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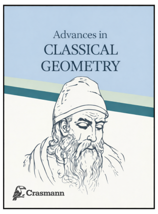

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Research Article

Orderings of Triangle Centers

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ABSTRACT

Triangle centers are usually studied individually or through special geometric relationships, but little attention has been given to global structure among them. In this paper we introduce several natural ways to order triangle centers, including the isosceles order, vertex order, side order, and trace order. These partial orders compare centers by their relative positions in families of triangles, such as acute triangles with a fixed shortest side. Using barycentric coordinates and symbolic computation, we determine ordering relations among many of the first 100 triangle centers listed in Kimberling's *Encyclopedia of Triangle Centers*. The results reveal surprising structural patterns and suggest new ways to organize and study triangle centers. Many new inequalities are also revealed. For example, in an acute triangle ABC, with shortest side BC, the Gergonne point is always closer to side BC than the nine-point center.

1. Introduction

Triangle centers form one of the richest collections of special points associated with a triangle. Thousands of such centers have been cataloged in Kimberling's *Encyclopedia of Triangle Centers* [2], and many remarkable geometric relationships between them are known. Most studies, however, focus on individual centers or small families of centers.

In this paper we investigate a different question: can triangle centers be ordered in a natural way?

When a triangle varies in shape, the positions of its centers move in complicated ways, often appearing chaotic. Nevertheless, when we restrict attention to certain families of triangles, clear ordering relationships emerge. We introduce several ways to compare triangle centers based on their geometric positions, including comparisons by distance from a vertex, distance from a side, and the position of cevian traces.

Let X_n denote the n-th triangle center cataloged in [2].

Figure 1 shows a random triangle. The black dots represent the triangle centers $X_1, X_2, X_3, \ldots, X_{10000}$, except that only triangle centers of $\triangle ABC$ are shown if they lie inside the triangle.

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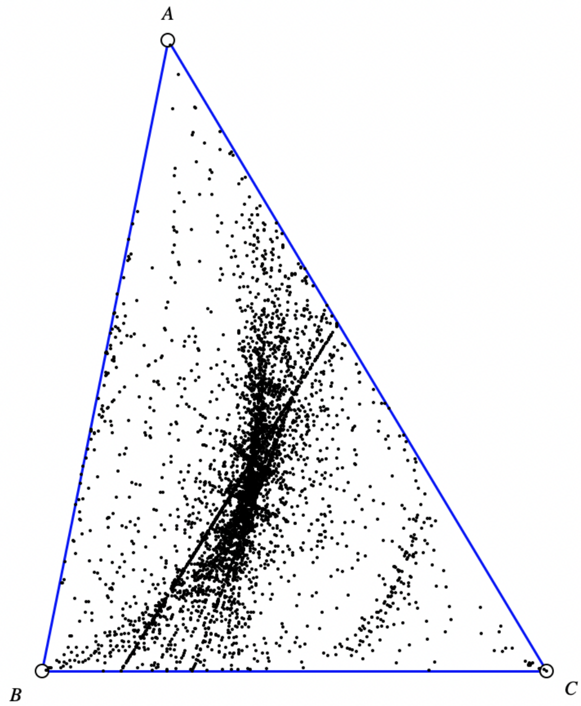

Figure 1: Triangle Centers inside $\triangle ABC$

Some patterns can be noticed. For example, a number of dots lie along straight lines. These correspond to known central lines [3] associated with a triangle, such as the Euler Line, the Nagel Line, and the Brocard Axis. Apart from this pattern, the dots appear to be randomly scattered throughout the interior of the triangle. As we vary the shape of the triangle, the dots move around seemingly chaotically as they change their order in relationship to each other. Sometimes the points move outside the triangle.

In this paper, we try to bring some order to this chaos. We explore various ways for ordering these centers.

To study these orderings we use barycentric coordinates together with symbolic computation in Mathematica. Section 2 develops several preliminary results concerning barycentric coordinates and distances to the sides of a triangle. Sections 3–6 introduce four different orderings on triangle centers and investigate their properties. In each case we determine ordering relationships among many of the first 100 triangle centers listed in [2].

The reader who just wants to learn about ordering triangle centers can skip Section 2 and proceed to Section 3.

2. Preliminary Remarks

We assume that the reader is familiar with barycentric coordinates. We use the standard convention that in $\triangle ABC$, $a = BC$, $b = CA$, and $c = AB$.

The complete Mathematica proofs of all the theorems in this paper can be found in the Mathematica notebook included in the supplementary material associated with this paper.

If a point P has barycentric coordinates $(p : q : r)$ and $p + q + r = 1$, then we say that the coordinates are *normalized*. (Yiu [5, §3.1.2] calls these absolute barycentric coordinates.)

Since barycentric coordinates use signed areas, if the normalized barycentric coordinates for a point P are $(p : q : r)$, the value of p is positive if P lies on the same side of BC as A, and negative if it lies on the opposite side. The value of q is positive if P lies on the same side of CA as B, and negative if it lies on the opposite side. The value of r is positive if P lies on the same side of AB as C, and negative if it lies on the opposite side. This gives us the following lemma.

Lemma 2.1. *The sidelines of a triangle divide the plane into seven regions. If the normalized barycentric coordinates for a point P are $(p : q : r)$, then the signs of p, q, and r are as shown in Figure 2.*

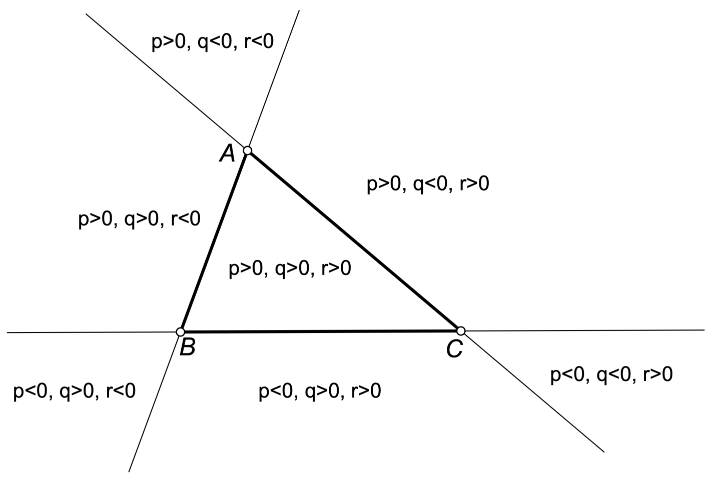

Figure 2: Seven regions formed by the sidelines of a triangle

In $\triangle ABC$, a point is said to be *inside* angle A if it lies in the open convex region bounded by rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. From Lemma 2.1, we get the following result.

Theorem 2.2. *Let P have barycentric coordinates $(u : v : w)$ with respect to $\triangle ABC$. Then P is inside angle A if and only if*

$$v(u+v+w) > 0 \qquad \text{and} \qquad w(u+v+w) > 0.$$

Proof. Since barycentric coordinates are homogeneous, the coordinates $(u : v : w)$ and $(uk : vk : wk)$ represent the same point for any $k \neq 0$. The normalized barycentric coordinates for P are therefore

$$\left(\frac{u}{u+v+w} : \frac{v}{u+v+w} : \frac{w}{u+v+w}\right).$$

Note that $u+v+w \neq 0$ since P is not on the line at infinity. From Figure 2, we see that for normalized coefficients, $P = (p : q : r)$ is inside angle A if and only if $q > 0$ and $r > 0$. In terms of u, v, and w, this means that $v/(u+v+w) > 0$ and $w/(u+v+w) > 0$. Multiplying each inequality by the positive quantity $(u+v+w)^2$ preserves the sense of the inequality and gives us the desired result. □

Theorem 2.3. *Let ABC be an isosceles triangle with base BC. Then X_n sometimes lies outside angle A when*
$n \in \{4, 5, 11, 13, 14, 16, 17, 18, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 33, 34, 36, 44,$
$46, 47, 48, 49, 50, 51, 52, 53, 54, 59, 62, 64, 66, 67, 68, 70, 73, 74, 77, 79, 80, 84, 87,$
$88, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100\}.$
The center X_{30} lies on the line at infinity.
In all other cases, X_n always lies inside angle A.

