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Relationships between a Central Triangle and its Reference Triangle

Stanley Rabinowitz[a] and Ercole Suppa[b]
[a] 545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net[2]
web: http://www.StanleyRabinowitz.com/
[b] Via B. Croce 54, 64100 Teramo, Italia
e-mail: ercolesuppa@gmail.com
web: https://www.esuppa.it

Abstract. Let D be a point in the plane of a triangle ABC. The lines from D to the vertices of $\triangle ABC$ form three associated triangles, called *side triangles*. Each side triangle is bounded by one side of the original (reference) triangle and two line segments from D. If we locate a triangle center (such as the incenter, centroid, orthocenter, etc.) in each of these triangles, the three triangle centers form another triangle called a *central triangle*. For each of 1000 different triangle centers, we compare the reference triangle to the central triangle. Using a computer, we determine how the two triangles are related. For example, we test to see if the two triangles are congruent, similar, have the same area, are perspective, or have the same perimeter. We identify new points from features such as the orthologic centers, cyclologic centers, and parallelogic centers of the two triangles. In addition to D being an arbitrary point, we also study the case where D is itself a triangle center of $\triangle ABC$.

Keywords. triangle centers, computer-discovered mathematics, Euclidean geometry, Rabinowitz-Suppa Triangles, GeometricExplorer.

Mathematics Subject Classification (2020). 51M04, 51-08.

[2]Corresponding author

1. Introduction

Let D be a point in the plane of a triangle ABC (called the *reference triangle*). The lines from D to the vertices of $\triangle ABC$ (the *spokes*) form three associated triangles, called *side triangles*. Each side triangle is bounded by one side of the reference triangle and two spokes. The three triangles, $\triangle DBC$, $\triangle DCA$, and $\triangle DAB$, are shown in Figure 1.

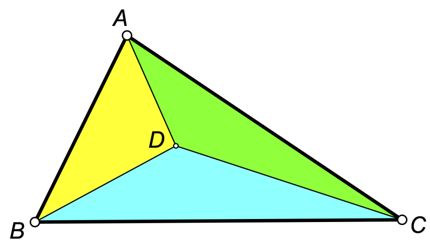


Figure 1. Three side triangles determined by D

If we locate a triangle center (such as the incenter, centroid, orthocenter, etc.) in each of these triangles, the three triangle centers form another triangle called a *central triangle*. The three centers will be named E, F, and G as shown in Figure 2.

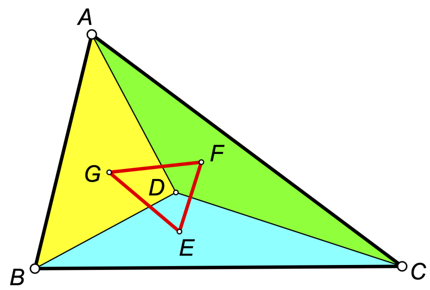


Figure 2. Central triangle EFG

The purpose of this paper is to determine interesting relationships between a reference triangle and its central triangle for various choices of D and the three centers.

2. Methodology

We used a computer program called GeometricExplorer to compare triangles with their central triangle. Starting with an arbitrary triangle ABC, we placed triangle centers in each of the three side triangles. Many interesting triangle centers have been cataloged in the Encyclopedia of Triangle Centers [10]. We use X_n to denote the n-th named center in this encyclopedia.

For each n from 1 to 1000, we determined the center X_n of each of the side triangles. The program then analyzes the central triangle formed by these three centers and reports if the central triangle is related to the reference triangle. Points at infinity were omitted. Centers X_{368} and X_{370} were also excluded because of the complexity of their barycentric coordinates. The types of relationships checked for are shown in the following table.

Relationships Checked For

notation	description
$[ABC]=[EFG]$	the triangles have the same area (This relationship is excluded if the triangles are congruent.)
$[ABC]=k[EFG]$	the area of ABC is k times the area of EFG † (This relationship is excluded if the triangles are homothetic.)
$ABC \equiv EFG$	the triangles coincide
$ABC \cong EFG$	the triangles are congruent (This relationship is excluded if the triangles coincide.)
$ABC \sim EFG$	the triangles are similar (This relationship is excluded if the triangles are homothetic.)
$\partial ABC = \partial EFG$	the triangles have the same perimeter (This relationship is excluded if the triangles are congruent.)
$\odot ABC \cong \odot EFG$	the triangles have congruent circumcircles (This relationship is excluded if the triangles are congruent.)
$\odot ABC \equiv \odot EFG$	the triangles have the same circumcircle
$\mathrm{h}[ABC]=\mathrm{h}[EFG]$	the distinct triangles have the same orthocenter
$\mathrm{i}[ABC]=\mathrm{i}[EFG]$	the distinct triangles have the same incenter
$\mathrm{k}[ABC]=\mathrm{k}[EFG]$	the distinct triangles have the same symmedian point
$\mathrm{m}[ABC]=\mathrm{m}[EFG]$	the distinct triangles have the same centroid
$\mathrm{n}[ABC]=\mathrm{n}[EFG]$	the distinct triangles have the same nine-point center
$\mathrm{o}[ABC]=\mathrm{o}[EFG]$	the distinct triangles have the same circumcenter (This relationship is excluded if the triangles have the same circumcircle.)
$\mathrm{persp}[ABC,EFG]$	the triangles are perspective (This relationship is excluded if the triangles are homothetic or bilogic.)
$\mathrm{cyclo}[ABC,EFG]$	the triangles are cyclologic
$\mathrm{ortho}[ABC,EFG]$	the triangles are orthologic (This relationship is excluded if the triangles are homothetic, bilogic, or are inversely similar.)
$\mathrm{homot}[ABC,EFG]$	the triangles are homothetic
$\mathrm{bilogic}[ABC,EFG]$	the triangles are bilogic
$\mathrm{euler}[ABC,EFG]$	the triangles are Eulerologic
$\mathrm{equi}[ABC,EFG]$	the triangles are equilogic
$\mathrm{para}[ABC,EFG]$	the triangles are parallelogic (This relationship is excluded if the triangles are inversely similar.)
$ABC \perp EFG$	the triangles are orthogonal
$P=\mathrm{conic}[ABCEFG]$	the triangles have a common circumconic with center P (This relationship is excluded if the conic is a circle.)
$\mathrm{eu}(ABC)=\mathrm{eu}(EFG)$	the Euler lines of the triangles coincide
$\mathrm{eu}(ABC) \parallel \mathrm{eu}(EFG)$	the triangles have parallel Euler lines (This relationship is excluded if the triangles are homothetic.)
$\mathrm{eu}(ABC) \perp \mathrm{eu}(EFG)$	the triangles have perpendicular Euler lines

† Only rational values of k were checked for with denominators less than 6.

