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Polynomial Relations Among Segments Determined by a Triangle Center and Its Cevian

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Abstract. For a triangle center P, let the cevian AP meet BC at D. We study polynomial relations among the four lengths AP, PD, BD, and DC. Explicit formulas in the side lengths and barycentric coordinates give short algebraic verifications of identities for familiar centers. We determine the least degree of a universal polynomial relation for each of the first eleven triangle centers.

Keywords. triangle centers, cevians, barycentric coordinates, polynomial identities.

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1. The four segments

Let ABC be a nondegenerate triangle, with $a = BC$, $b = CA$, $c = AB$. Let P lie inside ABC and let its cevian AP meet BC at D. We shall use the notation

$$x = AP, \qquad y = PD, \qquad z = BD, \qquad w = DC.$$

Thus $AD = x + y$ and $a = z + w$.

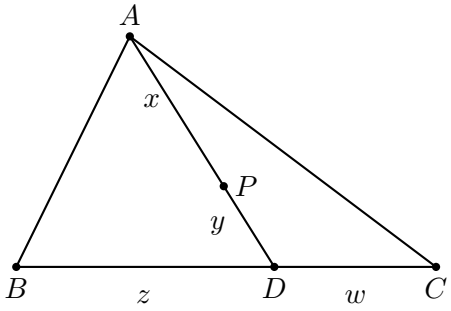


Figure 1. The four lengths $x = AP$, $y = PD$, $z = BD$, and $w = DC$.

Our purpose is to find simple polynomial relations among w, x, y, z when P is a familiar triangle center.

Suppose the barycentric coordinates of P are $(p : q : r)$, with $p, q, r > 0$, and put $h = p + q + r$. The trace D has barycentric coordinates $(0 : q : r)$. We write $s = (a + b + c)/2$ throughout.

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Theorem 1. *Let $P = (p : q : r)$ be interior to ABC, and define*

$$L = \sqrt{\frac{b^2r + c^2q}{q+r} - \frac{a^2qr}{(q+r)^2}}.$$

Then the four lengths in Figure 1 are

$$x = \frac{q+r}{h}L, \qquad y = \frac{p}{h}L, \qquad z = \frac{ar}{q+r}, \qquad w = \frac{aq}{q+r}.$$

Proof. The barycentric coordinates of D imply

$$BD = \frac{ar}{q+r}, \qquad DC = \frac{aq}{q+r}, \qquad \frac{AP}{PD} = \frac{q+r}{p}.$$

Stewart's theorem gives

$$AD^2 = \frac{b^2BD + c^2DC}{a} - BD \cdot DC = L^2.$$

Since $AD = AP + PD$, the result follows. □

Remark 2. The symbol L denotes the length AD of the complete cevian. It is introduced only to shorten the formulas; equivalently,

$$L^2 = \frac{b^2r + c^2q}{q+r} - \frac{a^2qr}{(q+r)^2}.$$

We follow Kimberling [4] in writing X_n for the nth center in the *Encyclopedia of Triangle Centers*. For concise barycentric formulas we use the Conway symbols

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \qquad S_B = \frac{c^2 + a^2 - b^2}{2}, \qquad S_C = \frac{a^2 + b^2 - c^2}{2}.$$

For use in the coordinates of the nine-point center, we define

$$S^2 = S_AS_B + S_BS_C + S_CS_A.$$

Table 1 gives explicit barycentric coordinates for the first 11 centers. Each triple may be multiplied by a common nonzero factor.

Center	Name	Barycentric coordinates $(p : q : r)$
X_1	Incenter	$(a : b : c)$
X_2	Centroid	$(1 : 1 : 1)$
X_3	Circumcenter	$(a^2S_A : b^2S_B : c^2S_C)$
X_4	Orthocenter	$(S_BS_C : S_CS_A : S_AS_B)$
X_5	Nine-point center	$(S^2 + S_BS_C : S^2 + S_CS_A : S^2 + S_AS_B)$
X_6	Symmedian point	$(a^2 : b^2 : c^2)$
X_7	Gergonne point	$((s-b)(s-c) : (s-c)(s-a) : (s-a)(s-b))$
X_8	Nagel point	$(s-a : s-b : s-c)$
X_9	Mittenpunkt	$(a(s-a) : b(s-b) : c(s-c))$
X_{10}	Spieker center	$(b+c : c+a : a+b)$
X_{11}	Feuerbach point	$((s-a)(b-c)^2 : (s-b)(c-a)^2 : (s-c)(a-b)^2)$

TABLE 1. Barycentric coordinates of the first eleven triangle centers.

