A Nonlinear Recurrence Yielding Binary Digits

Stanley Rabinowitz
Westford, MA 01886

Peter Gilbert
Digital Equipment Corporation
Nashua NH 03060

Graham and Pollak [2] considered the sequence

$$1, 2, 3, 4, 6, 9, 13, 19, 27, 38, 54, 77, \ldots$$

defined by the recurrence

$$u_1 = 1, \quad u_{n+1} = \lfloor \sqrt{2}(u_n + \frac{1}{2}) \rfloor, \quad n \geq 1,$$

where $\lfloor x \rfloor$ denotes the floor of x, the largest integer not larger than x. They discovered the unusual property that $u_{2n+1} - 2u_{2n-1}$ is just the nth digit in the binary expansion of $\sqrt{2}$. In discussing this result, Erdős and Graham [1] say "It seems clear that there must be similar results for $\sqrt{m}$ and other algebraic numbers but we have no idea what they are."

In this paper, we give a generalization of this result and obtain a recurrence relation which yields, in a similar manner, the nth digit in the binary expansion of any positive real number.

We begin by proving some properties of the floor function:

Definition. Let $\{x\}$ denote the fractional part of x, that is, $\{x\} = x - \lfloor x \rfloor$.

Lemma 1. $\big\lfloor a \lfloor x \rfloor + c \big\rfloor = \lfloor ax \rfloor$ if and only if $\lfloor \{ax\} - a\{x\} + c \rfloor = 0$.

Proof. The following equations are equivalent to each other in succession:

$$\begin{aligned}
\big\lfloor a\lfloor x\rfloor + c \big\rfloor &= \lfloor ax \rfloor \\
\lfloor ax - a\{x\} + c \rfloor &= \lfloor ax \rfloor \\
\big\lfloor \{ax\} + \lfloor ax \rfloor - a\{x\} + c \big\rfloor &= \lfloor ax \rfloor \\
\lfloor \{ax\} - a\{x\} + c \rfloor &= 0.
\end{aligned}$$

Lemma 2. If k is an integer, a is a real number in the range $1 < a < 2$, and $x = k/(a-1)$, then

$$\left\lfloor a\lfloor x \rfloor + \frac{a}{2} \right\rfloor = \lfloor ax \rfloor.$$

Proof. If $x = k/(a-1)$, then $ax = x + k$ and $\{ax\} = \{x\}$. Thus

$$f(x) = \{ax\} - a\{x\} + \frac{a}{2} = \{x\} - a\{x\} + \frac{a}{2} = \{x\}(1 - \frac{a}{2}) + (1 - \{x\})\frac{a}{2}.$$

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Clearly, $f(x) \geq 0$ and $f(x) < 1(1-\frac{a}{2})+1(\frac{a}{2}) = 1$. Hence $\lfloor f(x) \rfloor = 0$ and the result follows by Lemma 1.

Lemma 3. If k is an integer, a is a real number in the range $0 < a < 2$, and $x = k/(2-a)$, then

$$\left\lfloor a\lfloor x \rfloor + \frac{a}{2} \right\rfloor = \lfloor ax \rfloor.$$

Proof. From $x = k/(2-a)$ we get $2x = ax + k$. Taking the fractional part of both sides yields $\{2x\} = \{ax\}$. It is easy to show that $d = 2\{x\} - \{2x\}$ is always 0 or 1, so we may write $\{2x\}$ as $2\{x\} - d$. Thus

$$\begin{aligned} f(x) &= \{ax\} - a\{x\} + \frac{a}{2} = \{2x\} - \frac{a}{2}(\{2x\}+d) + \frac{a}{2} \\ &= \{2x\}(1-\frac{a}{2}) + \frac{a}{2}(1-d). \end{aligned}$$

Clearly $f(x) \geq 0$ and $f(x) < 1(1-\frac{a}{2}) + (\frac{a}{2})1 = 1$. Hence $\lfloor f(x) \rfloor = 0$ and the result follows by Lemma 1.

