Average Chord Lengths in a Triangle

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Abstract

Let P be a point inside a triangle T. We consider the average length of the chords of T through P, where the direction of the chord is chosen uniformly. An elementary formula is obtained in terms of the distances from P to the sides and vertices of the triangle. Several classical triangle centers give especially simple specializations. For example, if I is the incenter, then

$$M_T(I) = \frac{2r}{\pi} \log\left(\cot\frac{A}{4}\cot\frac{B}{4}\cot\frac{C}{4}\right).$$

Our main result is the sharp inequality

$$M_T(P) \le \frac{p}{\pi\sqrt{3}} \log(2+\sqrt{3}),$$

valid for every interior point P of every triangle of perimeter p. Equality holds only for an equilateral triangle, with P its center. The proof is an elementary symmetrization argument. We close with brief remarks relating the problem to the radial center of a convex body, the electrostatic potential center of a triangle, and dual quermassintegrals.

1 Introduction

Fix a point P inside a triangle T. Through P draw a line making an angle θ with a fixed direction, and let $\ell_P(\theta)$ be the length of the chord cut from T by this line. Since an unoriented line has period π, it is natural to define the average chord length through P by

$$M_T(P) = \frac{1}{\pi}\int_0^\pi \ell_P(\theta)\, d\theta.$$

The question is elementary to state: how does $M_T(P)$ depend on the triangle and on the location of P?

The corresponding integral has appeared in several other settings. In electrostatics it is the three-dimensional Coulomb (or Newtonian) potential of a uniformly charged triangular lamina, evaluated at a point of the lamina:

$$\int_T \frac{dA(X)}{|X-P|}.$$

Explicit potential formulas for polygons, and in particular for triangles, have long been known; see Duffin and McWhirter [3] and, for recent formulas emphasizing the centroid and incenter of a triangle, Böhm and Runge [2]. Our purpose here is different. We keep the geometric interpretation in terms of average chords, derive the needed formulas directly by elementary trigonometry and integration, and study their consequences for familiar triangle centers.

A first example already gives a pleasantly compact answer.

2 A motivating example: the incenter

Let I be the incenter of ABC, and let r be the inradius.

Proposition 2.1. *The average chord length through the incenter is*

$$\boxed{M_T(I) = \frac{2r}{\pi} \log\left(\cot\frac{A}{4}\cot\frac{B}{4}\cot\frac{C}{4}\right).}$$

Proof. Consider the rays from I which meet the side BC. The perpendicular distance from I to BC is r. If a ray makes angle ϕ with the perpendicular to BC, then its length from I to BC is $r\sec\phi$.

At B and C the angles of triangle IBC are $B/2$ and $C/2$. Consequently the total contribution of the rays meeting BC to the integral of the radial distance is

$$r\left(\log\cot\frac{B}{4} + \log\cot\frac{C}{4}\right).$$

Adding the analogous contributions from the other two sides, each of the three logarithms occurs twice. Dividing by π gives the formula. ☐

For an equilateral triangle of side a this becomes

$$M_T(I) = \frac{\sqrt{3}\,a}{\pi}\log(2+\sqrt{3}).$$

The incenter and centroid coincide in this case.

For a 3–4–5 triangle, $s = 6$ and $r = 1$, so the same formula may be written

$$M_T(I) = \frac{2}{\pi}(\operatorname{arsinh} 3 + \operatorname{arsinh} 2 + \operatorname{arsinh} 1) \approx 2.63781.$$

Thus even for a very familiar triangle the average chord length is not an especially obvious classical length.

An equivalent formula for the associated potential integral at the incenter can be extracted from the formulas of Böhm and Runge [2]. The derivation above is included because it shows directly why the quarter-angles occur.

3 A formula for an arbitrary interior point

Let $\rho_P(\theta)$ denote the distance from P to the boundary of T along the ray in direction θ.