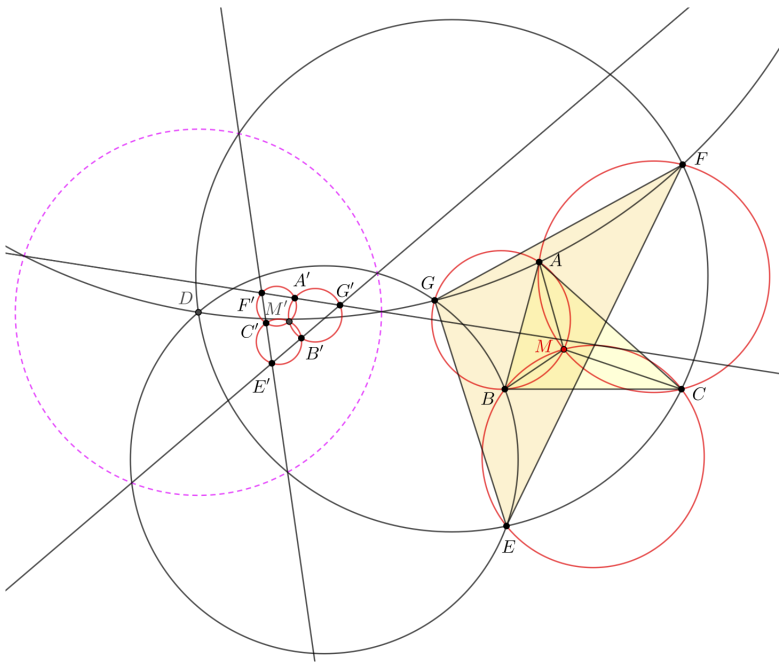
The terms in this table are defined, with additional detail, in the following subsections.

2.1. **Cyclologic Triangles.**

Let $T = ABC$ and $T' = A'B'C'$ be two triangles. We will show below that if circles $AB'C'$, $BC'A'$, and $CA'B'$ concur in a point P then the circles $A'BC$, $B'CA$, and $C'AB$ concur in a point P', and reciprocally. In this case, triangles T and T' are said to be *cyclologic triangles*. We write this as cyclo$[T, T']$. The points P and P' are called the *cyclologic centers* of the triangles [17].

Theorem 2.1. *Let A, B, C, E, F, G be six distinct points such that no three are collinear. If circles $\odot(AFG)$, $\odot(BEG)$, and $\odot(CEF)$ concur in a point D then circles $\odot(EBC)$, $\odot(FCA)$, and $\odot(GAB)$ concur in a point M.*

Proof. Let us apply a circular inversion $\mathbf{I}$ centered at point D with an arbitrary radius r. For any point X, let $X' = \mathbf{I}(X)$.



The circles $\odot(AFG)$, $\odot(BEG)$, and $\odot(CEF)$ all pass through the center of inversion D. Therefore, their images under $\mathbf{I}$ are straight lines. Clearly we have $\mathbf{I}(\odot(FGD)) \to F'G'$, $\mathbf{I}(\odot(EGD)) \to E'G'$, $\mathbf{I}(\odot(EFD)) \to E'F'$. These lines form the sides of a new triangle $E'F'G'$.

Since $A \in \odot(FGD)$, $B \in \odot(EGD)$, and $C \in \odot(EFD)$, their images A', B', C' must lie on the corresponding lines, i.e. $A' \in F'G'$, $B' \in E'G'$, and $C' \in E'F'$.

The circles $\odot(EBC)$, $\odot(FCA)$, and $\odot(GAB)$ generally do not pass through D. Thus, their images under $\mathbf{I}$ are again circles: $\odot(E'B'C')$, $\odot(F'C'A')$, and $\odot(G'A'B')$.

The configuration in the inverted plane is exactly that of **Miquel's Theorem** [36]. This classic result states that if we pick a point on each side of a triangle, the circles formed by a vertex and the two points on the adjacent sides are concurrent.

In our case, the circles $\odot(E'B'C')$, $\odot(F'C'A')$, and $\odot(G'A'B')$ meet at a unique point M'. Since the concurrence of circles is preserved by inversion, mapping the intersection point M' back to $M = \mathbf{I}(M')$, we conclude that the original circles $\odot(EBC)$, $\odot(FCA)$, and $\odot(GAB)$ must meet at point M. □

2.2. **Orthologic Triangles.**

Let $T = ABC$ and $T' = A'B'C'$ be two triangles. Through A, B, C draw perpendicular lines L_A, L_B, L_C to $B'C'$, $C'A'$ and $A'B'$, respectively. Through A', B', C' draw perpendicular lines $L_{A'}$, $L_{B'}$, $L_{C'}$ to BC, CA, and AB, respectively. Steiner proved in 1828 that if lines L_A, L_B, L_C concur in a point P then lines $L_{A'}$, $L_{B'}$, $L_{C'}$ concur in a point P', and reciprocally. See [27, Thm. 4] and [2, Thm. 1]. In this case, T and T' are said to be *orthologic triangles*. We write this as ortho$[T, T']$. The points P and $P'$are called the *orthologic centers* of the triangles [17]. Note that inversely similar triangles are orthologic [27, Thm. 26] and[20].

2.3. **Eulerologic Triangles.**

Two triangles $T = ABC$ and $T' = A'B'C'$ are *Eulerologic* if the Euler lines of $AB'C'$, $BC'A'$ and $CA'B'$ are concurrent [14]. In this case, the triangles T and T' are said to be *Eulerologic*. We write this as euler$[T, T']$. If euler$[T, T']$, it is not necessarily true that euler$[T', T]$. The points of concurrence of the Euler lines are called the *Eulerologic centers* [17].

2.4. **Parallelogic Triangles.**

Let $T = ABC$ and $T' = A'B'C'$ be two triangles. Through A, B, C draw parallel lines L_A, L_B, L_C to $B'C'$, $C'A'$ and $A'B'$, respectively. Through A', B', C' draw parallel lines $L_{A'}$, $L_{B'}$, $L_{C'}$ to BC, CA and AB, respectively. It has been proved that if lines L_A, L_B, L_C concur in a point P then lines $L_{A'}$, $L_{B'}$, $L_{C'}$ concur in a point P', and reciprocally [26]. See also [5], [19], [8], and [20]. In this case, T and T' are said to be *parallelogic triangles*. We write this as para$[T, T']$. The points P and P' are called the *parallelogic centers* of the triangles [18].

