Algorithmic Manipulation of Third-Order Linear Recurrences

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1. Introduction.

In [12] we showed how to algorithmically prove all polynomial identities involving a certain class of elements from second-order linear recurrences with constant coefficients. In this paper, we attempt to extend these results to third-order linear recurrences.

Let $\langle S_n \rangle$ be a sequence defined by the third-order linear recurrence

$$S_n = pS_{n-1} + qS_{n-2} + rS_{n-3} \tag{1}$$

where $r \neq 0$. We will consider three special such sequences, $\langle X_n \rangle$, $\langle Y_n \rangle$, and $\langle Z_n \rangle$ given by the following initial conditions:

$$\begin{aligned} X_0 &= 0, \quad X_1 = 0, \quad X_2 = 1; \\ Y_0 &= 0, \quad Y_1 = 1, \quad Y_2 = 0; \\ Z_0 &= 1, \quad Z_1 = 0, \quad Z_2 = 0. \end{aligned} \tag{2}$$

These initial conditions were chosen so that the three sequences form a basis for the set of all third-order linear recurrences with constant coefficients, and because they will allow us (in a future paper) to generalize our results to higher-order recurrences. These three sequences also have nice Binet forms.

Given any sequence $\langle S_n \rangle$ that satisfies recurrence (1), we can write its elements as a linear combination of X_n, Y_n, and Z_n, namely

$$S_n = S_2 X_n + S_1 Y_n + S_0 Z_n. \tag{3}$$

Thus, it suffices to show that we can algorithmically prove any identity involving X_n, Y_n, and Z_n.

The sequence $\langle S_n \rangle$ can be defined for negative values of n by using the recurrence (1) to extend the sequence backwards, or equivalently, by using the recurrence

$$S_{-n} = (-qS_{-n+1} - pS_{-n+2} + S_{-n+3})/r. \tag{4}$$

A short table of values of X_n, Y_n, and Z_n for small values of n is given below:

n	-2	-1	0	1	2	3	4	5
X_n	$-q/r^2$	$1/r$	0	0	1	p	p^2+q	$p^3+2pq+r$
Y_n	$(pq+r)/r^2$	$-p/r$	0	1	0	q	$pq+r$	$p^2q+pr+q^2$
Z_n	$(q^2-pr)/r^2$	$-q/r$	1	0	0	r	pr	$r(p^2+q)$

The characteristic equation for recurrence (1) is

$$x^3 - px^2 - qx - r = 0. \tag{5}$$

Reprinted from The Fibonacci Quarterly, 34(1996)447–464

Let the roots of this equation be r_1, r_2, and r_3, which we shall assume are distinct. The condition that these roots are distinct is that Δ, the discriminant, is nonzero. That is,

$$\Delta^2 = (r_1 - r_2)^2(r_2 - r_3)^2(r_3 - r_1)^2 = p^2q^2 - 27r^2 + 4q^3 - 4p^3r - 18pqr > 0. \tag{6}$$

The Binet forms for our sequences are given by:

$$\begin{aligned} X_n &= A_1r_1^n + B_1r_2^n + C_1r_3^n, \\ Y_n &= A_2r_1^n + B_2r_2^n + C_2r_3^n, \\ Z_n &= A_3r_1^n + B_3r_2^n + C_3r_3^n, \end{aligned} \tag{7}$$

where

$$\begin{aligned} A_1 &= \frac{1}{(r_1-r_2)(r_1-r_3)}, \quad & B_1 &= \frac{1}{(r_2-r_3)(r_2-r_1)}, \quad & C_1 &= \frac{1}{(r_3-r_1)(r_3-r_2)}; \\ A_2 &= \frac{-(r_2+r_3)}{(r_1-r_2)(r_1-r_3)}, \quad & B_2 &= \frac{-(r_3+r_1)}{(r_2-r_3)(r_2-r_1)}, \quad & C_2 &= \frac{-(r_1+r_2)}{(r_3-r_1)(r_3-r_2)}; \\ A_3 &= \frac{r_2r_3}{(r_1-r_2)(r_1-r_3)}, \quad & B_3 &= \frac{r_3r_1}{(r_2-r_3)(r_2-r_1)}, \quad & C_3 &= \frac{r_1r_2}{(r_3-r_1)(r_3-r_2)}. \end{aligned} \tag{8}$$

Another sequence of interest is

$$W_n = X_{n+2} + Y_{n+1} + Z_n = pX_{n+1} + 2qX_n + 3rX_{n-1} = (p^2+2q)X_n + pY_n + 3Z_n$$

because W_n has the Binet form

$$W_n = r_1^n + r_2^n + r_3^n. \tag{9}$$

We can solve the equations in (7) for the r_i^n. We get

$$\begin{aligned} r_1^n &= r_1^2X_n + r_1Y_n + Z_n \\ r_2^n &= r_2^2X_n + r_2Y_n + Z_n \\ r_3^n &= r_3^2X_n + r_3Y_n + Z_n. \end{aligned} \tag{10}$$

This idea was suggested by Murray Klamkin. It also follows from Lemma 1 of [11]. These equations let us convert an expression involving powers of r_i, where a variable n occurs in the exponents, to expressions involving X_n, Y_n, and Z_n.

From the relationship between the roots and coefficients of a cubic, we have

$$\begin{aligned} r_1 + r_2 + r_3 &= p \\ r_1r_2 + r_2r_3 + r_3r_1 &= -q \\ r_1r_2r_3 &= r. \end{aligned} \tag{11}$$

Thus any symmetric polynomial involving r_1, r_2, and r_3 can be expressed in terms of p, q, and r. An algorithmic method (Waring's Algorithm) for performing this transformation can be found on page 14 in [5].

An explicit formula for X_n in terms of p, q, and r was given in [13], namely

$$X_{n+2} = \sum_{a+2b+3c=n} \binom{a+b+c}{a \quad b \quad c} p^a q^b r^c. \tag{12}$$

Similar formulas for Y_n and Z_n can be obtained from the facts that $Y_n = X_{n+1} - pX_n$ and $Z_n = rX_{n-1}$.

Matrix formulations were given in [17] and [20]:

$$\begin{pmatrix} S_{n+2} \\ S_{n+1} \\ S_n \end{pmatrix} = \begin{pmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} S_2 \\ S_1 \\ S_0 \end{pmatrix}, \tag{13}$$

$$\begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix} = \begin{pmatrix} p & 1 & 0 \\ q & 0 & 1 \\ r & 0 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{14}$$

and

$$\begin{pmatrix} X_{n+2} & Y_{n+2} & Z_{n+2} \\ X_{n+1} & Y_{n+1} & Z_{n+1} \\ X_n & Y_n & Z_n \end{pmatrix} = \begin{pmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n. \tag{15}$$

2. The Basic Algorithms.

ALGORITHM "TribEvaluate":

Given an integer constant n, to evaluate X_n, Y_n, or Z_n numerically, apply the following algorithm:
STEP 1: [Make subscript positive]. If $n < 0$, apply algorithm "TribNegate" given below.
STEP 2: [Recurse]. If $n > 2$, apply the recursion:

$$S_n = pS_{n-1} + qS_{n-2} + rS_{n-3}.$$

This reduces the subscript by 1, so the recursion must eventually terminate. If n is 0, 1, or 2, use the values in display (2).
NOTE: While this may not be the fastest way to evaluate X_n, Y_n, and Z_n, it is nevertheless an effective algorithm.