The formulas apply when P is inside the triangle and its barycentric coordinates are defined. Thus, for the circumcenter and orthocenter, we consider triangles in which these centers are interior. We call a polynomial relation *universal* if it holds for every triangle in the applicable domain. Because scaling the triangle scales all four lengths equally, we need only consider homogeneous polynomials.

2. Eleven identities

The formulas in this section can be verified by substituting the barycentric coordinates from Table 1 into Theorem 1. Throughout the paper, when a proof is omitted, it is understood that the result follows by this substitution and elementary algebra.

Theorem 3. *The following identities hold for the indicated triangle centers.*

Center	*Name*	*Formula*
X_1	*Incenter*	$(x-y)zw = y^2(x+y)$
X_2	*Centroid*	$x = 2y, \qquad z = w$
X_3	*Circumcenter*	$zw = x^2 - y^2$
X_4	*Orthocenter*	$zw = y(x+y)$
X_5	*Nine-point center*	$(x+y)\big[(x-y)(x-3y) - (z-w)^2\big] + (x-y)zw = 0$
X_6	*Symmedian point*	$(2x-y)zw = y(x+y)^2$
X_7	*Gergonne point*	$(z+w)^2y^2(x+y)^2 = z^2w^2x(x+4y)$
X_8	*Nagel point*	$4xzw = (x+y)\big[(z+w)^2 - x^2\big]$
X_9	*Mittenpunkt*	$(x+y)(x-y)^2(z+w)^2\big[(x-y)zw - y^2(x+y)\big]$ $= 4x^3(x-2y)z^2w^2$
X_{10}	*Spieker center*	$(5x+y)zw = (x+y)\big[(z+w)^2 - (x-y)^2\big]$
X_{11}	*Feuerbach point*	$(x+y)(z+w)^2\Big[y(4x^2-3xy+y^2)z^2w^2$ $+ 2(x-y)^2(4x^3-2x^2y-xy^2+y^3)zw$ $+ y^2(x-y)^4(x+y)\Big]$ $= (x-y)^4(x+y)^2(z+w)^4$ $+ 16x^3(x-y)^2(x+y)z^2w^2 + 16x^3yz^3w^3$

Remark 4. The centroid equations are familiar; the circumcenter and orthocenter relations are also immediate from classical metric geometry. The interest here is the common four-length formulation and its low-degree classification, not a claim of priority for every individual identity.

3. Classification by degree

Theorem 5. *For each of the first eleven triangle centers, the least total degree of an identity given by a nonzero polynomial in x, y, z, w, holding for every triangle in the applicable domain, is as follows:*

Least degree	*Centers*
1	X_2
2	X_3, X_4
3	X_1, X_5, X_6, X_8, X_{10}
6	X_7
8	X_9
10	X_{11}

The upper bounds follow from Theorem 3. The lower bounds are proved in Appendix A.

4. A one-parameter family

The incenter, Nagel point, and Spieker center belong to the family

$$P_t = (a + ts : b + ts : c + ts), \qquad s = \frac{a+b+c}{2},$$

with $t = 0, -1, -2$, respectively. The centroid is the limiting point as $|t| \to \infty$.

Proposition 6. *Whenever P_t is a finite interior point, the four segments satisfy*

$$\big[(t^2 - t - 1)x + (t+1)^2 y\big] zw = (x+y)\left[\frac{t^2}{4}(z+w)^2 - \left(\frac{tx}{2} - (t+1)y\right)^2\right].$$

The cases $t = 0, -1, -2$ yield the incenter, Nagel, and Spieker identities in Theorem 3.

