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Inequalities For Distances Between Triangle Centers

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Abstract. In his seminal paper on triangle centers, Clark Kimberling made a number of conjectures concerning the distances between triangle centers. For example, if $D(i,j)$ denotes the distance between triangle centers X_i and X_j, Kimberling conjectured that $D(6,1) \leq D(6,3)$ for all triangles. We use symbolic mathematics techniques to prove these conjectures. In addition, we prove stronger results, using best-possible constants, such as $D(6,1) \leq (2-\sqrt{3})D(6,3)$.

Keywords. triangle geometry, triangle centers, inequalities, computer-discovered mathematics, Blundon's Fundamental Inequality GeometricExplorer.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

Let X_n denote the nth named triangle center as cataloged in the Encyclopedia of Triangle Centers [4]. Let X_iX_j denote the distance between X_i and X_j. We will also write this as $D(i,j)$.

In his seminal paper on triangle centers [3], Clark Kimberling made a number of conjectures concerning the distances between pairs of triangle centers. For example, Kimberling conjectured that $D(6,1) \leq D(6,3)$ for all triangles.

He also conjectured the truth of many chains of inequalities, such as the following.

$$X_3X_9 \leq X_3X_{10} \leq X_3X_2 \leq X_3X_{12} \leq X_3X_7 \leq X_3X_4.$$

Kimberling reached these conjectures by using a computer to examine 10,740 different shaped triangles and numerically computing the coordinates for the centers. Upon determining that the inequality held for each of these 10,740 triangles, he then conjectured that the inequality was true for all triangles.

With the advances in computers and symbolic algebra systems, it is now possible to prove these conjectures using exact symbolic computation.

2. Barycentric Coordinates

We use barycentric coordinates in this study. The barycentric coordinates for triangle centers X_1 through X_{20} in terms of the sides of the triangle, a, b, and c, are shown in Table 1, where

$$S = \frac{1}{2}\sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}.$$

Only the first barycentric coordinate is given, because if $f(a, b, c)$ is the first barycentric coordinate for a point P, then the barycentric coordinates for P are

$$\Big(f(a, b, c) : f(b, c, a) : f(c, a, b)\Big).$$

These were derived from [4].

Table 1. Barycentric coordinates for the first 20 centers

n	first barycentric coordinate for X_n
1	a
2	1
3	$a^2(a^2-b^2-c^2)$
4	$(a^2+b^2-c^2)(a^2-b^2+c^2)$
5	$c^4-a^2b^2+b^4-a^2c^2-2b^2c^2$
6	a^2
7	$(a+b-c)(a-b+c)$
8	$a-b-c$
9	$a(a-b-c)$
10	$b+c$
11	$(b-c)^2(-a+b+c)$
12	$(a+b-c)(a-b+c)(b+c)^2$
13	$a^4-2(b^2-c^2)^2+a^2(b^2+c^2+2\sqrt{3}S)$
14	$a^4-2(b^2-c^2)^2+a^2(b^2+c^2-2\sqrt{3}S)$
15	$a^2(\sqrt{3}(a^2-b^2-c^2)-2S)$
16	$a^2(\sqrt{3}(a^2-b^2-c^2)+2S)$
17	$(a^2+b^2-c^2+2\sqrt{3}S)(a^2-b^2+c^2+2\sqrt{3}S)$
18	$(a^2+b^2-c^2-2\sqrt{3}S)(a^2-b^2+c^2-2\sqrt{3}S)$
19	$a(a^2+b^2-c^2)(a^2-b^2+c^2)$
20	$3a^4-2a^2b^2-b^4-2a^2c^2+2b^2c^2-c^4$

To find the distance between two centers, we used the following formula which comes from [2].

Proposition 1. *Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates. Denote $x = u_1 - u_2$, $y = v_1 - v_2$ and $z = w_1 - w_2$. Then the distance between P and Q is*

$$\sqrt{-a^2yz - b^z x - c^2xy}.$$

3. Graphs

For n, i, and j ranging from 1 to 20, we used Algorithm B from [5] to check every inequality of the form $D(n,i) \le D(n,j)$. Algorithm B is based on Blundon's Fundamental Inequality [1]. Figure n shows a graph of the results. An arrow from node i to node j means that $D(n,i) \le D(n,j)$ for all triangles. No arrow means the inequality does not hold for all triangles. Since we used exact symbolic computations, these results are theorems and not conjectures. To avoid radicals, we replaced inequalities of the form $D(a,b) \le D(c,d)$ by the equivalent inequality $D(a,b)^2 \le D(c,d)^2$.

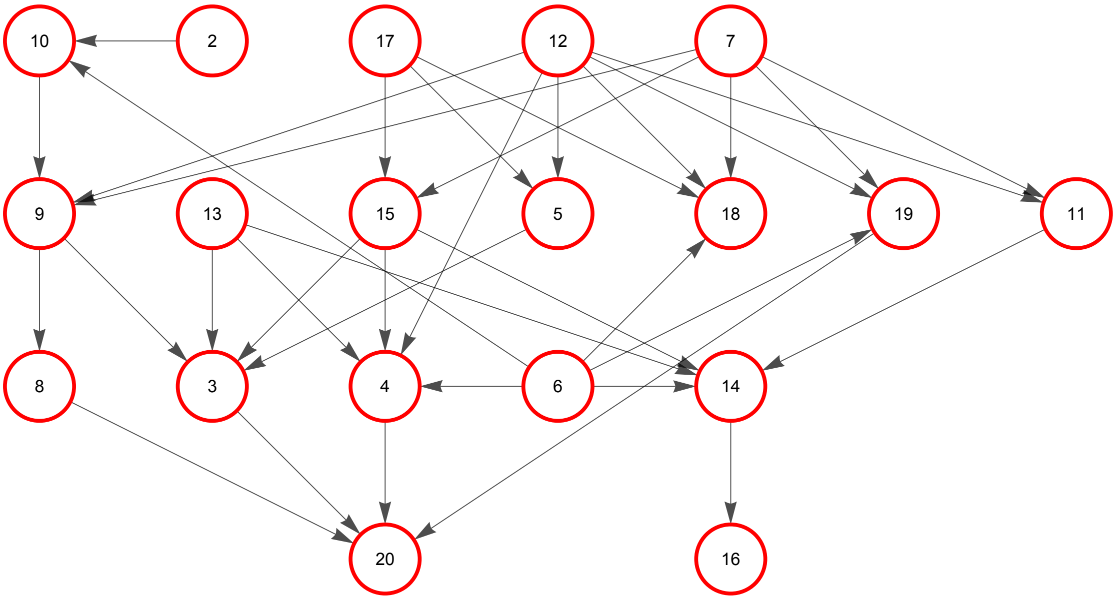

FIGURE 1. X_1 inequalities. An arrow from i to j means $X_1X_i \le X_1X_j$.

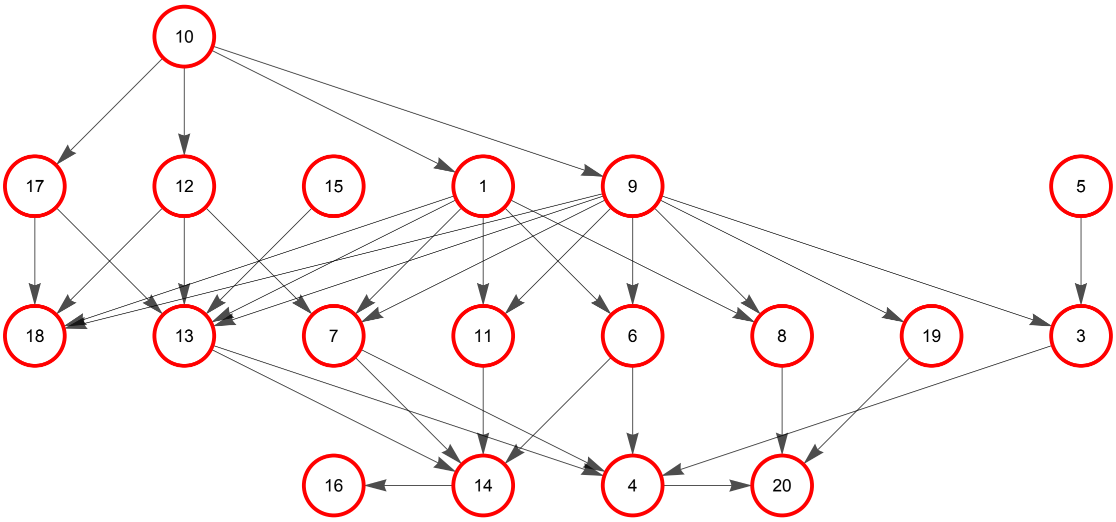

FIGURE 2. X_2 inequalities. An arrow from i to j means $X_2X_i \le X_2X_j$.

