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More Relationships between a Central Quadrilateral and its Reference Quadrilateral

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Abstract. The diagonals of a quadrilateral form four associated triangles, called *half triangles*. Each half triangle is bounded by two sides of the quadrilateral and one diagonal. If we locate a triangle center (such as the incenter, centroid, orthocenter, etc.) in each of these triangles, the four triangle centers form another quadrilateral called a *central quadrilateral*. For each of various shaped quadrilaterals, and each of 1000 different triangle centers, we compare the reference quadrilateral to the central quadrilateral. Using a computer, we determine how the two quadrilaterals are related. For example, we test to see if the two quadrilaterals are congruent, similar, have the same area, or have the same perimeter.

Keywords. triangle centers, quadrilaterals, computer-discovered mathematics, Euclidean geometry, GeometricExplorer, Baricentricas.

Mathematics Subject Classification (2020). 51M04, 51-08.

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1. Introduction

The diagonals of a quadrilateral (called the *reference quadrilateral*) form four associated triangles, called *half triangles*, shown in Figure 1. Each half triangle is bounded by two sides of the quadrilateral and one diagonal. The reference quadrilateral is always named $ABCD$. The four triangles (numbered 1 to 4) are shown in Figure 1.

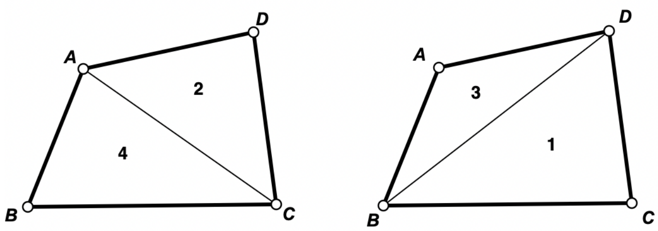


Figure 1. Half Triangles

The triangles have been numbered so that triangle 1 is opposite vertex A, triangle 2 is opposite vertex B, etc. The four triangles are $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$.

Triangle centers are selected in each triangle (for example, incenters, centroids, or orthocenters). The same type of triangle center is used with each half triangle. In order, the names of these points are E, F, G, and H, as shown in Figure 2. These four centers form a quadrilateral $EFGH$ that will be called the *central quadrilateral*. Quadrilateral $EFGH$ need not be convex.

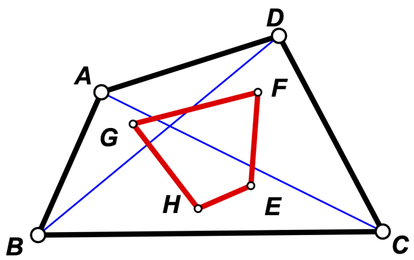


Figure 2. Central Quadrilateral

The purpose of this paper is to determine interesting relationships between a reference quadrilateral and its central quadrilateral. This paper extends our previous results found in [5].

2. Types of Quadrilaterals Studied

We are only interested in convex reference quadrilaterals that have a certain amount of symmetry. For example, we excluded bilateral quadrilaterals (those with two equal sides), bisect-diagonal quadrilaterals (where one diagonal bisects another), right kites, right trapezoids, and golden rectangles. The types of quadrilaterals we studied are shown in Table 1. The sides of the quadrilateral, in order, have lengths a, b, c, and d. The diagonals have lengths p and q. The measures of the angles of the quadrilateral, in order, are A, B, C, and D.

Table 1.

Types of Quadrilaterals Considered

Quadrilateral Type	**Geometric Definition**	**Algebraic Condition**
general	convex	none
cyclic	has a circumcircle	$A+C=B+D$
tangential	has an incircle	$a+c=b+d$
extangential	has an excircle	$a+b=c+d$
parallelogram	opposite sides parallel	$a=c$, $b=d$
equalProdOpp	product of opposite sides equal	$ac=bd$
equalProdAdj	product of adjacent sides equal	$ab=cd$
orthodiagonal	diagonals are perpendicular	$a^2+c^2=b^2+d^2$
equidiagonal	diagonals have the same length	$p=q$
Pythagorean	equal sum of squares, adjacent sides	$a^2+b^2=c^2+d^2$
kite	two pair adjacent equal sides	$a=b$, $c=d$
trapezoid	one pair of opposite sides parallel	$A+B=C+D$
rhombus	equilateral	$a=b=c=d$
rectangle	equiangular	$A=B=C=D$
Hjelmslev	two opposite right angles	$A=C=90°$
isosceles trapezoid	trapezoid with two equal sides	$A=B$, $C=D$
APquad	sides in arithmetic progression	$d-c=c-b=b-a$

The following combinations of entries in the above list were also considered: bicentric quadrilaterals (cyclic and tangential), exbicentric quadrilaterals (cyclic and extangential), bicentric trapezoids, cyclic orthodiagonal quadrilaterals, equidiagonal kites, equidiagonal orthodiagonal quadrilaterals, equidiagonal orthodiagonal trapezoids, harmonic quadrilaterals (cyclic and equalProdOpp), orthodiagonal trapezoids, tangential trapezoids, and squares (equiangular rhombi).

So, in addition to the general convex quadrilateral, a total of 27 other types of quadrilaterals were considered in this study.

A graph of the types of quadrilaterals considered is shown in Figure 3. An arrow from A to B means that any quadrilateral of type B is also of type A. For example: all squares are rectangles and all kites are orthodiagonal. If a directed path leads from a quadrilateral of type A to a quadrilateral of type B, then we will say that A is an *ancestor* of B. For example, an equidiagonal quadrilateral is an ancestor of a rectangle. In other words, all rectangles are equidiagonal.

Unless otherwise specified, when we give a theorem or table of properties of a quadrilateral, we will omit an entry for a particular shape quadrilateral if the property is known to be true for an ancestor of that quadrilateral.

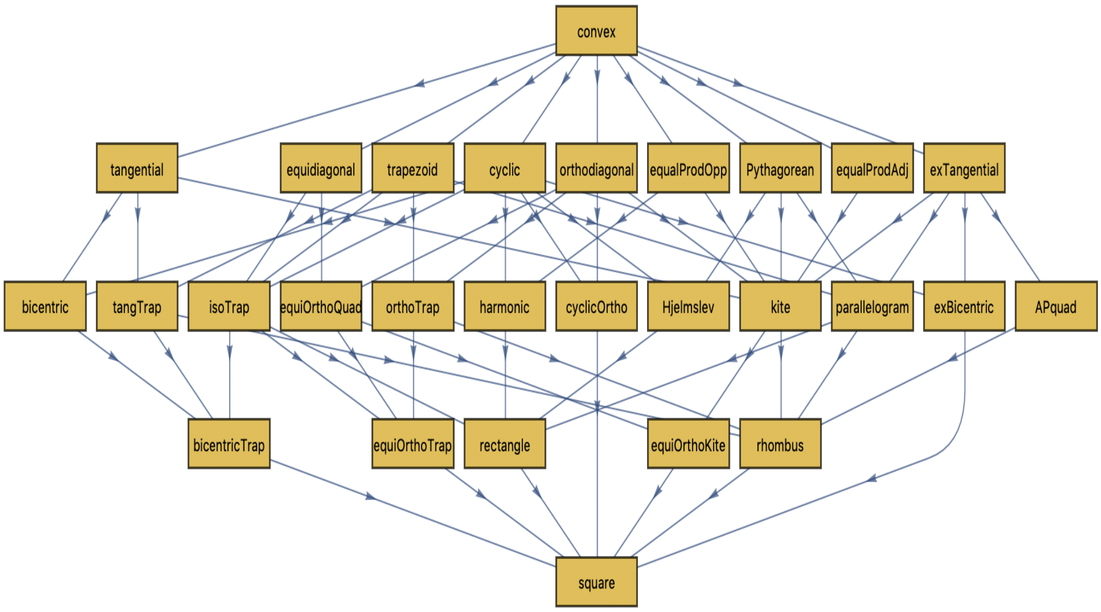


Figure 3. Quadrilateral Shapes

3. Centers

In this study, we will place triangle centers in the four half triangles. We use Clark Kimberling's definition of a triangle center [1].

A *center function* is a nonzero function $f(a,b,c)$ homogeneous in a, b, and c and symmetric in b and c. *Homogeneous* in a, b, and c means that

$$f(ta,tb,tc) = t^n f(a,b,c)$$

for some nonnegative integer n, all $t > 0$, and all positive real numbers (a,b,c) satisfying $a < b+c$, $b < c+a$, and $c < a+b$. *Symmetric* in b and c means that

$$f(a,c,b) = f(a,b,c)$$

for all a, b, and c.

