International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 10, 2025, pp. 325–348
Received 7 July 2025. Published on-line 21 Aug. 2025
web: http://www.journal-1.eu/

Geometric Properties of a Cassini Oval

Stanley Rabinowitz
545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com/

Abstract. We survey the literature to find geometrical properties of a Cassini oval. We also use a computer to find additional properties.

Keywords. Cassini oval, lemniscate of Bernoulli, computer-discovered mathematics.

Mathematics Subject Classification (2020). 51M04, 51-08.

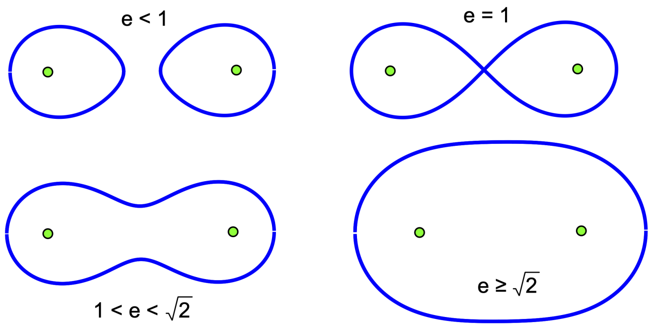
1. Introduction

A *Cassini oval* is a plane curve that is the locus of points such that the product of their distances from two fixed points is a constant, b^2. The two fixed points are known as the *foci* of the oval.

In this paper, we survey geometrical results about the Cassini oval as well as give additional results that were found by computer. If a result is stated without a reference or a proof, this means that the result was discovered by computer but we have no formal proof of the result.

Throughout this paper, the foci of the Cassini oval will always be labeled F and G, and they will be colored green. The midpoint of segment FG is called the center of the oval and will be named O. The rays OF and OG meet the oval at points X and Y, respectively, known as the *vertices* of the oval.

Let the length of OF be a. Let $e = b/a$. The shape of the Cassini oval varies depending on the values of a and b (see [25] and [1, article 246]). If $e < 1$, the curve consists of two disconnected loops, each of which contains a focus. When $e = 1$, the curve is the lemniscate of Bernoulli. When $e > 1$, the curve is a single, connected loop enclosing both foci. It is peanut-shaped for $1 < e < \sqrt{2}$ and convex for $e \geq \sqrt{2}$.



If the foci are located $(-a, 0)$ and $(a, 0)$, then, from the defining property, the equation of the oval is

$$\left((x-a)^2+y^2\right)\left((x+a)^2+y^2\right)=b^4. \tag{1}$$

Theorem 1. *Let P be any point on a Cassini oval with foci F and G and vertices X and Y as shown in Figure 1. Then $FP \cdot GP = FX \cdot FY$.*

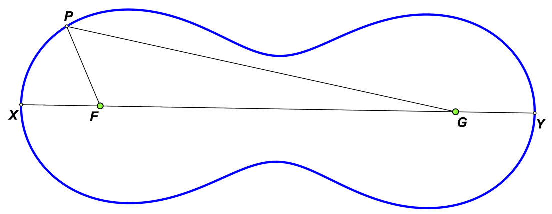


FIGURE 1. $FP \cdot GP = FX \cdot FY$

Proof. By the defining property of Cassini ovals, we have $FP \cdot GP = FX \cdot GX$. By symmetry, we see that $GX = FY$. Thus, $FP \cdot GP = FX \cdot FY$. □

Theorem 2. *Let P be any point on a Cassini oval with center O, foci F and G and vertices X and Y. A perpendicular at F meets the semicircle with diameter XY at P as shown in Figure 2. Then $OF = a$ and $PF = b$.*

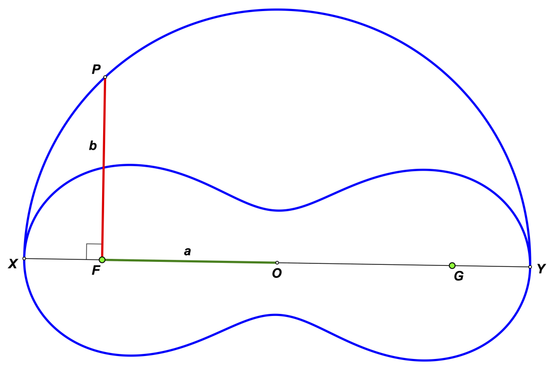


FIGURE 2. $OF = a$ and $PF = b$

Proof. By the defining property of Cassini ovals, we have $FX \cdot GX = b^2$. By symmetry, we see that $FX \cdot FY = b^2$. Since PF is an altitude of right triangle XPY, we have $PF^2 = FX \cdot FY$. Thus, $PF = b$. □

Note also that $OP = \sqrt{a^2+b^2}$, so that $OX = \sqrt{a^2+b^2}$ and $XF = \sqrt{a^2+b^2} - a$.

Theorem 3. *A Cassini oval has center O and foci F and G. Suppose a perpendicular to FG at O meets the curve at P as shown in Figure 3. Then $PF = b$.*

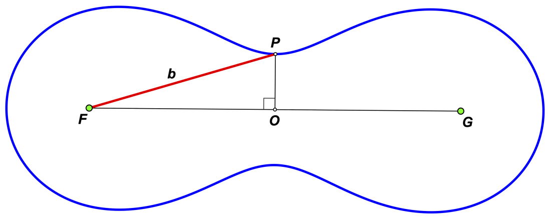


FIGURE 3. red length $= b$

Proof. Since the oval is symmetrical about the line PO, $PF = PG$. From $FP \cdot GP = b^2$, we can conclude that $FP = b$. □

Note also that since $OF = a$, we have $OP = \sqrt{b^2 - a^2}$.

Theorem 4. *A Cassini oval has center O and vertices X and Y. Suppose a perpendicular to XY at O meets the curve at a point P as shown in Figure 4. Then $PY = b\sqrt{2}$.*

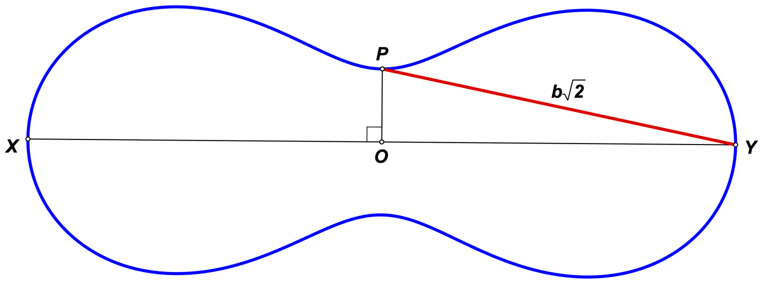


FIGURE 4. red length $= b\sqrt{2}$

Proof. From $OP = \sqrt{b^2 - a^2}$ and $OY = \sqrt{a^2 + b^2}$, we can conclude that $(PY)^2 = (OP)^2 + (OY)^2 = b^2 - a^2 + a^2 - b^2 = 2b^2$. Thus, $PY = b\sqrt{2}$. □

The following result was found by computer.

Theorem 5. *Let P be any point on a Cassini oval with foci F and G and vertices X and Y as shown in Figure 5. Then $(YP)^2 + (XP)^2 = (FP + GP)^2$.*

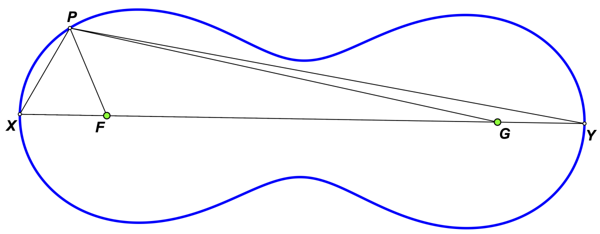


FIGURE 5. $(YP)^2 + (XP)^2 = (FP + GP)^2$

The following result was found by computer.