Note that inversely similar triangles are parallelogic [3] and [20].

2.5. **Equilogic Triangles.**

Two triangles $T = ABC$ and $T' = A'B'C'$ are *equilogic* if $AA' = BB' = CC'$ [15]. We write this as equi$[T, T']$.

2.6. **Orthogonal Triangles.**

Two triangles $T = ABC$ and $T' = A'B'C'$ are *orthogonal* if $AB \perp A'B'$, $BC \perp B'C'$, and $CA \perp C'A'$ [27, §7.1] We write this as $T \perp T'$.

2.7. **Perspective Triangles.**

Two triangles $T = ABC$ and $T' = A'B'C'$ are *perspective* if lines AA', BB', and CC' are concurrent or parallel. The point of concurrence (which could be a point at infinity) is called the *perspector*.

2.8. **Bilogic Triangles.**

Two triangles $T = ABC$ and $T' = A'B'C'$ are *bilogic* if they are perspective and orthologic [16]. The word bilogic was introduced by Neuberg [30] to be used to designate such triangles. Bilogic Triangles satisfy the theorem stated by Sondat in 1894 [28]. See also [30].

The program we used to find results about central triangles (GeometricExplorer) is a useful tool for discovering results, but it does not prove that these results are true. GeometricExplorer uses numerical coordinates (to 15 digits of precision) for locating all the points. Thus, a relationship found by this program does not constitute a proof that the result is correct, but gives us compelling evidence for the validity of the result.

To prove the results that we have discovered, we use geometric methods, when possible. If we could not find a purely geometrical proof, we turned to analytic methods using barycentric coordinates and performing exact symbolic computation using Mathematica.

These analytic proofs are given in Mathematica Notebooks included with the supplementary material accompanying the on-line publication of this paper. If we could not find a geometric or analytic proof of a relationship, the center is colored red in the relationship table.

3. Results for an arbitrary D

Our computer study found the following relationships between a triangle and its central triangle when the point D is an arbitrary point in the plane.

Central Triangles of $\triangle ABC$, D arbitrary	
Relationship	**centers**
homot$(-1:3)[ABC, EFG]$	2
bilogic$[ABC, EFG]$	2
$[ABC] = [EFG]$	4
conic$[ABCEFG]$	4
euler$[ABC, EFG]$	4
euler$[EFG, ABC]$	3, 4
cyclo$[ABC, EFG]$	3, 4, $\mathbb{C}$, 186, 265
ortho$[ABC, EFG]$	$\mathbb{S}$

The symbol $\mathbb{C}$ denotes the set of all triangle centers that lie on the circumcircle of the reference triangle. This list can be found in [34]. The first few are X_n for n =74, 98–112, 476, 477, 675, 681, 689, 691, 697, 699, 701, 703, 705, 707, 709, 711, 713, 715, 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 745, 747, 753, 755, 759, 761, 767, 769, 773, 777, 779, 781, 783, 785, 787, 789, 791, 793, 795, 797, 803, 805, 807, 809, 813, 815, 817, 819, 825, 827, 831, 833, 835, 839–843, 898, 901, 907, 915, 917, 919, 925, 927, 929–935, 953, and 972.

The symbol $\mathbb{S}$ denotes the set of all triangle centers that lie on the Euler line of the reference triangle and have fixed locations relative to X_2 and X_4. That is, those centers P such that the ratio X_4P/PX_2 is a constant as triangle ABC varies. Such points have constant Shinagawa coefficients. Shinagawa coefficients are defined in [10]. The first few n for which X_n has constant Shinagawa coefficients are n =2–5, 20, 140, 376, 381, 382, 546–550, 631, and 632. Such points have a first barycentric coordinate of the form $mS^2 + nS_BS_C$ where m and n are numeric constants, S is twice the area of $\triangle ABC$, and S_B and S_C are the Conway symbols [35] defined by $S_B = (c^2 + a^2 - b^2)/2$, and $S_C = (a^2 + b^2 - c^2)/2$.

If $P = (mS^2 + nS_BS_C : mS^2 + nS_CS_A : mS^2 + nS_AS_B)$, then we write

$$P = \text{Shin}(m, n).$$

Using barycentric coordinates, it is straightforward to prove the following result.

Theorem 3.1. *If $P = \text{Shin}(m, n)$, then $X_4P/PX_2 = 3m/n$, $X_3P/PX_4 = (m + n)/(2m)$, and $X_2P/X_3X_2 = 2n/(3m + n)$.*

The last equation is Lemma 3.2 in [41].

Conversely if $X_4P/PX_2 = k$, then $P = \text{Shin}(k, 3)$.

If $X_3P/PX_4 = k$, then $P = \text{Shin}(1, 2k - 1)$.

Centers with constant Shinagawa coefficients also have their first trilinear coordinate of the form $\cos B \cos C + k \cos A$ according to [23, p. 135].

3.1. Results involving X_2.

Theorem 3.2. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the centroids of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are homothetic. The ratio of similarity is $3:1$.*

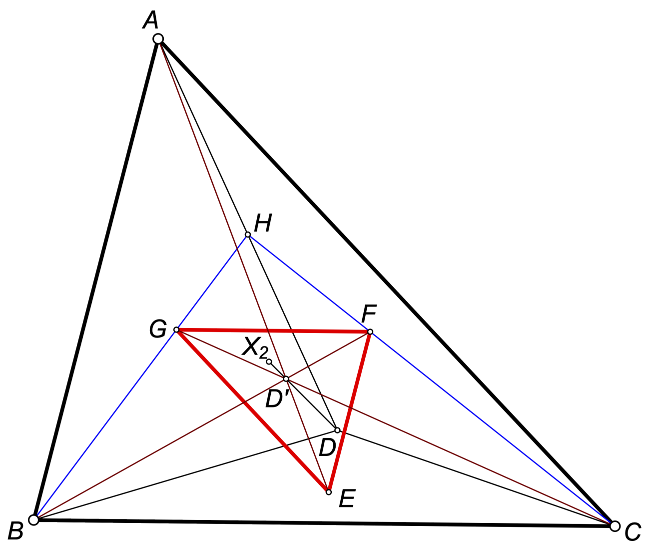


Figure 3. Central triangle using X_2 points

Proof. Let H be the midpoint of AD (Figure 3). Since F is the centroid of $\triangle CAD$, CF meets AD at H and $FH = \frac{1}{3}CH$. Since G is the centroid of $\triangle ABD$, BG meets AD at H and $GH = \frac{1}{3}BH$. From $FH = \frac{1}{3}CH$ and $GH = \frac{1}{3}BH$, we can conclude that $GF \parallel BC$ and $GF = \frac{1}{3}BC$. Similarly, $EG \parallel CA$ and $EG = \frac{1}{3}CA$; and $FE \parallel AB$ and $FE = \frac{1}{3}AB$. Thus triangles ABC and EFG are homothetic with ratio of similarity 3 : 1. □

In the supplemental notebooks, we find the center of the homothety, D', and show that the homothety that maps ABC into EFG is $h = \text{homot}(D', -1/3)$, where $D' = \text{homot}(X_2, 1/4)(D)$. See Figure 3.