Proof. We will use Mathematica. Let `Xn={p,q,r}` where $(p : q : r)$ denotes the barycentric coordinates for center X_n as found from [2].

The following Mathematica code creates a function called `sometimeOutsideQ` such that `sometimeOutsideQ[Xn]` returns `True` if and only if center X_n sometimes lies outside angle A.

```
triang = a>0 && b>0 && c>0 && a+b>c && b+c>a && c+a>b;
isosceles = (b == c);
constraint = triang && isosceles;
sometimeOutsideQ[{u_, v_, w_}] := Module[{inst},
   inst = FindInstance[(v(u+v+w) <= 0 || w(u+v+w) <= 0)
             && constraint, {a,b,c}];
   If[inst =!= {}, Return[True]];
   Return[False];
   ];
```

We then use this function on each X_n to determine those n for which X_n can lie outside angle A. □

Theorem 2.4. *Let ABC be an isosceles triangle with base BC and $b > a$. Then X_n lies inside angle A for $1 \leq n \leq 100$ except for X_n when*
$n \in \{18, 26, 30, 59, 68, 70, 87, 90, 91, 93, 96, 99, 100\}$.

Proof. The proof is the same as the proof of Theorem 2.3, except that the constraint is replaced by the following Mathematica code.

```
constraint = triang && isosceles && (b>a);
```

Again, each X_n is tested with routine `sometimeOutsideQ[Xn]`. □

Theorem 2.5. *Let ABC be an isosceles triangle with base BC. Then X_n coincides with vertex A when* $n \in \{59, 99, 100\}$.

Proof. We use the following Mathematica code.

```
ptA= {1,0,0};
Simplify[Cross[Xn, ptA] == {0,0,0}, constraint]
```

The `Simplify` command then returns `True` if and only if X_n coincides with vertex A. □

Corollary 2.6. *Let ABC be an isosceles triangle with base BC and $b > a$. Then for $1 \leq n \leq 100$, X_n always lies outside angle A only when X_n coincides with vertex A.*

In a 1–2–2 triangle, X_n lies outside angle A for $n \in \{18, 26, 68, 93\}$.
In a 11–16–16 triangle, X_n lies outside angle A or $n \in \{90, 91, 96\}$.
In a 7–16–16 triangle, X_{87} lies outside angle A.
In a 1–2–2 triangle, X_{87} coincides with vertex A.
In an isosceles triangle, X_{70} can lie outside vertex angle A but only if $a > b$.
In a 9–8–8 triangle, X_{70} lies outside angle A.

In $\triangle ABC$, a point is said to be *above* side BC if it lies on the same side of BC as vertex A. From Lemma 2.1, we get the following result.

Theorem 2.7. *Let P have barycentric coordinates $(u : v : w)$ with respect to $\triangle ABC$. Then P is above side BC if and only if*

$$u(u + v + w) > 0.$$

Proof. Since barycentric coordinates are homogeneous, the coordinates $(u : v : w)$ and $(uk : vk : wk)$ represent the same point for any $k \neq 0$. The normalized barycentric coordinates for P are therefore

$$\left(\frac{u}{u+v+w} : \frac{v}{u+v+w} : \frac{w}{u+v+w}\right).$$

Note that $u + v + w \neq 0$ since P is not on the line at infinity. From Figure 2, we see that for normalized coefficients, $P = (p : q : r)$ is above side BC if and only if $p > 0$. In terms of u, v, and w, this means that $u/(u + v + w) > 0$. Multiplying this inequality by the positive quantity $(u + v + w)^2$ preserves the sense of the inequality and gives us the desired result. □

Theorem 2.8. *Let ABC be an acute triangle with smallest side BC. Then X_n always lies above BC when* $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 19, 20, 21, 22, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 45, 48, 51, 53, 54, 55, 56, 57, 58, 59, 60, 61, 63, 65, 68, 69, 71, 72, 73, 75, 76, 77, 78, 79, 81, 82, 83, 85, 86, 89, 92, 95, 97, 99, 100\}$.
In addition, X_n always lies on or above BC when $n = 11$.
Center X_n lies on the line at infinity when $n = 30$.
Center X_n always lies below BC when $n \in \{16, 23, 36, 44, 50\}$.
Center X_n can lie above or below BC when $n \in \{14, 18, 24, 43, 46, 47, 49, 52, 62, 64, 66, 67, 70, 74, 80, 84, 87, 88, 90, 91, 93, 94, 96, 98\}$.

Theorem 2.9 (Distance From Point to Line)**.** *The distance from the point $P = (p : q : r)$ to the line $ux + vy + wz = 0$ is*

$$\sqrt{\frac{\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right)(pu + qv + rw)^2}{4(p + q + r)^2\left(a^2(u - v)(u - w) + (v - w)\left(b^2(v - u) + c^2(u - w)\right)\right)}}.$$

Proof. The equation of the line through P perpendicular to the line L with equation $ux + vy + wz = 0$ is given by formula (7) in [1]. The intersection of these two lines can be found using the intersection formula (5) in [1]. This gives us the coordinates for F, the foot of the perpendicular from P to L. Then, using the formula (9) from [1] for the distance between two points, we get the distance from P to F which is the distance from point P to line L. □

Corollary 2.10. *The distance from point $P = (p : q : r)$ to side BC of $\triangle ABC$ is*

$$\sqrt{\frac{p^2\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right)}{4a^2(p + q + r)^2}}.$$

Proof. This follows from the fact that the equation for the line BC is $x = 0$, so we can put $u = 1$, $v = 0$, and $w = 0$ in Theorem 2.9. □

Corollary 2.11. *The distance from point $P = (p : q : r)$ to side BC of $\triangle ABC$ is*

$$\frac{2K}{a}\sqrt{\frac{p^2}{(p+q+r)^2}} = \frac{2K}{a}\left|\frac{p}{p+q+r}\right|$$

where K is the area of $\triangle ABC$.

Proof. This follows because the formula

$$K = \frac{1}{4}\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$$

follows algebraically from the well-known Heron's formula, $K = \sqrt{s(s-a)(s-b)(s-c)}$ when $s = (a+b+c)/2$. □

Corollary 2.12. *The distance from vertex A to side BC of $\triangle ABC$ is*

$$\frac{2K}{a}$$

where K is the area of $\triangle ABC$.

Proof. The coordinates for point A are $(1 : 0 : 0)$, so let $p = 1$, $q = 0$, and $r = 0$ in Corollary 2.11. This also follows from the fact that the area of a triangle is half the base times the height. □

Definition. Let $d(P)$ denote the *signed distance* from a point P in the plane of $\triangle ABC$ to side BC. The sign is positive if P is above BC and negative if P is below BC.

Theorem 2.13. *The signed distance from point $P = (p : q : r)$ to side BC of $\triangle ABC$ is*

$$d(P) = \frac{2K}{a}\left(\frac{p}{p+q+r}\right).$$

Proof. The (unsigned) distance is

$$\frac{2K}{a}\left|\frac{p}{p+q+r}\right|$$

by Corollary 2.11. If P is above BC, then by Theorem 2.7, $p(p+q+r) > 0$. Dividing both sides by the positive quantity $(p+q+r)^2$ shows that $p/(p+q+r) > 0$. Thus, the formula for $d(p)$ is correct when P is above BC. Similarly, if P is below BC, Theorem 2.7 implies that $p(p+q+r) < 0$ or equivalently $p/(p+q+r) < 0$. Thus, the formula for $d(p)$ is correct when P is below BC. If P lies on BC, then $p = 0$ and the formula is also correct. □

3. The Isosceles Order

From the definition of a triangle center, it is easy to see that in an isosceles triangle, all triangle centers lie on the median to the base. At first glance, it appears that the order of these triangle centers along the median is essentially random and varies as the shape of the triangle changes. However, if we only consider "tall" isosceles triangles, that is, ones in which $b > a$, we find a little more order.