Theorem 6. *Let A be any point on a Cassini oval with foci F and G and center O. A line through A parallel to FG meets the oval again at B, C, and D as shown in Figure 6. Let $BD = x$. Then $AB \cdot BD = 2(a^2 - x^2)$.*

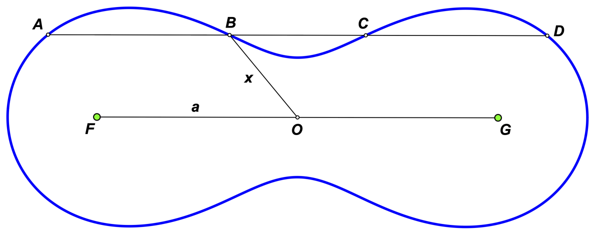


FIGURE 6. $AB \cdot BD = 2(a^2 - x^2)$

Proof. If the distance from A to FG is k, then the equation of line AD is $y = k$. Substituting $y = k$ in Equation (1), we get a quartic that we can solve for k. This gives us the coordinates of points A, B, C, and D from which a straightforward calculation shows that $AB \cdot BD = 2(a^2 - x^2)$. □

The following result comes from [2, article 5316].

Theorem 7. *Let P be any point on a Cassini oval with center O and foci F and G. Let PN be a normal to the oval with N on FG as shown in Figure 7. Let $OP = R$. Then* $PN = \dfrac{b^2R}{R^2 + a^2}$.

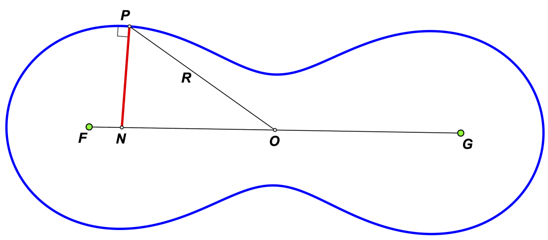


FIGURE 7. red length is $b^2R/(R^2 + a^2)$

The following result comes from [1, article 249] and [2, article 5315].

Theorem 8. *Let P be any point on a Cassini oval with center O and foci F and G. Let PN be a normal to the oval as shown in Figure 8. Then $\angle FPN = \angle OPG$.*

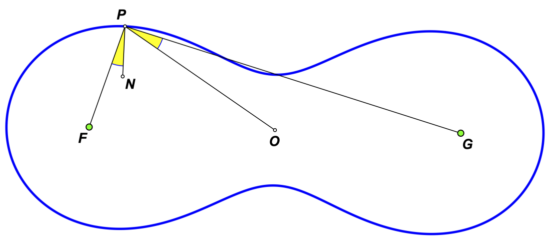


FIGURE 8. yellow angles are equal

Corollary 9. *Let P be any point on a Cassini oval with foci F and G. Let K be the symmedian point of $\triangle PFG$ as shown in Figure 9. Then PK is a normal to the oval.*

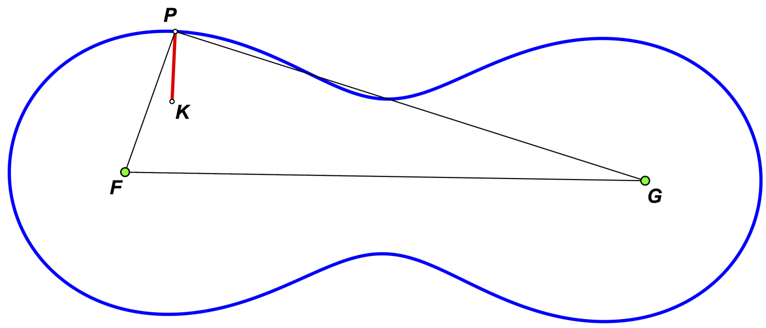


Figure 9. PK is a normal

Proof. This follows from Theorem 8 and the fact that in Figure 8, PO is a median of $\triangle PFG$. □

The following result comes from [1, article 249] and [4].

Theorem 10. *Let P be any point on a Cassini oval with foci F and G. Let PN be a normal to the oval with N on FG as shown in Figure 10. Then $NF/NG = (PF/PG)^2$.*

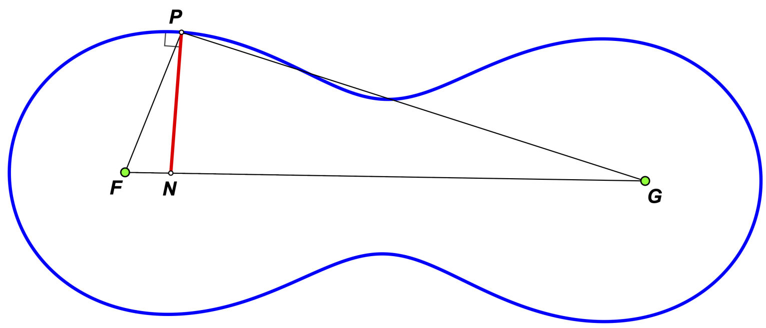


Figure 10. $NF/NG = (PF/PG)^2$

The following result was found by computer.

Theorem 11. *Let P be any point on a Cassini oval with center O and foci F and G. Let PQ be a tangent to the oval as shown in Figure 11. Then $\angle FPO + \angle GPQ = 90°$.*

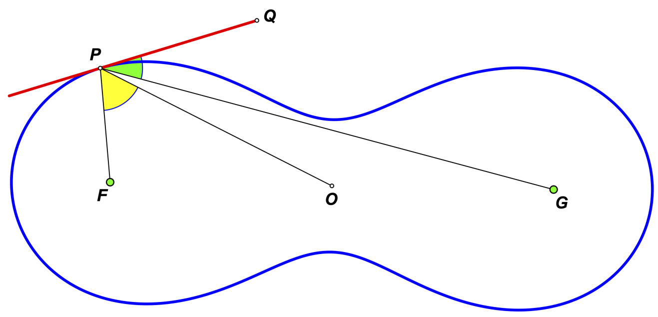
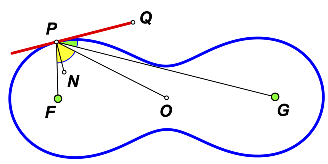


FIGURE 11. sum of colored angles is 90°

Proof. Let PN be the normal to the oval at P as shown in the figure to the right, so that $\angle NPQ = 90°$.
By Theorem 8, we have $\angle FPN = \angle OPG$.
Then $\angle QPG + \angle FPO = \angle QPG + \angle NPO + \angle FPN = \angle QPG + \angle NPO + \angle OPG = \angle NPQ = 90°$. □



The following result was found by computer.

Theorem 12. *Let P be any point on a Cassini oval with foci F and G and vertices X and Y. Then $\angle FPG + 2\angle GPY + 2\angle YXP = 180°$.*

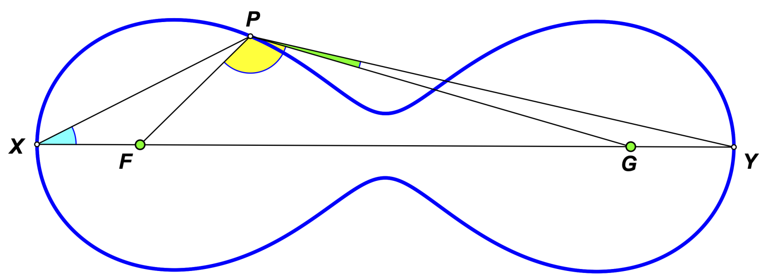


FIGURE 12. yellow angle + twice green angle + twice blue angle = 180°

The following result was found by computer. The result is so simple, that it is probably known, but I could not locate a reference.