Theorem. Given a positive real number w, let $b = (2^{m+1}+w)/(2^m+w)$ and $a = 2/b$, where $m = \lfloor \log_2 w \rfloor$. Define a sequence u_n by the recurrence

$$\begin{aligned} u_1 &= 1 \\ u_{n+1} &= \begin{cases} \lfloor a(u_n + 1/2) \rfloor, & \text{if } n \text{ is odd,} \\ \lfloor b(u_n + 1/2) \rfloor, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Then

(i)
$$\begin{aligned} u_{2n} &= \lfloor 2^{n-m-1}(2^m + w) \rfloor \\ u_{2n+1} &= \lfloor 2^{n-m-1}(2^{m+1} + w) \rfloor. \end{aligned}$$

(ii) $u_{2n+1} - 2u_{2n-1}$ is the nth digit in the binary expansion of w.
(The radix point appears after the $(m+1)$st digit.)

(iii) $u_{2n+2} - 2u_{2n}$ is the $(n+1)$st digit of the binary expansion of w.

(iv)
$$u_{2n+1} - u_{2n} = 2^{n-1}.$$

Proof. First note that $1 < b < 2$ and $1 < a < 2$. Also, a little algebra shows that $1 + w/2^m = 1/(b-1)$ and $2 + w/2^m = 2/(2-a)$.

We will prove property (i) by induction on the subscript of u. For $n = 0$, our formula specifies

$$u_1 = \lfloor 2^{-m-1}(2^m + w) \rfloor = \lfloor 2^{-1}(1 + \frac{w}{2^m}) \rfloor.$$

But m has been chosen so that $1 \leq w/2^m < 2$ which implies that $u_1 = 1$ and our formula checks for $n = 0$. There are now two cases, the subscript being even or odd.

If the result is true for u_{2n}, then we have $u_{2n} = \lfloor x \rfloor$ where

$$x = 2^{n-m-1}(2^m + w) = 2^{n-1}(1 + \frac{w}{2^m}) = 2^{n-1}\frac{1}{b-1}.$$

Then

$$\begin{aligned} u_{2n+1} &= \left\lfloor bu_{2n} + \frac{b}{2} \right\rfloor = \left\lfloor b\lfloor x \rfloor + \frac{b}{2} \right\rfloor \\ &= \lfloor bx \rfloor && \text{(by Lemma 2)} \\ &= \left\lfloor 2^{n-1}\frac{b}{b-1} \right\rfloor = \lfloor 2^{n-m-1}(2^{m+1} + w) \rfloor \end{aligned}$$

and the result is true for u_{2n+1}.

If the result is true for u_{2n+1}, then we have $u_{2n+1} = \lfloor x \rfloor$ where

$$x = 2^{n-m-1}(2^{m+1} + w) = 2^{n-1}(2 + \frac{w}{2^m}) = 2^{n}\frac{1}{2-a}.$$

Then

$$\begin{aligned} u_{2n+2} &= \left\lfloor au_{2n+1} + \frac{a}{2} \right\rfloor = \left\lfloor a\lfloor x \rfloor + \frac{a}{2} \right\rfloor \\ &= \lfloor ax \rfloor && \text{(by Lemma 3)} \\ &= \left\lfloor 2^{n}\frac{a}{2-a} \right\rfloor = \lfloor 2^{n-m}(2^{m} + w) \rfloor \end{aligned}$$

and the result is true for u_{2n+2}. This concludes the induction.

Property (ii) is a direct consequence of property (i). The nth binary digit of the number $w = d_1 d_2 d_3 \ldots d_m d_{m+1}.d_{m+2} d_{m+3} d_{m+4} \ldots$ can be found as follows:

$$2^{n-m-1} w = d_1 d_2 d_3 \ldots d_n.d_{n+1} d_{n+2} d_{n+3} \ldots$$

and

$$2^{n-m-2} w = d_1 d_2 d_3 \ldots d_{n-1}.d_n d_{n+1} d_{n+2} d_{n+3} \ldots$$

so

$$\begin{aligned} u_{2n+1} - 2u_{2n-1} &= \lfloor 2^{n-m-1}(2^{m+1} + w) \rfloor - 2\lfloor 2^{n-m-2}(2^{m+1} + w) \rfloor \\ &= \lfloor 2^{n-m-1} w \rfloor - 2\lfloor 2^{n-m-2} w \rfloor \\ &= d_1 d_2 \ldots d_n - d_1 d_2 \ldots d_{n-1} 0 = d_n. \end{aligned}$$

The proof of property (iii) is similar.