The key idea to algorithmically proving identities involving polynomials in X_{an+b}, Y_{an+b}, and Z_{an+b} is to first reduce them to polynomials in X_n, Y_n, and Z_n. To do that, we need reduction formulas for X_{m+n}, Y_{m+n}, and Z_{m+n}. Such formulas can be obtained from equations (7), (8), (10), and (11).

From equation (10), we can compute r_i^{n+m} by multiplying together r_i^n and r_i^m. Then equation (7) gives us X_{m+n}. Thus, $X_{n+m} = A_1(r_1^2 X_n + r_1 Y_n + Z_n)(r_1^2 X_m + r_1 Y_m + Z_m) +$

$B_1(r_2^2X_n + r_2Y_n + Z_n)(r_2^2X_m + r_2Y_m + Z_m) + C_1(r_3^2X_n + r_3Y_n + Z_n)(r_3^2X_m + r_3Y_m + Z_m)$. Substituting in the values of the A_1, B_1, and C_1 from equation (8) gives us an expression that is symmetric in r_1, r_2, and r_3. Applying Waring's Algorithm allows us to express this in terms of p, q, and r using equation (11). We can do the same for Y_{n+m} and Z_{n+m}. The results obtained are given by the following algorithm.

ALGORITHM "TribReduce" TO REMOVE SUMS IN SUBSCRIPTS.
Use the identities:

$$\begin{aligned} X_{m+n} &= (p^2+q)X_mX_n + p(X_nY_m + X_mY_n) + X_nZ_m + X_mZ_n + Y_mY_n \\ Y_{m+n} &= (pq+r)X_mX_n + q(X_nY_m + X_mY_n) + Y_nZ_m + Y_mZ_n \\ Z_{m+n} &= \qquad prX_mX_n + r(X_nY_m + X_mY_n) + Z_mZ_n. \end{aligned} \tag{16}$$

These are also known as the addition formulas.

From the table of initial values, we find that the reduction formulas can also be written in the form

$$\begin{aligned} X_{m+n} &= X_4X_mX_n + X_3(X_nY_m + X_mY_n) + X_nZ_m + X_mZ_n + Y_mY_n \\ Y_{m+n} &= Y_4X_mX_n + Y_3(X_nY_m + X_mY_n) + Y_nZ_m + Y_mZ_n \\ Z_{m+n} &= Z_4X_mX_n + Z_3(X_nY_m + X_mY_n) + Z_mZ_n. \end{aligned} \tag{17}$$

The matrix formulation is

$$X_{m+n} = \begin{pmatrix} X_m \\ Y_m \\ Z_m \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} X_4 & X_3 & X_2 \\ X_3 & X_2 & X_1 \\ X_2 & X_1 & X_0 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \\ Z_n \end{pmatrix} \tag{18}$$

with similar expressions for Y_{m+n} and Z_{m+n}.

If we allow subscripts on the right other than "n" and "m", simpler forms of the reduction formula can be found. For example, [18] gives the following:

$$S_{n+m} = X_mS_{n+2} + Y_mS_{n+1} + Z_mS_n. \tag{19}$$

Similar expressions can be found in [7] and [17]. In matrix form, they can be expressed as

$$\begin{pmatrix} S_{n+m} \\ S_{n+m-1} \\ S_{n+m-2} \end{pmatrix} = \begin{pmatrix} X_{m+1} & Y_{m+1} & Z_{m+1} \\ X_m & Y_m & Z_m \\ X_{m-1} & Y_{m-1} & Z_{m-1} \end{pmatrix} \begin{pmatrix} S_{n+1} \\ S_n \\ S_{n-1} \end{pmatrix}. \tag{20}$$

These formulations come from [18] and [20].

Algorithm "TribReduce" allows us to replace any term of the form S_{an+b} where a and b are positive integers by terms of the form S_n. To allow a and b to be negative integers as well, we can also use equation (16), however, then we will obtain expressions of the form S_{-n}. Since we would like to express these in the form S_n, we must find formulas for S_{-n}. The same procedure we used before works again. For example, from equation (10), we can compute r_i^{-n} as $1/r_i^n$. Equation (7) then gives $X_{-n} = A_1/(r_1^2X_n + r_1Y_n + Z_n) + B_1/(r_2^2X_n + r_2Y_n + Z_n) + C_1/(r_3^2X_n + r_3Y_n + Z_n)$. Again we apply Waring's Algorithm and we get the following result.

ALGORITHM "TribNegate" TO REMOVE NEGATIVE SUBSCRIPTS.

Use the identities:

$$\begin{aligned}
X_{-n} &= \frac{pX_nY_n - qX_n^2 + Y_n^2 - X_nZ_n}{r^n} \\
Y_{-n} &= \frac{(pq+r)X_n^2 - p^2X_nY_n - pY_n^2 - Y_nZ_n}{r^n} \\
Z_{-n} &= \frac{(q^2-pr)X_n^2 - (pq+r)X_nY_n - qY_n^2 + (p^2+2q)X_nZ_n + pY_nZ_n + Z_n^2}{r^n}.
\end{aligned} \tag{21}$$

If we allow subscripts on the right other than "n", simpler forms can be found. For example,

$$\begin{aligned}
X_{-n} &= (X_{n+1}Y_n - X_nY_{n+1})/r^n \\
Y_{-n} &= (X_nY_{n+2} - X_{n+2}Y_n)/r^n \\
Z_{-n} &= (X_{n+2}Y_{n+1} - X_{n+1}Y_{n+2})/r^n.
\end{aligned} \tag{22}$$

3. The Fundamental Identity Connecting X, Y, and Z.

The Fibonacci and Lucas numbers are connected by the fundamental identity

$$L_n^2 = 5F_n^2 + 4(-1)^n. \tag{23}$$

Furthermore, it can be shown that if $f(F_n, L_n)$ is any non-constant polynomial (with coefficients that are constants or of the form $(-1)^n$) that is 0 for all integral values of n, then this polynomial must be divisible by $L_n^2 - 5F_n^2 - 4(-1)^n$. That is, equation (23) is the unique identity connecting F_n and L_n.

A similar result holds for arbitrary second-order linear recurrences. For third-order linear recurrences, we believe there is also exactly one fundamental identity connecting X_n, Y_n, and Z_n. In this section, we will find such an identity, but we do not prove that this identity is unique.

To obtain an identity connecting X_n, Y_n, and Z_n, we can multiply together the equations in display (10). The result is a symmetric polynomial in r_1, r_2, and r_3 and can thus be expressed in terms of p, q, and r. The result is the following.

THE FUNDAMENTAL IDENTITY:

$$\begin{aligned}
r^n &= r^2X_n^3 + rY_n^3 + Z_n^3 + (q^2-2pr)X_n^2Z_n - qrX_n^2Y_n + prX_nY_n^2 \\
&\quad + (p^2+2q)X_nZ_n^2 - qY_n^2Z_n + pY_nZ_n^2 - (pq+3r)X_nY_nZ_n.
\end{aligned} \tag{24}$$

If we allow subscripts on the right other than "n", simpler forms of the fundamental identity can be found. For example, [15] gives the following equivalent formulation:

$$\begin{vmatrix} X_{n+2} & X_{n+1} & X_n \\ Y_{n+2} & Y_{n+1} & Y_n \\ Z_{n+2} & Z_{n+1} & Z_n \end{vmatrix} = r^n. \tag{25}$$

4. The Simplification Algorithm.

Let us be given a polynomial function of elements of the form X_w, Y_w, and Z_w, where the subscripts of X, Y, and Z are of the form $a_1n_1 + a_2n_2 + \cdots + a_kn_k + b$ where b and the a_i are integer constants and the n_i are variables. To put this expression in "canonical form", we apply the following algorithm:

ALGORITHM "TribSimplify" TO TRANSFORM AN EXPRESSION TO CANONICAL FORM.