A *triangle center* is an equivalence class $x : y : z$ of ordered triples (x,y,z) given by

$$x = f(a,b,c), \quad y = f(b,c,a), \quad z = f(c,a,b).$$

Tens of thousands of interesting triangle centers have been cataloged in the Encyclopedia of Triangle Centers [2]. We use X_n to denote the n-th named center in this encyclopedia.

Note that if the center function of a certain center is $f(a,b,c)$, then the trilinear coordinates of that point with respect to a triangle with sides a, b, and c are

$$\Big(f(a,b,c) : f(b,c,a) : f(c,a,b)\Big).$$

The barycentric coordinates for that point would then be

$$\Big(af(a,b,c) : bf(b,c,a) : cf(c,a,b)\Big).$$

4. METHODOLOGY

We used a computer program called GeometricExplorer to compare quadrilaterals with their central quadrilateral. Starting with each type of quadrilateral listed in Figure 3 for the reference quadrilateral, we placed triangle centers in each of the four half triangles.

For each n from 1 to 1000, we determined center X_n of each of the half triangles of the reference quadrilateral. The program then analyzes the central quadrilateral formed by these four centers and reports if the central quadrilateral is related to the reference quadrilateral. Points at infinity were omitted. The types of relationships checked for are shown in Table 4.

Relationships Checked For	
notation	**description**
$[ABCD] = [EFGH]$	the quadrilaterals have the same area (This relationship is excluded if the quadrilaterals are congruent.)
$[ABCD] = k[EFGH]$	the area of $ABCD$ is k times the area of $EFGH$ †
$ABCD \cong EFGH$	the quadrilaterals are congruent
$ABCD \sim EFGH$	the quadrilaterals are similar (This relationship is excluded if the quadrilaterals are homothetic.)
$\partial ABCD = \partial EFGH$	the quadrilaterals have the same perimeter (This relationship is excluded if the quadrilaterals are congruent.)
$\odot ABCD \cong \odot EFGH$	the quadrilaterals have congruent circumcircles (This relationship is excluded if the quadrilaterals are congruent.)
$\odot ABCD \equiv \odot EFGH$	the quadrilaterals have the same circumcircle
$\mathrm{o}(ABCD) = \mathrm{o}(EFGH)$	the quadrilaterals have the same circumcenter (This relationship is excluded if the quadrilaterals have the same circumcircle.)
$\mathrm{i}(ABCD) = \mathrm{i}(EFGH)$	the quadrilaterals have the same incenter
$\mathrm{dp}(ABCD) = \mathrm{dp}(EFGH)$	the quadrilaterals have the same diagonal point
$\mathrm{persp}(ABCD, EFGH)$	the quadrilaterals are perspective (This relationship is excluded if the quadrilaterals are homothetic.)
$\mathrm{homot}(ABCD, EFGH)$	the quadrilaterals are homothetic
$\mathrm{conic}(ABCD, EFGH)$	the quadrilaterals have a common noncircular circumconic
$\mathrm{hyperb}(ABCD, EFGH)$	the quadrilaterals have a common circumconic which is a rectangular hyperbola. (By definition, the center of this hyperbola is the Poncelet point (QA-P2) of both $ABCD$ and $EFGH$.)
$\mathrm{ctr1}[ABCD] = \mathrm{ctr2}[EFGH]$	the quadrilaterals have coincident centers
† Only rational values of k were checked for with denominators less than 10.	

The types of quadrilateral centers considered are shown in Table 2. For example, the relationship ponce[$ABCD$] = stein[$EFGH$] means that the Poncelet point of quadrilateral $ABCD$ coincides with the Steiner point of quadrilateral $EFGH$.

Table 2.

Quadrilateral Centers Considered		
name	**description**	**symbol**
vertex centroid	(QA-P1)	m
Poncelet point	(QA–P2), also known as the Euler-Poncelet point	ponce
Steiner point	(QA–P3), also known as the Gergonne-Steiner point	stein
diagonal point	intersection of the diagonals (QG–P1)	dp
The following centers are only defined for cyclic quadrilaterals.		
anticenter	intersection of the maltitudes (QA–P2)	anti
circumcenter	center of circumscribed circle (QA-P3)	o
centrocenter	center of circle through centroids of half triangles (QA-P7)	centro
orthocenter	center of circle through orthocenters of half triangles	h
The following centers are only defined for tangential quadrilaterals.		
incenter	center of inscribed circle	i

Some quadrilateral centers only exist for certain shape quadrilaterals. For example, the circumcenter, anticenter, orthocenter, and centrocenter only apply to cyclic quadrilaterals. The incenter only applies to tangential quadrilaterals. A code in parentheses represents the name for the point as listed in the Encyclopedia of Quadri-Figures [7].

When reporting perspectivities or homotheties, we will specify what type of point the perspector is. Only quadrangle points listed in [7] are detected. As of January 2025, only 44 points were listed.

For example, the property QA-Pi=persp($ABCD, EFGH$)=QA-Pj means that the perspector is the QA-Pi point of quadrilateral $ABCD$ and is the QA-Pj point of quadrilateral $EFGH$.

A similar notation is used when describing properties involving conics. We check to see if the center of the conic is one of the known quadrangle points. For example, the property QA-Pi=conic($ABCD, EFGH$) means that the center of the common circumconic is the QA-Pi point of quadrilateral $ABCD$.

The most common quadrangle points are described in the following table. See [7] for the definition of terms and more details. Note that a *quadrangle* is an unordered set of four points in the plane (no three of which are collinear).

Table 3.

Common Quadrangle Centers		
symbol	**common name**	**description**
QA-P1	Quadrangle Centroid	center of gravity of equal masses placed at the vertices
QA-P2	Euler-Poncelet Point	common point of the nine-point circles of the half triangles
QA-P3	Gergonne-Steiner Point	Common point of the four midray circles
QA-P4	Isogonal Center	homothetic center of $ABCD$ with the 2nd generation isogonal conjugate quadrangle
QA-P5	Isotomic Center	perspector of $ABCD$ with the isotomic conjugate quadrangle
QA-P6	Parabola Axes Crosspoint	intersection point of the axes of the two parabolas that can be constructed through A, B, C, and D
QA-P7	9-pt Homothetic Center	homothetic center of $ABCD$ with the quadrangle composed of four 2nd generation nine-point venters
QA-P9	QA Miquel Center	common point of the three Miquel circles of the half triangles
QA-P12	Orthocenter of the Diagonal Triangle	orthocenter of the triangle formed by the three diagonal points of $ABCD$
QA-P34	Euler-Poncelet Point of the Centroid Quadrangle	Euler-Poncelet point of the quadrangle formed by centroids of the half triangles

5. Barycentric Coordinates and Quadrilaterals

The program we used to find results about central quadrilaterals (GeometricExplorer) is a useful tool for discovering results, but it does not prove that these results are true. GeometricExplorer uses numerical coordinates (to 15 digits of precision) for locating all the points. Thus, a relationship found by this program does not constitute a proof that the result is correct, but gives us compelling evidence for the validity of the result.

If a theorem in this paper is accompanied by a figure, this means that the figure was drawn using either Geometer's Sketchpad or GeoGebra. In either case, we used the drawing program to dynamically vary the points in the figure. Noticing that the result remains true as the points vary offers further evidence that the theorem is true. But again, this does not constitute a proof.

To prove the results that we have discovered, we use geometric methods, when possible. If we could not find a purely geometrical proof, we turned to analytic methods using barycentric coordinates and performing exact symbolic computation using Mathematica and the package `baricentricas.m`[3].

These analytic proofs are given in Mathematica Notebooks included with the supplementary material accompanying the on-line publication of this paper.

If our only "proof" of a particular relationship is by using numerical calculations (and not using exact computation), then we have colored the center red in the table of relationships.

When proving results analytically, we used barycentric coordinates. We assume the reader is familiar with this coordinate system. Given a quadrilateral $ABCD$, we set up a barycentric coordinate system using $\triangle ABC$ as the reference triangle. We assign coordinates $(p : q : r)$ to point D as shown in Figure 4. Note that $AB = c$, $BC = a$, and $AC = b$.

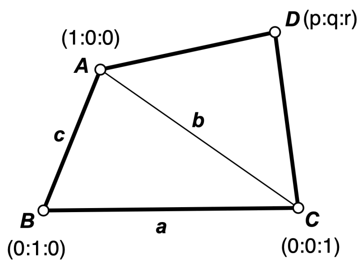


Figure 4. barycentric coordinate system for quadrilateral $ABCD$

The barycentric coordinates for the various triangle centers were found from [2]. To find the coordinates of a center $(u : v : w)$ with respect to a triangle XYZ, we use the function `CentroETCTriangulo` in the `baricentricas.m` package via the call `CentroETCTriangulo[{u,v,w},{ptX,ptY,ptZ}]` where `ptX`, `ptY`, and `ptZ`, are the barycentric coordinates for the vertices of $\triangle XYZ$.

