THE OCTAGON ANOMALY

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A lattice point in the plane is a point with integer coordinates. A lattice polygon is a simple polygon whose vertices are all lattice points.

In 1980, Arkinstall [1] proved the following result:

The Lattice Pentagon Theorem. A convex lattice pentagon must contain a lattice point in its interior.

In 1986, Rabinowitz [2] found similar results for other polygons. For example, he showed that a lattice heptagon and a lattice octagon must each contain at least 4 interior lattice points.

For a given n, there is some largest integer $g(n)$ such that all convex lattice n-gons must contain at least $g(n)$ interior lattice points. One might think that there would exist a lattice n-gon with p interior lattice points for all $p \geq g(n)$. However, a computer search [2] revealed that although a convex lattice octagon can have 4 interior lattice points, it can not have exactly 5 interior lattice points.

The Octagon Anomaly. A convex lattice octagon can have 4 interior lattice points or 6 interior lattice points, but it can not have exactly 5 interior lattice points. (See figure 1 in which the dots represent lattice points and the circles represent the vertices of the polygon.)

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. o o .          . o o . .
o . . o          o . . . o
o . . o          o . . . o
. o o .          . o o . .
```

Figure 1
Lattice octagons with 4 and 6
interior lattice points

The first non-computer proof of this result was given by Steinberg [4] in 1988. She was a high school student at the time and came up with the proof as part of a Westinghouse project. We give a simplified version of her proof below and then close with an open problem.

We start with a definition that will simplify the discussion.

Definition. Let K be a convex polygon with edge AB. Then $h(AB)$ denotes the open halfplane bounded by AB that is exterior to K.

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Lemma. Let K be a convex lattice polygon and let H be the convex hull of the lattice points inside K. Let AB be an edge of H. Then $h(AB)$ contains at most two vertices of K.

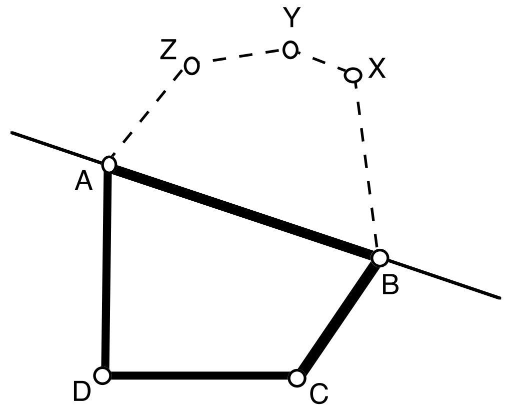

figure 2

Proof. Suppose this open halfplane contains 3 vertices of K, say X, Y, and Z (see figure 2). Consider the five points: A, B, X, Y, and Z. The point X can not be in the convex hull of the other four points because then X would be an interior point of K and not a vertex of K. A similar argument holds for Y and Z. Thus $ABXYZ$ is a convex lattice pentagon with no interior lattice points contradicting the Lattice Pentagon Theorem. This proves the Lemma.

Theorem. A convex lattice octagon can not contain exactly 5 interior lattice points.

Proof. Suppose we had a convex lattice octagon, K, that contained precisely 5 interior lattice points, A, B, C, D, and E. Let H denote the convex hull of these 5 points.

