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When Can a Triangle Center Coincide with a Vertex?

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Abstract. We examine the first 100 triangle centers from the Encyclopedia of Triangle Centers and determine when they can coincide with a vertex of the triangle.

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1. Motivation

Since the incenter of a triangle lies inside the incircle, it is clear that the incenter cannot coincide with a vertex. On the other hand, it is well known that the orthocenter of a right triangle coincides with the vertex opposite the hypotenuse.

This suggests that it would be interesting to determine which triangle centers could coincide with a vertex of the triangle. We examine the first 100 triangle centers from the Encyclopedia of Triangle Centers [1] and find conditions that would make the triangle center coincide with a vertex.

2. Methodology

Without loss of generality, we will find conditions that make a triangle center associated with $\triangle ABC$ coincide with vertex A. Let the sides of the triangle have lengths a, b, and c. Again, without loss of generality, we will assume that $a = 1$ and, if possible, we will find conditions on b and c so that the triangle center coincides with vertex A.

We will use barycentric coordinates. Let X_n denote the n-th named triangle center from the Encyclopedia of Triangle Centers [1]. From this encyclopedia, we can find the barycentric coordinates $(u : v : w)$ for X_n for each n. The condition that this triangle center coincide with vertex A is

$$v = 0 \text{ and } w = 0.$$

We thus get the barycentric coordinates of X_n and solve the system of equations

$$\begin{cases} a = 1 \\ v = 0 \\ w = 0 \end{cases}$$

for b and c. We used Mathematica to solve these equations.

3. Finding a Triangle Where X_n Coincides with Vertex A

We used the Mathematica command `FindInstance` to find an example of a triangle for which $a = 1$, $v = 0$, and $w = 0$ for center X_n. Table 1 on the next page gives the results found. If there is no entry for X_n, this means that X_n cannot coincide with a vertex of the triangle. Otherwise, we give values for b and c so that in the triangle with sides 1, b, and c, center X_n coincides with vertex A. Note that there may be other values of b and c that work; we only give one set.

The following constants are used within this table.

$k_1 \approx 0.7772523754$ is the smallest positive root of
$65536x^8 - 139264x^6 + 512x^4 + 131040x^2 - 57375 = 0$.
$k_2 \approx 0.9655712319$ is the positive root of $8x^3 + 4x^2 - 2x - 9 = 0$.
$k_3 \approx 1.183159543$ is the positive root of $512x^3 + 320x^2 - 584x - 605 = 0$.

4. Finding All Triangles Where X_n Coincides with Vertex A

We used the Mathematica command `Solve` to find all triangles for which $a = 1$, $b \leq c$, $v = 0$, and $w = 0$ for center X_n. Table 2 gives the general shape of a triangle for which X_n coincides with vertex A. In column 2, we express c in terms of b, when $a = 1$. Note that we must also have $\{1, b, c\}$ satisfying the triangle inequality. If center X_n does not appear in Table 2, but does appear in Table 1, this means that there is a unique shape triangle (up to similarity) as given in Table 1.

If two or more entries for X_n appear in Table 2, this means that there are several classes of triangles in which X_n coincides with vertex A. For example, X_{93} coincides with vertex A either when angle A is 30°, 90°, or 150°.

If the relationship between a, b, and c is simple, we give this relationship in column 3 (labeled "alternative description") in Table 2. For example, X_{88} coincides with vertex A in a triangle for which $b + c = 2a$. Such a triangle is known as an A. P. Triangle because its sides are in arithmetic progression. The relationship $a^2 = b^2 + c^2$ means that $\triangle ABC$ is a right triangle with $\angle A = 90°$.