[3]The package `baricentricas.m` written by F. J. G Capitán can be freely downloaded from `http://garciacapitan.epizy.com/baricentricas/`

When analyzing an initial quadrilateral with a special shape, we restrict the values of p, q, and r by specifying a condition that a, b, c, p, q, and r must satisfy using the conditions shown in the following table.

Geometrical condition	**Analytic condition**
A, B, C, D concyclic	$a^2qr + b^2pr + c^2pq = 0$
$AB + CD = BC + AD$	$2pr\,(a^2 - 2ac + b^2 + c^2) + p^2(a-b-c)(a+b-c) + r^2(a-b-c)(a+b-c) - 4cpq(a-c) + 4aqr(a-c) = 0$
$AB + BC = AD + DC$	$p^2\,(a^2 + 2ac - b^2 + c^2) + pr\,(2a^2 + 4ac + 2b^2 + 2c^2) + r^2\,(a^2 + 2ac - b^2 + c^2) + qr\,(4a^2 + 4ac) + pq\,(4ac + 4c^2) = 0$
$BC + CD = AB + AD$	$p^2\,(a^2 - 2ac - b^2 + c^2) + pr\,(2a^2 - 4ac + 2b^2 + 2c^2) + r^2\,(a^2 - 2ac - b^2 + c^2) + qr\,(4a^2 - 4ac) + pq\,(4c^2 - 4ac) = 0$
$AB = BC$	$a = c$
$AD \parallel BC$	$q + r = 0$
$AB \parallel CD$	$p + q = 0$
$AB \cdot CD = AC \cdot DA$	$p(b^2 - c^2)(qc^2 + rb^2) + a^2\,(c^2q(p+q) - b^2r(p+r)) = 0$
$AB \cdot AC = BC \cdot CD$	$a^4q(p+q) + a^2p\,(b^2(p+q) - qc^2) = b^2c^2(p+q+r)^2$
$AC \perp BD$	$b^2(p-r) = (a^2 - c^2)(p+r)$
$AB \perp BC$	$b^2 = a^2 + c^2$
$AC = BD$	$b^2\,(p^2 + (q+r)^2 + p(2q+3r)) = (p+r)(c^2p + a^2r)$
$AB^2 + AC^2 = BC^2 + CD^2$	$a^2\,(p^2 + p(q+2r) + r(2q+r)) = b^2\,(2p^2 + p(3q+2r) + (q+r)^2) + c^2\,(p^2 + p(q+2r) + (q+r)^2)$
kite	$p + 2q + r = 0 \ \wedge\ b^2 + q = 0 \ \wedge\ c^2 = a^2 + q + r$
parallelogram	$q + r = 0 \ \wedge\ p + q = 0$
rhombus	$q + r = 0 \ \wedge\ p + q = 0 \ \wedge\ a = c$
rectangle	$q + r = 0 \ \wedge\ p + q = 0 \ \wedge\ b^2 = a^2 + c^2$
isosceles trapezoid	$b^2p + (a^2 - c^2)\,q = 0$
harmonic quadrilateral	$ra^2 = pc^2 \ \wedge\ pb^2 + 2qa^2 = 0$
orthodiagonal quadrilateral	$a^2(p+r) + b^2(r-p) - c^2(p+r) = 0$

If a quadrilateral shape is formed by a combination of conditions, then the condition used to obtain that shape is the conjunction of the primitive conditions.

For example, a parallelogram has $AD \parallel BC$ and $AB \parallel CD$, so the condition on p, q, and r that makes $ABCD$ a parallelogram is $(q + r = 0) \wedge (p + q = 0)$. A rhombus is a parallelogram with the added condition $AB = BC$, so we add in the analytical condition $a = c$.

When checking to see if a point is a notable center of quadrilateral $ABCD$, we use the CT coordinates found from [7]. The coordinates were scaled to get rid of any fractions. Otherwise, applying these coordinates to certain shape quadrilaterals would produce divide-by-zero errors.

5.1. Example.

We now give an example of how barycentric coordinates can be used to prove the results in this paper. We show how to prove the following theorem using barycentric coordinates.

Theorem 5.1. *Let $ABCD$ be an orthodiagonal quadrilateral. Let E, F, G, and H be the X_5 -points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then the centroid of quadrilateral $ABCD$ coincides with the diagonal point of quadrilateral $EFGH$ (Figure 5).*

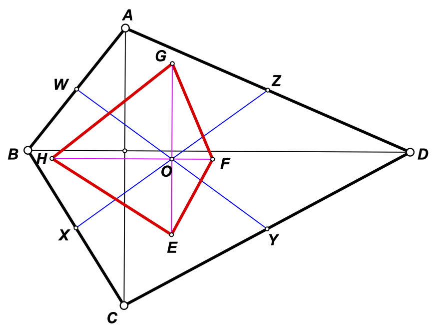


FIGURE 5. orthodiagonal quad with X_5-points $\implies$ m$[ABCD]$ = dp$[EFGH]$

Note that W, X, Y, and Z are the midpoints of the sides of quadrilateral $ABCD$, making O the centroid. We need to show that O coincides with the intersection of diagonals EG and FH of quadrilateral $EFGH$.

Proof. We begin by specifying the coordinates for the vertices of quadrilateral $ABCD$.

```
ptA = {1:0:0};
ptB = {0:1:0};
ptC = {0:0:1};
ptD = {p:q:r};
```

Then we use the function `CentroETCTriangulo` from the `baricentricas` package to create a routine that determines the center X_n of the four half triangles of quadrilateral $ABCD$.

```
CentralQuadrilateral[n_] :=
  {
  Simplificar[CentroETCTriangulo[ETC[[n, 2]], {ptB, ptC, ptD}]],
  Simplificar[CentroETCTriangulo[ETC[[n, 2]], {ptA, ptC, ptD}]],
  Simplificar[CentroETCTriangulo[ETC[[n, 2]], {ptA, ptB, ptD}]],
  Simplificar[CentroETCTriangulo[ETC[[n, 2]], {ptA, ptB, ptC}]]
  };
```

Then we use this routine to find the coordinates of E, F, G, and H.

```
{ptE, ptF, ptG, ptH} = CentralQuadrilateral[5];
```

The result from Mathematica shows that ptE $= \{x, y, z\}$ where

$$x = \left((b^2-c^2)^2 - a^2(b^2+c^2)\right)p^2 - 2a^4qr - a^2\left(a^2+b^2-c^2\right)pr - a^2\left(a^2-b^2+c^2\right)pq,$$

$$y = \left((a^2-c^2)^2 - b^2(c^2+a^2)\right)p^2 + \left(2a^4+b^4+c^4-2b^2c^2-3a^2c^2-3a^2b^2\right)pq$$
$$+\left(a^4+b^4+c^4-2b^2c^2-2a^2c^2\right)pr + a^2\left(a^2+b^2-c^2\right)qr,$$

$$z = \left((a^2-b^2)^2 - c^2(a^2+b^2)\right)p^2 + \left(a^4+b^4+c^4-2b^2c^2-2a^2b^2\right)pq$$
$$+\left(2a^4+b^4+c^4-2b^2c^2-3a^2c^2-3a^2b^2\right)pr + a^2qr\left(a^2-b^2+c^2\right)$$

with similarly complicated expressions for ptF, ptG, and ptH.

Next, we find the centroid of quadrilateral $ABCD$.

```
centroid = CentroidQuad[{ptA, ptB, ptC, ptD}];
```

using the routine

```
CentroidQuad[{P_, Q_, R_, S_}] := Punto[
   Recta[Medio[P, Q], Medio[R, S]],
   Recta[Medio[P, S], Medio[Q, R]]
   ];
```

giving the result

$$\text{centroid} = \{2p+q+r, p+2q+r, p+q+2r\}.$$

Then we find the diagonal point of quadrilateral $EFGH$,

```
dp = DiagonalPt[{ptE, ptF, ptG, ptH}];
```

using the routine

```
DiagonalPt[{P_, Q_, R_, S_}] := Punto[Recta[P, R], Recta[Q, S]];
```

giving a complicated expression for dp.

Now we write down the condition that these two points coincide recalling the fact that two barycentric coordinates represent the same point if they are proportional.

```
sameCondition = Cross[centroid, dp] == {0, 0, 0};
```

The resulting expression is quite complicated since the two points do not coincide in an arbitrary quadrilateral.