Theorem 13. *A Cassini oval has center O and foci F and G. Let PQ be a tangent to the oval that is parallel to FG and touches it at two points as shown in Figure 13. Then $OP = OG$.*

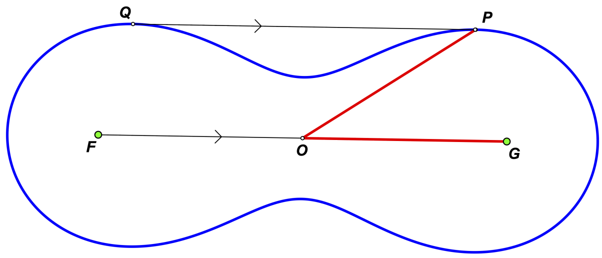


FIGURE 13. red lengths are equal

Proof. Place the Cassini oval in a Cartesian coordinate system so that O lies at the origin and $G = (a, 0)$. Then from the defining property of Cassini ovals, the equation of the oval is

$$\left((x-a)^2 + y^2\right)\left((x+a)^2 + y^2\right) = b^4.$$

Implicit differentiation with respect to x gives

$$\left((a+x)^2 + y^2\right)(2(x-a) + 2yy') + \left((x-a)^2 + y^2\right)(2(a+x) + 2yy') = 0.$$

A horizontal tangent is a point where $y' = 0$. Solving for y', we find

$$y' = \frac{x\left(a^2 - x^2 - y^2\right)}{y\left(a^2 + x^2 + y^2\right)}. \tag{2}$$

If $y' = 0$, then $x\left(a^2 - x^2 - y^2\right) = 0$. The condition $x = 0$ gives the horizontal tangent at 0, which does not touch the oval at two points. Thus, if $P = (x, y)$, then $x^2 + y^2 = a^2$. That is, the distance from P to the origin is a. Since $OG = a$, we have $OP = OG$. □

The following result was found by computer. The result is so simple, that it is probably known, but I could not locate a reference.

Theorem 14. *A Cassini oval has foci F and G. Let PQ be a tangent to the oval that is parallel to FG and touches it at two points as shown in Figure 14. Then $FP \perp PG$.*

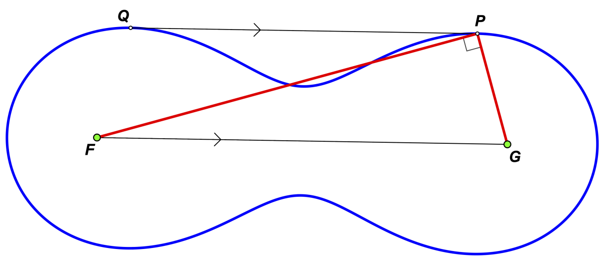


FIGURE 14. red lines are perpendicular

Proof. By Theorem 13, P lies on the semicircle with diameter FG. But an angle inscribed in a semicircle must be a right angle. □

In Theorem 11, if we let the tangent pass through G, we get the following result. This result is so simple, that it is probably known, but I could not locate a reference. See also [16].

Theorem 15. *A Cassini oval has center O and foci F and G. A tangent to the oval from G touches the oval at P as shown in Figure 15. Then $OP \perp FP$.*

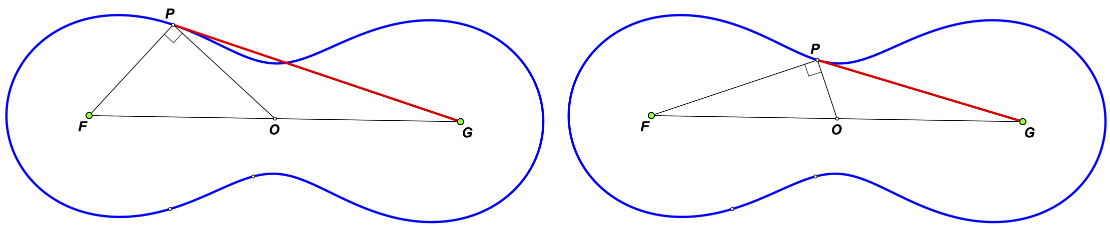


FIGURE 15. marked angles are right angles

An analytic proof can also be given.

Proof. Place the Cassini oval in a Cartesian coordinate system so that O lies at the origin, $G = (a, 0)$, and $F = (-a, 0)$. Let the coordinates of P be (x, y). From equation (2), we know that the slope of the tangent at P is

$$\frac{x\,(a^2 - x^2 - y^2)}{y\,(a^2 + x^2 + y^2)}.$$

The slope of the line joining P and G is $y/(x - a)$. Therefore,

$$\frac{x\,(a^2 - x^2 - y^2)}{y\,(a^2 + x^2 + y^2)} = \frac{y}{x - a}.$$

Cross multiply, bring all terms to the left, and factor. We get

$$\left(y^2 + (x - a)^2\right)\left(y^2 + x(x + a)\right) = 0.$$

The first factor is always positive and can never be 0, so $y^2 = -x(x + a)$. Let PH be the altitude of $\triangle PFO$ with H on FO. Then $PH = y$, $|HO| = -x$, and $|FH| = x + a$. Thus, $(PH)^2 = HO \cdot FH$ which makes $\triangle FPO$ a right triangle with right angle at P. □

In Theorem 11, if we let the tangent pass through Y, we get the following result.

Theorem 16. *A Cassini oval has center O, foci F and G, and vertices X and Y. A tangent to the oval from Y touches the oval at P as shown in Figure 16. Then $\angle FPO + \angle GPY = 90°$.*

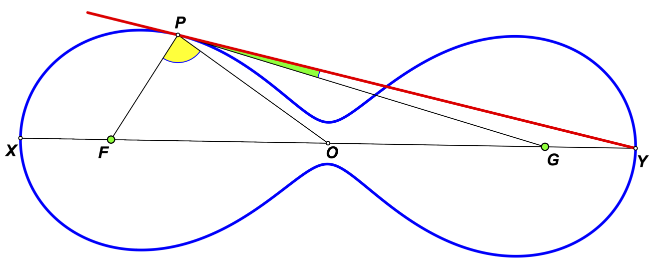


FIGURE 16. yellow angle + green angle = 90°

The following result was found by computer.

Theorem 17. *A Cassini oval has center O and vertices X and Y. A tangent to the oval from Y touches the oval at P as shown in Figure 17. Then $\angle OPY + 2\angle YXP = 90°$.*

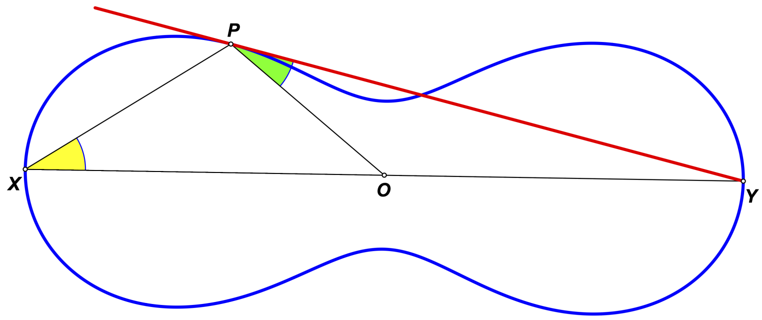
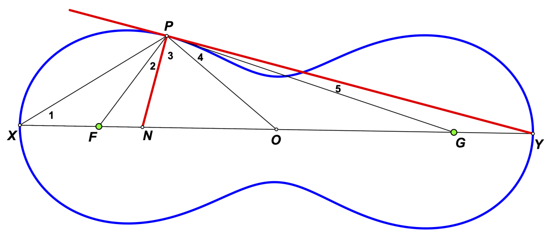


FIGURE 17. green angle + twice yellow angle = 90°

Proof. Let the foci be F and G and let PN be the normal at P. Number the angles as shown to the right. Then by Theorem 12, $\angle 2 + \angle 3 + \angle 4 + 2\angle 5 + 2\angle 1 = 180°$. Since PN is a normal to the oval and PY is a tangent, $PN \perp PY$ or $\angle 3+\angle 4+\angle 5 = 90°$. Subtracting gives $\angle 2 + \angle 5 + 2\angle 1 = 90°$. But $\angle 2 = \angle 4$ by Theorem 8, so we can conclude that$\angle 4 + \angle 5 + 2\angle 1 = 90°$. □



In Theorem 11, if we let the tangent pass through a point on OG, we get the following result.