5. Special triangles

Proposition 7 (Right triangles)**.** *If $\angle A = 90^\circ$ and $P = X_6$, then*

$$x = y, \qquad zw = 4y^2.$$

For $P = X_5$ in the same triangle, $x = y$ and $z = w$.

Proposition 8 (A linear side condition)**.** *If $3a = b + c$, then*

$$\begin{aligned} P = X_1 &: \quad x = 3y, \quad zw = 2y^2, \\ P = X_8 &: \quad x = y, \quad y^2 = z^2 + w^2, \\ P = X_{10} &: \quad 3x = 5y. \end{aligned}$$

Proposition 9 (Automedian triangles)**.** *Suppose $2b^2 = a^2 + c^2$, so that the squared side lengths form an arithmetic progression with b^2 as the middle term. For $P = X_6$,*

$$(2x - y)w = (x + y)z.$$

Proposition 10 (Isosceles triangles)**.** *Suppose $b = c$ and $P = X_5$ lies inside ABC. Then*

$$z = w, \qquad z^2 = (x+y)(3y - x).$$

6. Relation to earlier work

Classical cevian formulas and barycentric coordinates are developed in Altshiller-Court [3] and Yiu [6]. Abu-Saymeh and Hajja [1] investigated triangle centers with linear side intercepts; subsequent work [2] considered related cubic loci. Rabinowitz [5] studied linear relations between squares of cevian lengths. The present question is different: it asks for homogeneous polynomial relations among the four lengths associated with *one* center and *one* cevian. The formulas are not individually claimed to be new.

7. Concluding remarks

The elementary expressions in Theorem 1 allow most identities to be checked without lengthy synthetic proofs. The more distinctive result is the exact degree classification, whose lower bounds are proved in the appendix. In particular, the degrees 6, 8, and 10 for the Gergonne, Mittenpunkt, and Feuerbach points are substantially higher than those for the other eight centers. An interesting question is whether equally simple classifications hold for further centers.

One natural extension is the family of *power points* $P_n = (a^n : b^n : c^n)$. The values $n = 0, 1, 2$ give the centroid, incenter, and symmedian point. For $n = -2$, the third Brocard point X_{76} (see [4]), substitution gives the homogeneous quintic relation

$$xzw\big[(x+y)^2 + zw\big] = (z+w)^2 y(z^2+w^2).$$

This quintic is a relation of least degree. Indeed, applying the method of Appendix A to X_{76} in degree 4 gives parity matrices of sizes 19 and 16, with respective determinants 498244 and 252724 modulo 1,000,003. Both are nonzero, so no relation of degree less than five is possible. It would also be interesting to treat centers outside the triangle using directed lengths.

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Appendix A. Proof of Theorem 5

We prove that the degrees in Theorem 5 cannot be reduced. The explicit identities establish the upper bounds. The case X_2 needs no lower-bound argument, since degree one is the smallest possible for a nonzero universal polynomial relation.

A polynomial relation can be separated into homogeneous components: scaling the triangle by an arbitrary positive factor scales x, y, z, w by that factor, so each homogeneous component must vanish separately. It therefore suffices to exclude

homogeneous relations of smaller degree. If a relation of degree $e < d$ existed, multiplication by x^{d-1-e} would give a relation of degree $d-1$. Consequently, to prove a least degree d it is enough to exclude homogeneous relations of degree $d-1$.

Lemma 11 (Separation according to parity)**.** *For any of the centers in Table 1, put*

$$\alpha = \frac{q+r}{p+q+r}, \qquad \beta = \frac{p}{p+q+r}, \qquad Q = L^2.$$

The quantities α, β, z, w, Q are rational functions of a, b, c, and $x = \alpha L$, $y = \beta L$. Moreover, $L \notin \mathbb{R}(a,b,c)$. Let F be a homogeneous polynomial in x, y, z, w, and write $F = F_0 + F_1$, where F_0 contains monomials in which the combined exponent of x and y is even and F_1 contains those in which it is odd. If F vanishes for all triangles in the applicable domain, then F_0 and F_1 each vanish identically.