STEP 1: [Remove sums in subscripts]. Apply Algorithm "TribReduce" to remove any sums (or differences) in subscripts.

STEP 2: [Make multipliers positive]. All subscripts are now of the form cn where c is an integer. For any term in which the multiplier c is negative, apply Algorithm "TribNegate".

STEP 3: [Remove multipliers]. All subscripts are now of the form cn where c is a positive integer. For any term in which the multiplier c is not 1, apply Algorithm "TribReduce" successively until all subscripts are variables.

STEP 4: [Remove powers of Z]. If any term involves an expression of the form Z_n^k where $k > 2$ reduce the exponent by 1 by replacing Z_n^3 by its equivalent value as given by the fundamental identity (24), namely

$$\begin{aligned} Z_n^3 = r^n - r^2X_n^3 - rY_n^3 - (q^2 - 2pr)X_n^2Z_n + qrX_n^2Y_n - prX_nY_n^2 \\ - (p^2 + 2q)X_nZ_n^2 + qY_n^2Z_n - pY_nZ_n^2 + (pq + 3r)X_nY_nZ_n. \end{aligned} \tag{26}$$

Continue doing this until no Z_n term has an exponent larger than 2.

PROVING IDENTITIES.

To prove that an expression is identically 0, it suffices to apply algorithm "TribSimplify". If the resulting canonical form is 0, then the expression is identically 0. We believe that the converse is true as well; that is, an expression is identically 0 if and only if algorithm "TribSimplify" transforms it to 0. A formal proof can probably be given along the lines of [18], however, we do not do so. Suffice it to say that algorithm "TribSimplify" was checked on about 100 identities culled from the literature and it worked every time. A selection of these identities is given in the appendix. See also [6] for a related algorithm for trigonometric polynomials.

5. Other Algorithms.

These algorithms can be verified by applying algorithm "TribSimplify".

ALGORITHM "ConvertToX" TO CHANGE Y's AND Z's to X's

Use the identities:

$$\begin{aligned} Y_n &= -pX_n + X_{n+1} \\ Z_n &= rX_{n-1}. \end{aligned} \tag{27}$$

ALGORITHM "ConvertToY" TO CHANGE Z's AND X's to Y's

Use the identities:

$$\begin{aligned} Z_n &= (rY_{n+1} - qrY_{n-1})/(pq + r) \\ X_n &= (pY_{n+1} + rY_{n-1})/(pq + r). \end{aligned} \tag{28}$$

ALGORITHM "ConvertToZ" TO CHANGE X's AND Y's to Z's

Use the identities:

$$\begin{aligned} X_n &= Z_{n+1}/r \\ Y_n &= Z_{n-1} + qZ_n/r. \end{aligned} \tag{29}$$

ALGORITHM "Removepqr" TO REMOVE p's, q's, AND r's

Use the identities:

$$\begin{aligned} p &= (X_{n+1} - Y_n)/X_n \\ q &= (Y_{n+1} - Z_n)/X_n \\ r &= Z_{n+1}/X_n. \end{aligned} \tag{30}$$

ALGORITHM "TribShiftDown1" TO DECREASE A SUBSCRIPT BY 1

Use the identities:

$$\begin{aligned} X_{n+1} &= pX_n + Y_n \\ Y_{n+1} &= qX_n + Z_n \\ Z_{n+1} &= rX_n. \end{aligned} \tag{31}$$

These can be found in [10].

ALGORITHM "TribShiftUp1" TO INCREASE A SUBSCRIPT BY 1

Use the identities:

$$\begin{aligned} X_{n-1} &= Z_n/r \\ Y_{n-1} &= X_n - pZ_n/r \\ Z_{n-1} &= Y_n - qZ_n/r. \end{aligned} \tag{32}$$

SUBTRACTION FORMULAS

Use the identities:

$$\begin{aligned} X_{m-n} &= (rX_n(X_nY_m - X_mY_n) - (qX_n + Z_n)(X_nZ_m - X_mZ_n) \\ &\quad + (pX_n + Y_n)(Y_nZ_m - Y_mZ_n))/r^n \\ Y_{m-n} &= (r(pX_n + Y_n)(X_mY_n - X_nY_m) + (pq + r)X_n(X_nZ_m - X_mZ_n) \\ &\quad - (p(p+1)X_n - Z_n)(Y_nZ_m - Y_mZ_n))/r^n \\ Z_{m-n} &= (r^2X_mX_n^2 - qrX_n^2Y_m + prX_nY_mY_n + rY_mY_n^2 + q^2X_n^2Z_m - prX_n^2Z_m \\ &\quad - pqX_nY_nZ_m - rX_nY_nZ_m - qY_n^2Z_m - prX_mX_nZ_n - rX_nY_mZ_n \\ &\quad - rX_mY_nZ_n + p^2X_nZ_mZ_n + 2qX_nZ_mZ_n + pY_nZ_mZ_n + Z_mZ_n^2)/r^n. \end{aligned} \tag{33}$$

If we allow subscripts on the right other than simple variables, simpler subtraction formulas can be found. For example, [2] gives the following equivalent formulation:

$$X_{m-n} = \begin{vmatrix} Z_m & Y_m & X_m \\ Z_n & Y_n & X_n \\ Z_{n+1} & Y_{n+1} & X_{n+1} \end{vmatrix} / r^n$$

$$Y_{m-n} = \begin{vmatrix} Z_m & Y_m & X_m \\ Z_n & Y_n & X_n \\ Z_{n+2} & Y_{n+2} & X_{n+2} \end{vmatrix} / r^n \tag{34}$$

$$Z_{m-n} = \begin{vmatrix} Z_m & Y_m & X_m \\ Z_{n+1} & Y_{n+1} & X_{n+1} \\ Z_{n+2} & Y_{n+2} & X_{n+2} \end{vmatrix} / r^n.$$

DOUBLE ARGUMENT FORMULAS

Letting $m = n$ in equation (16) gives us the following.

$$\begin{aligned} X_{2n} &= (p^2+q)X_n^2 + 2pX_nY_n + Y_n^2 + 2X_nZ_n \\ Y_{2n} &= (pq+r)X_n^2 + 2qX_nY_n + 2Y_nZ_n \\ Z_{2n} &= prX_n^2 + 2rX_nY_n + Z_n^2. \end{aligned} \tag{35}$$

TO REMOVE SCALAR MULTIPLES OF ARGUMENTS IN SUBSCRIPTS

An expression of the form S_{kn} where k is a positive integer can be thought of as being of the form $S_{n+n+\cdots+n}$ where there are k terms in the subscript. This can be expanded out in terms of S_n by $k-1$ repeated applications of the reduction formula (16). For example, for $k = 3$ we get the following identities.