Theorem 3.3. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the centroids of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are orthologic with orthologic centers the orthocenters of $\triangle ABC$ and $\triangle EFG$.*

Proof. Note that $\triangle ABC$ and $\triangle EFG$ are homothetic by Theorem 3.2. The result then follows from the following theorem. □

Theorem 3.4. *Let ABC and EFG be two homothetic triangles. Then triangles ABC and EFG are orthologic with orthologic centers the orthocenters of $\triangle ABC$ and $\triangle EFG$.*

Proof. Since the triangles are homothetic, we must have $GF \parallel BC$. Thus, the line from A perpendicular to GF is also perpendicular to BC and is therefore an altitude of $\triangle ABC$ (segment AH in Figure 4). Hence, this line passes through the orthocenter of $\triangle ABC$. Similarly, the line from B perpendicular to EG and the line from C perpendicular to FE also pass through the orthocenter of $\triangle ABC$. Thus, triangles ABC and EFG are orthologic. Similar reasoning shows that the second orthologic center is the orthocenter of $\triangle EFG$. □

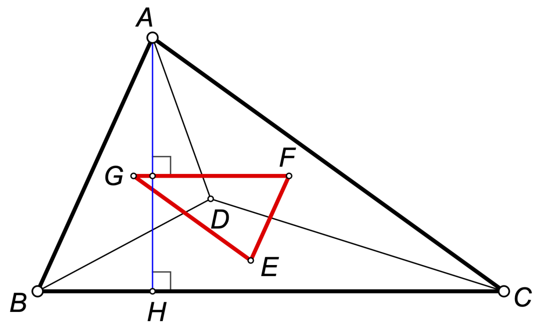


Figure 4. Central triangle using X_2 points

3.2. **Results involving X_3.**

Theorem 3.5. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the circumcenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Let O be the circumcenter of $\triangle ABC$. Then triangles ABC and EFG are orthologic (Figure 5). The orthologic centers are D and O.*

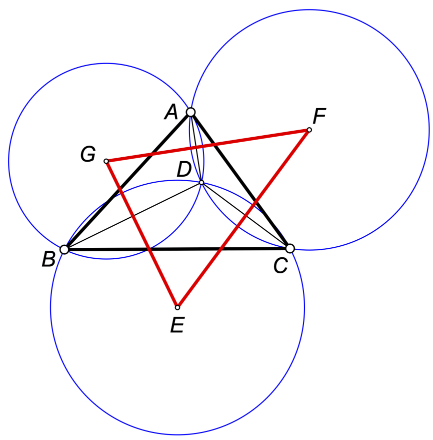


Figure 5. Central triangle using X_3 points

Proof. Since $GA = GD$ and $FA = FD$, we can conclude that $AD \perp FG$. Similarly, $BD \perp GE$ and $CD \perp EF$. Thus, $D = \text{ortho}[ABC, DEF]$.

Since $EB = EC$, the line from E perpendicular to BC is the perpendicular bisector of BC. Similarly, the line from F perpendicular to CA is the perpendicular bisector of CA and the line from G perpendicular to AB is the perpendicular bisector of AB. The three perpendicular bisectors meet at O, the circumcenter of $\triangle ABC$. Therefore, $O = \text{ortho}[DEF, ABC]$. $\square$

Theorem 3.6. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the circumcenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Let O be the circumcenter of $\triangle ABC$. Then triangles ABC and EFG are Eulerologic with $O = \text{euler}[EFG, ABC]$ (Figure 6).*

Proof. Triangle BCE is isosceles with $EB = EC$. The Euler line of an isosceles triangle is the altitude from the vertex. Since $OB = OC$, OE is the perpendicular

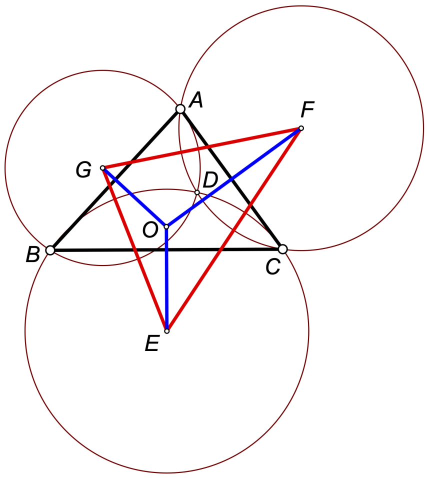


Figure 6. Central triangle using X_3 points

bisector of BC. Therefore, the Euler line of $\triangle BCE$ passes through O. Similarly, the Euler lines of triangles CAF and ABG also pass through O. □

3.3. **Results involving X_4.**

Theorem 3.7. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the orthocenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are orthologic with $D = \text{ortho}[EFG, ABC]$.*

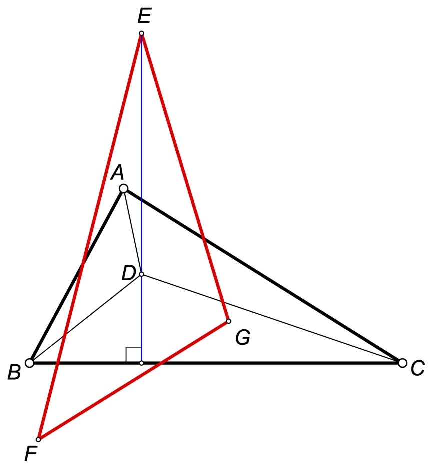


Figure 7. Central triangle using X_4 points

Proof. Since E is the orthocenter of $\triangle DBC$, $ED \perp BC$ (Figure 7). Thus, the line from E perpendicular to BC passes through D. Similarly, the line from F perpendicular to CA passes through D and the line from G perpendicular to AB passes through D. Therefore, triangles ABC and EFG are orthologic with $D = \text{ortho}[EFG, ABC]$. □

Theorem 3.8. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the orthocenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are Eulerologic with* $\text{euler}[ABC, EFG]$ *and* $D = \text{euler}[EFG, ABC]$.

