Central Angles Close to 180°

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Abstract

Some familiar triangle centers form angles remarkably close to 180°, no matter how the triangle is varied. We prove sharp bounds for these angles and classify the angles determined by the first ten centers in Kimberling's *Encyclopedia of Triangle Centers* that exceed 160° for every scalene triangle. A one-parameter inequality explains three unexpected constants, 1/3, 1/7, and 1/9.

Keywords. Triangle centers; Nagel line; incenter; symmedian point; Gergonne point; sharp angle inequalities; barycentric coordinates.
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1 Introduction

By a *central angle* we mean an angle formed by three triangle centers, with one of them as vertex. Angles are always measured in the ordinary, unsigned sense, from 0° to 180°.

This study grew out of a dynamic geometry sketch. Let G, K, and N denote the centroid, symmedian point, and Nagel point of a triangle. As the vertices were moved, the angle $\angle KGN$ seemed to remain surprisingly close to a straight angle (Figure 1). In particular, it appeared never to fall below 170°. How small can this angle actually be?

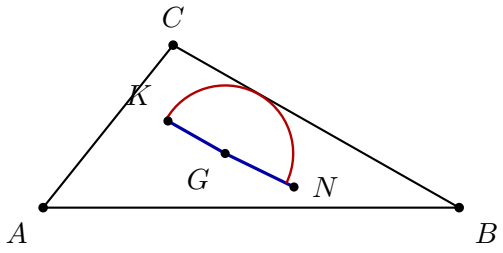


Figure 1: The nearly straight angle $\angle KGN$.

The first question has a pleasing answer.

Theorem 1.1 (A motivating example)**.** *For every scalene triangle,* $\angle KGN > 180° - \arcsin(1/7)$. *The bound is sharp, and* $\angle KGN < 180°$.

The moving sketch thus reveals a genuine geometric restriction. Which other triples of triangle centers have this property: *their angle is always nearly straight, but never exactly straight?*

To make the question manageable, we consider the first ten entries in Kimberling's *Encyclopedia of Triangle Centers* [2]. From now on, X_i denotes Kimberling's center number i. For example, $G = X_2$, $K = X_6$, and $N = X_8$. The notation $\angle X_iX_jX_k$ denotes an ordinary angle with vertex X_j. Interchanging the two endpoints does not change the angle, so these ten centers determine

$$10\binom{9}{2} = 360$$

choices of central angle with three distinct indices. Throughout the paper we consider nondegenerate scalene triangles; for some isosceles triangles, relevant centers coincide and their angles are undefined.

The principal classification is as follows.

Theorem 1.2 (Large-angle classification)**.** *Among the 360 angles formed by three distinct centers $X_1, \dots, X_{10}$, exactly nineteen exceed 160° for every scalene triangle. Eleven are identically 180°. The other eight appear in Table 1; every listed lower bound is sharp. Each of the remaining 341 angles is below 160° for at least one scalene triangle.*

Table 1: The eight nonconstant central angles always greater than 160°.

Angle(s)	Sharp lower bound
$\angle X_2X_1X_6$, $\angle X_6X_1X_8$, $\angle X_6X_1X_{10}$	$180^\circ - \arcsin(1/3)$
$\angle X_3X_1X_7$, $\angle X_5X_{10}X_9$	$180^\circ - \arcsin(1/3)$
$\angle X_6X_2X_8$, $\angle X_6X_2X_{10}$	$180^\circ - \arcsin(1/7)$
$\angle X_6X_{10}X_8$	$180^\circ - \arcsin(1/9)$

The sharp bounds are approximately 160.529°, 171.787°, and 173.621°. Three of the four rows in Table 1 come from a single family of inequalities; the remaining row has a different proof.

Let $I = X_1$ be the incenter. The centroid $G = X_2$, symmedian point $K = X_6$, and Nagel point $N = X_8$ are as above. On the line through I and G, define

$$Q_t = I + t(G - I), \qquad t \in \mathbb{R}. \tag{1}$$

Four special cases are $Q_0 = I$, $Q_1 = G$, $Q_{3/2} = X_{10}$ (the Spieker center), and $Q_3 = N$. These points lie on the Nagel line (Figure 2).