Theorem 18. *A Cassini oval has center O and foci F and G. Let Q be a point on OG. A tangent to the oval from Q touches the oval at P as shown in Figure 18. Then $\angle FPO - \angle GPY = 90°$.*

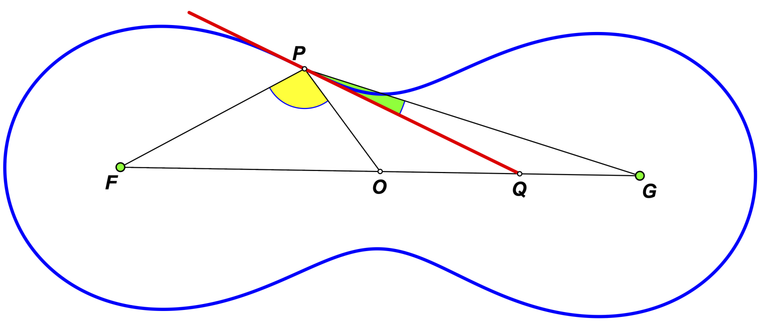


FIGURE 18. yellow angle − green angle = 90°

The following result comes from [1, article 248].

Theorem 19. *Let P be any point on a Cassini oval with foci F and G. Perpendiculars to PF and PG at F and G meet the tangent to the oval at P in points U and V as shown in Figure 19. Then $PU = PV$.*

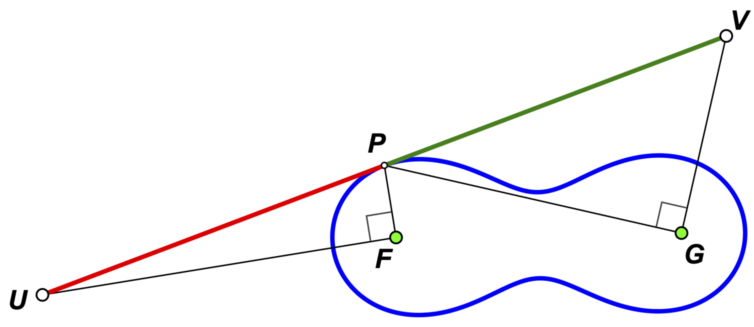


FIGURE 19. red length = green length

The following result comes from [8].

Theorem 20. *Let P and Q be two points on a Cassini oval with center O such that $OP \perp OQ$ and $a < b$ as shown in Figure 20. Let R be a point on the oval such that $\angle ROY = 45°$. Suppose $OP = r_1$, $OQ = r_2$, and $OR = r$. Then $r_1 r_2 = r^2 = \sqrt{b^4 - a^4}$.*

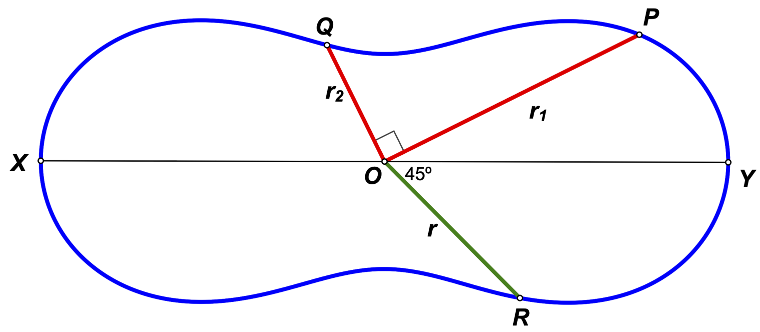


FIGURE 20. green length is mean proportional of red lengths

The following result comes from [1, article 252].

Theorem 21. *Let P be any point on a Cassini oval with foci F and G. Let D be the circle that passes through F, G and P. Let V_1V_2 be the diameter of the circle perpendicular to FG as shown in Figure 21. Tangents to the circle at F and G meet at F_1. Point F_2 is the reflection of F_1 about the center of the circle. A hyperbola has foci F_1 and F_2 and vertices V_1 and V_2. Then the tangent to the oval at P is also tangent to the hyperbola. The normal to the oval at P passes through F_1.*

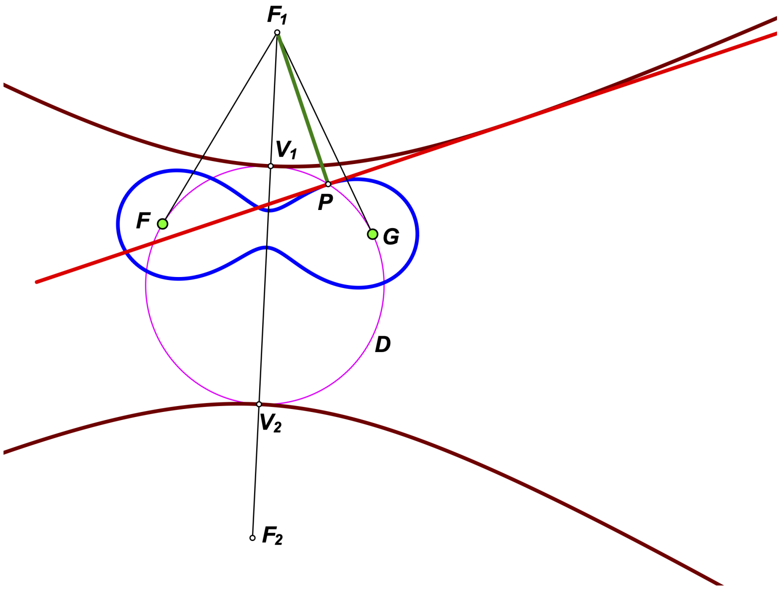


Figure 21. red line is tangent to both oval and hyperbola

The following result comes from [13, Theorem 7.19]. See also [18].

Theorem 22 (Steiner's Theorem)**.** *Let P be any point on a Cassini oval with foci F and G. The normal at point P meets FG at Q as shown in Figure 22. Let $\angle FPQ = \alpha$ and $\angle QPG = \beta$. Let $PF = r_1$ and $PG = r_2$. Then $\dfrac{r_1}{r_2} = \dfrac{\sin\alpha}{\sin\beta}$.*

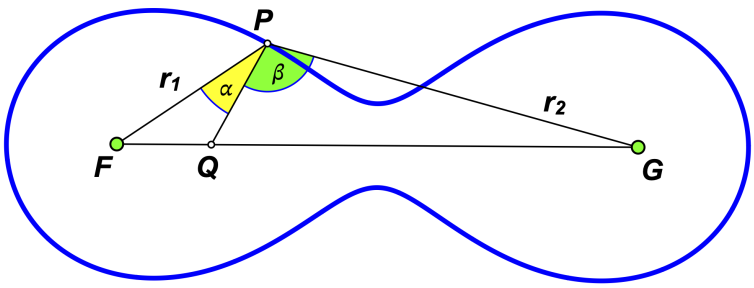


Figure 22. $\dfrac{r_1}{r_2} = \dfrac{\sin\alpha}{\sin\beta}$

The following result was found by computer. See [15].

Theorem 23. *A Cassini oval has foci F and G. A secant meets the oval at points A, B, C, and D as shown in Figure 23. Then $FA \cdot FB = GC \cdot GD$.*

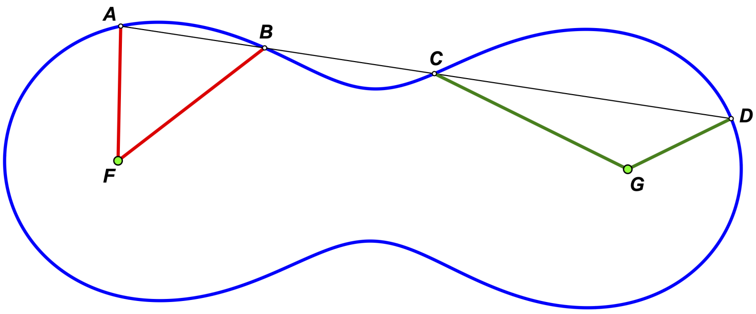


FIGURE 23. product of red segments = product of green segments

If we let points A and B coincide, we get the following result.