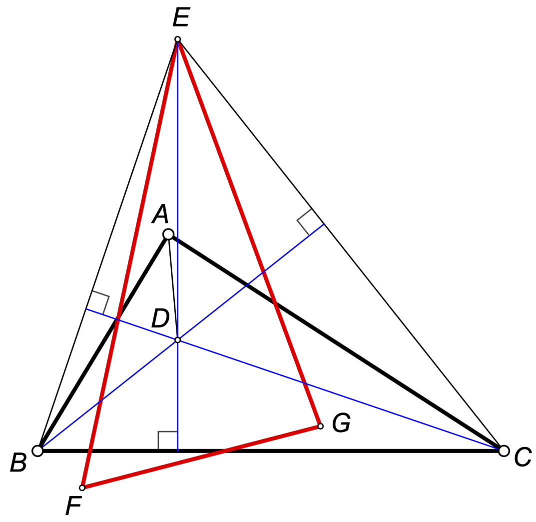


Figure 8. Central triangle using X_4 points

Proof. Three points and their orthocenter form an orthocentric system [37]. Since E is the orthocenter of $\triangle BCD$, this means that D is the orthocenter of $\triangle BCE$ (Figure 8). Similarly, D is the orthocenter of $\triangle CAF$ and D is the orthocenter of $\triangle ABG$. Thus, $D = \text{euler}[EFG, ABC]$. □

In the supplementary notebooks, we prove that euler$[ABC, EFG]$ and that the Eulerologic center euler$[ABC, EFG]$ is the barycentric quotient of the point D and the orthocorrespondent of D [38].

Theorem 3.9. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the orthocenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG have the same area (Figure 9).*

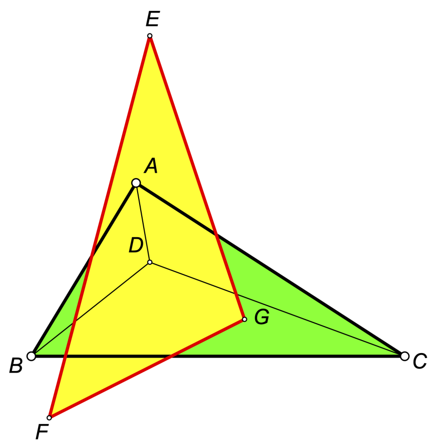


Figure 9. Central triangle using X_4 points

Proof. This is Theorem 3 from [24]. □

Theorem 3.10. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the orthocenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then there is a conic that passes through A, B, C, D, E, F, and G (Figure 10). The center of this conic is the Poncelet point* [31] *of quadrangle $ABCD$.*

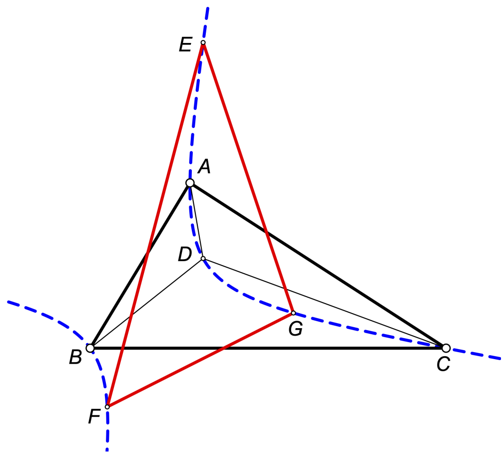


Figure 10. Central triangle using X_4 points

Proof. This is a special case of a result about quadrilaterals (Theorem 6.7 from [25]) in which two of the vertices of the quadrilateral coincide. ☐

When D is a triangle center of $\triangle ABC$, the center of the conic given by Theorem 3.10 is related to $\triangle ABC$ as shown in the following table.

Conic Centers for Specific D	
point D	**conic center**
X_1, X_7, X_8, X_9	X_{11}
X_2, X_{10}, X_{13}, X_{14}, X_{17}, X_{18}	X_{115}
X_3, X_6	X_{125}
X_5	X_{137}
X_{11}	not cataloged in [10] as of 2025
X_{12}	X_{52119}
X_{15}	X_{15609}
X_{16}	X_{15610}
X_{19}	X_{5521}
X_{20}	X_{122}

3.4. **Results involving $\mathbb{C}$.**

Recall that $\mathbb{C}$ is the set of triangle centers that lie on the circumcircle of the reference triangle.

Theorem 3.11. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let X be a triangle center that lies on the circumcircle of $\triangle ABC$. Let E, F, and G be the X-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are cyclologic.*

Figure 11 shows an example using Steiner (X_{99}) points. Note that the blue circles ($\odot AFG$, $\odot BEG$, and $\odot CEF$) meet at a point (shown in green).

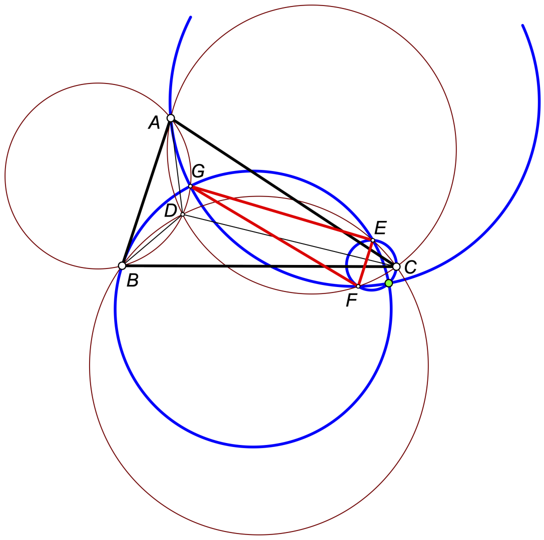


FIGURE 11. Central triangle using X_{99} points

This result follows immediately from the following generalization.

Theorem 3.12. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be points on $\odot BCD$, $\odot CAD$, and $\odot ABD$, respectively. Then triangles ABC and EFG are cyclologic.*

Proof. By construction, D lies on circles $\odot BCE$, $\odot CAF$, and $\odot ABG$, respectively. Thus, the circles BCE, CAF, and ABG all pass through D showing that $D = \text{cyclo}[EFG, ABC]$.
Since $\text{cyclo}[PQR, XYZ]$ implies that $\text{cyclo}[XYZ, PQR]$, we can conclude that $\text{cyclo}[ABC, EFG]$ and triangles ABC and EFG are cyclologic. □

3.5. Results involving $\mathbb{S}$.

Recall that $\mathbb{S}$ denotes the set of triangle centers that lie on the Euler line of the reference triangle and have constant Shinagawa coefficients.