Lemma 3.1. *For every interior point P,*

$$\boxed{M_T(P) = \frac{1}{\pi}\int_0^{2\pi} \rho_P(\theta)\,d\theta.}$$

Moreover,

$$\boxed{\pi M_T(P) = \int_T \frac{dA(X)}{|X-P|}.}$$

Proof. The chord through P in direction θ has length

$$\ell_P(\theta) = \rho_P(\theta) + \rho_P(\theta + \pi).$$

Therefore

$$\int_0^{\pi} \ell_P(\theta)\, d\theta = \int_0^{2\pi} \rho_P(\theta)\, d\theta.$$

For the second identity, use polar coordinates centered at P:

$$\int_T \frac{dA(X)}{|X - P|} = \int_0^{2\pi} \int_0^{\rho_P(\theta)} \frac{1}{r}\, r\, dr\, d\theta = \int_0^{2\pi} \rho_P(\theta)\, d\theta.$$

□

Write

$$d_a = \operatorname{dist}(P, BC), \qquad d_b = \operatorname{dist}(P, CA), \qquad d_c = \operatorname{dist}(P, AB).$$

Theorem 3.2. *For every interior point P,*

$$\boxed{M_T(P) = \frac{1}{\pi} \sum_{\text{cyc}} d_a \log \frac{PB + PC + a}{PB + PC - a}.}$$

Equivalently,

$$M_T(P) = \frac{2}{\pi} \sum_{\text{cyc}} d_a \operatorname{artanh} \frac{a}{PB + PC}.$$

Proof. Consider again the rays from P which meet BC. Put

$$\beta = \angle PBC, \qquad \gamma = \angle PCB.$$

Integrating $d_a \sec\phi$ over the angular interval from the ray PB to the ray PC gives

$$d_a \left(\log\cot\frac{\beta}{2} + \log\cot\frac{\gamma}{2} \right).$$

Let

$$\sigma = \frac{a + PB + PC}{2}$$

be the semiperimeter of triangle PBC. The standard half-angle formulas give

$$\cot\frac{\beta}{2}\cot\frac{\gamma}{2} = \frac{\sigma}{\sigma - a} = \frac{PB + PC + a}{PB + PC - a}.$$

Thus the contribution from the rays meeting BC is

$$d_a \log \frac{PB + PC + a}{PB + PC - a}.$$

Summing over the three sides and applying Lemma 3.1 proves the first formula. The second follows from

$$\log \frac{1 + t}{1 - t} = 2 \operatorname{artanh} t.$$

□

4 Some classical triangle centers

Theorem 3.2 can be specialized to any interior triangle center. We record several cases which simplify naturally.

Let G be the centroid, K the symmedian point, N the Nagel point, G_e the Gergonne point, and M the Mittenpunkt. As usual, let m_a, m_b, m_c denote the medians and h_a, h_b, h_c the altitudes.

4.1 The centroid

Since

$$GB = \frac{2}{3}m_b, \qquad GC = \frac{2}{3}m_c, \qquad d_a(G) = \frac{h_a}{3},$$

Theorem 3.2 gives the following compact expression.

Proposition 4.1.

$$\boxed{M_T(G) = \frac{1}{3\pi}\sum_{\text{cyc}} h_a \log\frac{2m_b + 2m_c + 3a}{2m_b + 2m_c - 3a}.}$$

4.2 The symmedian point

Put

$$S_2 = a^2 + b^2 + c^2.$$

The barycentric coordinates of K are

$$(a^2 : b^2 : c^2).$$

Hence

$$d_a(K) = \frac{2\Delta\, a}{S_2}.$$

A direct vector calculation also gives

$$KA = \frac{2bc\, m_a}{S_2},$$

and cyclically.

Proposition 4.2.

$$\boxed{M_T(K) = \frac{2\Delta}{\pi S_2}\sum_{\text{cyc}} a \log\frac{S_2 + 2(cm_b + bm_c)}{2(cm_b + bm_c) - S_2}.}$$

The denominator in each logarithm is positive. For example,

$$2(cm_b + bm_c) > S_2,$$

which is just the triangle inequality $KB + KC > a$ after multiplying by S_2/a.