The next step is to find the condition that ensures that $ABCD$ is orthodiagonal.

```
orthodiag = SonPerpendiculares[Recta[ptA, ptC], Recta[ptB, ptD]];
```

The condition found is

$$a^2(p+r) + b^2(r-p) - c^2(p+r) = 0.$$

Finally, we simplify sameCondition subject to this constraint.

```
Simplify[sameCondition, orthodiag]
```

Mathematica responds with

```
True
```

indicating that the points coincide. □

6. General Quadrilaterals

Our computer study found the following relationships between a general quadrilateral and its central quadrilateral.

Central Quadrilaterals of General Quadrilaterals	
Relationship	**centers**
$[ABCD] = 9[EFGH]$	2
$[ABCD] = [EFGH]$	4
QA-P2=hyperb$(ABCD, EFGH)$=QA-P2	4
m$[ABCD]$ = m$[EFGH]$	2
m$[ABCD]$ = ponce$[EFGH]$	5
ponce$[ABCD]$ = ponce$[EFGH]$	4
ponce$[EFGH]$ = stein$[ABCD]$	3
QA-P1=homot$[ABCD, EFGH]$=QA-P1	2

6.1. **Properties involving X_2.**

The following result comes from [4].

Theorem 6.1. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_2-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ are similar. The ratio of similitude is 3 (Figure 6).*

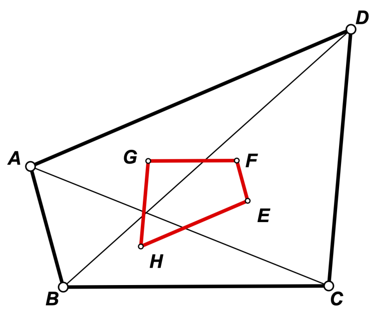


FIGURE 6. general quadrilateral with X_2-points $\implies$ $ABCD \sim EFGH$

Theorem 6.2. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_2-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then $[ABCD] = 9[EFGH]$ (Figure 6).*

Proof. This follows immediately from Theorem 6.1 since the ratio of the areas of two similar figures is equal to the square of the ratio of their sides. □

Theorem 6.3. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_2-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ are homothetic. The homothetic center is the centroid of quadrilateral $ABCD$ (Figure 7).*

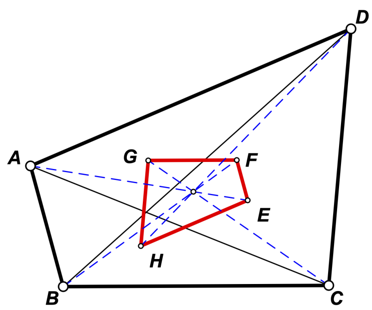


Figure 7. general quadrilateral with X_2-points $\implies$ homot$(ABCD, EFGH)$

Proof. From [8] we know that the lines from the vertices of a triangle to the centroid of the opposite half triangle meet in a point known as the centroid of the quadrilateral. Thus, the quadrilaterals are perspective. Since they are also similar, this means they are homothetic. □

Theorem 6.4. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_2-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ have the same centroid (Figure 7).*

Proof. This follows from Theorem 6.3 since a homothety maps the centroid of a figure into the centroid of the new figure and the center of the homothety is the centroid of quadrilateral $ABCD$. □

6.2. **Properties involving X_3.**

Theorem 6.5. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_3-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then*

$$\text{ponce}[EFGH] = \text{stein}[ABCD].$$

Our proof of Theorem 6.5 is analytical using barycentric coordinates.

Open Question 1. *Is there a purely geometrical proof of Theorem 6.5?*

6.3. **Properties involving X_4.**

The following result comes from [4].

Theorem 6.6. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_4-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ have the same area (Figure 8).*

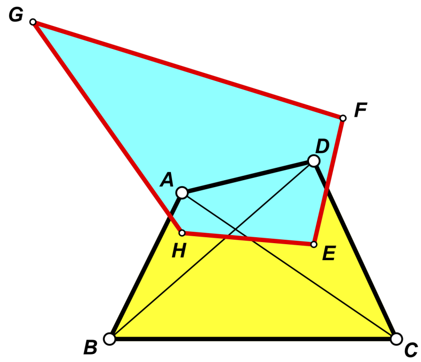


FIGURE 8. general quadrilateral with X_4-points $\implies$ $[ABCD] = [EFGH]$

The following result comes from [9].

Theorem 6.7. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_4-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ have a common circumconic (Figure 9). The conic is a rectangular hyperbola and the center of the conic, O, is the Euler-Poncelet Point (QA-P2) of both quadrilaterals $ABCD$ and $EFGH$.*

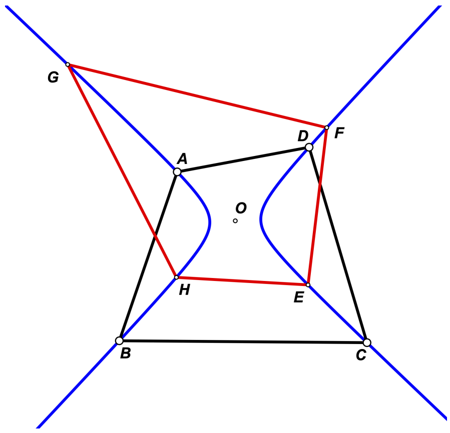


FIGURE 9. general quadrilateral with X_4-points $\implies$ hyperb$(ABCD, EFGH)$

Corollary 6.8. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_4-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then*

$$\text{ponce}[ABCD] = \text{ponce}[EFGH].$$

6.4. **Properties involving X_5.**

Theorem 6.9. *Let $ABCD$ be an arbitrary quadrilateral. Let E, F, G, and H be the X_5-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then*

$$\text{m}[EFGH] = \text{ponce}[ABCD].$$

Our proof of Theorem 6.9 is analytical using barycentric coordinates.

Open Question 2. *Is there a purely geometrical proof of Theorem 6.9?*

7. Tangential Quadrilaterals

A *tangential quadrilateral* in one in which a circle can be inscribed, touching all four sides. The center of this circle is called the *incenter* of the quadrilateral. The circle is called the *incircle*.

Our computer study found only one relationship between a tangential quadrilateral and its central quadrilateral (using any of the first 1000 centers) that was not true for quadrilaterals in general. It is listed in the following table.

Central Quadrilaterals of Tangential Quadrilaterals	
Relationship	**centers**
$\text{i}[ABCD] = \text{persp}[ABCD, GHEF]$	1

Theorem 7.1. *Let $ABCD$ be a tangential quadrilateral with incenter I. Let E, F, G, and H be the X_1-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $GHEF$ are perspective with perspector I (Figure 10).*

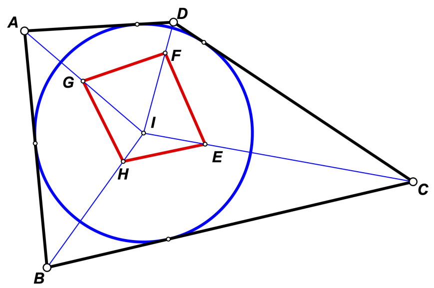


FIGURE 10. tangential quadrilateral with X_1-points $\implies$ persp$[ABCD, GHEF]$

Proof. The point G is the incenter of $\triangle ABD$, hence G lies on the angle bisector of $\angle BAD$. Thus, $G \in AI$. Similarly, $H \in BI$, $E \in CI$, and $F \in DI$. Therefore, AG, BH, CE, and DF concur in I. Hence, quadrilaterals $ABCD$ and $GHEF$ are perspective and the perspector is I. □

Open Question 3. *For the quadrilaterals in Theorem 7.1, how is the perspector related to quadrilateral $EFGH$?*

8. Extangential Quadrilaterals

An *extangential quadrilateral* with consecutive sides of lengths a, b, c, and d is one in which $a + b = c + d$.

Our computer study did not find any relationships between an extangential quadrilateral and its central quadrilateral (using any of the first 1000 centers) that was not true for quadrilaterals in general.

Central Quadrilaterals of exTangential Quadrilaterals
No new relationships were found.

9. EqualProdOp Quadrilaterals

An *equalProdOp quadrilateral* with consecutive sides of lengths a, b, c, and d is one in which $ac = bd$.

Our computer study did not find any relationships between an equalProdOp quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for quadrilaterals in general.

Central Quadrilaterals of EqualProdOp Quadrilaterals
No new relationships were found.

10. EqualProdAdj Quadrilaterals

An *equalProdAdj quadrilateral* with consecutive sides of lengths a, b, c, and d is one in which $ab = cd$.

Our computer study did not find any relationships between an equalProdAdj quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for quadrilaterals in general.