Corollary 24. *A Cassini oval has foci F and G. Let P be a point on the oval such that the tangent at P meets the oval again at points A and B as shown in Figure 24. Then $FP \cdot FA = GP \cdot GB$.*

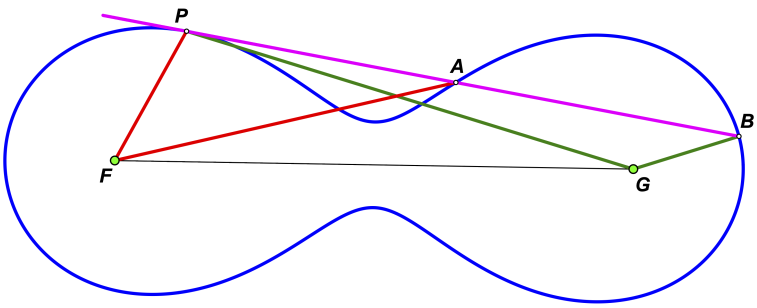


FIGURE 24. product of red segments = product of green segments

The following result comes from [7, p. 307].

Theorem 25. *A Cassini oval has center O. A secant meets the oval at points A, B, C, and D as shown in Figure 25. Let P be the midpoint of AB and let Q be the midpoint of CD Then $OP = OQ$.*

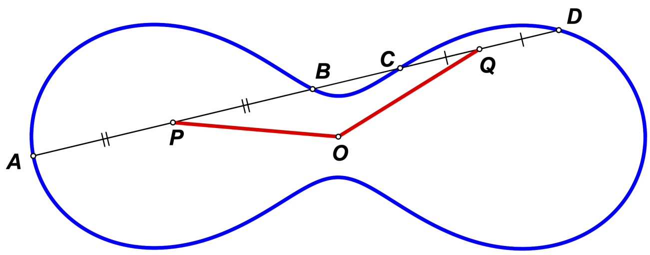


FIGURE 25. red lengths are equal

The following result comes from [19].

Theorem 26. *Let P be any point on a Cassini oval with foci F and G. The line PN is a normal to the oval as shown in Figure 26. Extend FP to Q so that $FP = PQ$. Then $\angle NPG = \angle FQG$.*

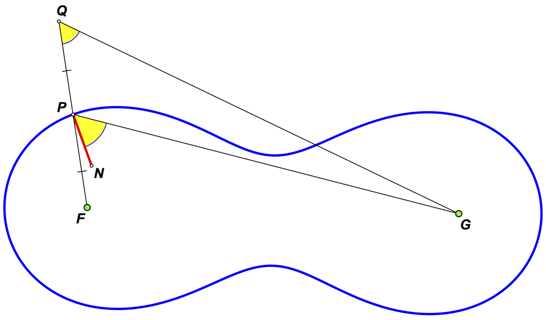


FIGURE 26. yellow angles are equal

In a Cassini oval, if a perpendicular to XY at O meets the oval, the two points of intersection are known as the *covertices* of the oval.

The following two results are due to Keita Miyamoto [12], [11].

Theorem 27. *Let PQ be a chord through the center O of a Cassini oval with vertices X and Y and covertices U and V. The perpendicular bisector of PQ meets the oval at R as shown in Figure 27. Let T be the reflection of R about XY. Then $\angle PTQ$ is invariant and $\angle PTQ = \angle UYV$.*

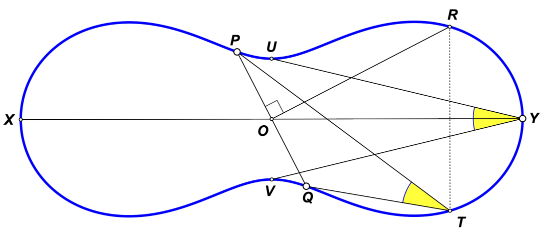


FIGURE 27. yellow angles are equal

Theorem 28. *Let PQ be a chord through the center O of a Cassini oval with vertices X and Y. The perpendicular bisector of PQ meets the oval at R as shown in Figure 28. Let T be the reflection of R about XY. Then circle $\odot PQT$ is tangent to the oval at T.*

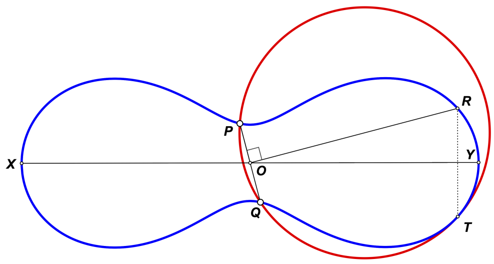


FIGURE 28. red circle is tangent to the oval

The following result comes from [24, pp. 189–190] according to [5].

Theorem 29. *A Cassini oval has foci F and G and vertices X and Y. A line through Y meets the circle with diameter FG at P and Q as shown in Figure 29. The circle centered at G with radius equal to PY meets the oval at R. Then $FR = YQ$.*

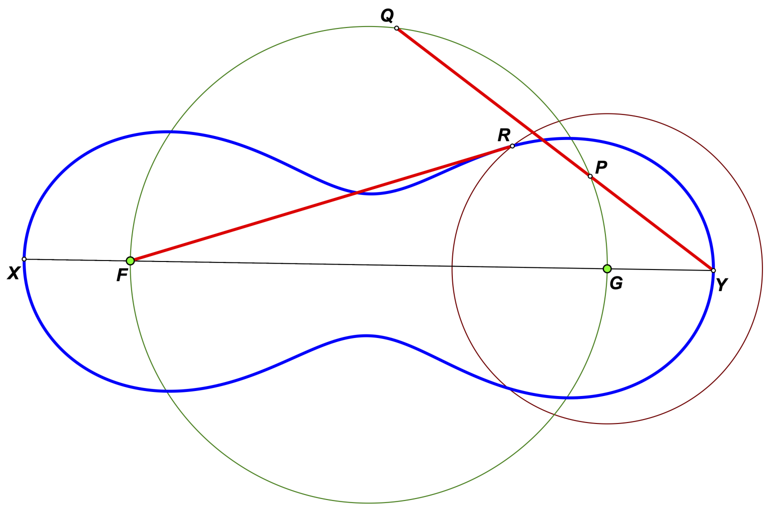


FIGURE 29. red lengths are equal

The following result comes from [17, pp. 14–15] according to [20].

Theorem 30. *Let P be any point on a Cassini oval with foci F and G and vertices X and Y. Let AB be a common tangent to circles $G(P)$ and $F(P)$ as shown in Figure 30. Reflect G about point B to get point M. Then $FM = XY$.*

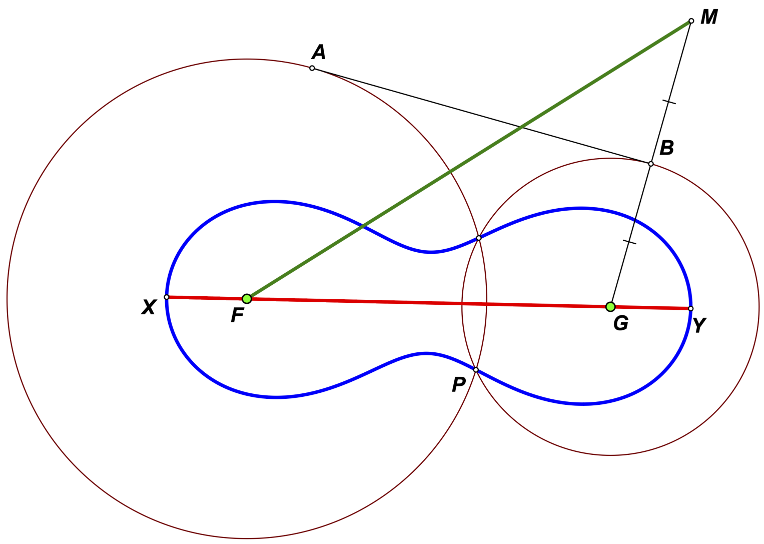


FIGURE 30. red length = green length

The following result comes from [21].