4.3 Nagel, Gergonne, and Mittenpunkt

It is convenient to put

$$x = s - a, \qquad y = s - b, \qquad z = s - c.$$

The following table lists barycentric coordinates and the distance to BC. The other sideline distances are obtained cyclically.

Point	Barycentric coordinates	d_a
Incenter I	$(a : b : c)$	r
Centroid G	$(1 : 1 : 1)$	$h_a/3$
Symmedian point K	$(a^2 : b^2 : c^2)$	$\dfrac{2\Delta a}{a^2 + b^2 + c^2}$
Nagel point N	$(x : y : z)$	$\dfrac{2rx}{a}$
Gergonne point G_e	$(yz : zx : xy)$	$\dfrac{2s\,yz}{a(4R + r)}$
Mittenpunkt M	$(ax : by : cz)$	$\dfrac{sx}{4R + r}$

Thus Theorem 3.2 immediately gives explicit formulas for the average chord through each of these points. For example, the Mittenpunkt satisfies

$$\boxed{M_T(M) = \frac{s}{\pi(4R + r)} \sum_{\text{cyc}} (s - a) \log \frac{MB + MC + a}{MB + MC - a}.}$$

The table is often more useful than expanding the remaining vertex distances into radicals.

4.4 A 3–4–5 example

For a concrete comparison, the following are the average chord lengths through these six familiar centers in a 3–4–5 triangle. The values are obtained directly from Theorem 3.2.

Point	Average chord length
Centroid G	2.63841
Incenter I	2.63781
Symmedian point K	2.57012
Mittenpunkt M	2.55819
Gergonne point G_e	2.55035
Nagel point N	2.42950

The centroid and incenter values are remarkably close, although the two points are distinct. By comparison, the sharp bound proved in the next section gives $2.90431\ldots$ for an equilateral triangle with the same perimeter 12.

5 A sharp bound for every interior point

We now prove the main result. Remarkably, the same sharp bound holds for every interior point, not merely for a distinguished center.

Theorem 5.1. *Let T be a triangle of perimeter p, and let P be any interior point of T. Then*

$$\boxed{M_T(P) \le \frac{p}{\pi\sqrt{3}} \log(2+\sqrt{3}).}$$

Equality holds if and only if T is equilateral and P is its center.

The proof uses only an elementary form of Steiner symmetrization. We include the details.

Lemma 5.2. *Fix a side BC of a triangle T and a point $P \in T$. Replace each segment of T parallel to BC by a segment of the same length whose midpoint lies on the line through P perpendicular to BC. The resulting set T^* is an isosceles triangle with the same base length and altitude as T, and $P \in T^*$. Moreover,*

$$\int_{T^*} \frac{dA(X)}{|X-P|} \ge \int_T \frac{dA(X)}{|X-P|},$$

and

$$p(T^*) \le p(T).$$

Equality in the first inequality holds only when T is already symmetric about the line through P perpendicular to BC.

Proof. Take BC horizontal and write $P=(0,t)$. At height y, the horizontal section of T is an interval I_y of some length $\ell(y)$. The corresponding section of T^* is the centered interval

$$I_y^* = \left[-\frac{\ell(y)}{2}, \frac{\ell(y)}{2}\right].$$

Since the section at the height of P is centered at P, we have $P \in T^*$. The section lengths are unchanged, so T^* has the same base length and altitude as T.

For fixed $y \ne t$, the function

$$x \longmapsto \frac{1}{\sqrt{x^2+(y-t)^2}}$$

is even and strictly decreasing as $|x|$ increases. Hence its integral over an interval of prescribed length is largest when the interval is centered at the origin. Integrating over y gives the first inequality. The exceptional level $y=t$ has measure zero. If P lies on the boundary, the same conclusion follows by approximation from interior points.

If equality holds, then for almost every y the midpoint of I_y must lie on $x=0$. The midpoints of the parallel sections of a triangle vary affinely with y, so the original triangle must itself be symmetric about $x=0$.