Central Quadrilaterals of EqualProdAdj Quadrilaterals
No new relationships were found.

11. Pythagorean Quadrilaterals

A *Pythagorean quadrilateral* with consecutive sides of lengths a, b, c, and d is one in which $a^2 + b^2 = c^2 + d^2$.

Our computer study did not find any relationships between a Pythagorean quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for quadrilaterals in general.

Central Quadrilaterals of Pythagorean Quadrilaterals
No new relationships were found.

12. Equidiagonal Quadrilaterals

An *equidiagonal quadrilateral* is a quadrilateral with two equal diagonals.

Our computer study did not find any relationships between an equidiagonal quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for quadrilaterals in general.

Central Quadrilaterals of Equidiagonal Quadrilaterals
No new relationships were found.

13. Orthodiagonal Quadrilaterals

An *orthodiagonal quadrilateral* is a quadrilateral in which the two diagonals are perpendicular.

Our computer study found a few relationships between an orthodiagonal quadrilateral and its central quadrilateral (using any of the first 1000 centers) that are not true for quadrilaterals in general. These are shown in the following table.

Central Quadrilaterals of Orthodiagonal Quadrilaterals	
Relationship	**centers**
stein$[ABCD]$ = dp$(EFGH)$	3
QA-P4=persp$[ABCD, GHEF]$=QA-P4	3
dp$(ABCD)$ = dp$(EFGH)$ = ponce$[ABCD]$	4
m$[ABCD]$ = dp$(EFGH)$	5
persp$[ABCD, GHEF]$	25, 68, 485, 486

Theorem 13.1. *Let $ABCD$ be an orthodiagonal quadrilateral. Let E, F, G, and H be the X_4 -points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ have the same diagonal point and* dp$(ABCD)$ = ponce$[ABCD]$ *(Figure 11).*

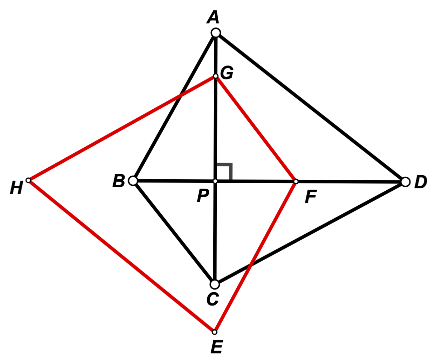


Figure 11. orthodiagonal quad with X_4-points $\implies$ dp$(ABCD)$ = dp$(EFGH)$

Proof. Let P be the diagonal point of quadrilateral $ABCD$. Since $ABCD$ is orthodiagonal, AP is an altitude of $\triangle ABD$. Since E is the orthocenter of $\triangle BCD$, E lies on this altitude. Similarly, G also lies on this altitude. In the same way, points F and H lie on BD. Thus, the diagonals EG and FH of central quadrilateral $EFGH$ meet at P and so $EFGH$ has diagonal point P. Hence, dp$(ABCD)$ = dp$(EFGH)$.

The point P is the foot of the C-altitude of $\triangle BCD$, so C lies on the ninepoint circle of $\triangle BCD$. Similarly P lies on the ninepoint circles of the other half triangles. Therefore, P = ponce$[ABCD]$ and dp$(ABCD)$ = ponce$[ABCD]$. □

Open Question 4. *Are there purely geometrical proofs for the results found in this section involving centers X_3, X_4, and X_5?*

14. Cyclic Quadrilaterals

An *cyclic quadrilateral* is a quadrilateral that can be inscribed in a circle.

Our computer study found many relationships between a cyclic quadrilateral and its central quadrilateral. They are summarized in the following tables. Properties that are true for quadrilaterals in general are excluded.

Centers that are colored blue in the following table represent centers for which the central quadrilateral is cyclic.

Central Quadrilaterals of Cyclic Quadrilaterals	
Relationship	**centers**
$[ABCD] = 4[EFGH]$	5, 550
$[ABCD] = 9[EFGH]$	376
$[ABCD] = 16[EFGH]$	140, 548
$[ABCD] = 25[EFGH]$	631
$[ABCD] = 36[EFGH]$	549
$[ABCD]/[EFGH] = 1/4$	382
$[ABCD]/[EFGH] = 9/4$	381
$[ABCD]/[EFGH] = 16/9$	546
$[ABCD]/[EFGH] = 100/9$	632
$[ABCD]/[EFGH] = 144/25$	547
$\text{conic}(ABCD, EFGH)$	6, 54, 64
$ABCD \cong EFGH$	4, 20
$\text{anti}[ABCD] = \text{anti}[EFGH]$	4
$\text{anti}[ABCD] = \text{centro}[EFGH]$	546
$\text{anti}[ABCD] = \text{m}[EFGH]$	381
$\text{anti}[ABCD] = \text{o}[EFGH]$	5
$\text{anti}[EFGH] = \text{centro}[ABCD]$	381
$\text{anti}[EFGH] = \text{h}[ABCD]$	382
$\text{anti}[EFGH] = \text{m}[ABCD]$	5
$\text{centro}[ABCD] = \text{centro}[EFGH]$	5
$\text{centro}[ABCD] = \text{o}[EFGH]$	2
$\text{h}[ABCD] = \text{o}[EFGH]$	4
$\text{m}[ABCD] = \text{o}[EFGH]$	140

Central Quadrilaterals of Cyclic Quadrilaterals (cont.)	
Relationship	**centers**
o$[ABCD]$ = h$[EFGH]$	$\mathbb{S}$
dp$(ABCD)$ = dp$(EFGH)$	6
dp$(ABDC)$ = dp$(EFHG)$	6, 15, 16, 61, 62, 371, 372
dp$(ACBD)$ = dp$(EGFH)$	6, 15, 16, 61, 62, 371, 372
$(ABCD) \equiv (EFGH)$	$\mathbb{C}$
o$[ABCD]$ = o$[EFGH]$	399
QA-P2=homot$[ABCD, EFGH]$=QA-P2	4
QA-P7=homot$[ABCD, EFGH]$=QA-P7	5
QA-P34=homot$[ABCD, EFGH]$=QA-P34	631
homot$[ABCD, EFGH]$	$\mathbb{S}$

The symbol $\mathbb{C}$ denotes the set of all triangle centers that lie on the circumcircle of the reference triangle.

The list of triangle centers that lie on the circumcircle of the reference triangle can be found in [11]. The first few are X_n for n =74, 98–112, 476, 477, 675, 681, 689, 691, 697, 699, 701, 703, 705, 707, 709, 711, 713, 715, 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 745, 747, 753, 755, 759, 761, 767, 769, 773, 777, 779, 781, 783, 785, 787, 789, 791, 793, 795, 797, 803, 805, 807, 809, 813, 815, 817, 819, 825, 827, 831, 833, 835, 839–843, 898, 901, 907, 915, 917, 919, 925, 927, 929–935, 953, and 972.

The triangle centers that do not lie on the circumcircle, for which the central quadrilateral of a cyclic quadrilateral is cyclic are X_n for n =1, 2, 4, 5, 13–16, 20, 23, 36, 40, 80, 125, 140, 165, 186, 265, 376, 381, 382, 399, 546–550, 631, 632. The only one where the circumcircle of the central quadrilateral is concentric with the circumcircle of the reference triangle is X_{399}.

The symbol $\mathbb{S}$ denotes the set of all triangle centers that lie on the Euler line of the reference triangle and have constant Shinagawa coefficients. Shinagawa coefficients are defined in [2]. The first few n for which X_n has constant Shinagawa coefficients are n =2, 3, 4, 5, 20, 140, 376, 381, 382, 546-550, 631, and 632.

The following are known facts about cyclic quadrilaterals.

Theorem 14.1. *The Gergonne-Steiner point (QA-P3) of a cyclic quadrilateral coincides with the circumcenter of that quadrilateral. That is,*

$$\text{stein}[ABCD] = \text{o}[ABCD].$$

Relationships of this form will be excluded from our tables.

Theorem 14.2. *The Euler-Poncelet point (QA-P2) of a cyclic quadrilateral coincides with the anticenter of that quadrilateral. That is,*

$$\text{ponce}[ABCD] = \text{anti}[ABCD].$$

Relationships of this form will be excluded from our tables.