Proof. The first assertions follow from Theorem 1 and Table 1; all the coordinate formulas are rational in a, b, c.
Suppose $L^2 = G^2$ for some $G \in \mathbb{R}(a,b,c)$. For each center in Table 1, the denominator of L^2 is not divisible by $b-c$. Since the reduced denominator of G^2 is the square of the reduced denominator of G, neither is the denominator of G. We may therefore specialize $c = b$. The trace D is then the midpoint of BC, so the specialization would give

$$b^2 - \frac{a^2}{4} = \left(G|_{c=b}\right)^2,$$

which is impossible: $(b - \frac{a}{2})(b + \frac{a}{2})$ is not a square in $\mathbb{R}(a,b)$. Thus $L \notin \mathbb{R}(a,b,c)$. Now put $A = F_0(\alpha L, \beta L, z, w)$ and $B = L^{-1} F_1(\alpha L, \beta L, z, w)$, both elements of $\mathbb{R}(a,b,c)$. Since F vanishes throughout the applicable nonempty open set of triangles, $A + BL = 0$ there. If B were not identically zero, we would have $L^2 = (A/B)^2$ on that open set, and hence as an identity of rational functions, a contradiction. Therefore $B = 0$ and then $A = 0$, as required. □

We next give finite exact computations excluding the required degrees. Work modulo the prime $M = 1{,}000{,}003$. For a fixed degree e, let $\mathcal{M}_0$ (respectively $\mathcal{M}_1$) be the monomials $x^i y^j z^k w^\ell$ of total degree e with $i + j$ even (respectively odd). For a triangle and a monomial in $\mathcal{M}_\varepsilon$, evaluate

$$\alpha^i \beta^j Q^{(i+j-\varepsilon)/2} z^k w^\ell.$$

This is rational, so it can be reduced modulo M whenever its denominator is not divisible by M. No square roots are involved. A nonsingular evaluation matrix for each parity implies, by Lemma 11, that no nonzero homogeneous relation of degree e can exist.
Here is a deterministic rule for selecting the rows. Enumerate all integer triangles (a,b,c) with $b < c$ and pairwise distinct sides, in increasing order of $a+b+c$ and then lexicographically by (a,b). Discard a triangle if the center is not interior or a denominator is zero modulo M. For each parity independently, retain its evaluation row only if that row increases the rank modulo M of the preceding retained rows. Stop when the number of rows equals the number of monomials. Order the monomials lexicographically by their nondecreasing sequences of indices from $\{0,1,2,3\}$, representing x, y, z, w. The determinants of the resulting square matrices are given in Table 2; in every case both are nonzero modulo M.

Center	Excluded degree	$\lvert\mathcal{M}_0\rvert$	$\det_0 \bmod M$	$\lvert\mathcal{M}_1\rvert$	$\det_1 \bmod M$
X_1	2	6	597968	4	43846
X_3	1	2	842108	2	57143
X_4	1	2	375000	2	885717
X_5	2	6	599104	4	606401
X_6	2	6	433646	4	234229
X_7	5	28	990368	28	583132
X_8	2	6	836358	4	335579
X_9	7	60	947751	60	640167
X_{10}	2	6	458227	4	742399
X_{11}	9	110	418225	110	759765

TABLE 2. Exact modular determinants proving the degree lower bounds. The entries are ordered by triangle center number.

Since a determinant that is nonzero modulo M is also nonzero over $\mathbb{Q}$, each pair of matrices has full column rank. Together with the parity lemma this excludes every polynomial relation of the specified degree. The degree-reduction observation at the beginning of the appendix completes the proof of Theorem 5. The same parity argument and deterministic row selection apply to X_{76}, using the polynomial barycentrics $(b^2c^2 : c^2a^2 : a^2b^2)$, and give the degree-four determinants reported in the concluding remarks. A Python script implementing the enumeration, row selection, modular determinants, and symbolic identity checks is available from the author.