For the perimeter assertion, let $a=BC$ and let h be the altitude from the opposite vertex. If δ is the horizontal displacement of that vertex from the midpoint of BC, the sum of the two non-base sides is

$$f(\delta) = \sqrt{h^2+\left(\delta+\frac{a}{2}\right)^2} + \sqrt{h^2+\left(\delta-\frac{a}{2}\right)^2}.$$

The function f is even and strictly convex, and therefore has its minimum at $\delta=0$. The symmetrized triangle has $\delta=0$, which proves $p(T^*) \le p(T)$, with equality if and only if $AB=AC$. Notice that this perimeter comparison depends only on the triangle; it is independent of the location of P. $\square$

We also need a simple compactness observation.

Lemma 5.3. *For every triangle T of area Δ and every point $P \in T$,*

$$\int_T \frac{dA(X)}{|X-P|} \le 2\sqrt{\pi\Delta}.$$

Proof. By Lemma 3.1,

$$V := \int_T \frac{dA(X)}{|X-P|} = \int_0^{2\pi} \rho_P(\theta)\, d\theta,$$

while

$$2\Delta = \int_0^{2\pi} \rho_P(\theta)^2\, d\theta.$$

Cauchy–Schwarz therefore gives

$$V^2 \le 2\pi \int_0^{2\pi} \rho_P(\theta)^2\, d\theta = 4\pi\Delta.$$

□

Proof of Theorem 5.1. Because both perimeter and $M_T(P)$ scale linearly, fix the perimeter p. It is convenient here to allow P temporarily to lie on the boundary of T; the integral

$$V(T,P) = \int_T \frac{dA(X)}{|X-P|}$$

is still finite and depends continuously on (T,P).

A maximizing pair (T,P) exists. Indeed, after translating P to the origin, all vertices lie in a fixed bounded disk. A maximizing sequence therefore has a convergent subsequence. A degenerate limiting triangle has area tending to zero, and Lemma 5.3 then forces $V(T,P) \to 0$, so a maximizer cannot be degenerate. Thus the maximum is attained on a nondegenerate triangle. The argument below will in fact force the maximizing point to be the interior center of an equilateral triangle.

Choose any side BC and form the triangle T^* of Lemma 5.2. Then

$$V(T^*,P) \ge V(T,P), \qquad p(T^*) \le p.$$

Set

$$\lambda = \frac{p}{p(T^*)} \ge 1$$

and enlarge T^* by a homothety centered at P with factor λ. Since

$$V(\lambda T^*,P) = \lambda V(T^*,P) \ge \lambda V(T,P),$$

maximality forces $\lambda = 1$ and

$$V(T^*,P) = V(T,P).$$

Thus equality holds throughout Lemma 5.2. The perimeter equality gives $AB = AC$, while equality in the integral shows that T is symmetric about the line through P perpendicular to BC; in particular, P lies on the perpendicular bisector of BC.

Apply the same argument to a second side. The triangle is then isosceles with respect to two different sides, and hence equilateral. The two symmetry axes meet at its center, so P is that center.

It remains only to evaluate $M_T(P)$ for an equilateral triangle. Let its side length be $a = p/3$. The distance from its center to a side is

$$d = \frac{a\sqrt{3}}{6} = \frac{p}{6\sqrt{3}}.$$

For the rays meeting one side,

$$\rho(\theta) = d \sec\theta, \qquad -\frac{\pi}{3} \le \theta \le \frac{\pi}{3}.$$

Thus

$$\int_0^{2\pi} \rho(\theta)\, d\theta = 3d \int_{-\pi/3}^{\pi/3} \sec\theta\, d\theta = 6d \log(2+\sqrt{3}) = \frac{p}{\sqrt{3}} \log(2+\sqrt{3}).$$

Division by π completes the proof. □

Corollary 5.4. *For every triangle of perimeter p, the average chord through each of the points*

$$I, \quad G, \quad K, \quad G_e, \quad N, \quad M$$

is at most

$$\frac{p}{\pi\sqrt{3}} \log(2+\sqrt{3}),$$

with equality only for an equilateral triangle.