Theorem 14.3. *The Quadrangle Nine-point Homothetic Center (QA-P7) of a cyclic quadrilateral coincides with the centrocenter of that quadrilateral. That is,*

$$QA\text{-}P7\,[ABCD] = \text{centro}[ABCD].$$

Theorem 14.4. *Let $ABCD$ be a cyclic quadrilateral. Let E, F, G, and H be the X_{399}-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ have concentric circumcircles and their radii are in the ratio $1:2$(Figure 12).*

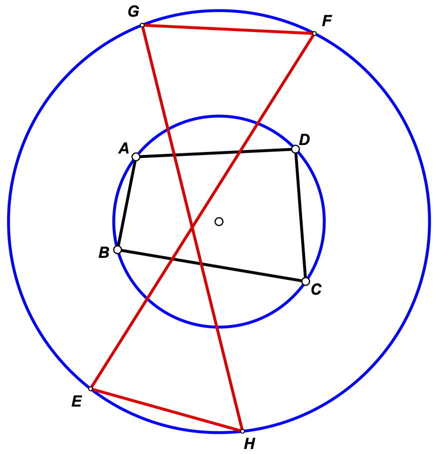


FIGURE 12. cyclic quad with X_{399}-points $\implies$ concentric$[1:2](ABCD, EFGH)$

An analytic proof of Theorem 14.4 is given in the supplementary material accompanying the on-line publication of this paper.

Theorem 14.5. *Let $ABCD$ be a cyclic quadrilateral. Let X be any triangle center that lies on the circumcircle of the reference triangle. Let E, F, G, H be the X-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ have the same circumcircle, i.e. $(ABCD) \equiv (EFGH)$.*

Proof. Let Γ be the circumcircle of quadrilateral $ABCD$. Since E is the X-point of $\triangle BCD$, E must lie on the circumcircle of $\triangle BCD$. Hence E lies on Γ. In the same way, F, G, and H must also lie on Γ. Therefore. $ABCD$ and $EFGH$ have the same circumcircle, Γ. □

Open Question 5. *Are there purely geometrical proofs for the results found in this section involving centers X_2, X_3, X_4, X_5, and X_6?*

15. BICENTRIC QUADRILATERALS

A *bicentric quadrilateral* is a quadrilateral that is both cyclic and tangential.

Our computer study found a few relationships between a bicentric quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for cyclic quadrilaterals in general.

Central Quadrilaterals of Bicentric Quadrilaterals	
Relationship	**centers**
persp$[ABCD, GHEF]$	35, 36, 55, 56, 999

16. Trapezoids

An *trapezoid* is a quadrilateral with a pair of parallel sides.

Our computer study found only one relationship between a trapezoid and its central quadrilateral (using any of the first 1000 centers) that is not true for quadrilaterals in general. It is shown in the following table.

Central Quadrilaterals of Trapezoids	
Relationship	**centers**
$ABCD \sim HGFE$	3

17. Tangential Trapezoids

A *tangential trapezoid* is a trapezoid that is also tangential.

Our computer study did not find any relationships between a tangential trapezoid and its central quadrilateral (using any of the first 1000 centers) that were not true for trapezoids or tangential quadrilaterals in general.

Central Quadrilaterals of Tangential Trapezoids
No new relationships were found.

18. Orthodiagonal Trapezoids

An *orthodiagonal trapezoid* is a trapezoid that is also orthodiagonal.

Our computer study did not find any relationships between an orthodiagonal trapezoid and its central quadrilateral (using any of the first 1000 centers) that were not true for trapezoids in general or for orthodiagonal quadrilaterals.

Central Quadrilaterals of Orthodiagonal Trapezoids
No new relationships were found.

19. Hjelmslev Quadrilaterals

A *Hjelmslev quadrilateral* is a quadrilateral with two right angles at opposite vertices. Hjelmslev quadrilaterals are necessarily cyclic.

Our computer study did not find any relationships between a Hjelmslev quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for cyclic quadrilaterals in general.

Central Quadrilaterals of Hjelmslev Quadrilaterals
No new relationships were found.

20. Isosceles Trapezoids

An *isosceles trapezoid* is a trapezoid with its nonparallel sides having the same length. Isosceles trapezoids are necessarily cyclic.

Our computer study found a few relationships between an isosceles trapezoid and its central quadrilateral (using any of the first 1000 centers) that were not true for cyclic quadrilaterals in general. They are given in the table below.

Central Quadrilaterals of Isosceles Trapezoids	
persp$[ABCD, HGFE]$	19, 25, 48, 49, 63, 69, 186, 264, 265, 304, 305, 317, 340, 847
QA-P9=persp$[ABCD, HGFE]$	24

21. Harmonic Quadrilaterals

A *harmonic quadrilateral* is a cyclic quadrilateral that is also an equalProdOpp quadrilateral.

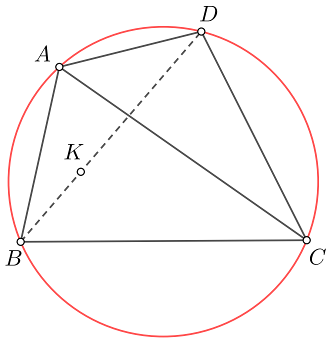
Our computer study found a few relationships between a harmonic quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for cyclic quadrilaterals in general. They are shown in the following table.

Central Quadrilaterals of Harmonic Quadrilaterals	
Relationship	**centers**
persp$[ABCD, EFGH]$	13–18, 61, 62, 371, 372, 395–398, 485, 590, 615
persp$[ABCD, GHEF]$	15, 16, 61, 62, 371, 372

When proving these results analytically using barycentric coordinates, we use the following result.

Theorem 21.1. *Let $ABCD$ be a harmonic quadrilateral. The barycentric coordinates of D with respect to $\triangle ABC$ are $(2a^2 : -b^2 : 2c^2)$, where $a = BC$, $b = CA$, and $c = AB$.*

Proof. It is known [12] that the point D is the second intersection of the B-symmedian with the circumcircle of $\triangle ABC$.



The coordinates of the symmedian point, K, are $(a^2 : b^2 : c^2)$, so the equation of the B-symmedian is

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ a^2 & b^2 & c^2 \end{vmatrix} = 0 \quad \Leftrightarrow \quad c^2x - a^2z = 0.$$

The equation of the circumcircle of the reference triangle ABC is well known [12] to be

$$a^2qr + b^2pr + c^2pq = 0.$$

Therefore, the coordinates of the point D are obtained by solving the system

$$\begin{cases} a^2qr + b^2pr + c^2pq = 0 \\ c^2x - a^2z = 0 \end{cases}$$

from which it is easily found that the coordinates of D are $(2a^2 : -b^2 : 2c^2)$. □

Open Question 6. *Are there purely geometrical proofs for the results found in this section involving centers X_3, X_4, X_5, and X_6?*

22. Cyclic Orthodiagonal Quadrilaterals

An *cyclic orthodiagonal quadrilateral* is a cyclic quadrilateral whose diagonals are perpendicular.

Our computer study found a few relationships between a cyclic orthodiagonal quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for cyclic quadrilaterals in general or for orthodiagonal quadrilaterals in general. These are shown in the following table.

Central Quads of Cyclic Orthodiagonal Quadrilaterals	
$[ABCD] = [EHGF]$	68
anti$[ABCD]$ = dp$[ABCD]$ = stein$[EFGH]$	51–53, 128–130, 137–139, 143
centro$[ABCD]$ = stein$[EFGH]$	568
anti$[ABCD]$ = dp$(ABCD)$ = dp$(EFGH)$	6, 24, 25, 68, 186, 378, 847, 933
m$[ABCD]$ = stein$[EFGH]$	389
m$[ABCD]$ = dp$[EFGH]$	182, 216, 343
persp$[ABCD, GHEF]$	186, 378, 571
QA-P9=persp$[ABCD, GHEF]$	24

23. Kites

A *kite* is a quadrilateral consisting of two adjacent sides of length a and the other two sides of length b. A kite is necessarily orthodiagonal.

Our computer study found a few relationships between a kite and its central quadrilateral (using any of the first 1000 centers) that were not true for orthodiagonal quadrilaterals in general.

Central Quadrilaterals of Kites	
$\mathrm{m}[ABCD] = \mathrm{stein}[EFGH]$	402, 618–620
$\mathrm{ponce}[ABCD] = \mathrm{stein}[EFGH]$	13, 14
$\mathrm{stein}[ABCD] = \mathrm{stein}[EFGH]$	616, 617

We can assume without loss of generality that D is the reflection of B about AC. Hence, the barycentric coordinates of D are $(a^2 + b^2 - c^2 : -b^2 : -a^2 + b^2 + c^2)$.

When proving these results analytically using barycentric coordinates, we use the following result.