$$\begin{aligned} X_{3n} &= (p^4+3p^2q+q^2+2pr)X_n^3 + 3(p^3+2pq+r)X_n^2Y_n + 3(p^2+q)X_nY_n^2 \\ &\quad + pY_n^3 + 3(p^2+q)X_n^2Z_n + 6pX_nY_nZ_n + 3Y_n^2Z_n + 3X_nZ_n^2 \\ Y_{3n} &= (p^3q+2pq^2+p^2r+2qr)X_n^3 + 3(p^2q+q^2+pr)X_n^2Y_n + 3(pq+r)X_nY_n^2 \\ &\quad + qY_n^3 + 3(pq+r)X_n^2Z_n + 6qX_nY_nZ_n + 3Y_nZ_n^2 \\ Z_{3n} &= (p^3r+2pqr+r^2)X_n^3 + 3r(p^2+q)X_n^2Y_n + 3prX_nY_n^2 + rY_n^3 + 3prX_n^2Z_n \\ &\quad + 6rX_nY_nZ_n + Z_n^3. \end{aligned}$$

In general, we have

$$S_{kn} = \sum_{a+b+c=k} \binom{k}{a\ \ b\ \ c} S_{2a+b} X_n^a Y_n^b Z_n^c \tag{36}$$

where $\binom{k}{a\ b\ c}$ denotes the trinomial coefficient $\frac{k!}{a!b!c!}$. Formula (36) can be proven by induction on k.

CHANGE OF BASIS (Shift Formulas)

ALGORITHM "TribShift" TO TRANSFORM AN EXPRESSION INVOLVING X_n, Y_n, Z_n INTO ONE INVOLVING X_{n+a}, Y_{n+b}, Z_{n+c}

Use identities such as:

$$X_n = \frac{1}{D}\left(\begin{vmatrix} qX_b + Z_b & Y_b \\ rX_c & Z_c \end{vmatrix} X_{n+a} - \begin{vmatrix} pX_a + Y_a & X_a \\ rX_c & Z_c \end{vmatrix} Y_{n+b} + \begin{vmatrix} pX_a + Y_a & X_a \\ qX_b + Z_b & Y_b \end{vmatrix} Z_{n+c}\right)$$

where

$$D = \begin{vmatrix} (p^2+q)X_a + pY_a + Z_a & pX_a + Y_a & X_a \\ (pq+r)X_b + qY_b & qX_b + Z_b & Y_b \\ prX_c + rY_c & rX_c & Z_c \end{vmatrix} \tag{37}$$

which can be obtained by solving the linear equations

$$\begin{aligned} X_{n+a} &= (p^2+q)X_aX_n + p(X_nY_a + X_aY_n) + X_nZ_a + X_aZ_n + Y_aY_n \\ Y_{n+b} &= (pq+r)X_bX_n + q(X_nY_b + X_bY_n) + Y_nZ_b + Y_bZ_n \\ Z_{n+c} &= prX_cX_n + r(X_nY_c + X_cY_n) + Z_cZ_n \end{aligned}$$

for X_n, Y_n, and Z_n.

One can change from the basis (X_n, Y_n, Z_n) to the basis $(X_{n+a}, X_{n+b}, X_{n+c})$ in a similar manner. Other combinations can be found in the same way. To change from an arbitrary basis to another, apply algorithm "TribReduce" to transform the given expression to the basis (X_n, Y_n, Z_n). Then use one of the above formulas.

6. Turning Squares into Sums.

For Lucas Numbers, there is the well-known formula

$$L_n^2 = L_{2n} - 2(-1)^n \tag{38}$$

which allows us to replace the square of a term with a sum of terms. To find an analog for third-order recurrences, we can proceed as follows.

Combining equations (21) and (35) gives us 6 equations in the 6 variables X_nY_n, Y_nZ_n, X_nZ_n, X_n^2, Y_n^2, and Z_n^2. We can then solve these equations for X_n^2, Y_n^2, and Z_n^2 in terms of X_{2n}, Y_{2n}, Z_{2n}, X_{-n}, Y_{-n}, and Z_{-n}. We get the following (computer-generated) result.

ALGORITHM "TribExpandSquares" TO TURN SQUARES INTO SUMS:

$$\begin{aligned} dX_n^2 = r^n[&2(p^4 + 5p^2q + 4q^2 + 6pr)X_{-n} + 2(p^3 + 4pq + 9r)Y_{-n} + 2(p^2+3q)Z_{-n}] \\ &+ 2(3pr - q^2)X_{2n} + (pq + 9r)Y_{2n} - 2(p^2+3q)Z_{2n} \end{aligned} \tag{39}$$

$$\begin{aligned} dY_n^2 = r^n[&2(p^6 + 6p^4q + 8p^2q^2 + 8p^3r + 16pqr + 9r^2)X_{-n} \\ &+ 2(p^5 + 5p^3q + 4pq^2 + 7p^2r + 3qr)Y_{-n} + 2(p^4 + 4p^2q + q^2 + 6pr)Z_{-n}] \\ &+ (9r^2 - p^2q^2 - 2q^3 + 2p^3r + 4pqr)X_{2n} + (p^3q + 3pq^2 + p^2r + 3qr)Y_{2n} \end{aligned} \tag{40}$$

$$
\begin{aligned}
&-2(p^4+4p^2q+q^2+6pr)Z_{2n}\\
dZ_n^2 = {}& r^n[2r(p^5+6p^3q+8pq^2+7p^2r+12qr)X_{-n}\\
&+2r(p^4+5p^2q+4q^2+6pr)Y_{-n}+2r(p^3+4pq+9r)Z_{-n}]\\
&-2r^2(p^2+3q)X_{2n}+r(p^2q+4q^2-3pr)Y_{2n}\\
&+(9r^2-p^2q^2-4q^3+2p^3r+10pqr)Z_{2n}
\end{aligned}
\tag{41}
$$

where $d = 27r^2 - p^2q^2 - 4q^3 + 4p^3r + 18pqr$.

These formulas are a bit outrageous. Are there any simpler formulas? Can these be put in simpler form? To be more specific, we ask the following.

Query. Is there a simpler formula than formula (41) that allows us to express Z_n^2 as a linear combination of terms, each of the form X_{an+b}, Y_{an+b}, or Z_{an+b}? The coefficients may include the constants p, q, and r as well as the non-linear expression r^n.

7. Turning Products into Simpler Products.

For Lucas Numbers, there is the well-known formula

$$L_mL_n = L_{m+n} + (-1)^nL_{m-n} \tag{42}$$

which allows you to turn products into sums. For third-order recurrences, there probably is no corresponding formula. However, there is a formula that allows us to turn products of three or more terms into sums of products consisting of just two terms.