Remark 5.5. For the incenter alone, the sharp inequality also has a short direct proof. Put

$$x = \frac{A}{2}, \qquad y = \frac{B}{2}, \qquad z = \frac{C}{2}, \qquad x+y+z = \frac{\pi}{2}.$$

Since

$$s = r(\cot x + \cot y + \cot z),$$

the incenter formula reduces the assertion to

$$\frac{\sum \log\cot(x/2)}{\sum \cot x} \le \frac{\log(2+\sqrt{3})}{\sqrt{3}}.$$

This follows immediately from Jensen's inequality applied to

$$h(x) = \log\cot\frac{x}{2} - c\cot x, \qquad c = \frac{\log(2+\sqrt{3})}{\sqrt{3}}.$$

Indeed,

$$h''(x) = \csc x\, \cot x \big(1 - 2c\csc x\big) < 0 \qquad (0 < x < \pi/2),$$

because $2c > 1$ and $\csc x \ge 1$. Thus h is strictly concave, and $h(\pi/6) = 0$.

6 The point of maximum average chord

For a fixed triangle, it is natural to ask which point P maximizes $M_T(P)$. By Lemma 3.1, this is exactly the problem of maximizing

$$\int_T \frac{dA(X)}{|X-P|}.$$

Abraham and Kovač [1] studied this point as the center of maximal electrostatic potential of a uniformly charged triangle. It is now catalogued as $X(5626)$; see also Nicollier [6]. In the more general language of convex geometry, it is the radial center of order 1; radial centers were introduced and studied by Moszyńska [5] and others.

If P is the maximizing point, Abraham and Kovač obtained the equivalent characterization

$$\boxed{\frac{1}{a}\log\frac{PB+PC+a}{PB+PC-a}=\frac{1}{b}\log\frac{PC+PA+b}{PC+PA-b}=\frac{1}{c}\log\frac{PA+PB+c}{PA+PB-c}.}$$

Thus the radial center has a particularly simple interpretation in the present setting: it is the point through which the average chord of the triangle is longest.

7 A connection with dual quermassintegrals

We finish by placing the preceding elementary problem in a broader context. If a star body $K\subset\mathbb{R}^2$ has radial function ρ_K, one standard normalization of the dual quermassintegrals is

$$\widetilde{W}_i(K)=\frac{1}{2}\int_0^{2\pi}\rho_K(\theta)^{2-i}\,d\theta.$$

These quantities belong to the dual Brunn–Minkowski theory developed by Lutwak; see [4].

Taking $K=T-P$ and $i=1$ gives

$$\boxed{M_T(P)=\frac{2}{\pi}\widetilde{W}_1(T-P).}$$

Thus the average chord considered here is, up to normalization, the first dual quermassintegral of the triangle translated so that P is the origin.

The elementary symmetrization argument used above also extends to other radial powers. For example, if G is the centroid and $0<q\le 2$, the quantity

$$\int_0^{2\pi}\rho_G(\theta)^q\,d\theta$$

is maximized, among triangles of fixed perimeter, by the equilateral triangle. Indeed,

$$\int_0^{2\pi}\rho_G(\theta)^q\,d\theta=q\int_T|X-G|^{q-2}\,dA(X),$$

and the kernel on the right is radially nonincreasing. Steiner symmetrization preserves the centroid, since it preserves the lengths of all parallel sections and moves their midpoints to the symmetry axis. Degenerate triangles cause no difficulty, since Hölder's inequality gives

$$\int_0^{2\pi}\rho_G^q\,d\theta\le(2\pi)^{1-q/2}(2\Delta)^{q/2}.$$

The case $q=1$ is the average-chord result, while $q=2$ is equivalent to the familiar maximal-area property of the equilateral triangle. We do not pursue these extensions here.

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