Theorem 23.1. *If $ABCD$ is a kite with $AB = AD$ and $CB = CD$, then the Steiner point of its central quadrilateral coincides with the midpoint of EG.*

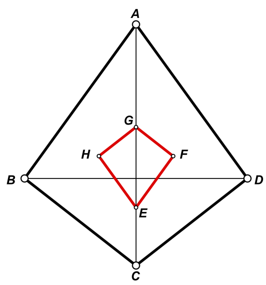


Figure 13. kite $\implies$ kite

Proof. From Theorem 6.27 of [4], the central quadrilateral $EFGH$ is a kite with $EF = EH$ and $GF = GH$ (Figure 13). Then, by Corollary 10.5 of [5], the Steiner point of $EFGH$ coincides with the midpoint of EG. □

24. AP Quadrilaterals

An *AP quadrilateral* is a quadrilateral whose sides (in order) form an arithmetic progression.

Our computer study found no relationships between an AP quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for quadrilaterals in general.

Central Quadrilaterals of AP Quadrilaterals
No new relationships were found.

25. Equidiagonal Orthodiagonal Quadrilaterals

An *equidiagonal orthodiagonal quadrilateral* is a quadrilateral in which the two diagonals are both equal and perpendicular.

Our computer study found a few relationships between an equidiagonal orthodiagonal quadrilateral and its central quadrilateral (using any of the first 1000 centers) that are not true for orthodiagonal quadrilaterals in general. These are shown in the following table.

Table 4.

Central Quads of Equidiagonal Orthodiagonal Quadrilaterals	
Relationship	**centers**
m$[ABCD]$ = m$[EFGH]$	489
stein$[ABCD]$ = stein$[EFGH]$	638
m$[ABCD]$ = stein$[EFGH]$	640
QA-P1=persp$[ABCD, GHEF]$	485
QA-P5=persp$[ABCD, GHEF]$	68
QA-P3=persp$[ABCD, EFGH]$	637
QA-P4=persp$[ABCD, EFGH]$	489

26. Exbicentric Quadrilaterals

An *exbicentric quadrilateral* is a cyclic quadrilateral that is also extangential.

Our computer study did not find any relationships between an exbicentric quadrilateral and its central quadrilateral (using any of the first 1000 centers) that were not true for cyclic quadrilaterals in general.

Central Quadrilaterals of Exbicentric Quadrilaterals
No new relationships were found.

27. Parallelograms

A *parallelogram* is a quadrilateral in which both pairs of opposite sides are parallel.

Our computer study found hundreds of relationships between a parallelogram and its central quadrilateral. Instead of listing all relationships found, we only list a few of the interesting relationships.

Central Quadrilaterals of Parallelograms	
Relationship	**centers**
QA-P1=conic$(ABCD, EFGH)$=QA-P1	7, 13, 14, 17, 18, 66, 330, 485, 486

28. Bicentric Trapezoids

A *bicentric trapezoid* is a trapezoid that is also bicentric.

A bicentric trapezoid is necessarily an isosceles trapezoid.

Our computer study found a few relationships between a bicentric trapezoid and its central quadrilateral (using any of the first 1000 centers) that are not true for bicentric quadrilaterals or isosceles trapezoids in general. These are shown in the following table.

Central Quadrilaterals of Bicentric Trapezoids	
Relationship	**centers**
persp$[ABCD, GHEF]$	35, 36, 55, 56, 145, 999
persp$[ABCD, HGFE]$	49, 63, 92, 186, 265, 304, 305, 317, 328, 563

29. Rhombi

A *rhombus* is a quadrilateral all of whose sides have the same length.

Our computer study found hundreds of relationships between a rhombus and its central quadrilateral. Instead of listing all relationships found, we only list a few of the interesting relationships.

Central Quadrilaterals of Rhombi	
Relationship	**centers**
$[ABCD] = 3[EFGH]$	13, 14
$[ABCD] = 4[EFGH]$	402, 620
$[ABCD] = 9[EFGH]$	290, 671, 903
$[EFGH] = 4[ABCD]$	446

Open Question 7. *Are there purely geometrical proofs for the results found in this section involving centers X_{13} and X_{14}?*

30. RECTANGLES

A *rectangle* is a quadrilateral all of whose angles are right angles.

Our computer study found hundreds of relationships between a rectangle and its central quadrilateral. Instead of listing all relationships found, we only list a few of the interesting relationships.

Central Quadrilaterals of Rectangles	
Relationship	**centers**
$[ABCD] = 25[EFGH]$	95
$[ABCD] = 2[EFGH]$	946
$[ABCD] = 4[EFGH]$	402
$[EFGH] = 9[ABCD]$	23
$[ABCD]/[EFGH] = 25/4$	233
QA-P1=hyperb($ABCD, EFGH$)=QA-P1	251, 315, 481, 850, 961, 998
$\partial[ABCD] = \partial[EFGH]$	46, 47, 117, 163, 579, 580, 920

When proving these results analytically using barycentric coordinates, we use the following result which is Theorem 6.32 in [4].

Theorem 30.1. *If $ABCD$ be a rectangle, then the central quadrilateral is also a rectangle.*

Figure 14 shows the case when $n = 46$. Because of this theorem, to prove that the reference quadrilateral and the central quadrilateral have the same perimeter ($\partial[ABCD] = \partial[EFGH]$), it is only necessary to prove that $AB+BC = EF+FG$.

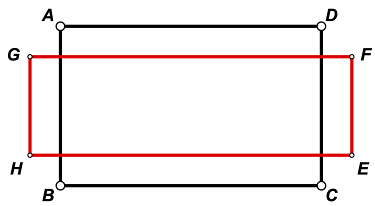


FIGURE 14. Rectangle with X_{46}-points $\implies \partial[ABCD] = \partial[EFGH]$

31. Squares

A *square* is a rectangle that is also a rhombus.

Our computer study found hundreds of relationships between a square and its central quadrilateral. Instead of listing all relationships found, we only list a few of the interesting relationships.

Central Quadrilaterals of Squares	
Relationship	**centers**
$[ABCD] = 3[EFGH]$	13, 14
$[ABCD] = 49[EFGH]$	183, 252
$[ABCD] = 8[EFGH]$	496, 613, 988
$[EFGH] = 25[ABCD]$	352

Open Question 8. *Are there purely geometrical proofs for the results found in this section involving centers X_{13} and X_{14}?*

32. Areas for Future Research

There are many avenues for future investigation.

32.1. Investigate other triangle centers.

In our study, we only investigated triangle centers X_n for $n \leq 1000$. Extend this study to larger values of n.

As an example, the following result was found by Ercole Suppa [6].

Theorem 32.1. *Let $ABCD$ be a cyclic quadrilateral. Let E, F, G, and H be the X_{1173}-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ have a common circumconic (Figure 15).*

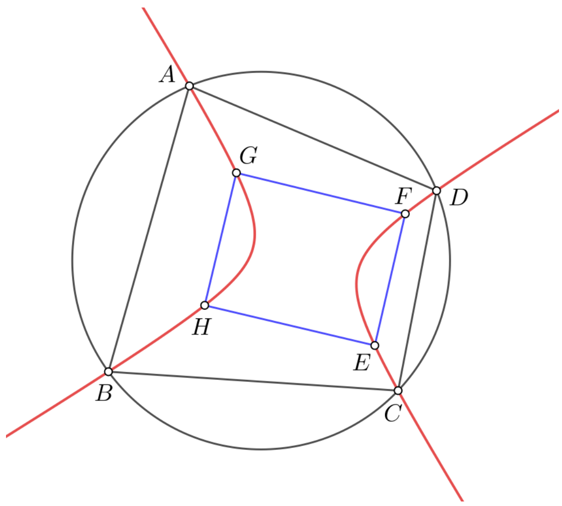


FIGURE 15. cyclic quadrilateral with X_{1173}-points $\implies$ conic$[ABCD, EFGH]$

32.2. **Use other shape quadrilaterals.**

In our investigation, we only studied 28 shapes of quadrilaterals as shown in Figure 3. There are many other shapes of quadrilaterals. Study these other shapes. For example, we say that a quadrilateral is *orthoptic* if its opposite sides are perpendicular. Figure 16 shows an orthoptic quadrilateral in which $AB \perp CD$ and $BC \perp AD$. The following result was found by computer.

Theorem 32.2. *Let $ABCD$ be an orthoptic quadrilateral. Let E, F, G, and H be the symmedian points (X_6-points) of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ are perspective. The perspector is the Euler-Poncelet point (QA-P2) of quadrilateral $ABCD$.*

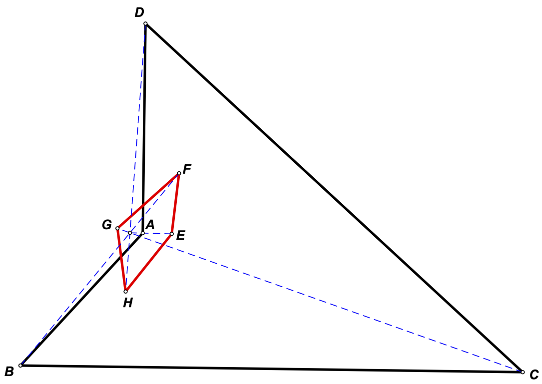


FIGURE 16. orthoptic quadrilateral with X_6-points $\implies$ persp$[ABCD, EFGH]$

Open Question 9. *Is there a purely geometrical proof of this result?*

An *orthocentric* quadrilateral is a quadrilateral in which each vertex is the orthocenter of the triangle formed by the other three vertices. The following result was found by computer.