To find a formula for $X_mX_nX_s$, we can proceed as follows. From equation (7), we have

$$X_mX_nX_s = (A_1r_1^m + A_2r_2^m + A_3r_3^m)(A_1r_1^n + A_2r_2^n + A_3r_3^n)(A_1r_1^s + A_2r_2^s + A_3r_3^s).$$

After expanding this out, replace any term of the form $r_1^ar_2^br_3^c$ (with $a, b, c > 0$) by $r^sr_1^{a-s}r_2^{b-s}r_3^{c-s}$, which is equivalent because $r_1r_2r_3 = r$. Since one of a, b, c is equal to s, this substitution turns this term into one involving the product of only two powers of the r_i. Use equation (10) to convert powers of r_1, r_2, and r_3 back to expressions involving X, Y, and Z. Then use Waring's algorithm and equations (8) and (11) to replace A_1, A_2, A_3, r_1, r_2, and r_3, by p, q, and r. We get the following (computer generated) result.

$$
\begin{aligned}
X_mX_nX_s = \big[&-c_8X_{m+n}X_s - c_8X_nX_{m+s} - c_8X_mX_{n+s} + c_6X_{m+n+s} - c_7X_{n+s}Y_m\\
&- c_7X_{m+s}Y_n - c_3X_sY_{m+n} - c_7X_{m+n}Y_s - c_6Y_{m+n}Y_s - c_3X_nY_{m+s}\\
&- c_6Y_nY_{m+s} - c_3X_mY_{n+s} - c_6Y_mY_{n+s} - c_5Y_{m+n+s} - c_6X_{n+s}Z_m\\
&+ c_5Y_{n+s}Z_m - c_6X_{m+s}Z_n + c_5Y_{m+s}Z_n - c_2X_sZ_{m+n} + c_5Y_sZ_{m+n}\\
&- c_6X_{m+n}Z_s + c_5Y_{m+n}Z_s + 3c_1Z_{m+n}Z_s - c_2X_nZ_{m+s} + c_5Y_nZ_{m+s}\\
&+ 3c_1Z_nZ_{m+s} - c_2X_mZ_{n+s} + c_5Y_mZ_{n+s} + 3c_1Z_mZ_{n+s}\\
&- 3c_1Z_{m+n+s} - r^s(-2c_8X_{m-s}X_{n-s} + c_9X_{n-s}Y_{m-s}\\
&+ c_9X_{m-s}Y_{n-s} - 2c_6Y_{m-s}Y_{n-s} + 2c_4X_{n-s}Z_{m-s} + 2c_5Y_{n-s}Z_{m-s}\\
&+ 2c_4X_{m-s}Z_{n-s} + 2c_5Y_{m-s}Z_{n-s} + 6c_1Z_{m-s}Z_{n-s})\big]/d^2
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 &= p^2q^2 + 4q^3 - 4p^3r - 18pqr - 27r^2\\
c_2 &= -2p^4q^2 - 13p^2q^3 - 20q^4 + 8p^5r + 56p^3qr + 90pq^2r + 54p^2r^2 + 135qr^2\\
c_3 &= p^3q^3 + 4pq^4 - 4p^4qr - 12p^2q^2r + 24q^3r - 24p^3r^2 - 135pqr^2 - 162r^3\\
c_4 &= p^4q^2 + 6p^2q^3 + 8q^4 - 4p^5r - 27p^3qr - 36pq^2r - 27p^2r^2 - 54qr^2\\
c_5 &= pc_1\\
c_6 &= qc_1\\
c_7 &= -3c_1r\\
c_8 &= -p^2q^4 - 4q^5 + 6p^3q^2r + 26pq^3r - 8p^4r^2 - 36p^2qr^2 + 27q^2r^2 - 54pr^3\\
c_9 &= -p^3q^3 - 4pq^4 + 4p^4qr + 15p^2q^2r - 12q^3r + 12p^3r^2 + 81pqr^2 + 81r^3
\end{aligned}
$$

and

$$d = 27r^2 - p^2q^2 - 4q^3 + 4p^3r + 18pqr.$$

These formulas can be simplified. Using the first formula in display (16), we can remove any terms of the form Y_mY_n. Using the second formula in display (16), we can remove any terms of the form $Y_nZ_m + Y_mZ_n$. Using the third formula in display (16), we can remove any terms of the form Z_mZ_n. Upon doing this, we get the following.

$$
\begin{aligned}
dX_mX_nX_s = {} & 2(q^2 - 3pr)[X_sX_{m+n} + X_nX_{s+m} + X_mX_{n+s} - 2r^sX_{m-s}X_{n-s}]\\
& - 2q[X_{m+n+s} - r^sX_{m+n-2s}] + 2p[Y_{m+n+s} - r^sY_{m+n-2s}]\\
& + 6[Z_{m+n+s} - r^sZ_{m+n-2s}]\\
& - (pq + 9r)[X_sY_{m+n} + X_nY_{s+m} + X_mY_{n+s}\\
& \qquad - r^s(X_{m-s}Y_{n-s} + X_{n-s}Y_{m-s})]\\
& + 2(p^2 + 3q)[X_sZ_{m+n} + X_nZ_{s+m} + X_mZ_{n+s}\\
& \qquad - r^s(X_{m-s}Z_{n-s} + X_{n-s}Z_{m-s})].
\end{aligned}
\tag{43}
$$

This can also be expressed in the following form:

ALGORITHM "TribShortenProducts" TO TURN PRODUCTS OF MANY TERMS INTO PRODUCTS OF TWO TERMS:

$$
\begin{aligned}
X_mX_nX_s = \big[& X_sC_{m+n} + X_nC_{s+m} + X_mC_{n+s}\\
& - r^s(X_{m-s}C_{n-s} + X_{n-s}C_{m-s})\\
& - 2qX_{m+n+s} + 2pY_{m+n+s} + 6Z_{m+n+s}\\
& - r^s(-2qX_{m+n-2s} + 2pY_{m+n-2s} + 6Z_{m+n-2s})\big]/d
\end{aligned}
\tag{44}
$$

where $d = 27r^2 - p^2q^2 - 4q^3 + 4p^3r + 18pqr$ and

$$C_n = 2(q^2 - 3pr)X_n - (pq + 9r)Y_n + 2(p^2 + 3q)Z_n.$$

For products of three terms not all involving X's, first apply algorithm "ConvertToX", formula (27), to change any Y or Z terms to X terms. For products of more than three terms, this procedure can be repeated, three terms at a time, until only products of two terms remain.

Formula (44) is still pretty messy. Can it be simplified? Can it be made to look symmetric under permutations of (m, n, s)?

8. Simson's Formula.

For Fibonacci numbers, there is the well-known Simson Formula, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = -(-1)^{n-1}. \tag{45}$$

The generalization of this to third-order recurrences is

$$\begin{vmatrix} X_{n+2} & X_{n+1} & X_n \\ X_{n+1} & X_n & X_{n-1} \\ X_n & X_{n-1} & X_{n-2} \end{vmatrix} = -r^{n-2} \tag{46}$$

which can be further generalized to

$$\begin{vmatrix} S_{n+4} & S_{n+3} & S_{n+2} \\ S_{n+3} & S_{n+2} & S_{n+1} \\ S_{n+2} & S_{n+1} & S_n \end{vmatrix} = r^n \begin{vmatrix} S_4 & S_3 & S_2 \\ S_3 & S_2 & S_1 \\ S_2 & S_1 & S_0 \end{vmatrix}. \tag{47}$$

These formulas come from [15].

9. Summations.

We can perform indefinite summations of expressions involving X_n, Y_n, and Z_n any time we can perform such summations with a^n instead, since by (7), these terms are actually exponentials with bases r_1, r_2, and r_3.

First, the expression is converted to exponential form using equation (7). Then it is summed. The result is converted back to X's, Y's, and Z's by using equation (10). Then r_1, r_2, and r_3 are converted to p, q, and r using equation (11).