Lemma 3.13. *Let ABC and XYZ be two triangles in the plane. Triangles ABC and XYZ are orthologic if and only if*

$$\left(AY^2 - AZ^2\right) + \left(BZ^2 - BX^2\right) + \left(CX^2 - CY^2\right) = 0.$$

Proof. This is proven in section 2.3 of [2]. □

Lemma 3.14. *Let D be an point on side BC of $\triangle ABC$ such that $BD : DC = m : n$ and $m + n = 1$. Then*

$$AD^2 = m \cdot AC^2 + n \cdot AB^2 - mn \cdot BC^2.$$

Proof. This follows directly from Stewart's Theorem [39]. □

Theorem 3.15. *Let ABC, UVW, and XYZ be triangles such that* $\text{ortho}[ABC, UVW]$ *and* $\text{ortho}[ABC, XYZ]$. *Let $E \in UX$, $F \in VY$, $G \in WZ$ such that $UE : EX = VF : FY = WG : GZ = m : n$, where m, n are real numbers (Figure 12). Then we have* $\text{ortho}[ABC, EFG]$.

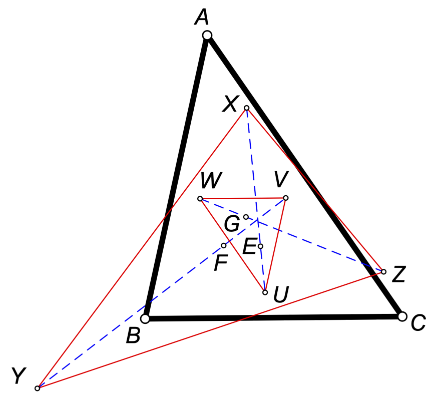


Figure 12.

Proof. Since $\text{ortho}[ABC, UVW]$, according to Lemma 3.13 we have

$$(AV^2 - AW^2) + (BW^2 - BU^2) + (CU^2 - CV^2) = 0, \tag{1}$$

and since $\text{ortho}[ABC, XYZ]$ we have

$$(AY^2 - AZ^2) + (BZ^2 - BX^2) + (CX^2 - CY^2) = 0. \tag{2}$$

Assume without loss of generality that $m+n = 1$. Lemma 3.14 applied to $\triangle AVY$ and $\triangle AWZ$ yields

$$AF^2 = m \cdot AY^2 + n \cdot AV^2 - mn \cdot VY^2 \tag{3}$$

and

$$AG^2 = m \cdot AZ^2 + n \cdot AW^2 - mn \cdot WZ^2 \tag{4}$$

Lemma 3.14 applied to $\triangle BUX$ and $\triangle BWZ$ yields

$$AF^2 = m \cdot AY^2 + n \cdot AV^2 - mn \cdot VY^2 \tag{5}$$

and

$$AG^2 = m \cdot AZ^2 + n \cdot AW^2 - mn \cdot WZ^2 \tag{6}$$

Using (1),(2),(3), and (4) we get

$$AF^2 - AG^2 = m\left(AY^2 - AZ^2\right) + n\left(AV^2 - AW^2\right) + mn\left(WZ^2 - VY^2\right). \tag{7}$$

Similarly, we get

$$BG^2 - BE^2 = m\left(BZ^2 - BX^2\right) + n\left(BW^2 - BU^2\right) + mn\left(UX^2 - WZ^2\right) \tag{8}$$

and

$$CE^2 - CF^2 = m\left(CX^2 - CY^2\right) + n\left(CU^2 - CV^2\right) + mn\left(VY^2 - UX^2\right). \tag{9}$$

Summing (7), (8), (9) and using (1), (2) we have

$$\begin{aligned}&\left(AF^2 - AG^2\right) + \left(BG^2 - BE^2\right) + \left(CE^2 - CF^2\right)\\ =&m\left(AY^2 - AZ^2 + BZ^2 - BX^2 + CX^2 - CY^2\right) +\\ &n\left(AV^2 - AW^2 + BW^2 - BU^2 + CU^2 - CV^2\right) +\\ &mn\left(WZ^2 - VY^2 + UX^2 - WZ^2 + VY^2 - UX^2\right) = 0.\end{aligned}$$

Therefore, from Lemma 3.13, we obtain ortho$[ABC, EFG]$. □

Theorem 3.16. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let $X \in \mathbb{S}$. Let E, F, and G be the X-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG are orthologic.*

Figure 13 shows an example using X_{140} points.

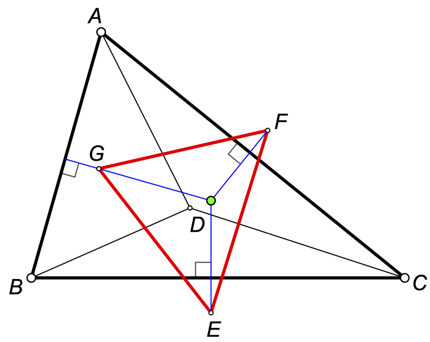


Figure 13. Central triangle using X_{140} points

Note that the line through E perpendicular to BC, the line through F perpendicular to CA, and the line through G perpendicular to AB, meet at a point (shown in green).

Some special cases of Theorem 3.16 are easy to prove geometrically.

Theorem 3.17. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the X_2-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG are orthologic.*

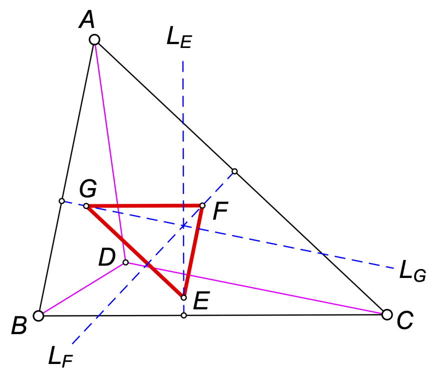


Figure 14. central triangle with X_2 points

Proof. By Theorem 3.2, triangles ABC and EFG are homothetic. Thus, $FG \parallel BC$. The line L_E through E perpendicular to BC is also perpendicular to FG (Figure 14). Similarly, The line L_F through F perpendicular to CA is also perpendicular to GE and the line L_G through G perpendicular to AB is also perpendicular to EF. The lines L_E, L_F, and L_G are therefore the altitudes of $\triangle EFG$ and hence are concurrent. Thus triangles EFG and ABC are orthologic. □

Theorem 3.18. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the X_4-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG are orthologic.*

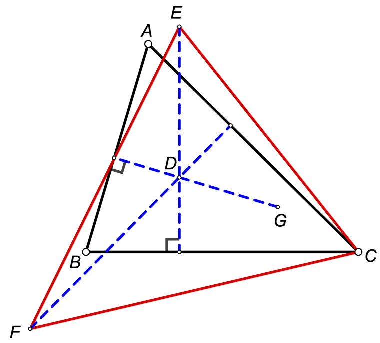


Figure 15. central triangle with X_4 points

Proof. Since E is the orthocenter of $\triangle BCD$, the perpendicular from E to BC passes through D. Similarly, the line from F perpendicular to CA passes through D and the line from G perpendicular to AB passes through D. Thus triangles EFG and ABC are orthologic. □

We can now give a complete synthetic proof of Theorem 3.16.