Definition. An isosceles triangle ABC with base BC is said to be *tall* if $b > a$.

Definition. We define the *isosceles order* on triangle centers, $\prec$, by

$P \prec Q$ if P is closer to A than Q

in all tall isosceles triangles ABC with base BC.

The symbols "$P \prec Q$" can be read as "P precedes Q" or less precisely as "P is less than Q".

Under this ordering, we find some order among the triangle centers as shown by the following theorem which involves some of the triangle centers from X_1 to X_{30}.

Theorem 3.1. *Using the isosceles order $\prec$, we have*

$$X_{20} \prec X_{22} \prec X_8 \prec X_3 \prec X_9 \prec X_{10} \prec X_{21} \prec X_2 \prec X_5 \prec X_{12} \prec X_{17} \prec X_1 \prec X_{13}$$

$$\prec X_7 \prec X_6 \prec X_4 \prec X_{27} \prec X_{19} \prec X_{28} \prec X_{25} \prec X_{11} \prec X_{14} \prec X_{16} \prec X_{23}.$$

Proof. Let d_n denote the square of the distance from X_n to A. Let us start with the first claim, $X_{20} \prec X_{22}$. Both X_{20} and X_{22} lie inside $\angle A$ (Theorem 2.4). Thus, all we need to show is that $d_{20} < d_{22}$. Using the distance formula [1, §2], we find that

$$d_{20} = -\frac{a^6 - 3a^2\left(b^2 - c^2\right)^2 + 2\left(b^2 - c^2\right)^2\left(b^2 + c^2\right)}{a^4 - 2a^2\left(b^2 + c^2\right) + \left(b^2 - c^2\right)^2}.$$

When $c = b$, this simplifies to

$$d_{20} = \frac{a^4}{(2b - a)(2b + a)}.$$

Similarly,

$$d_{22} = \frac{a^2b^2c^2\left(-a^8 + 2a^6\left(b^2 + c^2\right) - 2a^2\left(b^2 - c^2\right)^2\left(b^2 + c^2\right) + \left(b^4 - c^4\right)^2\right)}{\left(a^6 - a^4\left(b^2 + c^2\right) - a^2\left(b^4 + c^4\right) + \left(b^2 - c^2\right)^2\left(b^2 + c^2\right)\right)^2}$$

and when $c = b$, this simplifies to

$$d_{22} = \frac{a^4b^4(2b - a)(2b + a)}{\left(a^4 - 2a^2b^2 - 2b^4\right)^2}.$$

The inequality $d_{20} < d_{22}$ becomes

$$\frac{a^4}{(2b - a)(2b + a)} < \frac{a^4b^4(2b - a)(2b + a)}{\left(a^4 - 2a^2b^2 - 2b^4\right)^2}.$$

We could prove this inequality using algebra, standard inequality techniques, and some ingenuity, but instead, it is easier to use a symbolic algebra system to prove the inequality.

We will use the Mathematica command

```
Simplify[expr,constraint]
```

which simplifies an expression, equation, or inequality, subject to the stated constraint. If we let

```
isoConstraint = 0<a<b && a+b>c && b+c>a && c+a>b && b==c
```

and issue the Mathematica command

```
Simplify[d[20]<d[22],isoConstraint],
```

Mathematica responds with `True`, proving that the inequality is always true.

Using the `Simplify` command in Mathematica, all the other claims can be proved in the same manner. □

Figure 3 shows a tall isosceles triangle and the centers referenced in Theorem 3.1. As the shape of the triangle varies, the distances between these centers change relative to each other, however the order of the centers remains fixed.

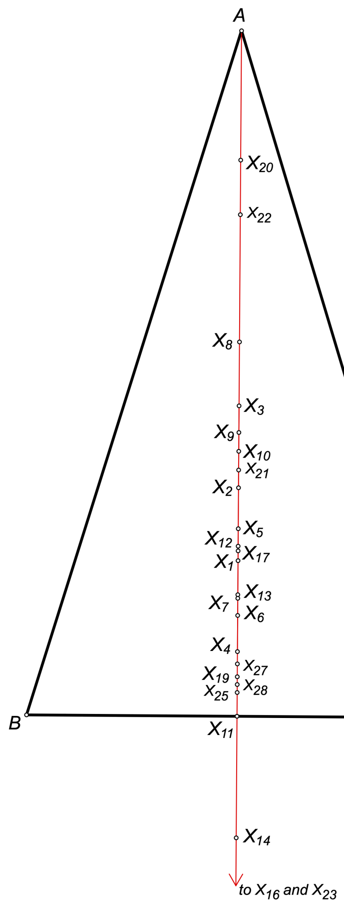

Figure 3: Centers in a tall isosceles triangle

You will note that triangle centers X_{15}, X_{18}, X_{24}, X_{26}, X_{29}, and X_{30} were not included in Theorem 3.1. This is because these centers are not always in the same order with respect to the other triangle centers or they lie outside angle A. Let us look a little closer at the location of these centers.

Center X_{15}.

Figure 4 shows that X_{15} is sometimes closer to A than X_5 and sometimes further away.

Center X_{15} is closer to A than X_{12} in a 1–4–4 triangle and further from A in a 1–8–8 triangle. We can, however, bound the location of X_{15} as the following result shows.

Theorem 3.2. *Let ABC be a tall isosceles triangle with base BC (Figure 5). Then*

$$X_2 \prec X_{15} \prec X_{17}.$$

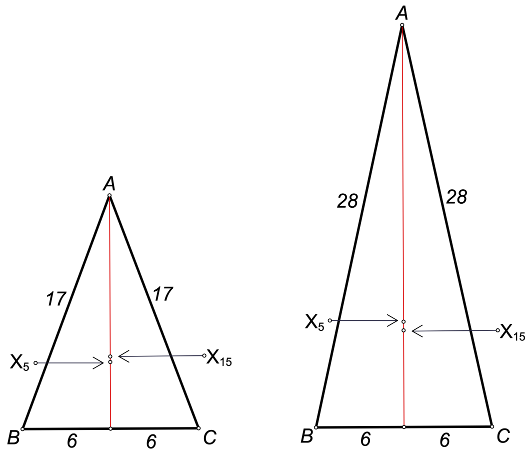

Figure 4: Centers X_{15} and X_5

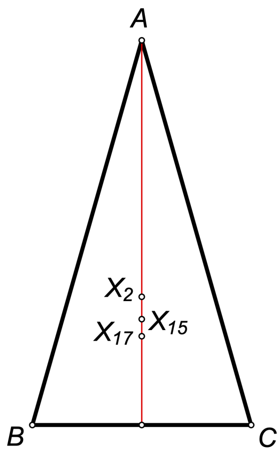

Figure 5: Centers X_2, X_{15}, and X_{17}

Proof. The result can be proven using the Mathematica `Simplify` command in the same way that it was used in the proof of Theorem 3.1. □

Theorem 3.3. *Triangle centers X_5 and X_{15} coincide in a 1–k–k triangle, where $k = \sqrt{2+\sqrt{3}}$.*

Proof. Two barycentric representations $p = (p_1 : p_2 : p_3)$ and $q = (q_1 : q_2 : q_3)$ represent the same point if their coordinates are proportional. In Mathematica, this condition is `Cross[p,q]=={0,0,0}`.