The following summations were found using this method.

$$\sum_{k=1}^{n} x^k X_k = \frac{-x^2 + x^{n+1}(X_{n+1} + xY_{n+1} + x^2 Z_{n+1})}{-1 + px + qx^2 + rx^3} \tag{48}$$

$$\begin{aligned}\sum_{k=0}^{n} X_{ak+b} = [&(Y_{a+b} - Y_{(n+1)a+b})\{rX_a^2 + (pX_a + Y_a)(Z_a - 1)\} \\ &+ (X_{a+b} - X_{(n+1)a+b})\{(Z_a - 1)^2 - rX_aY_a + qX_a(Z_a - 1)\} \\ &+ (Z_{a+b} - Z_{(n+1)a+b})\{(pX_a + Y_a)Y_a - qX_a^2 - X_a(Z_a - 1)\}]\end{aligned}$$

$$
\begin{aligned}
&/[r^2X_a^3 + rY_a^3 + (Z_a-1)^3 - qY_a^2(Z_a-1) \\
&+ X_a^2((q^2-2pr)(Z_a-1) - qrY_a) + pY_a(Z_a-1)^2 \\
&+ X_a((p^2+2q)(Z_a-1)^2 + prY_a^2 - Y_a(pq+3r)(Z_a-1))]
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
\sum_{k=1}^{n} kX_k = [&2-p+r-(n+1)(2r+q+1)X_{n+1} + n(2r+q+1)X_{n+2} \\
&+ (n+1)(p-r-2)Y_{n+1} - n(p-r-2)Y_{n+2} \\
&+ (n+1)(2p+q-3)Z_{n+1} - n(2p+q-3)Z_{n+2}]/(p+q+r-1)^2
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
\sum_{k=1}^{n} k^2X_k = [&(1+3q-pq+7r-3pr+r^2)\{-(n+1)^2X_{n+1} \\
&+ (2n^2+2n-1)X_{n+2} - n^2X_{n+3}\} \\
&+ (3-3p+p^2+q+6r-3pr-qr)\{-(n+1)^2Y_{n+1} \\
&+ (2n^2+2n-1)Y_{n+2} - n^2Y_{n+3}\} \\
&+ (6-8p+3p^2-3q+3pq+q^2+3r-pr)\{-(n+1)^2Z_{n+1} \\
&+ (2n^2+2n-1)Z_{n+2} - n^2Z_{n+3}\}]/(p+q+r-1)^3
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\sum_{k=0}^{n} X_kX_{n-k} = [&-(n+1)prX_n + (9r-npq-3nr)X_{n+1} + q(n-1)X_{n+2} - 3r(n+1)Y_n \\
&+ (np^2-p^2-3q+nq)Y_{n+1} - p(n-1)Y_{n+2} + (n+1)(p^2+4q)Z_n \\
&+ 2npZ_{n+1} - 3(n-1)Z_{n+2}]/(p^2q^2+4q^3-27r^2-4p^3r-18pqr).
\end{aligned} \tag{52}
$$

Most of the above formulas are special cases of formula (5.2) of [22].

10. The Tribonacci Sequence.

The Tribonacci Sequence, $\langle T_n \rangle$, may be defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \tag{53}$$

with initial conditions $T_0 = 0$, $T_1 = 1$, and $T_2 = 1$. A basis can be formed from (T_n, T_{n+1}, T_{n+2}).

For this sequence, we have $T_n = X_{n+1}$ with $p = q = r = 1$. To convert X's, Y's, and Z's to T's, use the identities

$$
\begin{aligned}
X_n &= T_{n+2} - T_{n+1} - T_n \\
Y_n &= 2T_n + T_{n+1} - T_{n+2} \\
Z_n &= 2T_{n+1} - T_{n+2}.
\end{aligned} \tag{54}
$$

The reduction formulas are:

$$
\begin{aligned}
T_{n+m} = &T_n(2T_{m+1} - T_{m+2}) + T_{n+1}(2T_m + T_{m+1} - T_{m+2}) \\
&- T_{n+2}(T_m + T_{m+1} - T_{m+2})
\end{aligned} \tag{55}
$$

and

$$T_{n-m} = T_n(T_{m+1}^2 - T_mT_{m+2}) + T_{n+1}(T_{m+2}^2 - T_mT_{m+1} - T_{m+2}T_m - T_{m+2}T_{m+1}) \\ + T_{n+2}(T_m^2 + T_mT_{m+1} + T_{m+1}^2 - T_{m+1}T_{m+2}). \tag{56}$$

A form of the addition formula was first found by Agronomof in 1914 [1].

The double argument formula is

$$T_{2n} = T_{n+2}^2 + T_{n+1}^2 + 4T_nT_{n+1} - 2T_nT_{n+2} - 2T_{n+1}T_{n+2}. \tag{57}$$

A form of this can also be found in [1].

The negation formula is

$$T_{-n} = T_{n+2}^2 + T_{n+1}^2 + T_n^2 - T_{n+2}(2T_{n+1} + T_n). \tag{58}$$

The fundamental identity connecting T_n, T_{n+1}, and T_{n+2} is

$$T_n^3 + 2T_{n+1}^3 + T_{n+2}^3 + 2T_nT_{n+1}(T_n + T_{n+1}) + T_nT_{n+2}(T_n - T_{n+2} - 2T_{n+1}) - 2T_{n+1}T_{n+2}^2 = 1. \tag{59}$$

The formula to expand squares is

$$T_n^2 = (5T_{2n+2} - 3T_{2n+1} - 4T_{2n} + 4T_{-n} + 10T_{-n-1} - 2T_{-n-2})/22. \tag{60}$$

The analog of Simson's formula is

$$\begin{vmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \end{vmatrix} = -1 \tag{61}$$

which was found by Miles [9] along with generalizations to higher order recurrences.

Miles [9] also generalized the relationship between Fibonacci numbers and binomial coefficients from Pascal's triangle,

$$F_{n+1} = \sum_{a+2b=n} \binom{a+b}{a},$$

to the following formula which relates Tribonacci numbers and trinomial coefficients from Pascal's pyramid:

$$T_{n+1} = \sum_{a+2b+3c=n} \binom{a+b+c}{a \quad b \quad c}. \tag{62}$$

The following summation was found using the methods of Section 9.

$$\sum_{k=1}^{n} T_k^2 = [1 + 4T_nT_{n+1} - (T_{n+1} - T_{n-1})^2]/4. \tag{63}$$

Appendix 1: Selected Identities.

We present below some selected identities culled from the literature. All these identities were successfully checked by algorithm “TribSimplify”. Recall that W_n is defined by equation (9).