Theorem 31. *Let P be any point on a Cassini oval with foci F and G. Let $PF = r_1$ and $PG = r_2$. Extend GP through P a distance r_1 to get point K. Extend FP through P a distance r_2 to get point M. Construct parallelogram $PKNM$ as shown in Figure 31. Then NP is normal to the oval.*

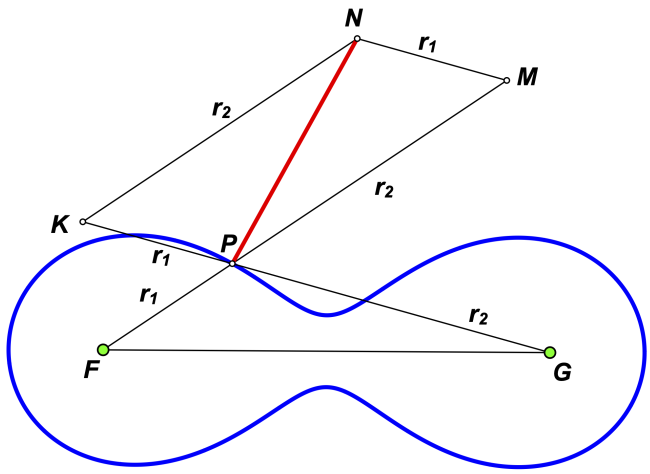


Figure 31. NP is normal to the oval

The following result comes from [22].

Theorem 32. *Let T be any point on a Cassini oval with foci F and G. Let L_1 be the line through F that is perpendicular to TF. Let L_2 be the line through G that is perpendicular to TG. A circle center T cuts L_2 at points P and Q as shown in Figure 32. Reflect P and Q about T to get points P' and Q'. Let $P'Q'$ meet L_1 at U. Then UT is tangent to the oval.*

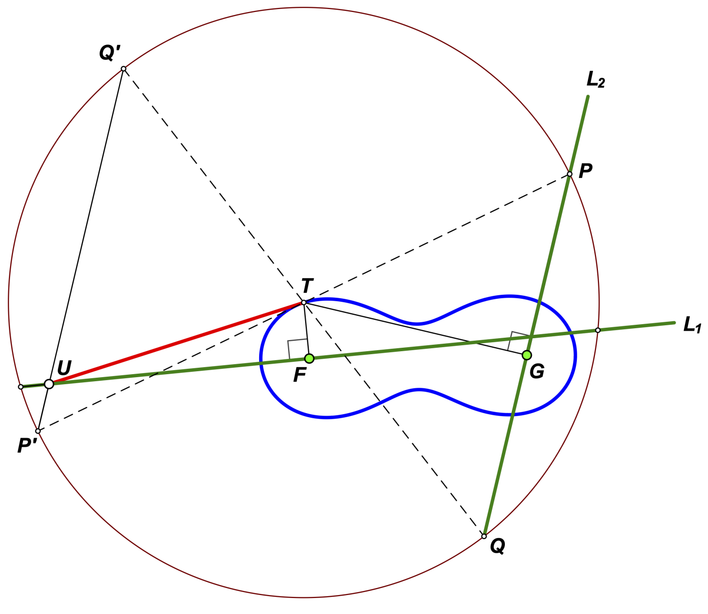


Figure 32. red line is tangent to the oval

The following result comes from [9].

Theorem 33. *A Cassini oval has center O and foci F and G. Let T be the tangent to the oval that is parallel to FG and touches the oval in two points. Let r be the distance from T to FG, Let P be a point on the circle with diameter FG as shown in Figure 33. The circle with center P and radius r cuts T at A and B. The orthogonal projection of A and B onto FG gives points A' and B'. Then the circle with diameter $A'B'$ is doubly tangent to the oval.*

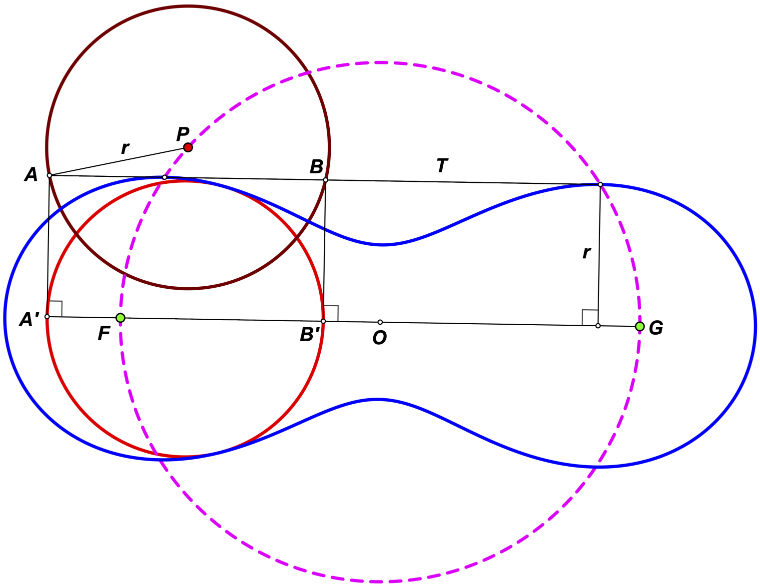


FIGURE 33. red circle is doubly tangent to the oval

Theorem 34. *A Cassini oval with two distinct loops has center O and foci F and G. The vertices are X, Z, W, and Y as shown in Figure 34. Then $(OY)^2 + (OZ)^2 = 2a^2$ and $(OY)^2 - (OZ)^2 = 2b^2$.*

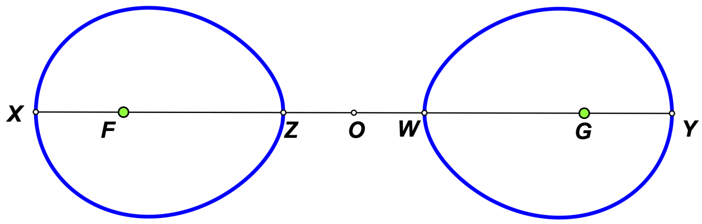


FIGURE 34. $(OY)^2 + (OZ)^2 = 2a^2$ and $(OY)^2 - (OZ)^2 = 2b^2$

Proof. Letting $y = 0$ in the equation of a Cassini oval,

$$\left((x-a)^2 + y^2\right)\left((x+a)^2 + y^2\right) = b^4,$$

gives the x-intercepts. We get $(x^2 - a^2)^2 = b^4$ or $x^2 - a^2 = \pm b^2$. Thus, we find $OY = \sqrt{a^2 + b^2}$ and $OZ = \sqrt{a^2 - b^2}$, and the result follows. □

The following result comes from [2, article 5314] and [6].

Theorem 35. *A Cassini oval with two distinct loops has center O. A chord through O meets one loop at P and P' as shown in Figure 35. A tangent from O touches the oval at T. Then $OP \cdot OP' = (OT)^2 = \sqrt{a^4 - b^4}$.*

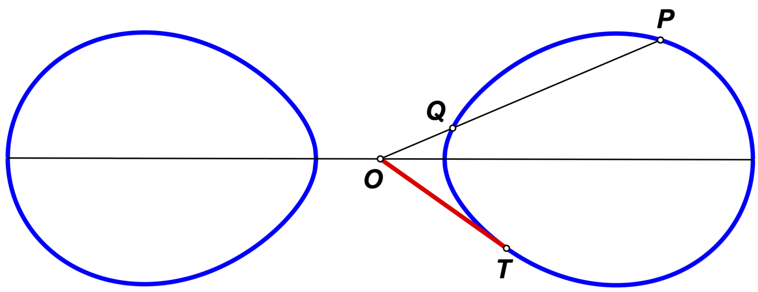


Figure 35. $OP \cdot OP' = (OT)^2 = \sqrt{a^4 - b^4}$

Corollary 36. *A Cassini oval with two distinct loops has center O and vertices X, Z, W, and Y as shown in Figure 36. OP is a tangent to the oval. OQ is a tangent to the circle with WY as diameter. Then $OP = OQ$.*

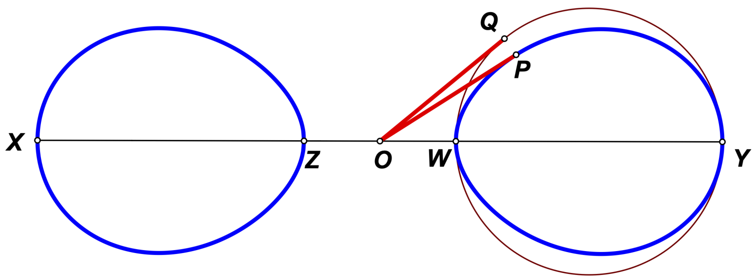


Figure 36. red lengths are equal

The following result was discovered by computer.