Theorem 1.3 (Nagel-line family)**.** *For every scalene triangle and $0 \le t < 3$,*

$$180^\circ - \arcsin\frac{1}{3+4t} < \angle KQ_tN < 180^\circ. \tag{2}$$

For each fixed t, the lower bound is sharp.

Let $O = X_3$ be the circumcenter and $D = X_7$ the Gergonne point.

Theorem 1.4 (Circumcenter–Gergonne angle)**.** *For every scalene triangle,*

$$180^\circ - \arcsin\frac{1}{3} < \angle OID < 180^\circ. \tag{3}$$

The lower bound is sharp.

Agaoka [1] and Nagydobai Kiss [3] have studied collinearities and affine relationships among classical triangle centers, including relationships used here. The present results concern sharp bounds for the angles they form. We give the proofs below, followed by an exact finite check that completes the classification.

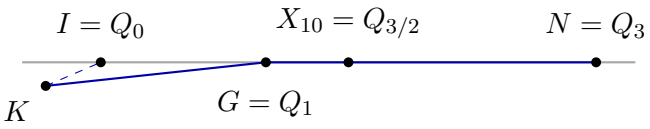


Figure 2: Points on the Nagel line and the symmedian point K.

2 A barycentric reduction

Let $a = BC$, $b = CA$, $c = AB$, $p = a + b + c$, $q = a^2 + b^2 + c^2$, and $\Delta = [ABC] > 0$. The relevant normalized barycentric coordinates are

$$I = \frac{(a, b, c)}{p}, \quad G = \frac{(1, 1, 1)}{3}, \quad K = \frac{(a^2, b^2, c^2)}{q}, \quad N = \frac{(p - 2a, p - 2b, p - 2c)}{p}. \tag{4}$$

Normalized barycentric triples represent points, so affine expressions such as $G - I$ have their usual meaning.

Our inequalities are unchanged by relabeling the vertices, so we may assume $a > b > c$ and introduce positive quantities

$$h = \frac{b + c - a}{2}, \qquad j = a - b, \qquad k = b - c. \tag{5}$$

Thus

$$a = 2h + 2j + k, \quad b = 2h + j + k, \quad c = 2h + j, \tag{6}$$

and, on writing $s = 3h + 2j + k = p/2$, Heron's formula gives

$$\Delta^2 = h(h + j)(h + j + k)s. \tag{7}$$

Set

$$T = \Delta\, jk(j + k) > 0. \tag{8}$$

For convenience, put

$$\begin{aligned} P_0 &= jk^2(j + k)(2j + k), \\ P_1 &= 2(j^4 + 2j^3k + 6j^2k^2 + 5jk^3 + k^4), \\ P_2 &= 3(2j + k)(j^2 + jk + 2k^2), \\ P_3 &= 4(j^2 + jk + k^2), \end{aligned}$$

and

$$S = P_0 + hP_1 + h^2P_2 + h^3P_3, \tag{9}$$

$$U = \frac{2}{3}(hj^2 + hjk + hk^2 + 2jk^2 + k^3) \cdot (6h^2 + 8hj + 4hk + 3j^2 + 3jk + k^2). \tag{10}$$

Lemma 2.1. *For $0 \le t < 3$, the angle $\angle KQ_tN$ is obtuse and*

$$\tan(180^\circ - \angle KQ_tN) = \frac{T}{S + tU}. \tag{11}$$

Proof. From (4), $N = I + 3(G - I)$, so the ray $Q_t N$ is directed along $G - I$ for $t < 3$. In normalized barycentric coordinates the differences $G - I$ and $K - I$ have coefficient triples

$$-\frac{u}{3p}, \qquad \frac{v}{pq}, \quad u = (3a - p, 3b - p, 3c - p), \quad v = (pa^2 - qa, pb^2 - qb, pc^2 - qc).$$