The following six identities come from Jarden [7]:

$$S_{n+m} = rX_mS_{n-1} + X_{m+1}(S_{n+1} - pS_n) + X_{m+2}S_n$$

$$X_{2n} = (2rX_{n-1} + qX_n)X_n + X_{n+1}^2$$

$$X_{2n+1} = rX_n^2 + (2X_{n+2} - pX_{n+1})X_{n+1}$$

$$X_{2n} = X_nW_n + r^nX_{-n}$$

$$W_{2n} = W_n^2 - 2r^nW_{-n}$$

$$X_{2n+1} = X_{n+1}W_n + r^nX_{1-n}$$

The following three identities come from Yalavigi [21]:

$$2W_{3n} = W_n(3W_{2n} - W_n^2) + 6r^n$$

$$W_{4n} = W_nW_{3n} - W_{2n}(W_n^2 - W_{2n})/2 + r^nW_n$$

$$W_{4n+4m} - W_{4n} = W_{n+m}W_{3n+3m} - W_nW_{3n} - W_{2n+2m}(W_{n+m}^2 - 2W_{2n+2m})/2$$
$$+W_{2n}(W_n^2 - 2W_{2n})/2 + r^n(W_{n+m} - W_n)$$

The following three identities come from Yalavigi [20]:

$$S_{m+n} = X_{m+2}S_n + Y_{m+2}S_{n-1} + Z_{m+2}S_{n-2}$$

$$S_{2n} = X_{n+2}S_n + Y_{n+2}S_{n-1} + Z_{n+2}S_{n-2}$$

$$S_{m+n} = X_{m+h+2}S_{n-h} + Y_{m+h+2}S_{n-h-1} + Z_{m+h+2}S_{n-h-2}$$

The following identity comes from Shannon and Horadam [15]:

$$Y_n = qX_{n-1} + rX_{n-2}$$

The following ten identities come from Carlitz [4]: Both ρ_n and σ_n satisfy third-order linear recurrences with $r = 1$ and the same p and q with initial conditions $\rho_0 = 1$, $\rho_1 = \rho_2 = 0$, $\sigma_0 = 3$, $\sigma_1 = p$, $\sigma_2 = p^2 + 2q$. In particular, with $r = 1$, we have $\sigma_n = W_n$ and $\rho_n = Z_n$.

$$2\rho_m\rho_n - \rho_{m+1}\rho_{n-1} - \rho_{m-1}\rho_{n+1} = \sigma_{m-3}\sigma_{n-3} - \sigma_{m+n-6} - \sigma_{m-3}\rho_{n-3} - \sigma_{n-3}\rho_{m-3} + 2\rho_{m+n-6}$$

$$\sigma_{m+3n} - \sigma_{m+2n}\sigma_n + \sigma_{m+n}\sigma_{-n} - \sigma_m = 0$$

$$\sigma_{2n} = \sigma_n^2 - 2\sigma_{-n}$$

$$\sigma_{3n} = \sigma_n^3 - 3\sigma_n\sigma_{-n} + 3$$

$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{3-n}$$

$$\rho_n^2 - \rho_{n+1}\rho_{n-1} = \rho_{2n-6} - \rho_{n-3}\sigma_{n-3} + \sigma_{3-n}$$

$$\rho_m\sigma_n = \rho_{m+n} + \rho_{m-n}\sigma_{-n} - \rho_{m-2n}$$

$$\sigma_m\sigma_n = \sigma_{m+n} + \sigma_{m-n}\sigma_{-n} - \sigma_{m-2n}$$

$$\rho_{2n} = \rho_n\sigma_n - \sigma_{-n} + \rho_{-n}$$

$$\rho_{3n} = \rho_n\sigma_n^2 - \sigma_n\sigma_{-n} + \rho_{-n}\sigma_n - \rho_n\sigma_{-n} + 1$$

The following nine identities come from Waddill [17]: In their notation, we have $U_n = X_{n+1}$.

$$S_{n+m} = U_{n-k}S_{m+k+1} + Y_{n-k+1}S_{m+k} + rU_{n-k-1}S_{m+k-1}$$

$$S_{n+m} = U_{m-k}S_{n+k+1} + Y_{m-k+1}S_{n+k} + rU_{m-k-1}S_{n+k-1}$$

$$S_n^2 + qS_{n-1}^2 + 2rS_{n-1}S_{n-2} = S_2S_{2n-2} + (qS_1 + rS_0)S_{2n-3} + rS_1S_{2n-4}$$

$$U_{2n-1} = U_n^2 + qU_{n-1}^2 + 2rU_{n-1}U_{n-2}$$

$$U_{2n-1} = U_{n+1}U_{n-1} + rU_{n-1}U_{n-2} + U_n^2 - pU_nU_{n-1}$$

$$qU_{2n-1} = U_{n+1}^2 - pU_{n+1}U_n + (r - pq)U_nU_{n-1} + qU_n^2 - pr(U_nU_{n-2} + U_{n-1}^2) - qrU_{n-1}U_{n-2} - r^2(U_{n-1}U_{n-3} + U_{n-2}^2)$$

$$U_{3n-1} = U_{n-1}(U_{n+1}^2 + Y_{n+2}U_n + rU_{n-1}U_n) + Y_n(U_nU_{n+1} + Y_{n+1}U_n + rU_{n-1}^2) + rU_{n-2}(U_{n-1}U_{n+1} + Y_nU_n + rU_{n-2}U_{n-1})$$

$$\begin{vmatrix} S_{n+m+h} & S_{n+j+h} & S_{n+h} \\ S_{n+m+k} & S_{n+j+k} & S_{n+k} \\ S_{n+m} & S_{n+j} & S_n \end{vmatrix} = r^n \begin{vmatrix} U_{h-1} & U_h \\ U_{k-1} & U_k \end{vmatrix} \cdot \begin{vmatrix} S_{m+2} & S_{m+1} & S_m \\ S_{j+2} & S_{j+1} & S_j \\ S_2 & S_1 & S_0 \end{vmatrix}$$

$$\begin{vmatrix} S_{5n} & S_{4n} & S_{3n} \\ S_{4n} & S_{3n} & S_{2n} \\ S_{3n} & S_{2n} & S_n \end{vmatrix} = r^n \begin{vmatrix} U_{2n-1} & U_{2n} \\ U_{n-1} & U_n \end{vmatrix} \cdot \begin{vmatrix} S_{2n+2} & S_{2n+1} & S_{2n} \\ S_{n+2} & S_{n+1} & S_n \\ S_2 & S_1 & S_0 \end{vmatrix}$$

The following five identities were found by Zeitlin [23]:

$$S_{n+6}^2 = (p^2 + q)S_{n+5}^2 + (q^2 + qp^2 + rp)S_{n+4}^2 + (2r^2 + rp^3 + 4pqr - q^3)S_{n+3}^2 + (r^2p^2 - rpq^2 - r^2q)S_{n+2}^2 + (r^2q^2 - r^3p)S_{n+1}^2 - r^4S_n^2$$

$$S_{2n+6} - (p^2 + 2q)S_{2n+4} + (q^2 - 2rp)S_{2n+2} - r^2S_{2n} = 0$$

$$r^nS_{-n} = S_0(W_n^2 - W_{2n})/2 - W_nS_n + S_{2n}$$

$$(n-1)X_{n+1} = p\sum_{j=0}^{n+2} X_j X_{n+2-j} + 2q\sum_{j=0}^{n+1} X_j X_{n+1-j} + 3r\sum_{j=0}^{n} X_j X_{n-j}$$

$$\sum_{k=0}^{n} X_k X_{n-k} = \frac{(9r+pq)(n-1)X_{n+1} - (6q+2p^2)nY_{n+1} + (4q^2-3pr+p^2q)(n+1)X_n}{27r^2 - p^2q^2 - 4q^3 + 4p^3r + 18pqr}$$

See [19] for other identities.

Appendix 2: Selected Tribonacci Identities.

We present below selected identities from the literature in which $p = q = r = 1$. All these identities were successfully checked by algorithm "TribSimplify".