Theorem 37. *A Cassini oval with two distinct loops has center O and vertices X, Z, W, and Y as shown in Figure 37. WT is a tangent.. Then $\angle FTO - \angle WTG = 90°$.*

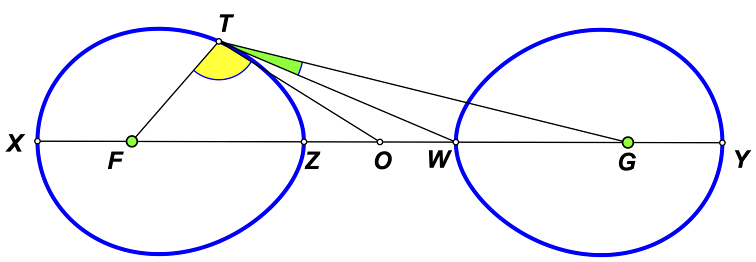


Figure 37. yellow angle – green angle = 90°

The following result comes from [3].

Theorem 38. *Let P be a variable point on a Cassini oval with two distinct loops and center O. Suppose a fixed secant through O meets the oval in four points A, B, C, and D as shown in Figure 38. Then $\angle DPC - \angle BPA$ does not depend on the location of P.*

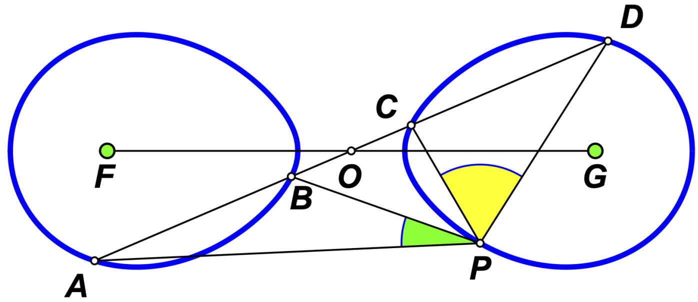


FIGURE 38. yellow angle minus green angle is constant

The following result was discovered by computer.

Theorem 39. *A Cassini oval with two distinct loops has vertices X, Z, W, and Y as shown in Figure 39. P is any point on the oval. Then $\angle XPZ - \angle WPY = 90°$.*

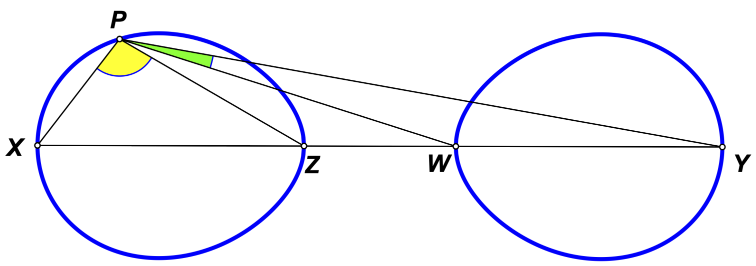


FIGURE 39. yellow angle – green angle = 90°

The following result was discovered by computer.

Theorem 40. *A Cassini oval with two distinct loops has foci F and G and vertices X and Y. Common tangents to the two loops touch the loops at P and Q as shown in Figure 40. Then $\angle PGQ = 2\angle PXQ$.*

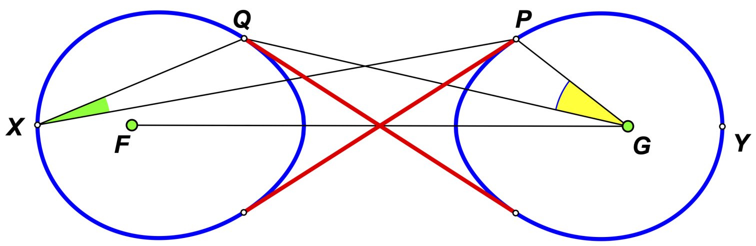


FIGURE 40. yellow angle is twice green angle

The following result was discovered by computer.

Theorem 41. *A Cassini oval with two distinct loops has center O and foci F and G. PQ and RS are common tangents to the two loops as shown in Figure 41. Then $\angle PGR + \angle SOP = 90°$.*

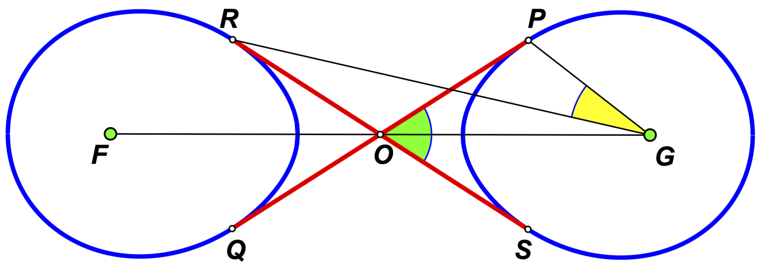


FIGURE 41. colored angles are complementary

The following result comes from [14].

Theorem 42. *A Cassini oval with two distinct loops has center O and foci F and G. A variable secant through O meets one loop at A and B as shown in Figure 42. $\odot ABF$ meets FO at K. Then the location of K is independent of the secant.*

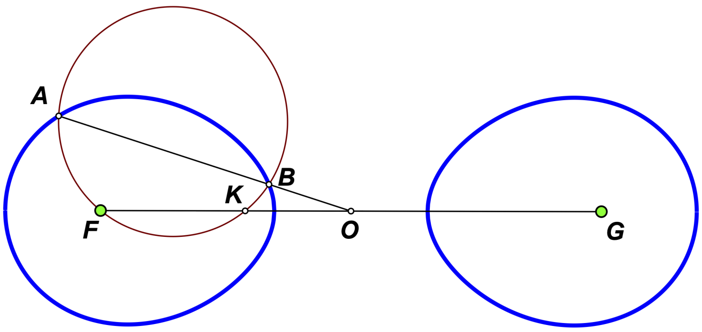


FIGURE 42. point K is invariant

The point K is known as a *singular focus* of the oval, according to [7].

The following result comes from [7, p. 324].

Theorem 43. *A Cassini oval with two distinct loops has center O and singular focus K as shown in Figure 43. Then, $OK = \sqrt{a^4 - b^4}/a$.*

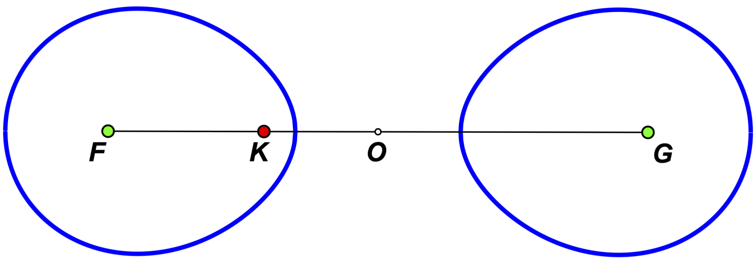


FIGURE 43. $OK = \sqrt{a^4 - b^4}/a$

If we let A and B coincide in Theorem 42, we get the following result.