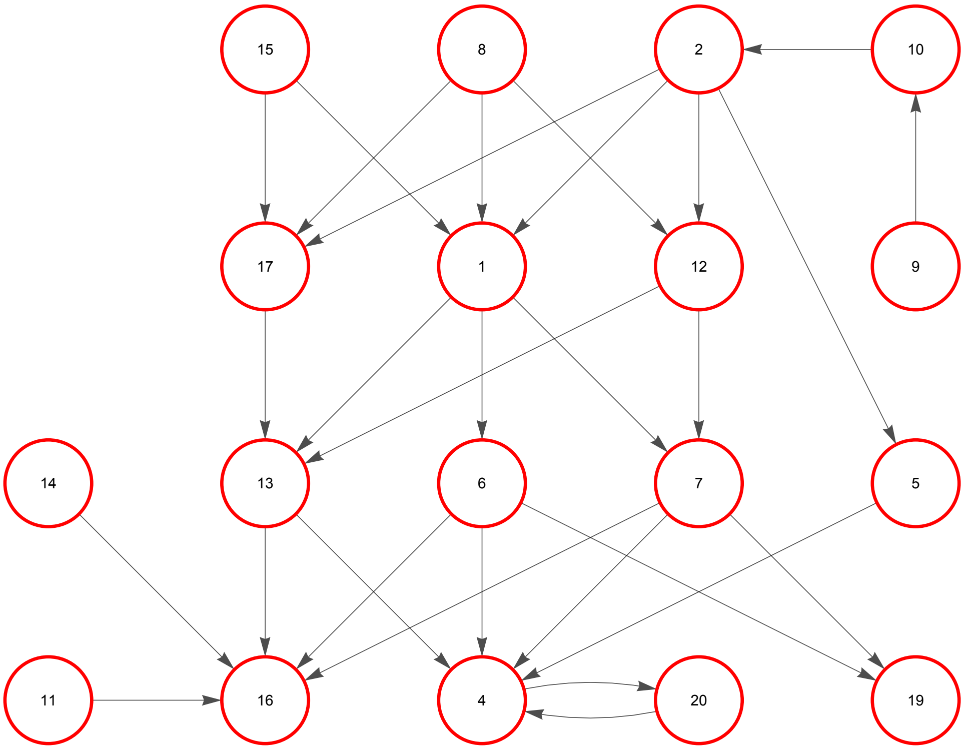

FIGURE 3. X_3 inequalities. An arrow from i to j means $X_3X_i \le X_3X_j$.

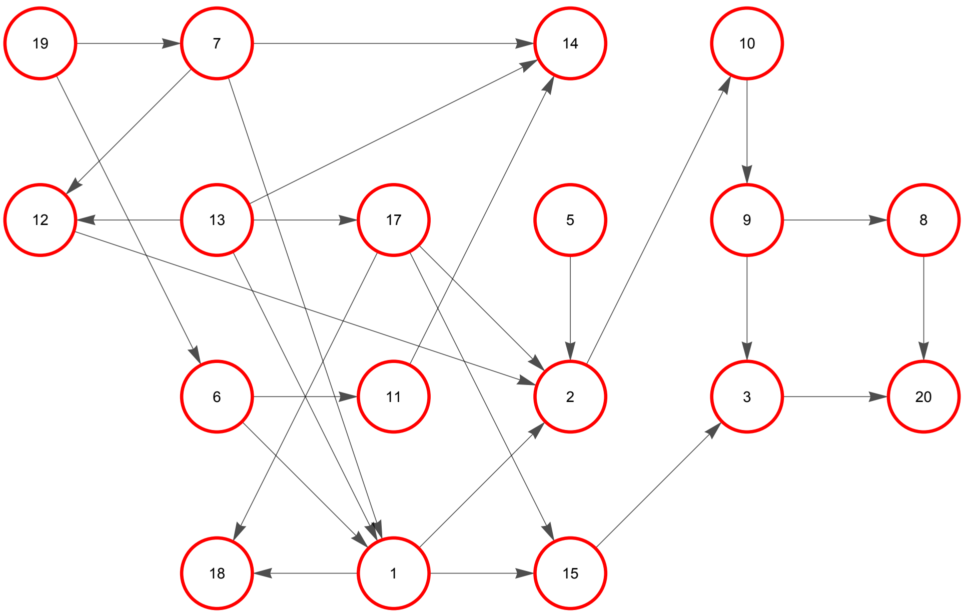

FIGURE 4. X_4 inequalities. An arrow from i to j means $X_4X_i \le X_4X_j$.

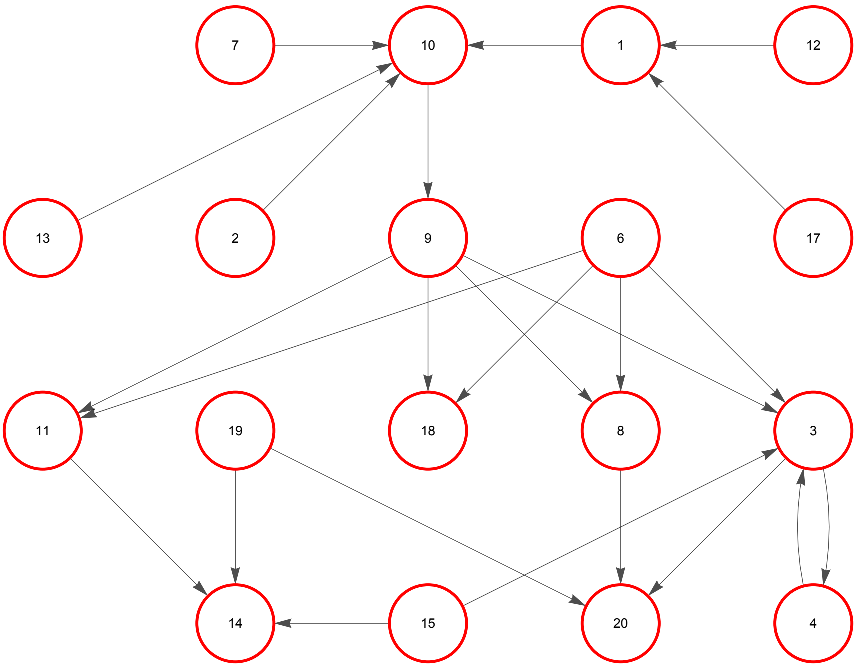

FIGURE 5. X_5 inequalities. An arrow from i to j means $X_5X_i \leq X_5X_j$.

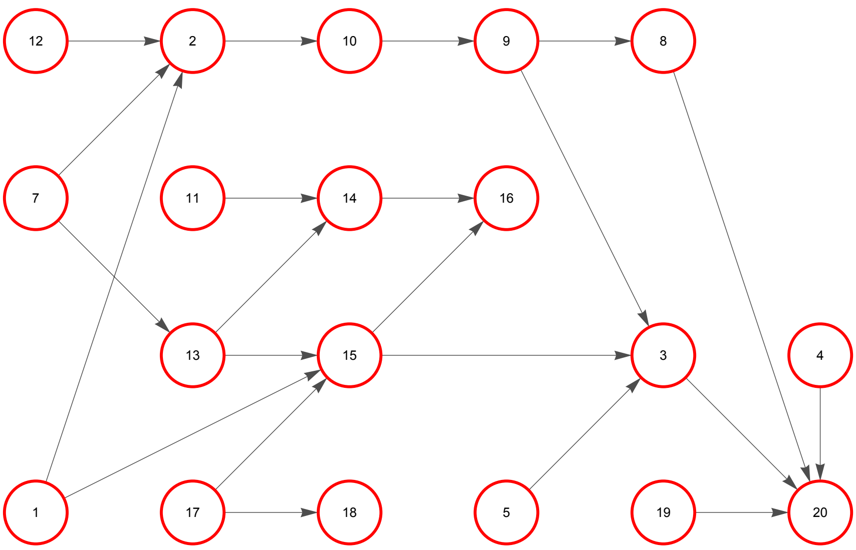

FIGURE 6. X_6 inequalities. An arrow from i to j means $X_6X_i \leq X_6X_j$.

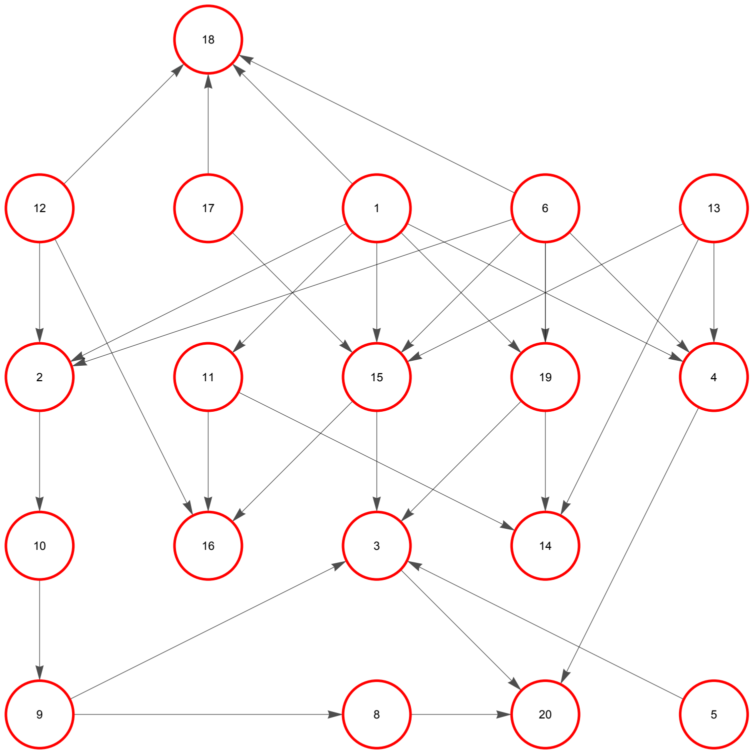

FIGURE 7. X_7 inequalities. An arrow from i to j means $X_7X_i \le X_7X_j$.