Property (iv) follows from the fact that property (i) may be rewritten as

$$\begin{aligned} u_{2n} &= 2^{n-1} + \lfloor 2^{n-m-1} w \rfloor \\ u_{2n+1} &= 2^{n} + \lfloor 2^{n-m-1} w \rfloor. \end{aligned}$$

We should note that the result of Graham and Pollak follows from our result when $w = \sqrt{2}$. In that case, $m = 0$ and $a = b = \sqrt{2}$. It should also be noted that when we speak of the nth digit of a number, we start counting at the leftmost non-zero digit. If $m < 0$, there will be $|m| - 1$ zeroes after the radix point before the first non-zero binary digit occurs.

To avoid having to consider the even and odd cases separately, our theorem may be rephrased in the following form (by setting $v_n = u_{2n-1}$):

Theorem (alternate formulation). Let w be a positive real number and let a and b be defined as before. Define a sequence v_n by the recurrence

$$\begin{aligned} v_1 &= 1 \\ v_{n+1} &= \left\lfloor b \left\lfloor a v_n + \frac{a}{2} \right\rfloor + \frac{b}{2} \right\rfloor . \end{aligned}$$

Then $v_{n+1} - 2v_n$ is the nth digit in the binary expansion of w.

If we are only interested in the digits comprising the fractional part of w, we have a slightly simpler result, not involving the variable m. The proof mimics the proof of the main theorem and we leave the details as an exercise for the reader.

Theorem. Given a positive real number w, let $b = (2 + w)/(1 + w)$ and $a = 2/b$. Define a sequence u_n by the recurrence

$$\begin{aligned} u_1 &= \lfloor 2 + w \rfloor \\ u_{n+1} &= \begin{cases} \lfloor a(u_n + 1/2) \rfloor, & \text{if } n \text{ is odd,} \\ \lfloor b(u_n + 1/2) \rfloor, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Then $u_{2n+2} - 2u_{2n}$ is the nth digit to the right of the radix point in the binary expansion of w.

The reader may wonder how the authors came up with the values of a, b, and m in the main theorem. We found that if we let $u_{n+1} = \lfloor a(u_n + \frac{1}{2}) \rfloor$ and then tried various a's other than $\sqrt{2}$, the quantity $u_{2n+2} - 2u_{2n}$ did not always yield binary digits (0's or 1's). Then we tried changing a to two values, a and b, for the odd and even values of n. Using a computer, we varied a and b and printed out those cases where $u_{2n+2} - 2u_{2n}$ always generated binary digits. We were rewarded by finding that in such cases, $ab = 2$; and when $1 < a < 3/2$, we found that the value of w that was generated appeared to have the value $2(a - 1)/(2 - a)$. This gave us a conjecture on how to get the binary digits for any w between 1 and 2 by picking $a = 2(1 + w)/(2 + w)$ and $b = 2/a$. Finally, we realized that if w was not between 1 and 2, we could multiply it by 2^m for some m to bring it into that range and then apply the simpler version of the theorem.

There are other open questions that the reader might wish to pursue. For example, suppose we define u_{n+1} in terms of three quantities, a, b, and c, depending upon whether n is congruent to 0, 1, or 2 (mod 3). For what choices of a, b, and c will $u_{2n+2} - 2u_{2n}$ yield binary digits? Perhaps one should look at $u_{2n+2} - 3u_{2n}$ instead (or maybe $u_{3n+3} - 3u_{3n}$) and try to get digits in base 3.

REFERENCES

[1] P. Erdős and R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*. L'Enseignement Mathématique. Genève: 1980, p. 96.

[2] R. L. Graham and H. O. Pollak, "Note on a Nonlinear Recurrence Related to $\sqrt{2}$", *Mathematics Magazine*. **43**(1970)143-145.