TABLE 1

center	b	c	**center**	b	c
X_4	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	X_{59}	$\frac{3}{4}$	$\frac{3}{4}$
X_5	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	X_{64}	$\frac{1}{2}$	$\frac{1}{2}\sqrt{4\sqrt{3}-3}$
X_{13}	$\frac{1}{4}$	$\frac{1}{8}(\sqrt{61}-1)$	X_{66}	$\frac{1}{2}$	$\frac{\sqrt[4]{15}}{2}$
X_{14}	$\frac{1}{4}$	$\frac{1}{8}(\sqrt{61}+1)$	X_{67}	$\frac{7}{16}$	$\frac{1}{16}\sqrt{\frac{1}{2}\left(49+\sqrt{254941}\right)}$
X_{17}	$\frac{5}{32}$	$\frac{1}{32}\sqrt{\frac{3}{2}(691-5\sqrt{1357})}$	X_{68}	$\frac{3}{8}$	$\frac{\sqrt{119}-3}{8\sqrt{2}}$
X_{18}	$\frac{5}{32}$	$\frac{1}{32}\sqrt{\frac{3}{2}(691+5\sqrt{1357})}$	X_{70}	$\frac{1}{4}$	k_1
X_{19}	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	X_{74}	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2\sqrt{7}-1}$
X_{24}	$\frac{3}{8}$	$\frac{\sqrt{55}}{8}$	X_{79}	$\frac{7}{16}$	$\frac{1}{32}\left(\sqrt{877}-7\right)$
X_{25}	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	X_{80}	$\frac{7}{16}$	$\frac{1}{32}\left(\sqrt{877}+7\right)$
X_{26}	$\sqrt{1-\frac{1}{\sqrt{2}}}$	$\sqrt{1-\frac{1}{\sqrt{2}}}$	X_{84}	$\frac{1}{2}$	k_2
X_{27}	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	X_{87}	$\frac{17}{8}$	$\frac{17}{9}$
X_{28}	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	X_{88}	$\frac{37}{64}$	$\frac{91}{64}$
X_{29}	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	X_{90}	$\frac{3}{8}$	k_3
X_{33}	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	X_{91}	$\frac{3}{8}$	$\frac{\sqrt{119}-3}{8\sqrt{2}}$
X_{34}	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	X_{92}	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
X_{46}	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	X_{93}	$\frac{1}{4}$	$\frac{1}{4}\sqrt{\frac{3}{2}\left(11-\sqrt{21}\right)}$
X_{47}	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	X_{94}	$\frac{7}{16}$	$\frac{1}{32}\left(\sqrt{877}-7\right)$
X_{49}	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	X_{95}	$\frac{5}{16}$	$\frac{1}{16}\sqrt{153+16\sqrt{114}}$
X_{51}	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	X_{96}	$\frac{7}{32}$	$\frac{\sqrt{1999}-7}{32\sqrt{2}}$
X_{52}	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	X_{97}	$\frac{5}{16}$	$\frac{1}{16}\sqrt{153+16\sqrt{114}}$
X_{53}	$\frac{5}{16}$	$\frac{\sqrt{231}}{16}$	X_{98}	$\frac{1}{2}$	$\frac{\sqrt{2+\sqrt{7}}}{2}$
X_{54}	$\frac{5}{16}$	$\frac{1}{16}\sqrt{153+16\sqrt{114}}$	X_{99}	$\frac{3}{4}$	$\frac{3}{4}$
			X_{100}	$\frac{3}{4}$	$\frac{3}{4}$

Table 2

center	conditions on b and c when $a = 1$	alternative description
X_4	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{13}	$c = \frac{1}{2}(\sqrt{4 - 3b^2} - b)$	$\angle A = 120°$
X_{14}	$c = \frac{1}{2}(\sqrt{4 - 3b^2} + b)$	$\angle A = 60°$
X_{17}	$c = \frac{1}{\sqrt{2}}\sqrt{2 + b^2 - b\sqrt{12 - 3b^2}}$	$\angle A = 150°$
X_{18}	$c = \frac{1}{\sqrt{2}}\sqrt{2 + b^2 + b\sqrt{12 - 3b^2}}$	$\angle A = 30°$
X_{19}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{24}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{25}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{27}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{28}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{29}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{33}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{34}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{53}	$c = \sqrt{1 - b^2}$	$\angle A = 90°$
X_{54}	$c = \sqrt{\frac{1}{2} + b^2 + \frac{1}{2}\sqrt{1 + 8b^2}}$	$(b^2 - c^2)^2 = a^2(b^2 + c^2)$
X_{59}	$c = b$	isosceles triangle
X_{64}	$c = \sqrt{b^2 - 1 + 2\sqrt{1 - b^2}}$	$(a^2 - b^2 + c^2)^2 = 4a^2(a^2 - b^2)$
X_{66}	$c = \sqrt[4]{1 - b^4}$	$a^4 = b^4 + c^4$
X_{67}	$c = \frac{1}{\sqrt{2}}\sqrt{b^2 \pm \sqrt{4 - 3b^4}}$	$b^4 + c^4 - a^4 = b^2c^2$
X_{68}	$c = \sqrt{1 - b\sqrt{2 - b^2}}$	$\angle A = 45°$
X_{68}	$c = \sqrt{1 + b\sqrt{2 - b^2}}$	$\angle A = 135°$
X_{70}	Note 1	
X_{74}	$c = \sqrt{-\frac{1}{2} + b^2 + \frac{1}{2}\sqrt{9 - 8b^2}}$	$(b^2 - c^2)^2 + a^2(b^2 + c^2) = 2a^4$