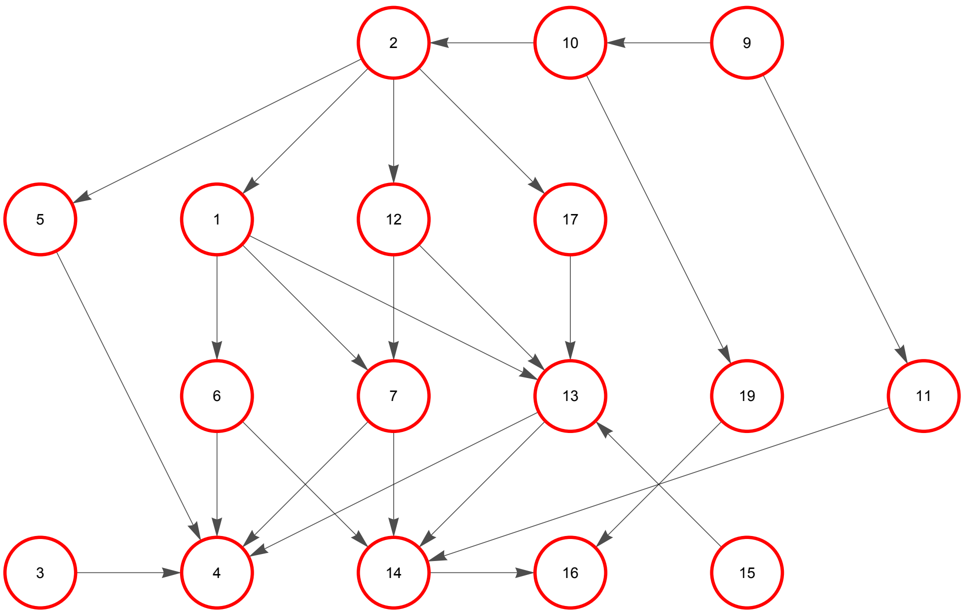

FIGURE 8. X_8 inequalities. An arrow from i to j means $X_8X_i \le X_8X_j$.

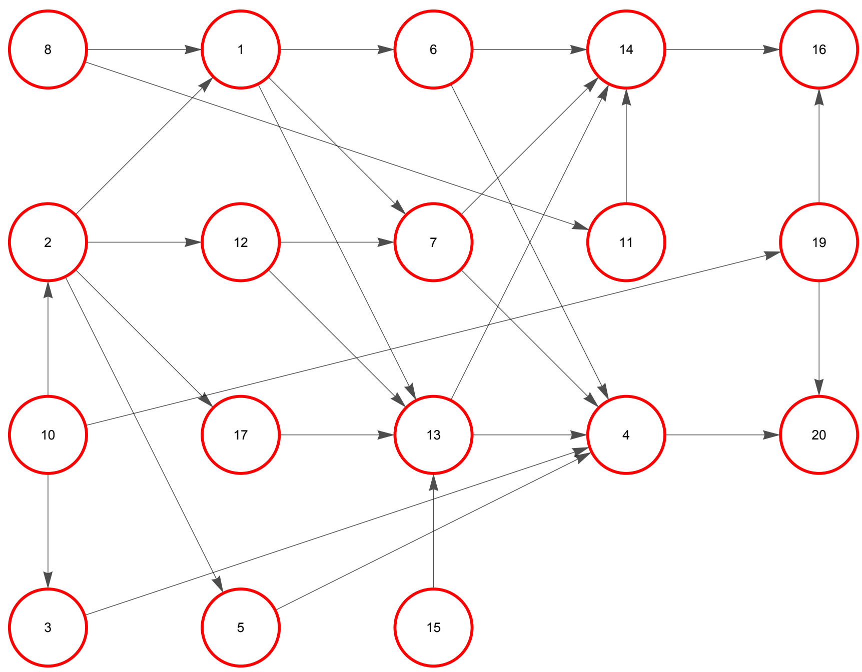

FIGURE 9. X_9 inequalities. An arrow from i to j means $X_9X_i \leq X_9X_j$.

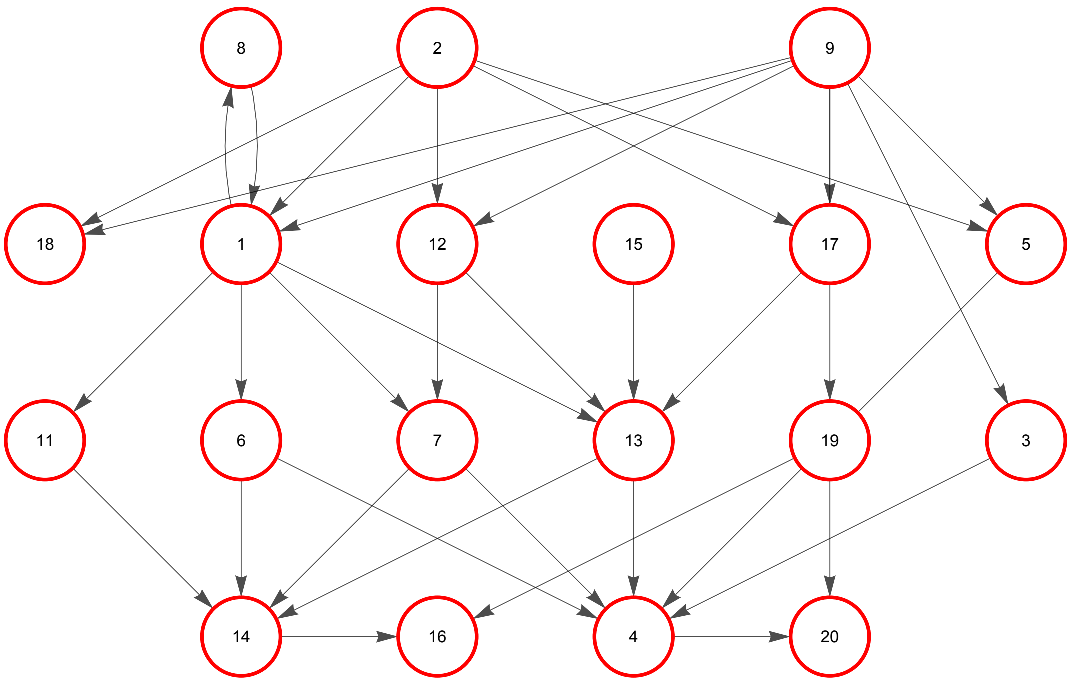

FIGURE 10. X_{10} inequalities. An arrow from i to j means $X_{10}X_i \leq X_{10}X_j$.

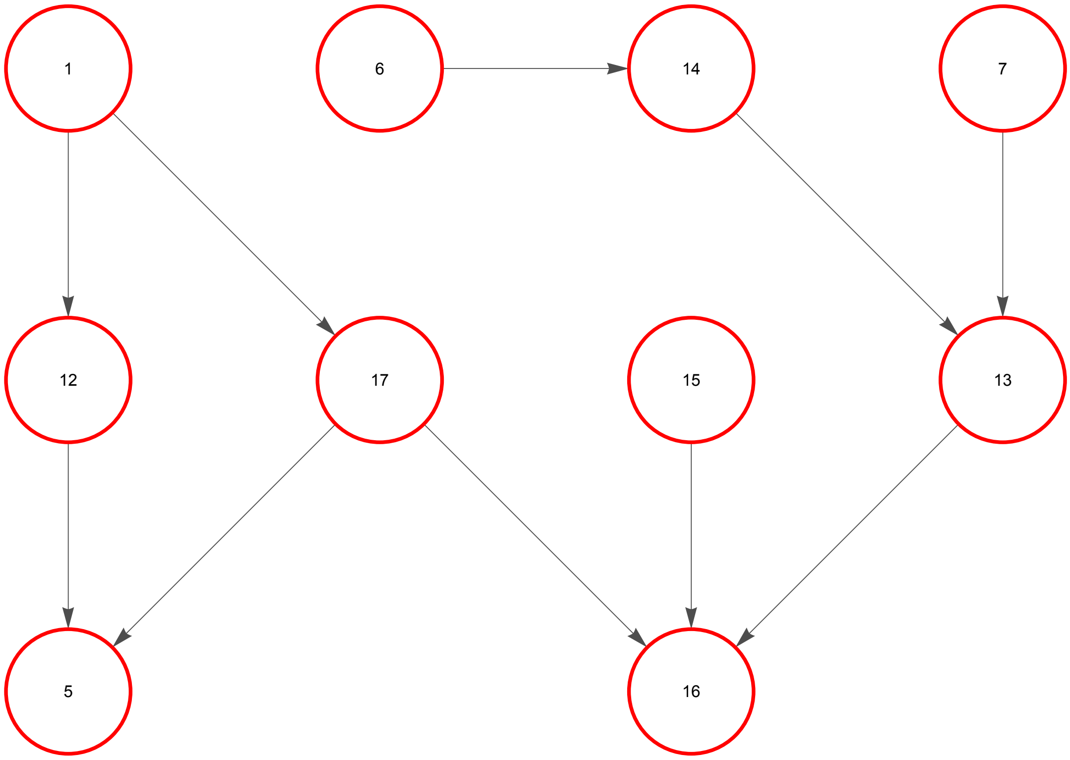

Figure 11. X_{11} inequalities. An arrow from i to j means $X_{11}X_i \leq X_{11}X_j$.