For any triple $z = (z_1, z_2, z_3)$ with $z_1 + z_2 + z_3 = 0$, its squared Euclidean norm is

$$F(z) = -a^2 z_2 z_3 - b^2 z_3 z_1 - c^2 z_1 z_2. \tag{12}$$

Polarizing this form yields

$$\langle u, v \rangle_F = 2pS, \qquad F(u) = \frac{6p}{q} U.$$

Consequently,

$$(K - Q_t) \cdot (G - I) = -\frac{2(S + tU)}{3pq} < 0.$$

The absolute planar cross product of these vectors is

$$|(K - Q_t) \times (G - I)| = \frac{2\Delta\, jk(j + k)}{3pq} = \frac{2T}{3pq},$$

by the Vandermonde determinant of the three barycentric triples $(1, 1, 1)$, (a, b, c), (a^2, b^2, c^2). Their cross product divided by the negative dot product proves (11), and the dot product shows that the angle is obtuse. □

3 A short proof of the Nagel-line bound

The next estimate is the key to the proof.

Lemma 3.1 (Two-term estimate)**.** *Set*

$$A = 2jh(h + j)(h + j + k)(j + k), \qquad B = jk^2(j + k)s. \tag{13}$$

Then $A, B > 0$, $AB = 2T^2$, *and*

$$S + tU > (1 + t)A + (1 + 2t)B \qquad (t \geq 0). \tag{14}$$

Proof. Heron's formula gives $AB = 2T^2$. For the inequality, put $w = j(j + k)$. Direct expansion gives

$$\begin{aligned} S - A - B = h\big[&2h^2(w + 2k^2) + h(2j + k)(w + 6k^2) \\ &+ k^2(7w + 2k^2)\big] > 0, \end{aligned}$$

and

$$\begin{aligned} U - A - 2B = \frac{2}{3}\big[&3h^3(w + 2k^2) + h^2(2j + k)(w + 10k^2) \\ &+ hk^2(11w + 5k^2) + k^4(2j + k)\big] > 0. \end{aligned}$$

Adding the first inequality to t times the second yields (14). □

Proof of Theorem 1.3. Lemmas 2.1 and 3.1 and AM–GM give

$$\begin{aligned} S + tU &> (1+t)A + (1+2t)B \\ &\geq 2\sqrt{(1+t)(1+2t)AB} \\ &= 2\sqrt{2(1+t)(1+2t)}\,T. \end{aligned}$$

Writing $\theta = 180^\circ - \angle KQ_tN \in (0, 90^\circ)$, we find

$$\tan\theta < \frac{1}{\sqrt{8(1+t)(1+2t)}}.$$

Since $1 + 8(1+t)(1+2t) = (3+4t)^2$,

$$\sin\angle KQ_tN = \sin\theta < \frac{1}{3+4t},$$

which proves (2).

To see that the bound is sharp, fix t and let

$$j = 1, \qquad k = \varepsilon, \qquad h = \frac{1+2t}{1+t}\,\varepsilon^2, \qquad \varepsilon \longrightarrow 0^+.$$

Equations (9)–(10) imply

$$\lim_{\varepsilon\to 0^+} \frac{T^2}{(S+tU)^2} = \frac{1}{8(1+t)(1+2t)}.$$

It follows that $\sin\angle KQ_tN \to 1/(3+4t)$. The sides given by (6) are scalene and satisfy the strict triangle inequalities for every sufficiently small positive ε. □

Corollary 3.2. *For each nondegenerate scalene triangle,*

$$\begin{aligned} \sin\angle X_2X_1X_6 &< \tfrac{1}{3}, \\ \sin\angle X_6X_2X_8 &< \tfrac{1}{7}, \\ \sin\angle X_6X_{10}X_8 &< \tfrac{1}{9}. \end{aligned}$$

The angles are obtuse, and all three constants are sharp.