Using this fact and the barycentric coordinates for X_5 and X_{15} from [2], we find that `Cross[X5,X15]=={0,0,0}` when $a = 1$ and $k = \sqrt{2+\sqrt{3}}$. □

Theorem 3.4. *Triangle centers X_{12} and X_{15} coincide in a 1–k–k triangle, where $k \approx 7.25054$ is the positive real root of $2x^6 - 13x^5 - 11x^4 + 6x^2 + x - 1 = 0$.*

Proof. This example is obtained using the Mathematica command

```
FindInstance[Cross[X12,X15]=={0,0,0} && b==c>a==1, {a,b,c}]
```

where `X12` and `X15` are the barycentric coordinates for centers X_{12} and X_{15}, respectively. □

Center X_{18}.

Figure 6 shows that X_{18} is sometimes closer to A than X_{20} and sometimes further away.

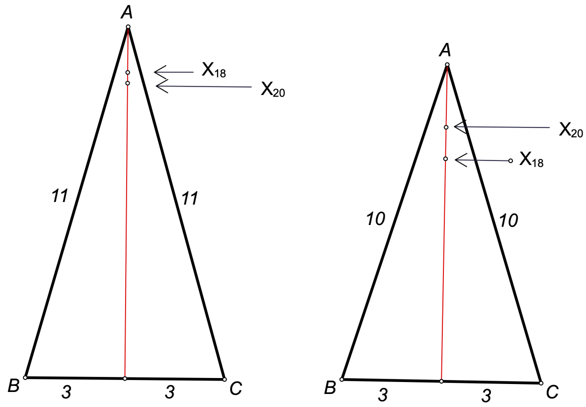

Figure 6: Centers X_{18} and X_{20}

Figure 7 shows that X_{18} and X_{26} can lie outside angle A. Sometimes X_{18} is closer to A than X_{26} and sometimes it is further away.

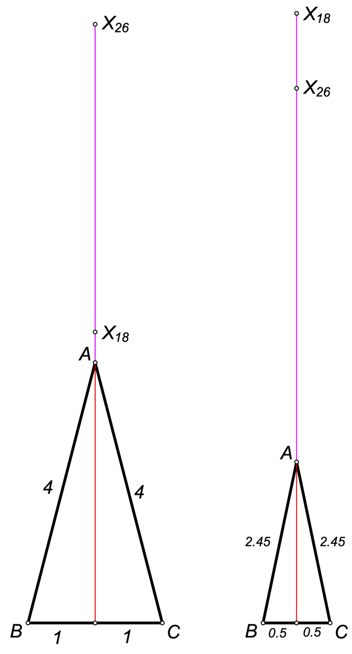

Figure 7: Centers X_{18} and X_{26}

Center X_{24}.

Figure 8 shows that X_{24} is sometimes closer to A than X_{11} and sometimes further away.

Theorem 3.5. *Triangle centers X_{11} and X_{24} coincide in a k–1–1 triangle, where $k = \sqrt{2-\sqrt{2}}$.*

Proof. The proof is the same as the proof of Theorem 3.3. □

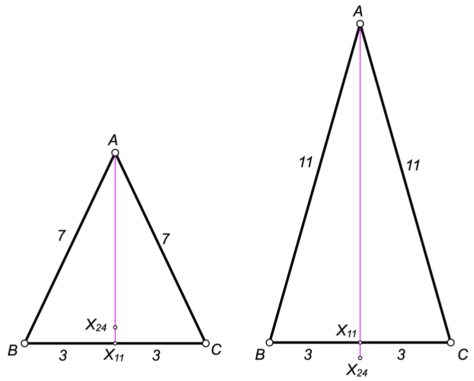

Figure 8: Centers X_{11} and X_{24}

We can, however, bound the location of X_{24} as the following result shows.

Theorem 3.6. *Let ABC be a tall isosceles triangle with $b = c$. Then*

$$X_{25} \prec X_{24} \prec X_{14}.$$

Proof. The proof is the same as the proof of Theorem 3.2. □

Center X_{26}.

Figure 7 shows that center X_{26} can sometimes lie outside angle A.

If X_{26} lies inside angle A, then it lies between A and X_{20} (Figure 9).

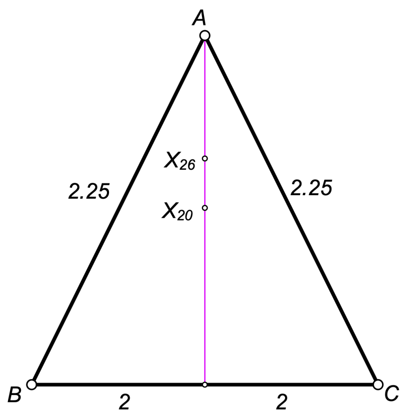

Figure 9: Centers X_{20} and X_{26}

Theorem 3.7. *Triangle center X_{26} coincides with vertex A in a k–1–1 triangle, where $k = \sqrt{2-\sqrt{2}}$.*

Proof. The proof is the same as the proof of Theorem 3.3. □

Center X_{29}.

Center X_{29} is closer to A than X_6 in a 1–2–2 triangle and further from A in a 7–8–8 triangle. We can, however, bound the location of X_{29} as the following result shows.

Theorem 3.8. *Let ABC be a tall isosceles triangle with $b = c$. Then*

$$X_7 \prec X_{29} \prec X_4.$$

Proof. The proof is the same as the proof of Theorem 3.2. □

Theorem 3.9. *The triangle centers X_{29} and X_6 coincide in a 1–k–k triangle, where $k = (1 + \sqrt{17})/4$.*

Proof. The proof is the same as the proof of Theorem 3.3. □

Center X_{30}.

Theorem 3.10. *Center X_{30} lies on the line at infinity.*

Proof. From [2], we have

$$X_{30} = \left(2a^4 - a^2\left(b^2 + c^2\right) - \left(b^2 - c^2\right)^2 :\right.$$

$$2b^4 - b^2\left(c^2 + a^2\right) - \left(c^2 - a^2\right)^2 :$$

$$\left.2c^4 - c^2\left(a^2 + b^2\right) - \left(a^2 - b^2\right)^2\right).$$

Algebraically, the sum of the coefficients is 0 which means that the point lies on the line at infinity. □

Center X_{11}.

Theorem 3.11. *The Feuerbach point, X_{11}, of an isosceles triangle coincides with the midpoint of the base.*

Proof. Without loss of generality, assume $b = c$. From [2], we find that the barycentric coordinates for X_{11} are

$$\Big((b-c)^2(b+c-a) : (a-c)^2(c+a-b) : (a-b)^2(a+b-c)\Big).$$

Letting $c = b$ gives

$$X_{11} = \left(0 : a(a-b)^2 : a(a-b)^2\right) = (0 : 1 : 1),$$

so X_{11} is the midpoint of BC. □

Using the information obtained above, we can draw a graph showing the ordering relationship between various centers in a tall isosceles triangle. The graph is shown in Figure 10, where an arrow from m to n means that $X_m \prec X_n$ in the isosceles order.

If we look at the first 100 triangle centers in [2], instead of the first 30, we get the following result.

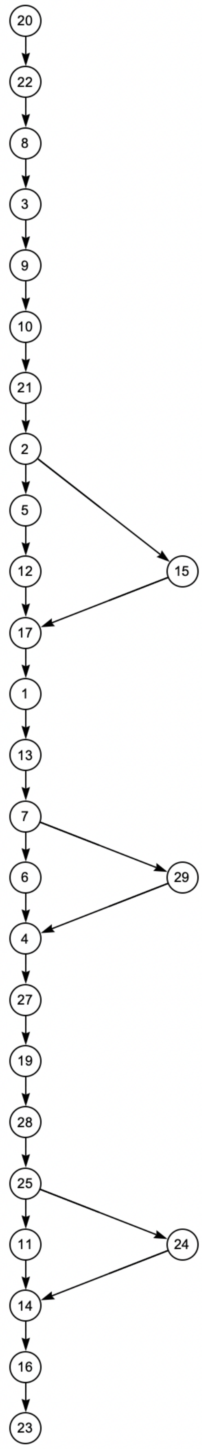

Figure 10: An arrow from m to n means $X_m \prec X_n$ in the isosceles order.