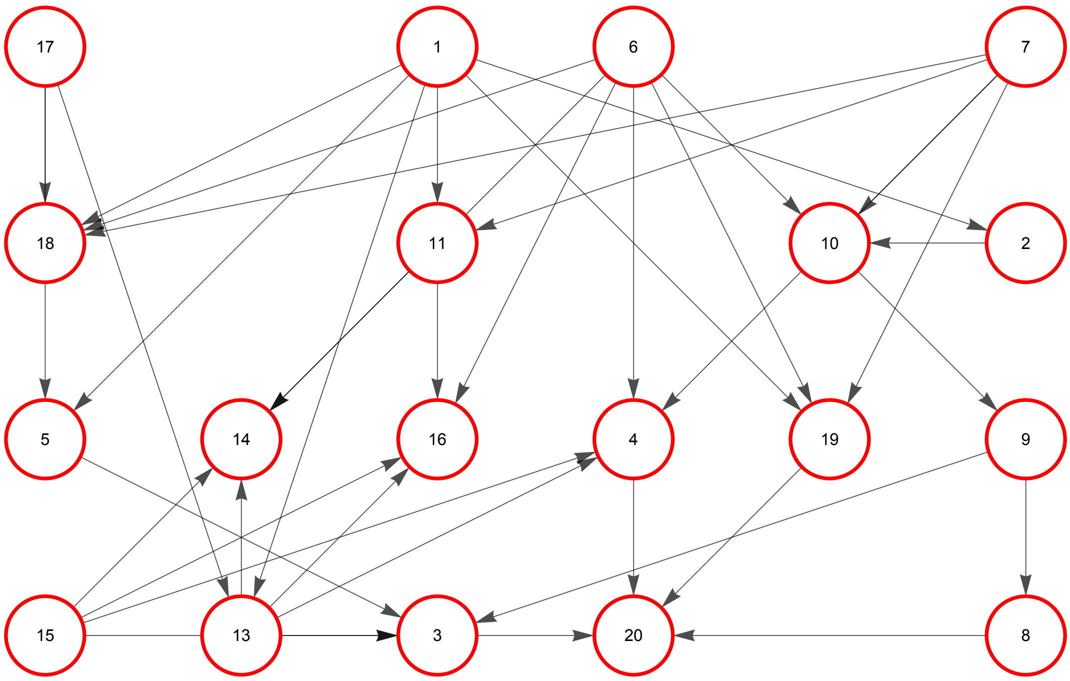

Figure 12. X_{12} inequalities. An arrow from i to j means $X_{12}X_i \leq X_{12}X_j$.

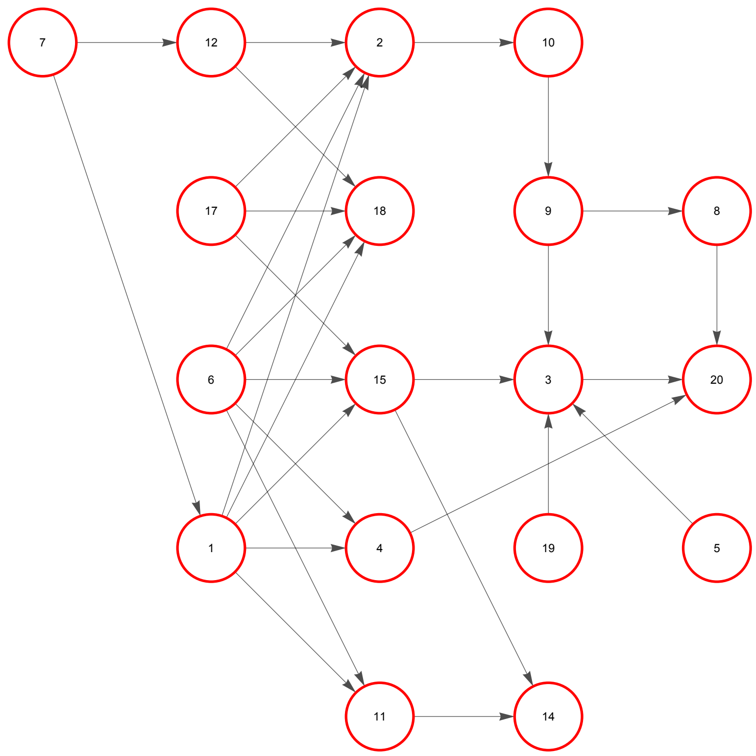

FIGURE 13. X_{13} inequalities. An arrow from i to j means $X_{13}X_i \leq X_{13}X_j$.

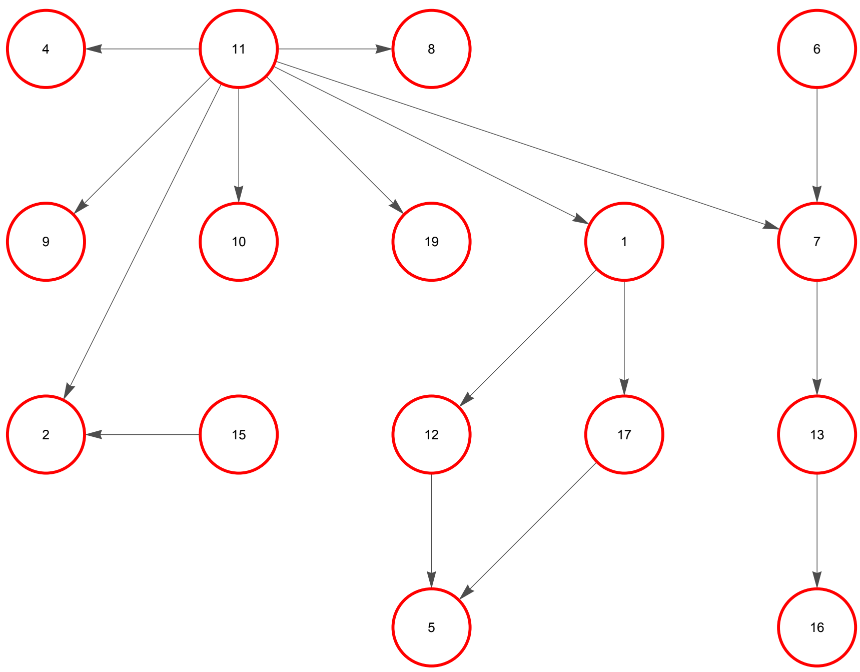

FIGURE 14. X_{14} inequalities. An arrow from i to j means $X_{14}X_i \leq X_{14}X_j$.

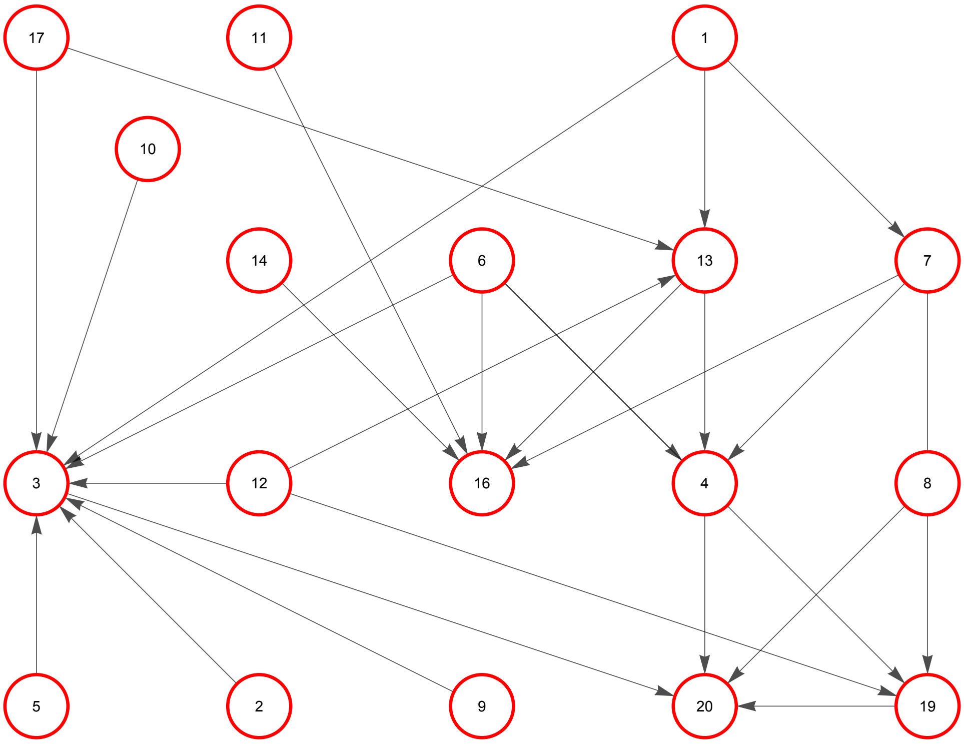

Figure 15. X_{15} inequalities. An arrow from i to j means $X_{15}X_i \leq X_{15}X_j$.

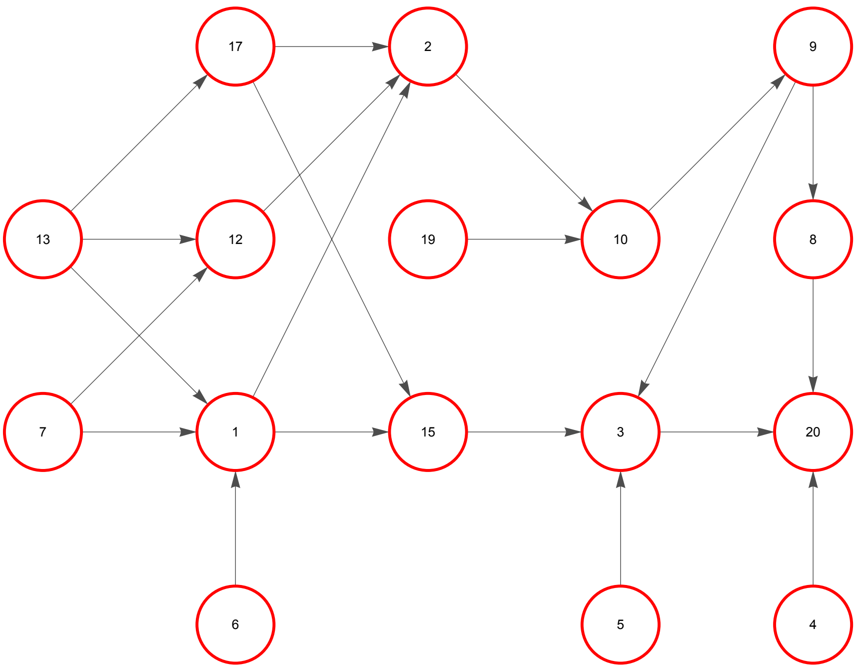

Figure 16. X_{16} inequalities. An arrow from i to j means $X_{16}X_i \leq X_{16}X_j$.