Proof. Take $t = 0, 1, 3/2$ in Theorem 1.3. At $t = 0$, $\angle KIN = \angle GIK$, since G and N lie on the same ray from I. □

4 The circumcenter–incenter–Gergonne angle

The remaining nonconstant case has a different limiting configuration. We again use h, j, k, s, T from (5)–(8) and put

$$\begin{aligned} V = {} & (2h^2 + 2hj + j^2)k^3 \\ & + (4h^3 + 9h^2j + 10hj^2 + 4j^3)k^2 \\ & + j(h+j)(4h^2 + 11hj + 5j^2)k \\ & + 2j^2(h+j)^2(2h+j). \end{aligned} \tag{15}$$

Every term in V is positive.

Lemma 4.1. *The angle $\angle OID$ is obtuse, and*

$$\tan(180^\circ - \angle OID) = \frac{T}{V}. \tag{16}$$

Moreover, $V^2 > 8T^2$.

Proof. Homogeneous barycentric coordinates for O and D are

$$O = \big(a^2(b^2 + c^2 - a^2) : \text{cyc.}\big), \quad D = \big((p - 2b)(p - 2c) : \text{cyc.}\big). \tag{17}$$

Normalize these triples and use the metric (12). Their dot product at I is negative, and division of the absolute cross product by its negative gives (16). Appendix A gives an explicit coefficient expansion of $V^2 - 8T^2$. Its coefficients are nonnegative and its constant term is positive, proving the assertion. □

Proof of Theorem 1.4. By Lemma 4.1,

$$\tan(180^\circ - \angle OID) = \frac{T}{V} < \frac{1}{\sqrt{8}}.$$

Converting to sine gives $\sin \angle OID < 1/3$; its obtuseness yields (3). For sharpness, set

$$j = 1, \qquad h = \frac{\sqrt{3} - 1}{2}, \qquad k \longrightarrow \infty.$$

Directly from (15) and (7), $T^2/V^2 \to 1/8$, and hence $\angle OID \to 180^\circ - \arcsin(1/3)$. □

The homothety of factor $-1/2$ about the centroid G maps

$$(O, I, D) \longmapsto (X_5, X_{10}, X_9).$$

Thus $\angle X_5X_{10}X_9 = \angle OID$. Figure 3 illustrates this relation.

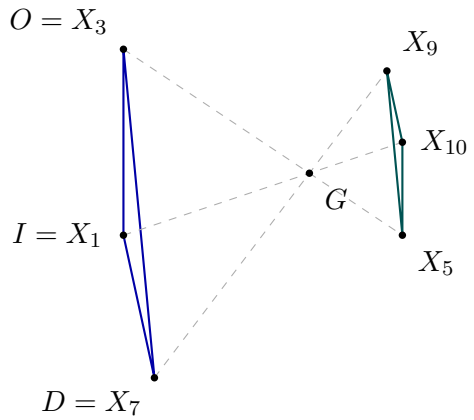


Figure 3: The homothety about G with factor $-1/2$.

5 Classification among the first ten centers

Eleven of the nineteen universal large angles are always straight. They arise from these ordered collinear configurations:

$X_3,\ X_2,\ X_5,\ X_4$	(Euler line),
$X_1,\ X_2,\ X_{10},\ X_8$	(Nagel line),
$X_6,\ X_1,\ X_9,$	
$X_7,\ X_2,\ X_9,$	
$X_4,\ X_{10},\ X_9.$	

In each row the order is valid up to reversal for every scalene triangle. The first two lines contribute four straight angles each, and the last three contribute one each, for a total of eleven. The barycentric coordinates in Appendix B verify these collinearities and the order statements.

All eight nonconstant angles in Table 1 follow from Theorems 1.3 and 1.4. Indeed, the first and third rows of the table are equal-angle configurations on the Nagel line; the fourth is the Spieker-center case of Theorem 1.3; and the second comes from the complementary homothety about G.