Theorem 32.3. *Let $ABCD$ be an orthocentric quadrilateral. Let E, F, G, and H be the Feuerbach points (X_{11}-points) of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilateral $EFGH$ is cyclic and the center of the circumcircle of $EFGH$ coincides with the centroid of quadrilateral $ABCD$ (Figure 17).*

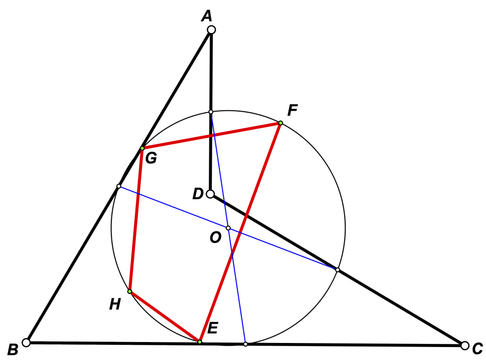


FIGURE 17. orthocentric quad with X_{11}-points $\implies$ m$[ABCD] =$ o$[EFGH]$

Open Question 10. *Is there a purely geometrical proof of this result?*

32.3. **Check for other quadrilateral centers.**

In our study, when we checked to see if some center of the central quadrilateral coincides with some center of the reference quadrilateral, we only checked the common centers listed in Table 2. Additional centers could be investigated, such as the Miquel point (QL-P1), the area centroid (QG-P4), the Morley Point (QL-P2), the Newton Steiner point (QL-P7), and the various quasi points.

32.4. **Investigate centers lying on quadrilateral lines.**

We could also check to see if some center of the central quadrilateral lies on some notable line of the reference triangle, such as the Newton line (QL-L1), the Steiner line (QL-L2), etc., or, in the case of cyclic quadrilaterals, the Euler line.

32.5. **Examine other properties.**

There are many other properties between two quadrilaterals that can be studied. For example, two polygons are *orthogonal* if their corresponding sides are perpendicular.

The following result was found by computer.

Theorem 32.4. *Let $ABCD$ be a trapezoid. Let E, F, G, H be the X_3-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $HGFE$ are orthogonal (Figure 18).*

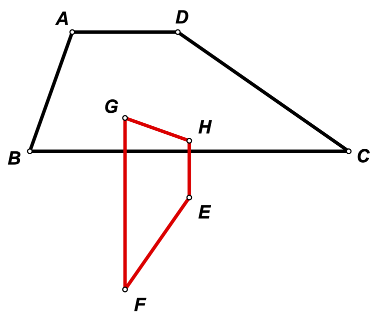


Figure 18. trapezoid with X_3-points $\implies$ ortho$(ABCD, EFGH)$

Open Question 11. *Is there a center, X, such that quadrilaterals $ABCD$ and $EFGH$ have a common inconic?*

Open Question 12. *Is there a center, X, and a tangential quadrilateral $ABCD$, such that the central quadrilateral $EFGH$ formed with X-points is also tangential and $ABCD$ and $EFGH$ have a common incircle? What about concentric incircles?*

32.6. **Investigate patterns in the center functions.**

Many properties found are true for triangle centers X_n for a list of values for n. What significance do these values have? Specifically, investigate the center functions associated with these centers to see if some pattern can be found.

For example, it has been found that if $ABCD$ is cyclic, then $ABCD$ and $EFGH$ have a common non-circular circumconic for centers X_n when n =4, 6, 54, 64, 1173, 11738, 3426, 3431, 11270, 13472, 13603, 14483, 14487, 14490, 14528, 16835, 11738... What is the significance of these values of n?

Dylan Wyrzykowski [13] has found the pattern with the following theorem.

Theorem 32.5. *Let $ABCD$ be a cyclic quadrilateral. Let E, F, G, and H be the X_n-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively, where the isogonal conjugate of X_n lies on the Euler line and has constant Shinagawa coefficients. Then quadrilaterals $ABCD$ and $EFGH$ have a common circumconic.*

We found in Section 14, that in a cyclic quadrilateral, $ABCD$ and $EFGH$ are homothetic for centers X_n when n =2, 4, 5, 20, 140, 376, 381, 382, 546- 550, 631, and 632. What is the pattern giving rise to these values of n?

We found the following result.

Theorem 32.6. *Let $ABCD$ be a cyclic quadrilateral. Let X be a triangle center whose (trilinear) center function is of the form $\cos B \cos C + k \cos A$, where k is some constant, not necessarily an integer. Let E, F, G, and H be the X-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ are homothetic.*

Note. These are the points on the Euler line that have constant Shinagawa coefficients.

For similar results involving these points, see [4].

We found in Section 21, that if quadrilateral $ABCD$ is harmonic, then using centers X_n, we have persp[$ABCD, GHEF$] when n=15, 16, 61, 62, 371, and 372. What is the pattern in these numbers? We found the following result by computer.

Theorem 32.7. *Let $ABCD$ be a harmonic quadrilateral. Let X be a triangle center whose center function is of the form $a\,(k(a^2 - b^2 - c^2) - S)$, where k is some constant and $S = 2[ABC]$. Let E, F, G, and H be the X-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $GHEF$ are perspective.*

Another result, found by computer, involving a set of centers meeting a pattern is the following.

A *power point* of a triangle is a triangle center whose center function is of the form $f(a, b, c) = a^k$, where k is a constant (not necessarily an integer).

Theorem 32.8. *Let $ABCD$ be a parallelogram. Let X be some power point of a triangle. Let E, F, G, and H be the X-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then quadrilaterals $ABCD$ and $EFGH$ have a common diagonal point.*

32.7. **Ask about uniqueness.** Find an entry in one of our tables where there is only one center giving a particular relationship for a certain type of quadrilateral. For example, for a general quadrilateral, $m[ABCD] = m[EFGH]$ seems to be true only when $n = 2$. Is this because we only searched the first 1000 values of n? Expand the search and find other values of n for which the relationship is true or prove that the result is unique. For example, we can state the following.

Conjecture 1. *Let $ABCD$ be an arbitrary quadrilateral. Let X denote a triangle center. Let E, F, G, and H be the X-points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then the centroid of $ABCD$ coincides with the centroid of $EFGH$ if and only if $X = X_2$.*

32.8. **Use notable points that are not triangle centers.**

There are other points associated with a triangle that are not triangle centers. Look for properties when some of these points are used. For example, the following result was found by computer.

Theorem 32.9. *Let $ABCD$ be a square. Let E, F, G, and H be the first Brocard points of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively (Figure 19). Then*

$$[ABCD] = 5[EFGH].$$

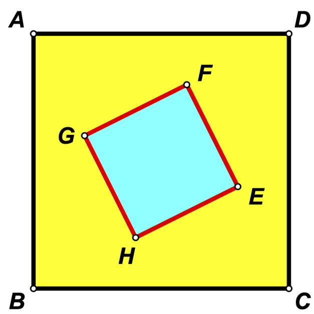


FIGURE 19. square with Brocard points $\implies$ $[ABCD] = 5[EFGH]$

Open Question 13. *Is there a purely geometrical proof of this result?*

32.9. **Place different centers in different half triangles.**

Would we find any interesting results if we place X_n-points in triangles ABC and ACD, but place X_m points in triangles ABD and BCD, with $m \neq n$?

32.10. **Investigate some QL-properties.**

If the lines AE, BF, CG, and DH do not concur, then these four lines (with their points of intersection) form a figure known as a *complete quadrilateral*. A complete quadrilateral has many notable points associated with it, such as the Miquel Point, the Morley Point, the Clawson Center, and the Newton-Steiner Point. For a more extensive list see the section on Quadrilateral Points in [7]. Investigate whether any of these points coincide with notable points associated with the reference quadrilateral, $ABCD$.

32.11. **Work in 3-space.**

If point D is moved off the plane of $\triangle ABC$, then the reference quadrilateral becomes a tetrahedron and the half triangles become the faces of the reference tetrahedron. The central quadrilateral becomes the central tetrahedron. Investigate how the central tetrahedron is related to the reference tetrahedron. Some results can be found in [3].

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