Theorem 3.12. *Let ABC be a tall isosceles triangle with $b = c$. Then, using the isosceles order $\prec$, we have*

$$X_{20} \prec X_{22} \prec X_{40} \prec X_{72} \prec X_{63} \prec X_8 \prec X_3 \prec X_9 \prec X_{95} \prec X_{77} \prec X_{21} \prec X_2$$

$$\prec X_{45} \prec X_{38} \prec X_{55} \prec X_{37} \prec X_{12} \prec X_{17} \prec X_1 \prec X_{61} \prec X_{60} \prec X_{81} \prec X_7 \prec X_{82}$$

$$\prec X_{89} \prec X_6 \prec X_{65} \prec X_{33} \prec X_{51} \prec X_{57} \prec X_4 \prec X_{27} \prec X_{19} \prec X_{28} \prec X_{25} \prec X_{34}$$

$$\prec X_{64} \prec X_{11} \prec X_{98} \prec X_{74} \prec X_{67} \prec X_{88} \prec X_{14} \prec X_{80} \prec X_{36} \prec X_{16} \prec X_{44} \prec X_{23}.$$

Not all centers from X_1 through X_{100} appear in this list because some centers vary their order relative to other centers. The complete order for all centers that lie in angle A is shown in Figure 11.

Question 3.13. Is there a simple reason that center X_1 is a cutpoint [4] for the graph shown in Figure 11?

Question 3.14. As more centers are added to the graph in Figure 11, does X_1 remain a cutpoint?

4. The Vertex Order

Definition. We define the *vertex order* on triangle centers, $\prec$, by

$$P \prec Q \text{ if } P \text{ is closer to } A \text{ than } Q$$
for all acute triangles ABC with shortest side BC.

Theorem 4.1. *Using the vertex order $\prec$, we have (Figure 12)*

$$X_3 \prec X_9 \prec X_{10} \prec X_2 \prec X_1 \prec X_6 \prec X_4 \prec X_{19} \prec X_{16}.$$

Proof. Let us start with the first claim, $X_3 \prec X_9$. Let v_n denote the square of the distance from X_n to A. It suffices to show that $v_3 < v_9$ in all acute triangles ABC with shortest side BC.

We get the barycentric coordinates for X_3 and X_9 from [2]. The value of v_n is found using the distance formula [1, §2]. We shall write this as `v[n]` in Mathematica.

The condition that a triangle with smallest side a is acute (`vertexConstraint`) can be written in Mathematica as

```
acuteConstraint = b^2<a^2+c^2 && c^2<a^2+b^2 && a^2<b^2+c^2
                              && a>0 && b>0 && c>0;
vertexConstraint = acuteConstraint && a<b && a<c;
```

We can then issue the Mathematica command

```
Simplify[v[3]<v[9], vertexConstraint]
```

and Mathematica responds with `True`, thus proving the inequality.

Using this `Simplify` command in Mathematica, all the other claims can be proven in the same manner. □

In Figure 12, the spacing between the circular bands vary as we vary the shape of the acute triangle. However, the order of the bands remains the same.

Not all centers from X_1 through X_{20} appear in Theorem 4.1 because some centers vary their order in relationship with other centers.

The complete order for all centers from X_1 to X_{20} is shown in Figure 13.

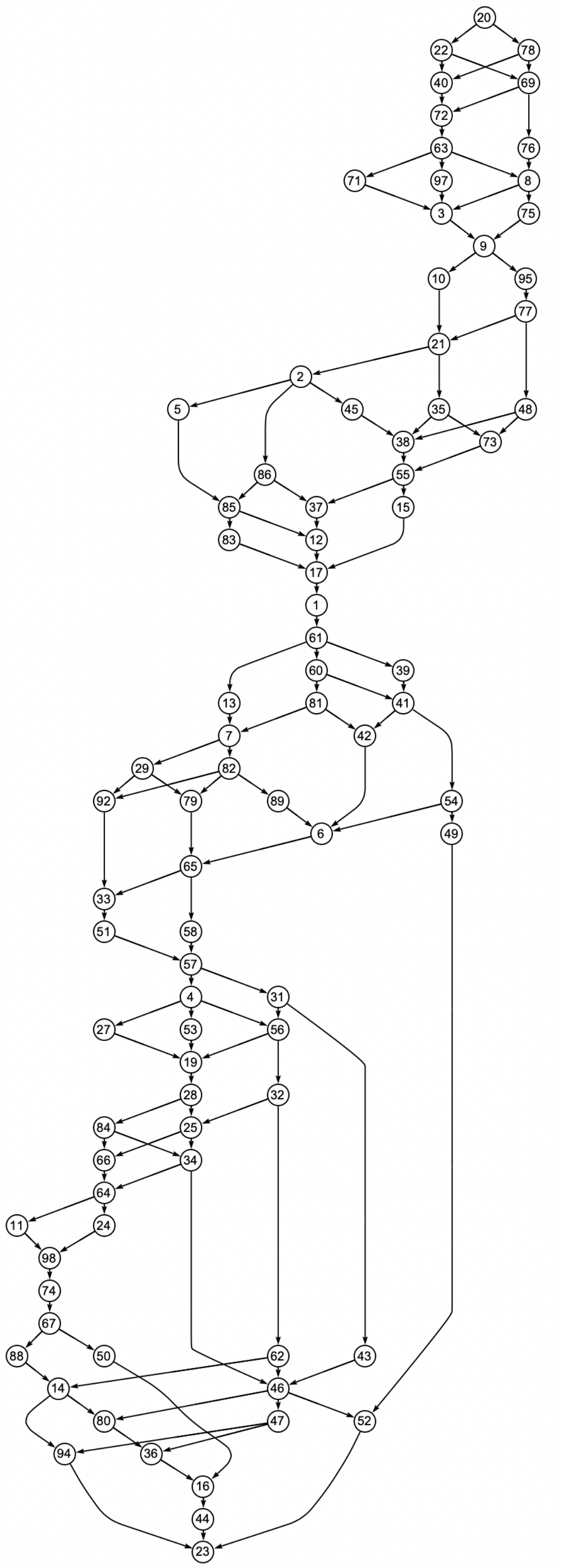

Figure 11: An arrow from m to n means $X_m \prec X_n$ in the isosceles order.

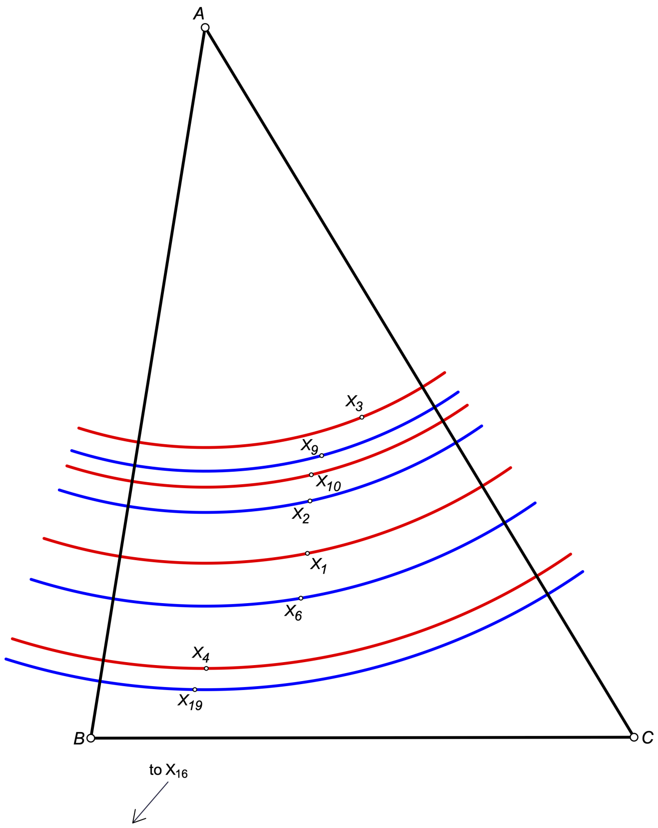

Figure 12: Ordered centers in an acute triangle with shortest side BC

The center X_{18} is not included in this graph because its distance from vertex A varies wildly. Depending on the shape of the triangle, X_{18} can get closer to vertex A than either X_3 or X_8 and can get farther away from A than X_{16}.