It remains to exclude the other 341 possibilities. For this purpose, three triangles suffice:

$$(a, b, c) = (13, 11, 6), \quad (106, 56, 51), \quad (14, 13, 3). \tag{18}$$

For each of the 341 other angles, at least one of these three triangles gives an angle strictly less than

$$\arccos(-15/16) = 159.635\ldots^\circ < 160^\circ. \tag{19}$$

Appendix B gives a rational-arithmetic test for this assertion. This completes the proof of Theorem 1.2.

6 Concluding remarks

The formula $1/(3 + 4t)$ explains the three sharp constants $1/3$, $1/7$, and $1/9$. Together with the three witness triangles in (18), it helps classify all 360 central angles. A geometric explanation of the two AM–GM terms, or a similar classification for more triangle centers, would be of interest.

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A A positivity certificate for $V^2 - 8T^2$

For reference, write

$$V^2 - 8T^2 = \sum_{m=0}^{6} C_m k^m,$$

where h, j are positive and

$$\begin{aligned}
C_0 &= 4j^4(h+j)^4(2h+j)^2,\\
C_1 &= 4j^3(h+j)^3(2h+j)(4h^2+11hj+5j^2),\\
C_2 &= j^2(h+j)^3(48h^3+128h^2j+125hj^2+41j^3),\\
C_3 &= 2j(h+j)(16h^5+88h^4j+155h^3j^2+135h^2j^3+76hj^4+22j^5),\\
C_4 &= 16h^6+88h^5j+213h^4j^2+216h^3j^3\\
&\qquad +110h^2j^4+60hj^5+26j^6,\\
C_5 &= 2\big(8h^5+26h^4j+26h^3j^2+j^3(h-j)^2+3j^5\big),\\
C_6 &= (2h^2+2hj-j^2)^2.
\end{aligned}$$

The displayed forms show $C_m > 0$ for $0 \le m \le 5$ and $C_6 \ge 0$, proving strict positivity.

B Exact certificate for the finite classification

For the finite check, we use the following barycentric coordinates. Write $p = a+b+c$; affine relations refer to normalized coordinates.

Center	Homogeneous barycentric coordinates or affine relation
X_1	(a, b, c)
X_2	$(1, 1, 1)$
X_3	$(a^2(b^2+c^2-a^2) : \text{cyc.})$
X_4	$3X_2 - 2X_3$ (normalized affine coordinates)
X_5	$(3X_2 - X_3)/2$ (normalized affine coordinates)
X_6	(a^2, b^2, c^2)
X_7	$((p-2b)(p-2c) : \text{cyc.})$
X_8	$(p-2a, p-2b, p-2c)$
X_9	$(a(p-2a), b(p-2b), c(p-2c))$
X_{10}	$(p-a, p-b, p-c)$

The Mittenpunkt is X_9. Homogeneous triples are normalized before affine operations.

The following exact test verifies the three witness triangles in (18). For a triple $z = (z_1, z_2, z_3)$ with zero sum, set

$$F(z) = -a^2 z_2 z_3 - b^2 z_3 z_1 - c^2 z_1 z_2.$$

For an angle with vertex X_m and endpoints X_i, X_k, set $u = X_i - X_m$ and $v = X_k - X_m$ using normalized barycentric vectors. Then $F(u), F(v) > 0$, and

$$d = \tfrac{1}{2}(F(u+v) - F(u) - F(v))$$

is the dot product of the corresponding Euclidean vectors. Thus $\angle X_i X_m X_k < \arccos(-15/16)$ holds exactly when

$$d \ge 0 \quad \text{or} \quad \big(d < 0 \text{ and } 256d^2 < 225F(u)F(v)\big).$$

Applying this test to all 360 angles and the three triangles in (18) leaves nineteen angles without a witness: the eleven straight angles in Section 5 and the eight in Table 1. Thus the other 341 angles each fail the 160° bound for at least one witness triangle. Every comparison is exact and uses rational arithmetic only.

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