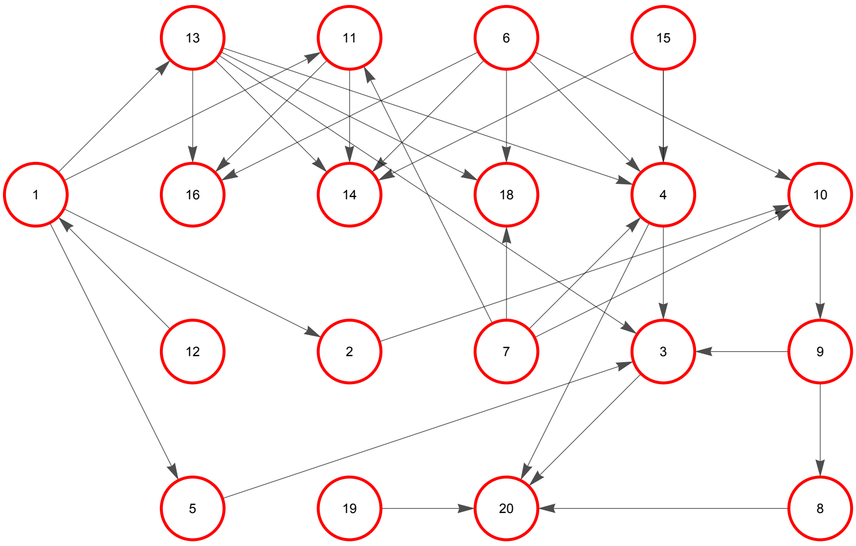

FIGURE 17. X_{17} inequalities. An arrow from i to j means $X_{17}X_i \leq X_{17}X_j$.

There were no inequalities found for $n = 18$. In other words, there were no inequalities of the form $D(18, i) \leq D(18, j)$ for any i and j with $1 \leq i \leq 20$, $1 \leq j \leq 20$, $i \neq 18$, $j \neq 18$, and $i \neq j$.

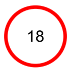

FIGURE 18. There are no inequalities of the form $X_{18}X_i \leq X_{18}X_j$.

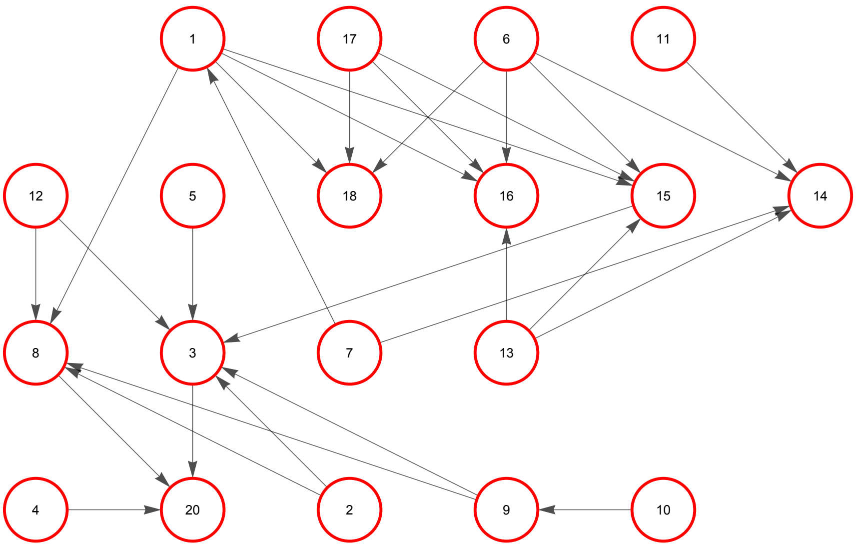

FIGURE 19. X_{19} inequalities. An arrow from i to j means $X_{19}X_i \leq X_{19}X_j$.

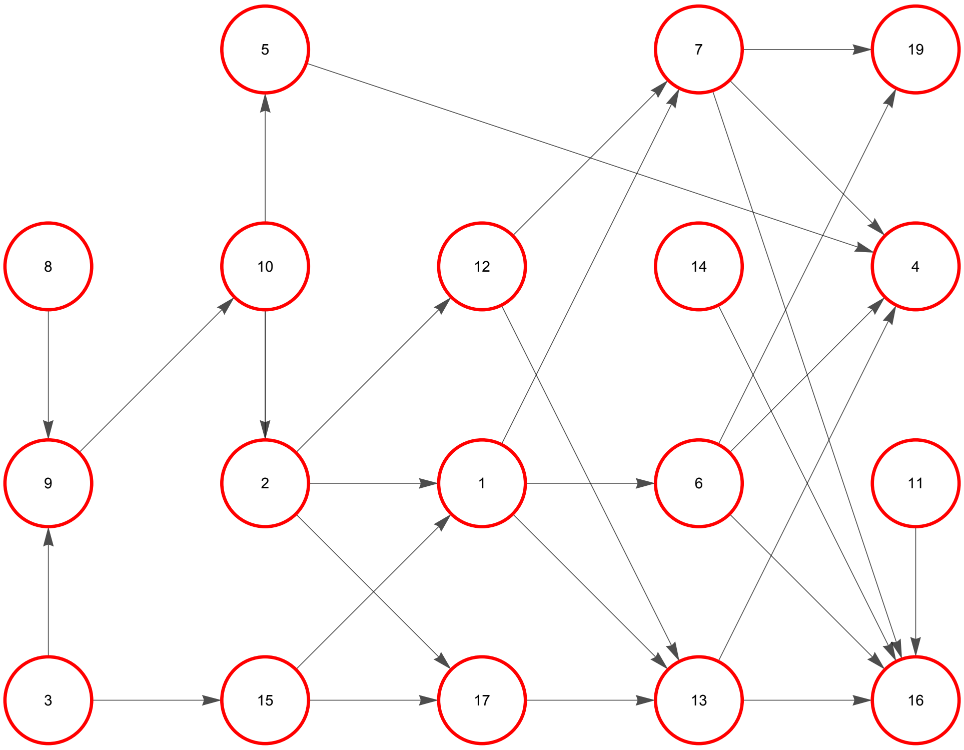

Figure 20. X_{20} inequalities. An arrow from i to j means $X_{20}X_i \le X_{20}X_j$.

Examining these graphs, we note that there are a few loops. An arrow from i to j and an arrow from j to i in Figure n means that $D(n,i) \le D(n,j)$ and $D(n,j) \le D(n,i)$. This implies that $D(n,i) = D(n,j)$. Three such equalities were noticed: $D(1,10) = D(8,10)$, $D(3,4) = D(3,20)$, and $D(3,5) = D(4,5)$.

These equalities were noticed by Kimberling in [3, Table 5.4]. These correspond to the (now) well-known facts that in all triangles, X_{10} is the midpoint of $\overline{X_1X_8}$, X_3 is the midpoint of $\overline{X_4X_{20}}$, and X_5 is the midpoint of $\overline{X_3X_4}$.

Since we only investigated inequalities between distances formed by three triangle centers, this does not mean that we can conclude that there aren't any other equalities of the form $D(i_1,i_2) = D(i_3,i_4)$, where i_1, i_2, i_3, and i_4 are all distinct.

To check for such equalities, we ran a separate Mathematica program that examined all distances of the form $D(i,j)$ where i and j are distinct integers between 1 and 20, looking for duplicate distances. No new equalities were found. This lets us state the following result.

Proposition 2. *The only pairs of centers from among the first 20 centers that have equal distances are the following.*

$$D(1,10) = D(8,10)$$
$$D(3,4) = D(3,20)$$
$$D(3,5) = D(4,5)$$

4. Bounds

Some of the inequalities from Section 3 can be strengthened. For example, from Figure 6, one can see that $D(6,2) \le D(6,10)$. However, the stronger inequality

$$D(6,2) \le \frac{1}{3}\left(1+\sqrt{2}\right) D(6,10)$$

is true. To find the best such inequalities, we applied Algorithm K from [5] to every inequality of the form

$$D(n,i) \le kD(n,j) \quad \text{or} \quad D(n,i) \ge kD(n,j)$$

for n, i, and j ranging from 1 to 10 with $i < j$ to find the smallest (resp. largest) constant k making the inequality true. The results are given below, shown as lower and upper bounds for $\dfrac{D(n,i)}{D(n,j)}$. Lower bounds of 0 and upper bounds of ∞ are omitted.

For example, $0 \le \dfrac{D(1,2)}{D(1,4)} \le \infty$ would mean that Algorithm K proved that there is no constant $k > 0$ such that $k \le \dfrac{D(1,2)}{D(1,4)}$ is true for all triangles, and that there is no constant k such that $\dfrac{D(1,2)}{D(1,4)} \le k$ is true for all triangles.