Direct substitution confirms the following two results.

Theorem 4.2. *Center X_{18} coincides with vertex A in a triangle with sides $a = 8$, $b = 15$, and $c = \sqrt{(353 + 15\sqrt{93})/2}$.*

Theorem 4.3. *Center X_{18} lies on the line at infinity in a triangle with sides $a = 16513$, $b = 42189$, and $c = 7\sqrt{35\left(940379 + 2\sqrt{10302477117}\right)}$.*

5. The Side Order

Definition. We define the *side order* on triangle centers, $\prec$, by

$$P \prec Q \text{ if } P \text{ is further from } BC \text{ than } Q$$

in all acute triangles ABC with smallest side BC.

By "P is further from BC than Q" we mean that $d(P) > d(Q)$ where $d(X)$ denotes the signed distance from X to BC.

Under this ordering, we find some order among triangle centers as shown by the following theorem which involves some of the triangle centers from X_1 to X_{40}.

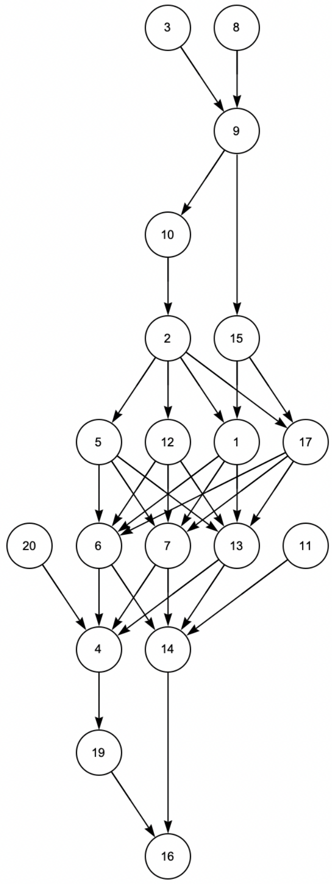

Figure 13: An arrow from m to n means $X_m \prec X_n$ in the vertex order.

Theorem 5.1. *Using the side order, $\prec$, we have*

$$X_{26} \prec X_{20} \prec X_{22} \prec X_{40} \prec X_8 \prec X_9 \prec X_{10} \prec X_2 \prec X_{37} \prec X_1 \prec X_7 \prec X_{29}$$

$$\prec X_{33} \prec X_4 \prec X_{27} \prec X_{19} \prec X_{28} \prec X_{25} \prec X_{34} \prec X_{24} \prec X_{36} \prec X_{16}.$$

Proof. Let us start with the first claim, $X_{26} \prec X_{20}$. We need to show that $d(X_{26}) > d(X_{20})$ in all acute triangles ABC with shortest side BC.

We get the barycentric coordinates for X_{26} and X_{20} from [2]. From Theorem 2.13, we find their signed distances from BC which we shall call `d[26]` and `d[20]` in Mathematica.

The condition that a triangle with smallest side a is acute (`sideConstraint`) can be written in Mathematica as

```
acuteConstraint = b^2<a^2+c^2 && c^2<a^2+b^2 && a^2<b^2+c^2
                                 && a>0 && b>0 && c>0;
sideConstraint = acuteConstraint && a<b && a<c;
```

We can then issue the Mathematica command

```
Simplify[d[26]>d[20], sideConstraint]
```

and Mathematica responds with `True`, thus proving the inequality is always true.

Using this `Simplify` command in Mathematica, all the other claims can be proven in the same manner. □

What this means, graphically, is that in Figure 14, as we vary the shape of $\triangle ABC$ (keeping it acute with smallest side BC), the spacing between the horizontal lines changes, but the order of the horizontal lines remains fixed.

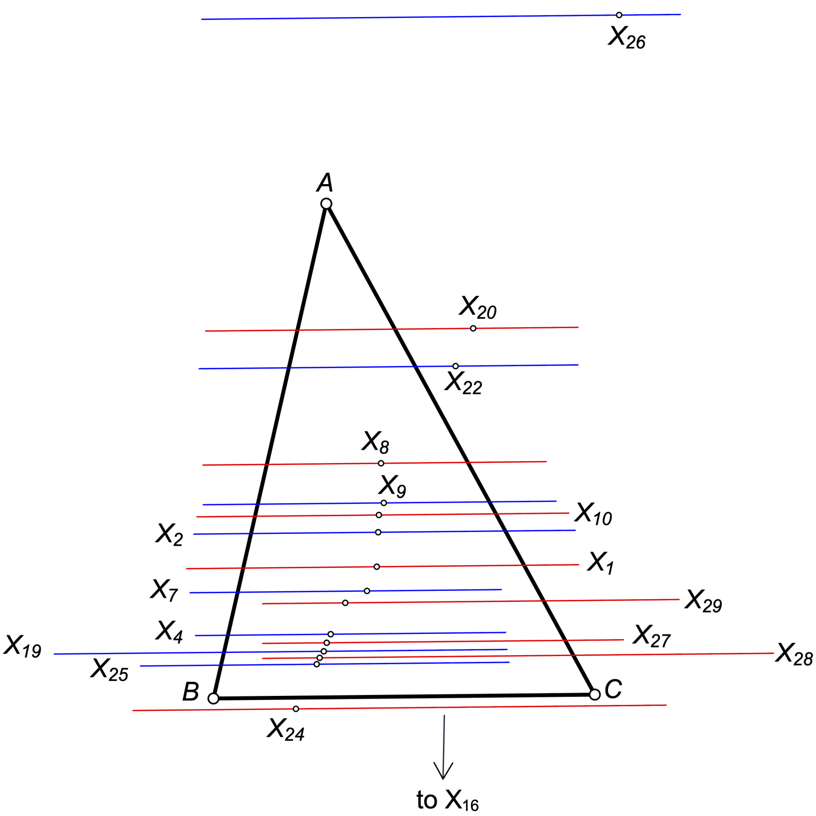

Figure 14: Centers in an acute triangle with shortest side BC

Using Mathematica to compare pairs of triangle centers, we can draw a graph showing the ordering relationship between various centers using the side order. The graph is shown in Figure 15, where an arrow from m to n means that $X_m \prec X_n$ in the side order.

The center X_{18} is not included in this graph because it can get arbitrarily far away from BC in either direction. The center X_{30} is not included in this graph because it lies on the line at infinity.

6. The Trace Order

A *cevian* of a triangle is a line through one of the vertices. The *trace* of the cevian is the point where that line meets the side opposite to that vertex.

Figure 16 shows a cevian through vertex A and its trace T_a. Note that the trace can lie anywhere on line BC and is not necessarily between B and C.