Theorem 44. *A Cassini oval with two distinct loops has center O, foci F and G, and singular focus K as shown in Figure 44. The tangent to the oval from O touches it at T. Then $\odot FTK$ is tangent to the oval.*

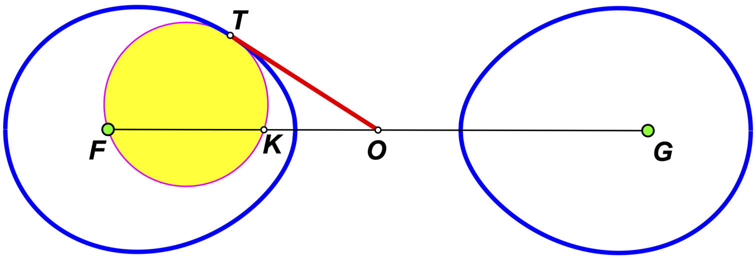


FIGURE 44. yellow circle is tangent to the oval

The following result comes from [7, p. 324] and [14].

Theorem 45. *A Cassini oval with two distinct loops has center O and singular foci K and K'. Let P be any point on the oval as shown in Figure 45. Then $a\sqrt{PK \cdot PK'} = b \cdot PO$.*

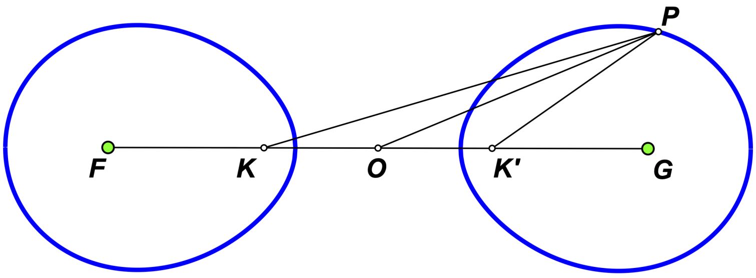


FIGURE 45. $a\sqrt{PK \cdot PK'} = b \cdot PO$

The following result was found by Keita Miyamoto [10].

Theorem 46. *Let L be a Cassini oval. Suppose that a circle (C_1) intersects L at 4 points P_1, P_2, P_3, P_4, and a circle (C_2) intersects L at 4 points P_3, P_4, P_5, P_6, and a circle (C_3) intersects L at 4 points P_5, P_6, P_7, P_8 as shown in Figure 46. Then, P_1, P_2, P_7, P_8 are concyclic.*

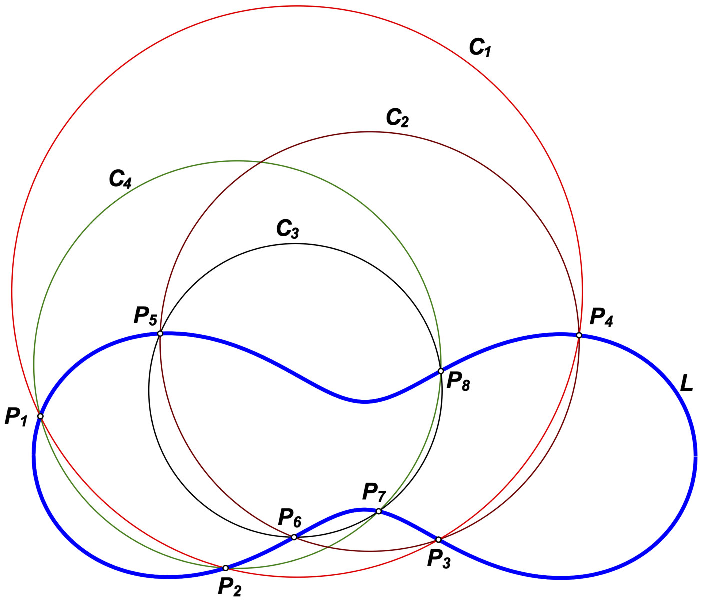


FIGURE 46. P_1, P_2, P_7, P_8 are concyclic

References

[1] Alfred Barnard Basset, An Elementary Treatise on Cubic and Quartic Curves. Deighton Bell, 1901. https://books.google.com/books?id=ym47AQAAIAAJ

[2] George S. Carr, Formulas and Theorems in Pure Mathematics, 2nd edition. Chelsea, 1970.

[3] R. W. Genese, Problem 9846, The Educational Times **42**(1889). https://books.google.com/books?id=ezA6AQAAMAAJ

[4] D. T. Griffiths, Problem 11633. Mathematical Questions and Solutions, from The Educational Times, **58**(1893). https://books.google.com/books?id=iP1RAQAAIAAJ

[5] Izidor Hafner and Marko Razpet, "1. Constructing a Point on a Cassini Oval" Wolfram Demonstrations Project, 2018. https://demonstrations.wolfram.com/1ConstructingAPointOnACassiniOval/

[6] Harry Hart, The Cassinian. Messenger of Mathematics, **6**(1877)169–172. https://books.google.com/books?id=8c7XwRLjQOUC

[7] Harold Hilton, Plane Algebraic Curves, Clarendon Press, 1920. https://books.google.com/books?id=-m47AQAAIAAJ

[8] Percy Alexander MacMahon, The Cassinian. Messenger of Mathematics, **12**(1883)118–120. https://books.google.com/books?id=YIZaAAAAYAAJ

[9] Keita Miyamoto, *Proposed Problem.* Romantics of Geometry Facebook group. June 15, 2025. https://www.facebook.com/groups/parmenides52/posts/23885777311109317/

[10] Keita Miyamoto, personal correspondence, July 2, 2025.

[11] Keita Miyamoto, *6 points lie on the same conic in a Cassinian oval configuration.* AoPS online. July 11, 2025. https://artofproblemsolving.com/community/c6h3607547p35301186

[12] Keita Miyamoto, *Proposed Problem.* Romantics of Geometry Facebook group. July 16, 2025. https://www.facebook.com/groups/parmenides52/posts/24130711623282550/

[13] Alexander Ostermann and Gerhard Wanner, Geometry by Its History, Springer, 2012.

[14] Arthur William Panton, Problem 2580, Mathematical Questions with Their Solutions, from the Educational Times **9**(1868). https://books.google.com/books?id=34xKAAAAYAAJ

[15] Stanley Rabinowitz, *Proposed Problem.* Romantics of Geometry Facebook group. June 18, 2025. https://www.facebook.com/groups/parmenides52/posts/23909635815390133/

[16] Stanley Rabinowitz, *Proposed Problem.* Romantics of Geometry Facebook group. June 24, 2025. https://www.facebook.com/groups/parmenides52/posts/23956451350708579/

[17] Marko Razpet, "Ellipse and Cassini Oval", Presek, **36**(1), 2008–2009.

[18] Marko Razpet and Izidor Hafner, "Steiner Formula for Cassini Oval" Wolfram Demonstrations Project, 2018. https://demonstrations.wolfram.com/SteinerFormulaForCassiniOval/

[19] Marko Razpet and Izidor Hafner, "2. Normal and Tangent to a Cassini Oval' Wolfram Demonstrations Project, 2018. https://demonstrations.wolfram.com/2NormalAndTangentToACassiniOval/

[20] Marko Razpet and Izidor Hafner, "2. Constructing a Point on a Cassini Oval" Wolfram Demonstrations Project, 2018. https://demonstrations.wolfram.com/2ConstructingAPointOnACassiniOval/

[21] Marko Razpet and Izidor Hafner, "1. Normal and Tangent to a Cassini Oval" Wolfram Demonstrations Project, 2018. https://demonstrations.wolfram.com/1NormalAndTangentToACassiniOval/

[22] Marko Razpet and Izidor Hafner, "3. Normal and Tangent to a Cassini Oval" Wolfram Demonstrations Project, 2018. https://demonstrations.wolfram.com/3NormalAndTangentToACassiniOval/

[23] Marko Razpet and Izidor Hafner, "3. Constructing a Point on a Cassini Oval" Wolfram Demonstrations Project, 2018.

https://demonstrations.wolfram.com/3ConstructingAPointOnACassiniOval/
[24] A. A. Savelov, Ravninske krivulje (Plane Curves), Zagreb: Ŝkolska knjiga, 1979.
[25] Wikipedia contributors. "Cassini oval." Wikipedia, The Free Encyclopedia, June 19, 2025. https://en.wikipedia.org/w/index.php?title=Cassini_oval&oldid=1296409367