Theorem 1. *The following bounds involving distances from X_1 hold for all triangles.*

$$\frac{D(1,2)}{D(1,3)} \le \frac{2}{3}$$
$$\frac{2\sqrt{2}}{3} \le \frac{D(1,2)}{D(1,6)}$$
$$\frac{1}{3} \le \frac{D(1,2)}{D(1,7)}$$
$$\frac{D(1,2)}{D(1,8)} = \frac{1}{3}$$
$$\frac{1}{3} \le \frac{D(1,2)}{D(1,9)} \le \frac{2}{3}$$
$$\frac{D(1,2)}{D(1,10)} = \frac{2}{3}$$
$$\frac{1}{2} \le \frac{D(1,3)}{D(1,4)}$$
$$2 \le \frac{D(1,3)}{D(1,5)}$$

$$1+\sqrt{3} \le \frac{D(1,3)}{D(1,6)}$$
$$\frac{3}{2}+\sqrt{2} \le \frac{D(1,3)}{D(1,7)}$$
$$\frac{1}{2} \le \frac{D(1,3)}{D(1,8)}$$
$$1 \le \frac{D(1,3)}{D(1,9)}$$
$$1 \le \frac{D(1,3)}{D(1,10)}$$
$$1 \le \frac{D(1,4)}{D(1,6)}$$
$$2 \le \frac{D(1,4)}{D(1,7)}$$
$$\frac{1}{3} \le \frac{D(1,6)}{D(1,7)}$$

$$\frac{D(1,6)}{D(1,8)} \le \frac{1}{2\sqrt{2}}$$
$$\frac{D(1,6)}{D(1,9)} \le \frac{1}{2}$$
$$\frac{D(1,6)}{D(1,10)} \le \frac{1}{\sqrt{2}}$$
$$\frac{D(1,7)}{D(1,8)} \le 1$$
$$\frac{D(1,7)}{D(1,9)} \le 1$$
$$\frac{D(1,7)}{D(1,10)} \le 2$$
$$1 \le \frac{D(1,8)}{D(1,9)} \le 2$$
$$\frac{D(1,8)}{D(1,10)} = 2$$
$$1 \le \frac{D(1,9)}{D(1,10)} \le 2$$

Theorem 2. *The following bounds involving distances from X_2 hold for all triangles.*

$$\frac{D(2,1)}{D(2,3)} \le 2$$
$$\frac{D(2,1)}{D(2,4)} \le 1$$
$$\frac{D(2,1)}{D(2,5)} \le 4$$
$$6\sqrt{2}-8 \le \frac{D(2,1)}{D(2,6)} \le 1$$
$$\frac{1}{4} \le \frac{D(2,1)}{D(2,7)} \le 1$$
$$\frac{D(2,1)}{D(2,8)} = \frac{1}{2}$$
$$\frac{1}{2} \le \frac{D(2,1)}{D(2,9)} \le 2$$
$$\frac{D(2,1)}{D(2,10)} = 2$$
$$\frac{D(2,3)}{D(2,4)} = \frac{1}{2}$$
$$\frac{D(2,3)}{D(2,5)} = 2$$
$$\frac{1}{2} \le \frac{D(2,3)}{D(2,6)}$$
$$\frac{1}{2} \le \frac{D(2,3)}{D(2,7)}$$
$$\frac{1}{4} \le \frac{D(2,3)}{D(2,8)}$$
$$1 \le \frac{D(2,3)}{D(2,9)}$$
$$1 \le \frac{D(2,3)}{D(2,10)}$$
$$\frac{D(2,4)}{D(2,5)} = 4$$
$$1 \le \frac{D(2,4)}{D(2,6)}$$
$$1 \le \frac{D(2,4)}{D(2,7)}$$
$$\frac{1}{2} \le \frac{D(2,4)}{D(2,8)}$$
$$2 \le \frac{D(2,4)}{D(2,9)}$$
$$2 \le \frac{D(2,4)}{D(2,10)}$$
$$\frac{1}{4} \le \frac{D(2,5)}{D(2,6)}$$
$$\frac{1}{4} \le \frac{D(2,5)}{D(2,7)}$$
$$\frac{1}{8} \le \frac{D(2,5)}{D(2,8)}$$
$$\frac{1}{2} \le \frac{D(2,5)}{D(2,9)}$$
$$\frac{1}{2} \le \frac{D(2,5)}{D(2,10)}$$
$$\frac{1}{2} \le \frac{D(2,6)}{D(2,7)} \le \frac{3}{2}$$
$$\frac{1}{2} \le \frac{D(2,6)}{D(2,8)} \le \frac{4+3\sqrt{2}}{8}$$
$$1 \le \frac{D(2,6)}{D(2,9)} \le 3$$
$$2 \le \frac{D(2,6)}{D(2,10)} \le 2+\frac{3}{\sqrt{2}}$$
$$\frac{1}{2} \le \frac{D(2,7)}{D(2,8)} \le 2$$
$$\frac{D(2,7)}{D(2,9)} = 2$$
$$2 \le \frac{D(2,7)}{D(2,10)} \le 8$$
$$1 \le \frac{D(2,8)}{D(2,9)} \le 4$$
$$\frac{D(2,8)}{D(2,10)} = 4$$
$$1 \le \frac{D(2,9)}{D(2,10)} \le 4$$

Theorem 3. *The following bounds involving distances from X_3 hold for all triangles.*

$$1 \le \frac{D(3,1)}{D(3,2)} \le 3$$
$$\frac{1}{3} \le \frac{D(3,1)}{D(3,4)} \le 1$$
$$\frac{2}{3} \le \frac{D(3,1)}{D(3,5)} \le 2$$
$$\sqrt{3}-1 \le \frac{D(3,1)}{D(3,6)} \le 1$$
$$\frac{1}{17}\left(7+4\sqrt{2}\right) \le \frac{D(3,1)}{D(3,7)} \le 1$$
$$1 \le \frac{D(3,1)}{D(3,8)}$$
$$1 \le \frac{D(3,1)}{D(3,9)}$$
$$1 \le \frac{D(3,1)}{D(3,10)}$$
$$\frac{D(3,2)}{D(3,4)} = \frac{1}{3}$$
$$\frac{D(3,2)}{D(3,5)} = \frac{2}{3}$$
$$\frac{1}{3} \le \frac{D(3,2)}{D(3,6)} \le 1$$

$$\frac{1}{3} \le \frac{D(3,2)}{D(3,7)} \le 1$$
$$\frac{1}{3} \le \frac{D(3,2)}{D(3,8)}$$
$$1 \le \frac{D(3,2)}{D(3,9)}$$
$$1 \le \frac{D(3,2)}{D(3,10)}$$
$$\frac{D(3,4)}{D(3,5)} = 2$$
$$1 \le \frac{D(3,4)}{D(3,6)} \le 3$$
$$1 \le \frac{D(3,4)}{D(3,7)} \le 3$$
$$1 \le \frac{D(3,4)}{D(3,8)}$$
$$3 \le \frac{D(3,4)}{D(3,9)}$$
$$3 \le \frac{D(3,4)}{D(3,10)}$$
$$\frac{1}{2} \le \frac{D(3,5)}{D(3,6)} \le \frac{3}{2}$$

$$\frac{1}{2} \le \frac{D(3,5)}{D(3,7)} \le \frac{3}{2}$$
$$\frac{1}{2} \le \frac{D(3,5)}{D(3,8)}$$
$$\frac{3}{2} \le \frac{D(3,5)}{D(3,9)}$$
$$\frac{3}{2} \le \frac{D(3,5)}{D(3,10)}$$
$$C_1 \le \frac{D(3,6)}{D(3,7)} \le C_2$$
$$1 \le \frac{D(3,6)}{D(3,8)}$$
$$1 \le \frac{D(3,6)}{D(3,9)}$$
$$1 \le \frac{D(3,6)}{D(3,10)}$$
$$1 \le \frac{D(3,7)}{D(3,8)}$$
$$1 \le \frac{D(3,7)}{D(3,9)}$$
$$1 \le \frac{D(3,7)}{D(3,10)}$$
$$\frac{1}{2} \le \frac{D(3,9)}{D(3,10)} \le 1$$

where $C_1 \approx 0.9002270330$ is the second largest root of

$$6137x^5 - 14689x^4 + 14429x^3 - 9547x^2 + 3698x - 100$$

and $C_2 \approx 1.100851119$ is the largest root of the same polynomial.