Proof. From Theorem 3.17, we have that if E_2, F_2, and G_2 are the X_2-points of triangles BCD, CAD, and ABD, respectively, then triangles ABC and $E_2F_2G_2$ are orthologic.

From Theorem 3.18, we have that if E_4, F_4, and G_4 are the X_4-points of triangles BCD, CAD, and ABD, respectively, then triangles ABC and $E_4F_4G_4$ are orthologic.

Since X lies on the Euler line of a triangle, $E \in E_2E_4$, $F \in F_2F_4$, and $G \in G_2G_4$ (Figure 16).

Since $X \in \mathbb{S}$, from Theorem 3.1, we see that there are constants m and n such that $X_4X : XX_2 = 3m : n$. From Theorem 3.15, we can conclude that triangles ABC and EFG are orthologic. □

We can also find one of the orthologic centers.

Theorem 3.19. *Let D be any point in the plane of $\triangle ABC$ (not on the boundary). Let $X = \mathrm{Shin}(m, n)$ be a point in $\mathbb{S}$. Let E, F, and G be the X-points of triangles BCD, CAD, and ABD, respectively. Let $O_2 = \mathrm{ortho}[EFG, ABC]$. Then $O_2 = \mathrm{homot}(O, \frac{m+n}{3m+n})(D)$ where O is the circumcenter of $\triangle ABC$ (Figure 17).*

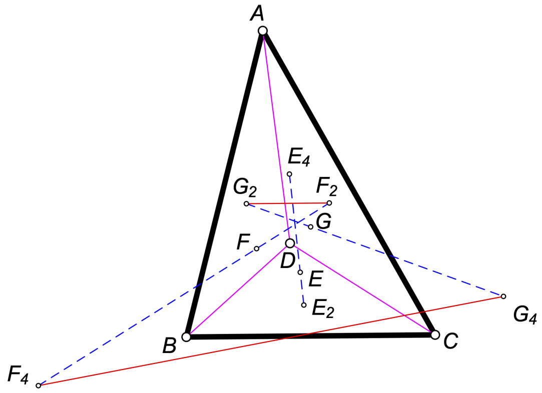


Figure 16.

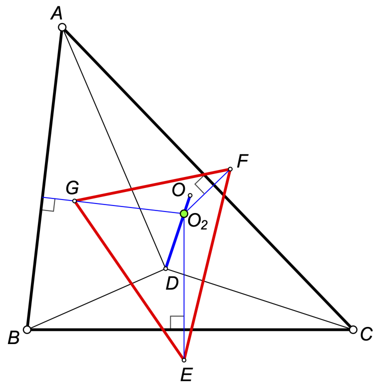


Figure 17.

The results in this section apply when D is any point in the plane of $\triangle ABC$ (not on the boundary). If we further restrict point D to lie *inside* $\triangle ABC$, then we get a few additional results. They are shown in the following table.

Central Triangles of $\triangle ABC$, D inside $\triangle ABC$	
Relationship	**centers**
cyclo$[ABC, EFG]$	13, 14, 15, 16

Figure 18 shows an example with $n = 13$. The cyclologic center of cyclo$[ABC, EFG]$ is shown in green.

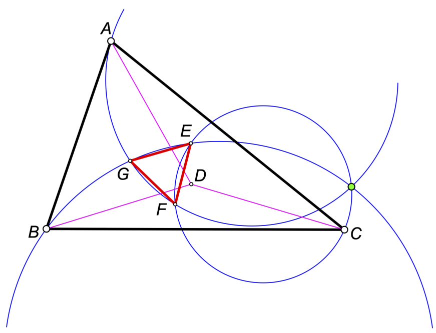


Figure 18. central triangle with X_{13} points

Theorem 3.20. *Let D be any point inside $\triangle ABC$. Let E, F, and G be the X_{13}-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG are cyclologic. The cyclologic center, $P = \text{cyclo}[EFG, ABC]$, is the X_{13}-point of $\triangle ABC$.*

Proof. It suffices to show that $\odot BCE$ passes through P. First, assume neither $\angle B$ nor $\angle C$ is larger than 120° (Figure 19). Since E is the X_{13}-point of $\triangle BDC$, and E lies on the same side of BC as A, we must have $\angle BEC = 120°$ [40]. Since P is the X_{13}-point of $\triangle ABC$, P must lie on the same side of BC as A and $\angle BPC = 120°$. Thus, quadrilateral $BPEC$ is cyclic and $\odot BCE$ passes through P.

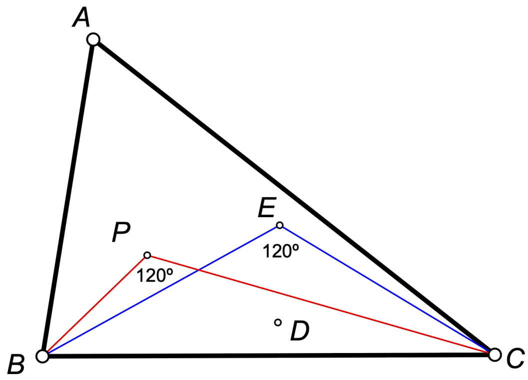


FIGURE 19.

Now assume, without loss of generality, that $\angle B$ is larger than 120° (Figure 20).

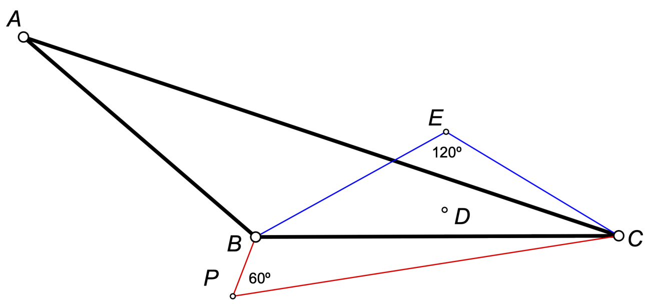


FIGURE 20.