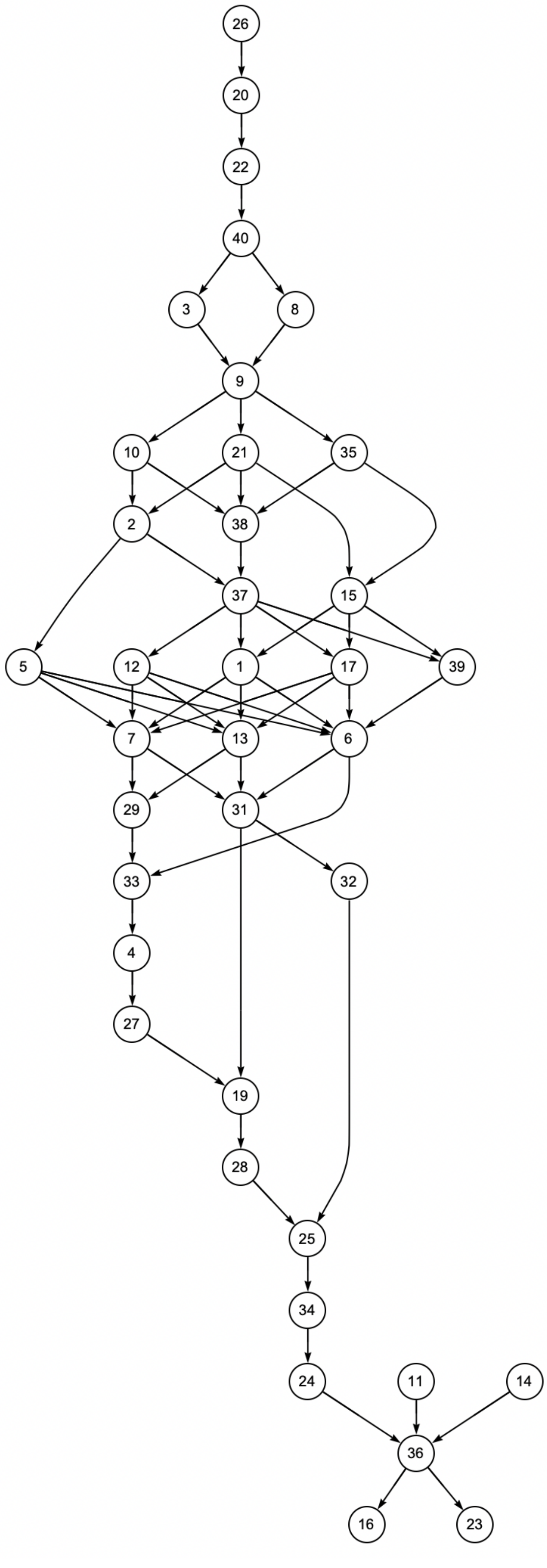

Figure 15: An arrow from m to n means $X_m \prec X_n$ in the side order.

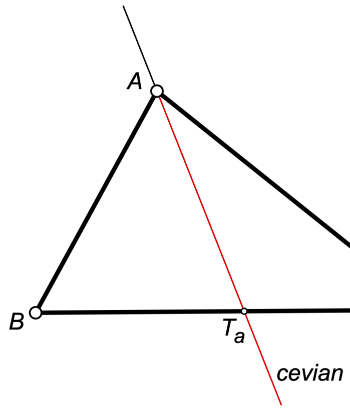

Figure 16: Cevian with trace T_a

If P is a point in the plane of $\triangle ABC$, distinct from A, then the trace of the cevian AP is called the *A-trace* of P and will be denoted by $\text{Atrace}(P)$. In Figure 17, $T_P = \text{Atrace}(P)$.

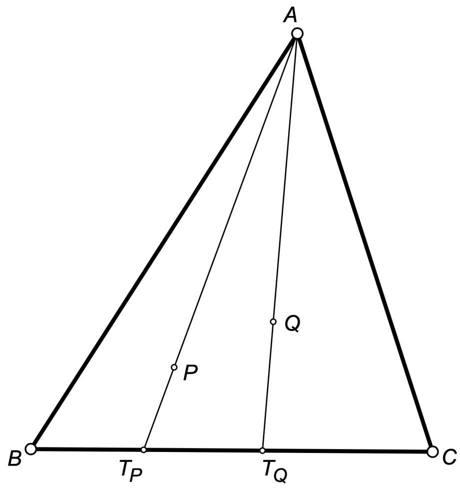

Figure 17: $P \prec$Q

Lemma 6.1. *If P has barycentric coordinates $(p : q : r)$, then the distance from $\text{Atrace}(P)$ to C is*

$$\left|\frac{aq}{q+r}\right|.$$

***Proof*.** Using the intersection formula (5) in [1], we find that point $T_P = \text{Atrace}(P)$ has coordinates $(0 : q : r)$. Using the distance formula (9) in [1], we find that the square of the distance from T_P to C is $a^2q^2/(q+r)^2$. □

Lemma 6.2. *Let ABC be a triangle, and let P be a point on the line BC with barycentric coordinates $P = (0 : v : w)$. Then P lies on the extension of BC beyond C if and only if*

$$v(v+w) < 0.$$

Proof. The normalized coordinates for P are

$$\left(0 : \frac{v}{v+w} : \frac{w}{v+w}\right).$$

Looking at Figure 2, we see that a point with normalized coordinates $(0 : p : q)$ lies on the extension of BC beyond C if and only if $q < 0$. Thus, P lies on the extension of BC beyond C if and only if $v/(v+w) < 0$. Multiplying both sides of this inequality by the positive quantity $(v+w)^2$, we see that the condition is $v(v+w) < 0$. □

Definition. If X lies on line BC, then the *signed distance* from X to C is the actual distance (taken as positive) if X lies on $\overrightarrow{CB}$ and the negative of the distance if X lies on the extension of BC beyond C.

Lemma 6.3. *If P has barycentric coordinates $(p : q : r)$, then the signed distance from Atrace(P) to C is*

$$\frac{aq}{q+r}.$$

Proof. By Lemma 6.1, the unsigned distance is

$$\left|\frac{aq}{q+r}\right| = a\left|\frac{q}{q+r}\right|.$$

By Lemma 6.2, if P lies on the extension of BC beyond C, then q and $q+r$ must have opposite signs, so the signed distance is negative and equal to $aq/(q+r)$. Similarly, if P lies on $\overrightarrow{CB}$, then the signed distance is positive, and by Lemma 6.2, q and $q+r$ have the same sign, making the signed distance equal to $aq/(q+r)$. □

We say that a point P on line BC is *to the right of* C if P lies on the extension of BC beyond C.

Definition. We define the *trace order* on triangle centers, $\prec$, by

$P \prec Q$ if Atrace(P) is further from C than Atrace(Q)
in all acute triangles ABC with $a < b < c$.

By "U is further from C than V" we mean that $d(U) > d(V)$ where $d(X)$ denotes the signed distance from X to C.

Considering only the triangle centers from X_1 through X_{30}, we get the following result.

Theorem 6.4. *Using the trace order $\prec$, we have*

$$X_{20} \prec X_{22} \prec X_3 \prec X_8 \prec X_9 \prec X_{21} \prec X_{10} \prec X_2 \prec X_1 \prec X_{17} \prec X_{12}$$

$$\prec X_7 \prec X_{13} \prec X_{29} \prec X_4 \prec X_{27} \prec X_{19} \prec X_{28} \prec X_{25} \prec X_{24} \prec X_{23}.$$

Proof. Let d_n denote the signed distance from Atrace(X_n) to C. From Lemma 6.3, the Mathematica function for d_n can be written as

```
d[{p_, q_, r_}] := a*q/(q+r);
d[n_] := d[x[n]];
```

where `x[n]` represents the barycentric coordinates for X_n. Let us start with the first claim, $X_{20} \prec X_{22}$. The barycentric coordinates `x[20]` and `x[22]` are obtained from [2]. We can then execute the Mathematica commands

```
acuteConstraint = b^2<a^2+c^2 && c^2<a^2+b^2 && a^2<b^2+c^2
                                && a>0 && b>0 && c>0;
traceConstraint = acuteConstraint && a<b<c;
Simplify[d[20] > d[22], traceConstraint]
```

and Mathematica responds with `True`, proving that Atrace(X_{20}) is always further from C than Atrace(X_{22}). The other claims are proven in the same manner. □

Figure 18 shows some of the centers from X_1 through X_{19}. As we vary the shape of the acute triangle ABC (keeping $a < b < c$), the spacing of the traces will vary, but their order remains fixed.