Theorem 4. *The following bounds involving distances from X_4 hold for all triangles.*

$$\frac{D(4,1)}{D(4,2)} \le 1$$
$$\frac{D(4,1)}{D(4,3)} \le \frac{2}{3}$$
$$\frac{D(4,1)}{D(4,5)} \le \frac{4}{3}$$
$$1 \le \frac{D(4,1)}{D(4,6)}$$
$$1 \le \frac{D(4,1)}{D(4,7)} \le 2$$
$$\frac{D(4,1)}{D(4,8)} \le 1$$
$$\frac{D(4,1)}{D(4,9)} \le 1$$
$$\frac{D(4,1)}{D(4,10)} \le 1$$
$$\frac{D(4,2)}{D(4,3)} = \frac{2}{3}$$
$$\frac{D(4,2)}{D(4,5)} = \frac{4}{3}$$
$$1 \le \frac{D(4,2)}{D(4,6)}$$
$$1 \le \frac{D(4,2)}{D(4,7)}$$
$$\frac{1}{3} \le \frac{D(4,2)}{D(4,8)} \le 1$$
$$\frac{2}{3} \le \frac{D(4,2)}{D(4,9)} \le 1$$
$$\frac{2}{3} \le \frac{D(4,2)}{D(4,10)} \le 1$$
$$\frac{D(4,3)}{D(4,5)} = 2$$
$$\frac{3}{2} \le \frac{D(4,3)}{D(4,6)}$$
$$\frac{3}{2} \le \frac{D(4,3)}{D(4,7)}$$
$$\frac{1}{2} \le \frac{D(4,3)}{D(4,8)} \le \frac{3}{2}$$
$$1 \le \frac{D(4,3)}{D(4,9)} \le \frac{3}{2}$$
$$1 \le \frac{D(4,3)}{D(4,10)} \le \frac{3}{2}$$
$$\frac{3}{4} \le \frac{D(4,5)}{D(4,6)}$$
$$\frac{3}{4} \le \frac{D(4,5)}{D(4,7)}$$
$$\frac{1}{4} \le \frac{D(4,5)}{D(4,8)} \le \frac{3}{4}$$
$$\frac{1}{2} \le \frac{D(4,5)}{D(4,9)} \le \frac{3}{4}$$
$$\frac{1}{2} \le \frac{D(4,5)}{D(4,10)} \le \frac{3}{4}$$
$$\frac{D(4,6)}{D(4,7)} \le C_3$$
$$\frac{D(4,6)}{D(4,8)} \le 1$$
$$\frac{D(4,6)}{D(4,9)} \le 1$$
$$\frac{D(4,6)}{D(4,10)} \le 1$$
$$\frac{D(4,7)}{D(4,8)} \le 1$$
$$\frac{D(4,7)}{D(4,9)} \le 1$$
$$\frac{D(4,7)}{D(4,10)} \le 1$$
$$1 \le \frac{D(4,8)}{D(4,9)} \le 2$$
$$1 \le \frac{D(4,8)}{D(4,10)} \le 2$$
$$1 \le \frac{D(4,9)}{D(4,10)} \le \frac{10}{9}$$

where $C_3 \approx 1.104068697$ is the positive root of $8x^4 - 36x^3 + 113x^2 - 69x - 25$.

Theorem 5. *The following bounds involving distances from X_5 hold for all triangles.*

$$\frac{D(5,1)}{D(5,2)} \le 3$$
$$\frac{D(5,1)}{D(5,3)} \le 1$$
$$\frac{D(5,1)}{D(5,4)} \le 1$$
$$\frac{D(5,1)}{D(5,8)} \le 1$$
$$\frac{D(5,1)}{D(5,9)} \le 1$$
$$\frac{D(5,1)}{D(5,10)} \le 1$$
$$\frac{D(5,2)}{D(5,3)} = \frac{1}{3}$$
$$\frac{D(5,2)}{D(5,4)} = \frac{1}{3}$$
$$\frac{1}{3} \le \frac{D(5,2)}{D(5,6)}$$
$$\frac{1}{3} \le \frac{D(5,2)}{D(5,7)}$$
$$\frac{1}{9} \le \frac{D(5,2)}{D(5,8)} \le 1$$
$$\frac{1}{3} \le \frac{D(5,2)}{D(5,9)} \le 1$$
$$\frac{1}{3} \le \frac{D(5,2)}{D(5,10)} \le 1$$
$$\frac{D(5,3)}{D(5,4)} = 1$$
$$1 \le \frac{D(5,3)}{D(5,6)}$$
$$1 \le \frac{D(5,3)}{D(5,7)}$$
$$\frac{1}{3} \le \frac{D(5,3)}{D(5,8)} \le 3$$
$$1 \le \frac{D(5,3)}{D(5,9)} \le 3$$
$$1 \le \frac{D(5,3)}{D(5,10)} \le 3$$
$$1 \le \frac{D(5,4)}{D(5,6)}$$
$$1 \le \frac{D(5,4)}{D(5,7)}$$
$$\frac{1}{3} \le \frac{D(5,4)}{D(5,8)} \le 3$$
$$1 \le \frac{D(5,4)}{D(5,9)} \le 3$$
$$1 \le \frac{D(5,4)}{D(5,10)} \le 3$$
$$\frac{D(5,6)}{D(5,8)} \le 1$$
$$\frac{D(5,6)}{D(5,9)} \le C_4$$
$$\frac{D(5,6)}{D(5,10)} \le C_5$$
$$\frac{D(5,7)}{D(5,8)} \le 1$$
$$\frac{D(5,7)}{D(5,9)} \le 1$$
$$\frac{D(5,7)}{D(5,10)} \le 1$$
$$1 \le \frac{D(5,8)}{D(5,9)} \le 3$$
$$1 \le \frac{D(5,8)}{D(5,10)} \le 3$$
$$1 \le \frac{D(5,9)}{D(5,10)} \le 7 - 4\sqrt{2}$$

where $C_4 \approx 1.053322135$ is the positive root of

$$6137x^5 + 5335x^4 + 678x^3 - 3702x^2 - 9479x - 1225$$

and $C_5 \approx 1.194505073$ is the positive root of

$$x^4 + 2x^3 + 22x^2 - 30x - 1.$$

Theorem 6. *The following bounds involving distances from X_6 hold for all triangles.*

$$\frac{D(6,1)}{D(6,2)} \le 9-6\sqrt{2}$$
$$\frac{D(6,1)}{D(6,3)} \le 2-\sqrt{3}$$
$$\frac{1}{2} \le \frac{D(6,1)}{D(6,7)}$$
$$\frac{D(6,1)}{D(6,8)} \le \frac{2\sqrt{2}-1}{7}$$
$$\frac{D(6,1)}{D(6,9)} \le \frac{1}{3}$$
$$\frac{D(6,1)}{D(6,10)} \le \sqrt{2}-1$$
$$\frac{D(6,2)}{D(6,3)} \le \frac{2}{3}$$
$$1 \le \frac{D(6,2)}{D(6,7)}$$
$$\frac{1}{3} \le \frac{D(6,2)}{D(6,8)} \le \frac{5+4\sqrt{2}}{21}$$
$$\frac{1}{2} \le \frac{D(6,2)}{D(6,9)} \le \frac{3}{4}$$
$$\frac{2}{3} \le \frac{D(6,2)}{D(6,10)} \le \frac{1+\sqrt{2}}{3}$$
$$\frac{1}{2} \le \frac{D(6,3)}{D(6,4)}$$
$$2 \le \frac{D(6,3)}{D(6,5)}$$
$$C_6 \le \frac{D(6,3)}{D(6,7)}$$
$$\frac{1}{2} \le \frac{D(6,3)}{D(6,8)}$$
$$1 \le \frac{D(6,3)}{D(6,9)}$$
$$1 \le \frac{D(6,3)}{D(6,10)}$$
$$\frac{D(6,7)}{D(6,8)} \le \frac{1}{2}$$
$$\frac{D(6,7)}{D(6,9)} \le \frac{1}{2}$$
$$\frac{D(6,7)}{D(6,10)} \le \frac{4}{5}$$
$$1 \le \frac{D(6,8)}{D(6,9)} \le 2$$
$$3-\sqrt{2} \le \frac{D(6,8)}{D(6,10)} \le 2$$
$$1 \le \frac{D(6,9)}{D(6,10)} \le \frac{8}{5}$$

where $C_6 \approx 7.8631112181$ is the largest root of
$6967296x^{20}+2015974656x^{18}-160813808784x^{16}+603818269839x^{14}-894980577861x^{12}$
$+677249814873x^{10}-274035844587x^{8}+60418557684x^{6}-6782842860x^{4}+290960784x^{2}$
$+7744$.

Theorem 7. *The following bounds involving distances from X_7 hold for all triangles.*

$$
\begin{array}{r|c|l}
\dfrac{D(7,1)}{D(7,2)} \le \dfrac{3}{4} & \dfrac{1}{3} \le \dfrac{D(7,2)}{D(7,8)} \le \dfrac{2}{3} & 1 \le \dfrac{D(7,3)}{D(7,10)} \\
\dfrac{D(7,1)}{D(7,3)} \le \dfrac{2}{17}\left(5-2\sqrt{2}\right) & \dfrac{D(7,2)}{D(7,9)} = \dfrac{2}{3} & 1 \le \dfrac{D(7,4)}{D(7,6)} \\
\dfrac{D(7,1)}{D(7,4)} \le 1 & \dfrac{2}{3} \le \dfrac{D(7,2)}{D(7,10)} \le \dfrac{8}{9} & \dfrac{D(7,6)}{D(7,8)} \le \dfrac{1}{3} \\
\dfrac{D(7,1)}{D(7,8)} \le \dfrac{1}{2} & \dfrac{1}{2} \le \dfrac{D(7,3)}{D(7,4)} & \dfrac{D(7,6)}{D(7,9)} \le \dfrac{1}{3} \\
\dfrac{D(7,1)}{D(7,9)} \le \dfrac{1}{2} & 2 \le \dfrac{D(7,3)}{D(7,5)} & \dfrac{D(7,6)}{D(7,10)} \le \dfrac{4}{9} \\
\dfrac{D(7,1)}{D(7,10)} \le \dfrac{2}{3} & C_7 \le \dfrac{D(7,3)}{D(7,6)} & 1 \le \dfrac{D(7,8)}{D(7,9)} \le 2 \\
\dfrac{D(7,2)}{D(7,3)} \le \dfrac{2}{3} & \dfrac{1}{2} \le \dfrac{D(7,3)}{D(7,8)} & \dfrac{4}{3} \le \dfrac{D(7,8)}{D(7,10)} \le 2 \\
2 \le \dfrac{D(7,2)}{D(7,6)} & 1 \le \dfrac{D(7,3)}{D(7,9)} & 1 \le \dfrac{D(7,9)}{D(7,10)} \le \dfrac{4}{3}
\end{array}
$$

where $C_7 \approx 7.9776615835$ is the largest root of

$$
\begin{aligned}
&833089536x^{28} + 220028016384x^{26} - 19474287964848x^{24} + 139707882692901x^{22} \\
&\quad - 410390834384412x^{20} + 732430210466916x^{18} - 892396597211316x^{16} \\
&\quad + 782711166381062x^{14} - 492062343977916x^{12} + 216425700787620x^{10} \\
&\quad - 65960002546284x^{8} + 14226627485565x^{6} - 2259294716376x^{4} + 253570773456x^{2} \\
&\quad - 14637417984.
\end{aligned}
$$