<u>Case 1</u>: Region H forms a straight line.

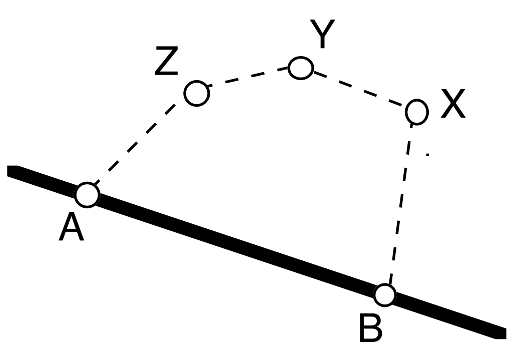

figure 3

Of the 8 vertices of K, at most 2 of them can lie on H. Of the remaining 6 vertices at least 3 of them, say X, Y, and Z, must lie in one open halfplane bounded by H. But then $ABXYZ$ is a convex lattice pentagon with no interior lattice points contradicting the Lattice Pentagon Theorem.

Case 2: Region H forms a triangle, say $\triangle ABC$.

By the Lemma, $h(AB)$, $h(BC)$, and $h(CA)$ each can contain at most 2 of the vertices of K. Thus the exterior of H contains at most 6 of the vertices of K. This is a contradiction since all 8 of the vertices of K must lie in the exterior of H.

Case 3: Region H forms a quadrilateral, say $ABCD$.

Lattice point E can lie inside this quadrilateral or on one of the edges (say CD). Of the four halfplanes, $h(AB)$, $h(BC)$, $h(CD)$, $h(DA)$, each one can contain at most two of the vertices of K. Since K has exactly 8 vertices, we conclude that each halfplane contains exactly two vertices of K and the intersections of two halfplanes contain no vertices of K. Let X and Y be the two vertices of K in $h(AB)$ but not in $h(DA)$ or $h(BC)$ (see figure 4). (Points X and Y might lie on CB or DA extended.) Then $AEBXY$ is a convex lattice pentagon containing no interior lattice points, contradicting the Lattice Pentagon Theorem.

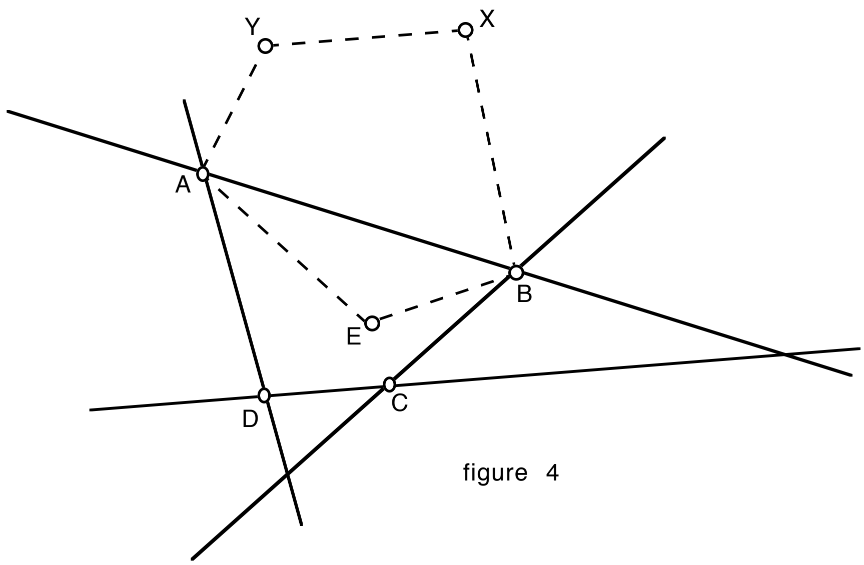

figure 4

Case 4: Region H forms a pentagon. Then H is a convex lattice pentagon containing no interior lattice points, contradicting the Lattice Pentagon Theorem.

Thus our assumption that the lattice octagon K has exactly 5 interior lattice points has been proven to be incorrect. Thus the Octagon Anomaly has been proven.

An open problem.

We leave the reader with an open problem about 9-gons from [3]. We suspect that this conjecture can be proved using the Lattice Pentagon Theorem and simple combinatorial arguments, however, the reader may wish to consult [3] for additional tools.

Conjecture (The Nonagon Anomaly). A convex lattice nonagon can have 7 interior lattice points or 10 interior lattice points, but it can not have either 8 or 9 interior lattice points.

```
. . o o .        . o o . . .
. o . . o        o . . . o .
. . . . o        . . . . . o
o . . o .        o . . . o .
o o . . .        . o o . . .
```

Figure 5
Lattice nonagons with 7 and 10
interior lattice points

References

[1] J. R. Arkinstall, "Minimal Requirements for Minkowski's Theorem in the Plane I", *Bulletin of the Australian Mathematical Society*. **22**(1980)259–274.

[2] Stanley Rabinowitz, *Convex Lattice Polytopes, Ph. D. Dissertation*. Polytechnic University. Brooklyn, NY: 1986.

[3] Stanley Rabinowitz, "On the Number of Lattice Points Inside a Convex Lattice n-gon", *Congressus Numerantium*. **73**(1990)99–124.

[4] Nicole Steinberg, *The Octagon Anomaly*. Westinghouse Report (unpublished). Brooklyn, NY: 1988.