The following three identities come from Agronomof [1]:

$$T_{n+m} = T_{m+1}T_n + (T_m + T_{m-1})T_{n-1} + T_m T_{n-2}$$

$$T_{2n} = T_{n-1}^2 + T_n(T_{n+1} + T_{n-1} + T_{n-2})$$

$$T_{2n-1} = T_n^2 + T_{n-1}(T_{n-1} + 2T_{n-2})$$

The following three identities come from Lin [8]: In their notation, we have $U_n = Y_{n+2}$, with $p = q = r = 1$.

$$U_{4n+1}U_{4n+3} + U_{4n+2}U_{4n+4} = T_{4n+4}^2 - T_{4n+2}^2$$

$$U_{n+1}^2 + U_{n-1}^2 = 2(T_n^2 + T_{n+1}^2)$$

$$T_{n+1}^2 - T_n^2 = U_{n+1}U_{n-1}$$

The following five identities were found by Zeitlin [23]:

$$\begin{aligned} T_{n+6+a}T_{n+6+b} = {} & 2T_{n+5+a}T_{n+5+b} + 3T_{n+4+a}T_{n+4+b} + 6T_{n+3+a}T_{n+3+b} \\ & - T_{n+2+a}T_{n+2+b} - T_{n+a}T_{n+b} \end{aligned}$$

$$\begin{aligned} -(1-2x-3x^2-6x^3+x^4+x^6)\sum_{k=0}^{n} T_k^2 x^k = {} & T_{n+1}^2 x^{n+1} + (T_{n+2}^2 - 2T_{n+1}^2)x^{n+2} \\ & + (T_{n+3}^2 - 2T_{n+2}^2 - 3T_{n+1}^2)x^{n+3} \\ & + (T_{n+4}^2 - 2T_{n+3}^2 - 3T_{n+2}^2 - 6T_{n+1}^2)x^{n+4} \\ & - T_{n-1}^2 x^{n+5} - T_n^2 x^{n+6} - x + x^2 + x^3 + x^4 \end{aligned}$$

$$8\sum_{k=0}^{n} T_k^2 = T_{n+5}^2 - T_{n+4}^2 - 4T_{n+3}^2 - 10T_{n+2}^2 - 9T_{n+1}^2 - T_n^2 + 2$$

$$T_{-n} = -W_n T_n + T_{2n}$$

$$22\sum_{j=0}^{n-2} T_j T_{n-2-j} = 5(n-1)T_n - 2(n-1)T_{n-1} - 4nT_{n-2}$$

The following two identities come from Shannon and Horadam [14]:

$$(S_n S_{n+4})^2 + (2(S_{n+1} + S_{n+2})S_{n+3})^2 = (S_n^2 + 2(S_{n+1} + S_{n+2})S_{n+3})^2$$

$$4(S_{n+2}S_{n-1} - S_{n+2}^2) = S_{n-1}^2 - S_{n+3}^2$$

The following eleven identities come from Waddill and Sacks [16]: In their notation, we have $K_n = X_{n+1}$, $L_n = Y_{n+1}$, and $R_n = S_{n-1} + S_{n-2}$, with $p = q = r = 1$.

$$L_n = K_{n-1} + K_{n-2}$$

$$S_{n+h} = K_{h+1}S_n + L_{h+1}S_{n-1} + K_h S_{n-2}$$

$$S_{2n} = K_{n+1}S_n + L_{n+1}S_{n-1} + K_n S_{n-2}$$

$$S_{2n-1} = K_n S_n + (K_{n-1} + K_{n-2})S_{n-1} + K_{n-1}S_{n-2}$$

$$S_{n+h} = K_{h+m+1}S_{n-m} + L_{h+m+1}S_{n-m-1} + K_{h+m}S_{n-m-2}$$

$$S_n^2 + S_{n-1}^2 + 2S_{n-1}S_{n-2} = S_2 S_{2n-2} + R_2 S_{2n-3} + S_1 S_{2n-4}$$

$$\begin{vmatrix} S_n & S_{n+h} & S_{n+h+k} \\ S_{n+t} & S_{n+h+t} & S_{n+h+k+t} \\ S_{n+m} & S_{n+h+m} & S_{n+h+k+m} \end{vmatrix} = \begin{vmatrix} K_h & K_{h+k} \\ L_{h+1} & L_{h+k+1} \end{vmatrix} \cdot \begin{vmatrix} S_t & S_{t+1} & S_{t+2} \\ S_m & S_{m+1} & S_{m+2} \\ S_0 & S_1 & S_2 \end{vmatrix}$$

$$\begin{vmatrix} K_n & K_{n+h} & K_{n+h+k} \\ K_{n+t} & K_{n+h+t} & K_{n+h+k+t} \\ K_{n+m} & K_{n+h+m} & K_{n+h+k+m} \end{vmatrix} = \begin{vmatrix} K_h & K_{h-1} \\ K_{h+k} & K_{h+k-1} \end{vmatrix} \cdot \begin{vmatrix} K_m & K_t \\ K_{m-1} & K_{t-1} \end{vmatrix}$$

$$\begin{vmatrix} K_{n+1} & K_n & K_{n+h} \\ K_{n+h+1} & K_{n+h} & K_{n+2h} \\ K_{n+2h+1} & K_{n+2h} & K_{n+3h} \end{vmatrix} = K_{h-1} \cdot \begin{vmatrix} K_h & K_{h-1} \\ K_{2h} & K_{2h-1} \end{vmatrix}$$

$$\begin{vmatrix} K_n & K_{n+h} & K_{n+m} \\ K_{n+h} & K_{n+2h} & K_{n+h+m} \\ K_{n+m} & K_{n+h+m} & K_{n+2m} \end{vmatrix} = -\begin{vmatrix} K_h & K_m \\ K_{h-1} & K_{m-1} \end{vmatrix}^2$$

$$\begin{vmatrix} S_{n+h+k+t} & S_{n+h+k} & S_{n+h+k+m} \\ R_{n+h+t} & R_{n+h} & R_{n+h+m} \\ S_{n+t} & S_n & S_{n+m} \end{vmatrix} = \begin{vmatrix} K_{h+k-1} & K_{h+k} \\ L_{h-1} & L_h \end{vmatrix} \cdot \begin{vmatrix} S_t & S_{t+1} & S_{t+2} \\ S_m & S_{m+1} & S_{m+2} \\ S_0 & S_1 & S_2 \end{vmatrix}$$

Errata.

Computer verification of the various identities encountered in the references consulted during this research revealed a number of typographical errors in the literature. We list the corrections below to set the record straight.

In [4], equation (1.15) should be the same as equation (4.1). Also, equation (1.16) should be the same as equation (3.14).

In [10], equation (2.1) should read "$J_{n+1} = PJ_n + K_n$".

In [13], in equation (1.4), "$t_2 = P^2 + Q$" should be "$t_2 = P^2 + 2Q$". Equation (2.2) should read "$t_n = Ps_{n-1} + 2Qs_{n-2} + 3Rs_{n-3}$".

In [16], the last term of equation (21) should be "$K_{h+k}P_{n-2}$", not "$K_{n+k}P_{n-1}$". Also, the final subscript in equation (41) should be "$h-1$", not "$n-1$". In equation (42), "P_{n+h+m}" should be "R_{n+h+m}" and "K_{n+k}" should be "K_{h+k}".

Acknowledgment.

I would like to thank Paul S. Bruckman, A. F. Horadam, Murray S. Klamkin, Tony Shannon, Lawrence Somer, and David Zeitlin for their fruitful discussions and advice. I also gratefully acknowledge the suggestions made by an anonymous referee.

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AMS Classification Numbers: 11Y16, 11B37