Theorem 8. *The following bounds involving distances from X_8 hold for all triangles.*

$$\frac{D(8,1)}{D(8,2)}=\frac{3}{2}$$
$$\frac{D(8,1)}{D(8,4)}\le 1$$
$$\frac{D(8,1)}{D(8,5)}\le\frac{4}{3}$$
$$\frac{2}{7}\left(4-\sqrt{2}\right)\le\frac{D(8,1)}{D(8,6)}\le 1$$
$$\frac{1}{2}\le\frac{D(8,1)}{D(8,7)}\le 1$$
$$2\le\frac{D(8,1)}{D(8,9)}$$
$$\frac{D(8,1)}{D(8,10)}=2$$
$$\frac{D(8,2)}{D(8,4)}\le\frac{2}{3}$$
$$\frac{D(8,2)}{D(8,5)}\le\frac{8}{9}$$
$$\frac{4}{21}\left(4-\sqrt{2}\right)\le\frac{D(8,2)}{D(8,6)}\le\frac{2}{3}$$
$$\frac{1}{3}\le\frac{D(8,2)}{D(8,7)}\le\frac{2}{3}$$
$$\frac{4}{3}\le\frac{D(8,2)}{D(8,9)}$$
$$\frac{D(8,2)}{D(8,10)}=\frac{4}{3}$$
$$\frac{D(8,3)}{D(8,4)}\le\frac{1}{2}$$
$$\frac{D(8,3)}{D(8,5)}\le 2$$
$$\frac{4}{3}\le\frac{D(8,4)}{D(8,5)}\le 4$$
$$1\le\frac{D(8,4)}{D(8,6)}$$
$$1\le\frac{D(8,4)}{D(8,7)}$$
$$2\le\frac{D(8,4)}{D(8,9)}$$
$$2\le\frac{D(8,4)}{D(8,10)}$$
$$C_8\le\frac{D(8,5)}{D(8,6)}$$
$$\frac{1}{8}\left(3+2\sqrt{2}\right)\le\frac{D(8,5)}{D(8,7)}$$
$$\frac{3}{2}\le\frac{D(8,5)}{D(8,9)}$$
$$\frac{3}{2}\le\frac{D(8,5)}{D(8,10)}$$
$$\frac{2}{3}\le\frac{D(8,6)}{D(8,7)}\le\frac{7}{6}$$
$$2\le\frac{D(8,6)}{D(8,9)}$$
$$2\le\frac{D(8,6)}{D(8,10)}\le 2+\frac{1}{\sqrt{2}}$$
$$2\le\frac{D(8,7)}{D(8,9)}$$
$$2\le\frac{D(8,7)}{D(8,10)}\le 4$$
$$\frac{D(8,9)}{D(8,10)}\le 1$$

where $C_8\approx 0.6817039304$ is the smallest positive root of

$$896x^4-2184x^3+1924x^2-758x+121.$$

Theorem 9. *The following bounds involving distances from X_9 hold for all triangles.*

$$\frac{3}{2} \le \frac{D(9,1)}{D(9,2)} \le 3$$
$$\frac{D(9,1)}{D(9,4)} \le 1$$
$$\frac{D(9,1)}{D(9,5)} \le 2$$
$$\frac{2}{3} \le \frac{D(9,1)}{D(9,6)} \le 1$$
$$\frac{1}{2} \le \frac{D(9,1)}{D(9,7)} \le 1$$
$$1 \le \frac{D(9,1)}{D(9,8)}$$
$$2 \le \frac{D(9,1)}{D(9,10)}$$
$$\frac{D(9,2)}{D(9,4)} \le \frac{1}{3}$$
$$\frac{D(9,2)}{D(9,5)} \le \frac{2}{3}$$
$$\frac{1}{4} \le \frac{D(9,2)}{D(9,6)} \le \frac{1}{2}$$
$$\frac{D(9,2)}{D(9,7)} = \frac{1}{3}$$
$$\frac{1}{3} \le \frac{D(9,2)}{D(9,8)}$$
$$\frac{4}{3} \le \frac{D(9,2)}{D(9,10)}$$
$$\frac{D(9,3)}{D(9,4)} \le \frac{1}{2}$$
$$\frac{D(9,3)}{D(9,5)} \le 2$$
$$1 \le \frac{D(9,3)}{D(9,10)}$$
$$2 \le \frac{D(9,4)}{D(9,5)} \le 4$$
$$1 \le \frac{D(9,4)}{D(9,6)}$$
$$1 \le \frac{D(9,4)}{D(9,7)}$$
$$1 \le \frac{D(9,4)}{D(9,8)}$$
$$10 \le \frac{D(9,4)}{D(9,10)}$$
$$C_9 \le \frac{D(9,5)}{D(9,6)}$$
$$\frac{1}{2} \le \frac{D(9,5)}{D(9,7)}$$
$$\frac{1}{2} \le \frac{D(9,5)}{D(9,8)}$$
$$\frac{5}{2} + \sqrt{2} \le \frac{D(9,5)}{D(9,10)}$$
$$\frac{2}{3} \le \frac{D(9,6)}{D(9,7)} \le \frac{4}{3}$$
$$1 \le \frac{D(9,6)}{D(9,8)}$$
$$\frac{8}{3} \le \frac{D(9,6)}{D(9,10)}$$
$$1 \le \frac{D(9,7)}{D(9,8)}$$
$$4 \le \frac{D(9,7)}{D(9,10)}$$

where $C_9 \approx 0.4870156430$ is the smallest positive root of

$$3072x^5 + 9304x^4 - 35096x^3 + 40708x^2 - 25350x + 6137.$$

Theorem 10. *The following bounds involving distances from X_{10} hold for all triangles.*

$$\frac{D(10,1)}{D(10,2)} = 3$$
$$\frac{D(10,1)}{D(10,4)} \leq 1$$
$$\frac{D(10,1)}{D(10,5)} \leq 2$$
$$2-\sqrt{2} \leq \frac{D(10,1)}{D(10,6)} \leq 1$$
$$\frac{1}{3} \leq \frac{D(10,1)}{D(10,7)} \leq 1$$
$$\frac{D(10,1)}{D(10,8)} = 1$$
$$1 \leq \frac{D(10,1)}{D(10,9)}$$
$$\frac{D(10,2)}{D(10,4)} \leq \frac{1}{3}$$
$$\frac{D(10,2)}{D(10,5)} \leq \frac{2}{3}$$
$$\frac{1}{3}\left(2-\sqrt{2}\right) \leq \frac{D(10,2)}{D(10,6)} \leq \frac{1}{3}$$
$$\frac{1}{9} \leq \frac{D(10,2)}{D(10,7)} \leq \frac{1}{3}$$
$$\frac{D(10,2)}{D(10,8)} = \frac{1}{3}$$
$$\frac{1}{3} \leq \frac{D(10,2)}{D(10,9)}$$
$$\frac{D(10,3)}{D(10,4)} \leq \frac{1}{2}$$
$$\frac{D(10,3)}{D(10,5)} \leq 2$$
$$2 \leq \frac{D(10,3)}{D(10,9)}$$
$$2 \leq \frac{D(10,4)}{D(10,5)} \leq 4$$
$$1 \leq \frac{D(10,4)}{D(10,6)}$$
$$1 \leq \frac{D(10,4)}{D(10,7)}$$
$$1 \leq \frac{D(10,4)}{D(10,8)}$$
$$9 \leq \frac{D(10,4)}{D(10,9)}$$
$$C_{10} \leq \frac{D(10,5)}{D(10,6)}$$
$$\frac{1}{2} \leq \frac{D(10,5)}{D(10,7)}$$
$$\frac{1}{2} \leq \frac{D(10,5)}{D(10,8)}$$
$$\frac{3}{2}+\sqrt{2} \leq \frac{D(10,5)}{D(10,9)}$$
$$\frac{5}{9} \leq \frac{D(10,6)}{D(10,7)} \leq \frac{4}{3}$$
$$1 \leq \frac{D(10,6)}{D(10,8)} \leq 1+\frac{1}{\sqrt{2}}$$
$$\frac{5}{3} \leq \frac{D(10,6)}{D(10,9)}$$
$$1 \leq \frac{D(10,7)}{D(10,8)} \leq 3$$
$$3 \leq \frac{D(10,7)}{D(10,9)}$$
$$1 \leq \frac{D(10,8)}{D(10,9)}$$

where $C_{10} \approx 0.4870156430$ is the smallest positive root of

$$50x^4 - 72x^3 + 22x^2 - 2x + 1.$$

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