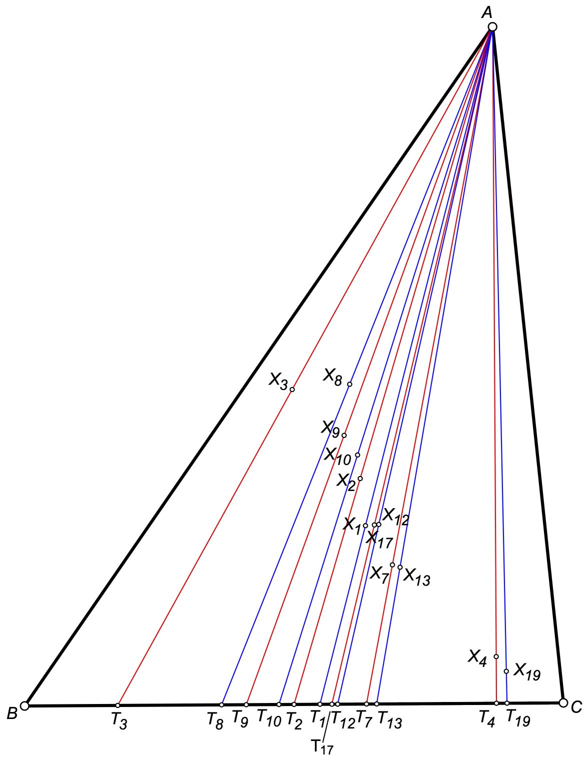

Figure 18: Centers in an acute triangle and their traces

Figure 19 shows the full order.

The center X_{26} does not appear in this graph because its A-trace can be arbitrarily far from C in either direction. The center X_{30} does not appear in this graph because it lies on the line at infinity.

Question 6.5. Is there a simple reason that center X_1 is a cutpoint for the graph shown in Figure 19?

Question 6.6. As more centers are added to the graph in Figure 19, does X_1 remain a cutpoint?

Figure 18 suggests that the traces of all centers lie on line segment BC. This is not the case.

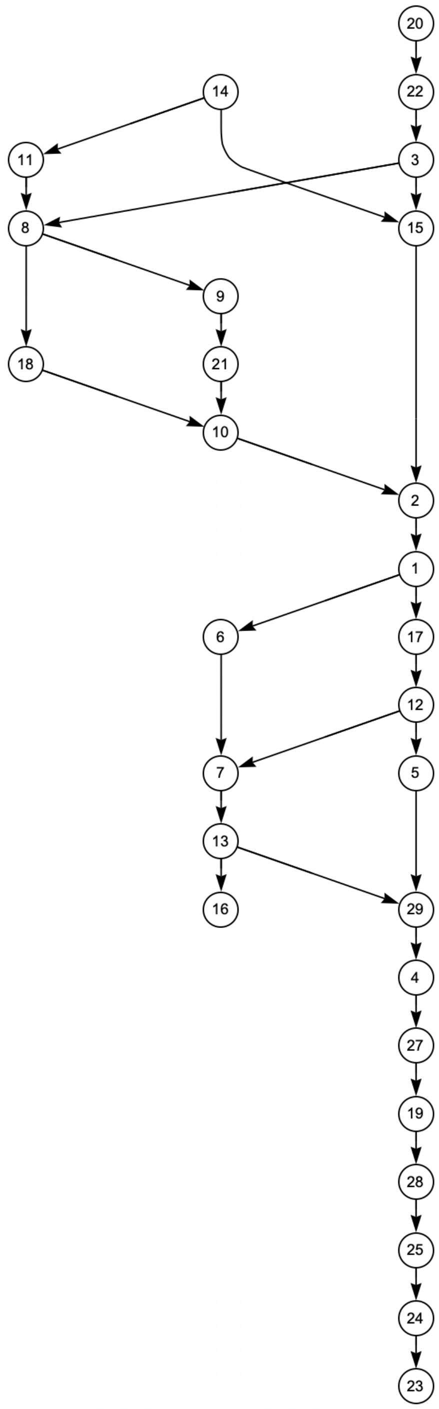

Figure 19: An arrow from m to n means $X_m \prec X_n$ under the trace order.

Theorem 6.7. *Using the trace order $\prec$, we have $C \prec X_{24}$.*

Proof. We execute the Mathematica command

```
Simplify[d[24] > d[ptC], traceConstraint]
```

where `ptC` = $\{0, 0, 1\}$ and Mathematica responds with `True`, confirming that $C \prec X_{24}$. □

The following theorem is proven in the same manner.

Theorem 6.8. *Using the trace order $\prec$, we have $X_{650} \prec B$.*

Theorem 6.9. *The A-trace of the center X_{23} can occur to the right of C.*

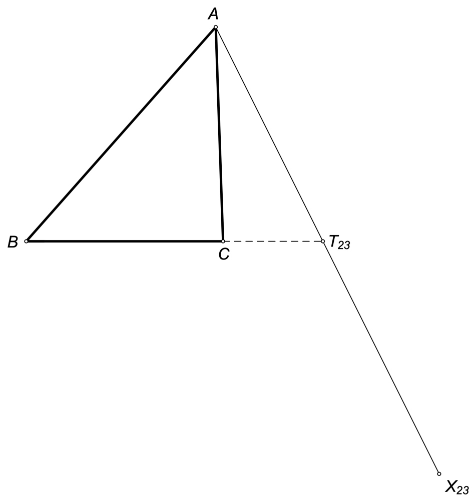

Figure 20: An 11–12–16 triangle

Proof. See figure 20 showing an 11–12–16 triangle in which T_{23} is to the right of point C. Note that angle C is about 88°. □

Although not shown in Figure 20, the same triangle shows that Atrace(X_i) can lie to the right of C for $i \in \{23, 36, 44, 50, 64, 66, 84, 99, 100\}$.

Theorem 6.10. *Of the first 29 triangle centers, only X_{16}, X_{23}, and X_{26} can have an A-trace to the right of C.*

Proof. From Lemma 6.2, the Mathematica condition for the A-trace to be to the right of C is

```
toTheRightOfC[{p_, q_, r_}] := q(q+r) < 0
```

and we can then use the Mathematica command

```
FindInstance[toTheRightOfC[x[i]] && traceConstraint, {a,b,c}]
```

(where `x[i]` denotes the coordinates for center X_i) to determine if Atrace(X_i) can be to the right of C. As we vary i from 1 to 29, we find that the only values for i for which Atrace(X_i) can be to the right of C are 16, 23, and 26. □

Theorem 6.11. *In a triangle with sides $a = 1$, $b = 2$, and $c = \sqrt{(1+\sqrt{61})/2}$, the A-trace of center X_{23} coincides with C. In other words, X_{23} lies on side AC for that triangle.*

Proof. From [2], we find that the barycentric coordinates for X_{23} are

$$\left(a^2\left(a^4-b^4+b^2c^2-c^4\right):b^2\left(-a^4+a^2c^2+b^4-c^4\right):c^2\left(-a^4+a^2b^2-b^4+c^4\right)\right).$$

Letting $a=1$, $b=2$, and $c=\sqrt{(1+\sqrt{61})/2}$ transforms this to

$$\left(\frac{3}{2}\left(\sqrt{61}-19\right):0:\frac{3}{2}\left(11+\sqrt{61}\right)\right)$$

which shows that X_{23} lies on AC. □

7. Areas for Future Research

Take some line associated with a triangle, such as the Euler line, the Brocard Axis, or a symmedian. Project centers onto this line using a point projection, a parallel projection, or an orthogonal projection. Investigate the order of the traces on this line for special types of triangles, such as acute triangles or triangles with a 60° angle.

8. Concluding Remarks

These ordering relations suggest that triangle centers possess an unexpected global structure, analogous to the partial orders that arise in other areas of discrete geometry.

We hope this paper has made the reader see that there is more order amongst triangle centers than they may have thought.

9. Funding

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