O(n^3) Bounds for the Area of a Convex Lattice n-gon

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A lattice point in the plane is a point with integer coordinates. A lattice polygon is a polygon whose vertices are all lattice points. A polygon with n vertices will be referred to as an n-gon.

Recently, Simpson [12] conjectured that for a convex lattice n-gon with area A, we must have $A \geq cn^3$ for some constant $c > 0$.

I. Bárány has informed me [7] that this result is already known – namely that Arnol'd [2] proved in 1980 that

$$A \geq \frac{n^3}{2 \cdot 16^3}.$$

It is the purpose of this note to give a better bound for A.

Theorem. If A is the area of a convex lattice n-gon, then

$$A > \frac{n^3}{8\pi^2}. \tag{1}$$

Proof. Let $K = P_1P_2 \ldots P_n$ be a convex lattice n-gon with area A. Let the area of $\triangle P_{i-1}P_iP_{i+1}$ be A_i, where $P_{n+1} \equiv P_1$ and let

$$f(K) = \frac{1}{A^n} \prod_{i=1}^{n} A_i.$$

By a result of Rényi and Sulanke [10], we have $f(K)$ is maximal when and only when K is an affine transformation of R_n, a regular n-gon. It is straightforward to show that this maximum value is

$$f(R_n) = \left(\frac{4\sin^2 \frac{\pi}{n}}{n}\right)^n$$

so that $f(K) \leq f(R_n)$. But since $\sin x < x$ for $x > 0$, we have

$$\prod_{i=1}^{n} A_i < A^n \left(\frac{4\pi^2}{n^3}\right)^n.$$

By the pigeonhole principle, we can conclude that there is some i such that

$$A_i < \frac{4\pi^2 A}{n^3}.$$

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From Pick's Formula ([5], p. 209), it follows that the area of a lattice triangle is not less than 1/2. Hence $A > A_i n^3/4\pi^2 \geq n^3/8\pi^2$. This concludes the proof.

Let $A(n)$ be the smallest possible area for a convex lattice n-gon. Then, since $2A(n)$ must be an integer, we can round our lower bound for $2A$ up to the next larger integer and write

$$\left\lceil \frac{n^3}{4\pi^2} \right\rceil \leq 2A(n) \leq 2\binom{\lceil n/2 \rceil}{3} + n - 2 \tag{2}$$

where the upper bound comes from [12].

Let $g(n)$ be the smallest number of lattice points that can be in the interior of a convex lattice n-gon. The functions $A(n)$ and $g(n)$ are related by the formula

$$A(n) = g(n) + n/2 - 1$$

(Proposition 7.2.5 of [8] and Theorem 1 of [12]). Thus

$$\left\lceil \frac{n^3}{8\pi^2} - \frac{n}{2} + 1 \right\rceil \leq g(n) \leq \binom{\lceil n/2 \rceil}{3}. \tag{3}$$

This proves Rabinowitz's conjecture [9], that there exists a constant $c > 0$ such that $g(n) > cn^3$.

We can compare our bounds for $2A(n)$ against the actual values obtained by Simpson [12] and Rabinowitz [9]:

n	lower bound for $2A(n)$	actual value of $2A(n)$	upper bound for $2A(n)$
3	1	1	1
4	2	2	2
5	4	5	5
6	6	6	6
7	9	13	13
8	13	14	14
9	19	21	27
10	26	28	28
11	34	[39,43]	49
12	44	48	50
13	56	65	81
14	70	80	82
15	86	[99,109]	125
16	104	118	126
17	125	[147,173]	183
18	148	174	184

19	174	[209,241]	257
20	203	242	258
21	235	[285,327]	349
22	270	328	350

The square brackets define a closed interval known to contain the value.

Related inequalities of interest can be found in [1], [3], [4], [6], and [11].

Open Questions.

1. What is the exact value of $A(11)$?

2. Can the bounds for $A(n)$ in equation (2) be improved?

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