Why are the Exponents the Same?

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It is shown that certain Diophantine equations, involving sums of powers of variables with relatively prime exponents, are easy to solve.

Why is it that the most interesting, unsolved, or unsolvable Diophantine equations have all the exponents the same?

When taking a course in number theory, or when looking through standard books on Diophantine equations (such as [1], [2], [4], [5]), one comes across equations such as

$$x^2 + y^2 = z^2 + w^2,$$
$$x^2 - Dy^2 = z^2$$
$$x^4 + y^4 = z^4 + w^4,$$
$$x^4 + y^4 = cz^4$$
$$x^4 + y^4 + z^4 + w^4 = t^4,$$

and

$$x^n + y^n = z^n$$

in which all the exponents are equal. One also frequently runs across Diophantine equations like

$$x^2 + y^2 = z^4$$

and

$$x^4 + y^4 = z^2$$

in which most of the exponents are the same.

Why is it that we do not run across equations like

$$x^3 + y^7 = z^{17} \tag{1}$$

more often? Are such equations that much harder?

The answer is just the opposite. Such equations are usually easier to solve. We will illustrate this fact in this note. These facts are known in the literature (see [3] problem 64, [6], [7], [8], [9]), but appear to be little-known in the classroom.

For example, to find solutions to equation (1), we can let

$$x = X^{28}, \quad y = Y^{12}, \quad \text{and} \quad z = Z^5,$$

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to get
$$X^{84} + Y^{84} = Z^{85}. \tag{2}$$
Some solutions to this equation are given by the two parameter family

$$\begin{aligned} X &= m(m^{84} + n^{84}) \\ Y &= n(m^{84} + n^{84}) \\ Z &= \ \ (m^{84} + n^{84}) \end{aligned}$$

where m and n are arbitrary integers. These solutions can readily be verified by substituting into equation (2).

This same trick works any time we can get the Diophantine equation into the form of equation (2) with all exponents of n on the left and an exponent of $n+1$ on the right.

Lemma. *Let k and n be positive integers and let $a_0, a_1, \ldots, a_k$ be non-zero integers. Then the Diophantine equation*

$$a_1 y_1^n + a_2 y_2^n + \cdots + a_k y_k^n = a_0 y_0^{n+1} \tag{3}$$

has infinitely many integral solutions $(y_0, y_1, \ldots, y_k)$.

Proof. Infinitely many solutions are given by

$$y_i = \frac{c_i}{a_0}(a_1 c_1^n + a_2 c_2^n + \cdots + a_k c_k^n), \quad i = 0, 1, \ldots, k,$$

where the c_j are arbitrary integral multiples of a_0 for $j = 1, 2, \ldots, k$ and $c_0 = 1$. This is easily verified by substituting into equation (3) and verifying that the result is an identity. The requirement that the c_j (other than c_0) be multiples of a_0 ensures that all the y_i are integers.

To transform equation (1) into the form specified in this lemma, we found a common multiple of 3 and 7 that was one less than a multiple of 17. When the exponents are relatively prime, this can always be done. The following result (from [8]) shows when we can get such equations into the proper form.

Theorem. *Let k be a positive integer and let $a_0, a_1, \ldots a_k$ be arbitrary non-zero integers. Let $r_0, r_1, \ldots, r_k$ be positive integers such that $\gcd(r_i, r_0) = 1$ for $i = 1, 2, \ldots, k$. Then the Diophantine equation*
$$a_1 x_1^{r_1} + a_2 x_2^{r_2} + \cdots + a_k x_k^{r_k} = a_0 x_0^{r_0} \tag{4}$$
has infinitely many integral solutions $(x_0, x_1, \ldots, x_k)$.

Proof. Let r be the least common multiple of $r_1, r_2, \ldots, r_k$.

It follows from $\gcd(r_i, r_0) = 1$ for $i = 1, 2, \ldots, k$ that $\gcd(r, r_0) = 1$. Thus there are positive integers p and q such that

$$qr_0 - pr = 1.$$

Then any solution $(y_0, y_1, \ldots, y_k)$ of

$$a_1 y_1^{pr} + a_2 y_2^{pr} + \cdots + a_k y_k^{pr} = a_0 y_0^{qr_0} \tag{5}$$

provides a solution $(x_0, x_1, \ldots, x_k)$ to equation (4) with $x_i = y_i^{pr/r_i}$, $i = 1, 2, \ldots, k$ and $x_0 = y_0^q$. But equation (5) has infinitely many solutions by the lemma, since $qr_0 = pr + 1$. Therefore equation (4) has infinitely many solutions.

References

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