$$

or

$$\frac{[x-(p+t)]^2}{(p+t)^2}+\frac{y^2}{4pt}=1.$$

This is exactly the same as the equation for the ellipse except that $p < 0$. Therefore the rest of the proof is exactly the same and as $t \to \infty$, the curve approaches the curve $y^2 = 8px$ (which in this case is a parabola opening to the left).