Note 1: $c = \sqrt{d}$ where d is a root of
$x^4 - 2(b^2 + 1)x^3 + 2b^4x^2 + 2(1 - b^6)x + b^8 - 2b^6 + 2b^2 - 1 = 0$

TABLE 2 (continued)

center	conditions on b and c when $a=1$	alternative description
X_{79}	$c=\frac{1}{2}\left(\sqrt{4-3b^2}-b\right)$	$\angle A=120^\circ$
X_{80}	$c=\frac{1}{2}\left(\sqrt{4-3b^2}+b\right)$	$\angle A=60^\circ$
X_{84}	Note 2	
X_{87}	$c=\frac{b}{b-1}$	$\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$
X_{88}	$c=2-b$	$b+c=2a$
X_{90}	Note 3	
X_{91}	$c=\sqrt{1-b\sqrt{2-b^2}}$	$\angle A=45^\circ$
X_{91}	$c=\sqrt{1+b\sqrt{2-b^2}}$	$\angle A=135^\circ$
X_{92}	$c=\sqrt{1-b^2}$	$\angle A=90^\circ$
X_{93}	$c=\sqrt{1-b^2}$	$\angle A=90^\circ$
X_{93}	$c=\frac{1}{\sqrt{2}}\sqrt{2+b^2+b\sqrt{12-3b^2}}$	$\angle A=30^\circ$
X_{93}	$c=\frac{1}{\sqrt{2}}\sqrt{2+b^2-b\sqrt{12-3b^2}}$	$\angle A=150^\circ$
X_{94}	$c=\frac{1}{2}\left(\sqrt{4-3b^2}-b\right)$	$\angle A=120^\circ$
X_{94}	$c=\frac{1}{2}\left(\sqrt{4-3b^2}+b\right)$	$\angle A=60^\circ$
X_{95}	$c=\sqrt{\frac{1}{2}+b^2+\frac{1}{2}\sqrt{1+8b^2}}$	$(b^2-c^2)^2=a^2(b^2+c^2)$
X_{96}	$c=\sqrt{1-b\sqrt{2-b^2}}$	$\angle A=45^\circ$
X_{96}	$c=\sqrt{1+b\sqrt{2-b^2}}$	$\angle A=135^\circ$
X_{96}	$c=\sqrt{\frac{1}{2}+b^2+\frac{1}{2}\sqrt{1+8b^2}}$	$(b^2-c^2)^2=a^2(b^2+c^2)$
X_{97}	$c=\sqrt{\frac{1}{2}+b^2+\frac{1}{2}\sqrt{1+8b^2}}$	$(b^2-c^2)^2=a^2(b^2+c^2)$
X_{98}	$c=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{-4b^4+4b^2+1}}$	$b^4+c^4=a^2(b^2+c^2)$
X_{99}	$c=b$	isosceles triangle
X_{100}	$c=b$	isosceles triangle

Note 2: c is the real root of $x^3-(b-1)x^2-(b-1)^2x+b^3+b^2-b-1=0$

Note 3: c is a root of $x^3-(b-1)x^2-(b^2+1)x+b^3+b^2-b-1=0$

REFERENCES

[1] Clark Kimberling, *Encyclopedia of Triangle Centers.* http://faculty.evansville.edu/ck6/encyclopedia/ETC.html