In this case, P lies on the opposite side of BC from A, and $\angle CPB = 60°$ [40]. Again, we find that quadrilateral $BECP$ is cyclic and $\odot BCE$ passes through point P. □

Theorem 3.21. *Let D be any point inside $\triangle ABC$. Let E, F, and G be the X_{14}-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG are cyclologic. The cyclologic center, $P = \text{cyclo}[EFG, ABC]$, is the X_{14}-point of $\triangle ABC$.*

Proof. The geometric proof is similar to the proof of Theorem 3.20 and is omitted. Figure 21 shows the points D, E, and P in the case when $\triangle ABC$ is acute. □

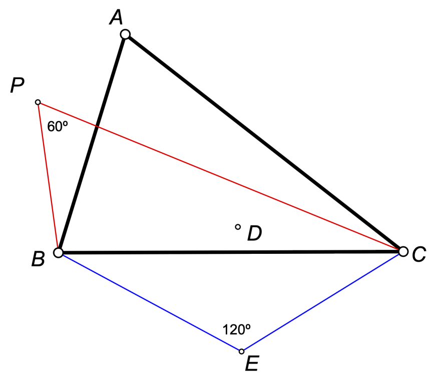


FIGURE 21.

The list of triangle centers that must lie inside the reference triangle can be found in [21]. The first few are X_n for n =1, 2, 6, 7, 8, 9, 10, 12, 21, 31, 32, 37, 38, 39, 41, 42, 45, 55, 56, 57, 58, 60, 65, 75, 76, 81, 82, 83, 85, 86, and 89.

This gives us the following result.

Theorem 3.22. *Let d be 1, 2, 6, 7, 8, 9, or 10. Let n be 13, 14, 15, or 16. Let D be the X_d-point of $\triangle ABC$. Let E, F, and G be the X_n-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG are cyclologic.*

Theorem 3.23. *Let n be 13, 14, 15, or 16. Let D be the X_{11}-point of $\triangle ABC$. Let E, F, and G be the X_n-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG are cyclologic if D is not on the boundary of $\triangle ABC$.*

This follows from Theorem 2 of [21] which says that X_{11} must always lie inside or on the boundary of the reference triangle.

4. RESULTS WHEN $D = X_1$

When $D = X_1$, the central triangle is sometimes known as the *BCI Triangle* [33].

Our computer study found a few additional results when D is the incenter of $\triangle ABC$. They are shown in the following table.

Central Triangles of $\triangle ABC$, $D = X_1$	
Relationship	**centers**
cyclo$[ABC, EFG]$	5, 51–54, 95–97, 128–130, 134, 137–139, 143, 252
$\odot ABC \equiv \odot EFG$	3
ortho$[ABC, EFG]$	157
persp$[ABC, EFG]$	1, 3, 13–16, 19, 63, 326, 399, 920, 921
euler$[ABC, EFG]$	3, 930
euler$[EFG, ABC]$	930

For the cases when triangles ABC and EFG are perspective, we list the perspectors in the following table.

Perspectors of perspective triangles ABC and EFG	
centers	**perspector**
3	X_1
1	X_{1127} [13]
399	X_{3065}
13	X_{66889}
14	X_{66890}
15	X_{39152}
16	X_{39153}
19, 63, 326, 920, 921	not cataloged in [10] as of 2025

Is it a coincidence that X_1, X_{3065}, X_{39152}, and X_{39153} all lie on cubic K206 [6]?

We highlight a few of these results.

Theorem 4.1. *Let D be the incenter (X_1) of $\triangle ABC$. Let E, F, and G be the circumcenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG have the same circumcircle (Figure 22).*

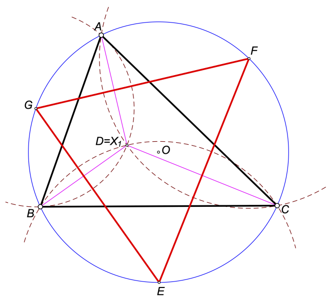
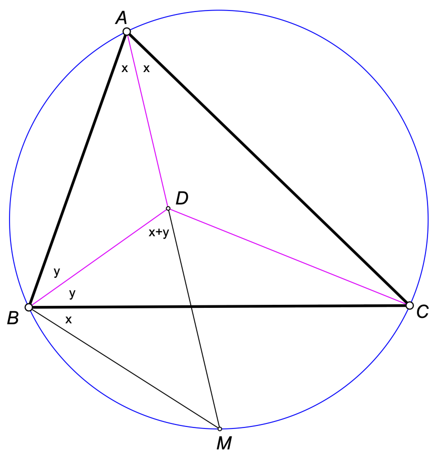


Figure 22. Central triangle using X_3 points

Proof. Since D is the incenter of $\triangle ABC$, AD is the angle bisector of $\angle BAC$. Thus, AD meets the circumcircle of $\triangle ABC$ at M, the midpoint of arc BC opposite A.



We have $\angle BAM = \angle MAC = x$ since AD is an angle bisector and $\angle MBC = x$ since it is inscribed in arc MC. We have $\angle DBA = \angle CBD = y$ since BD is an angle bisector. Then from $\triangle ABD$, we see that $\angle BDM = \angle BAD + \angle DBA =$

$x+y$. Therefore, $\angle MBD = \angle BDM$ (both are equal to $x+y$). Hence $MB = MD$. Since $MB = MC$, M must be the circumcenter of $\triangle BDC$. In other words, $M = E$, proving that E lies on the circumcircle of $\triangle ABC$. In the same manner, we see that F and G also lie on the circumcircle of $\triangle ABC$. □

Theorem 4.2. *Let D be the incenter (X_1) of $\triangle ABC$. Let E, F, and G be the circumcenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are perspective (Figure 23). The point D is the perspector.*

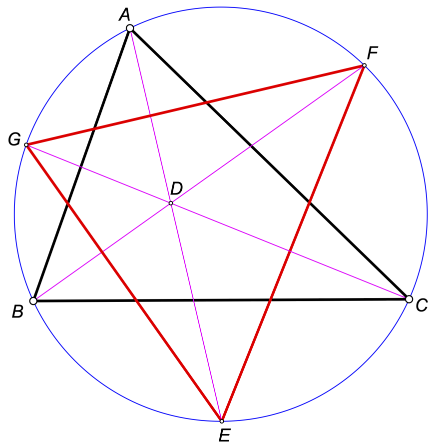


FIGURE 23. Central triangle using X_3 points

Proof. From the proof of Theorem 4.1, we found that E is the midpoint of arc BC. Thus, AE is the angle bisector of $\angle BAC$ and passes through D. In the same way, BF and CG pass through D. □

Theorem 4.3. *Let D be the incenter (X_1) of $\triangle ABC$. Let E, F, and G be the X_{157}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are orthologic (Figure 24).*

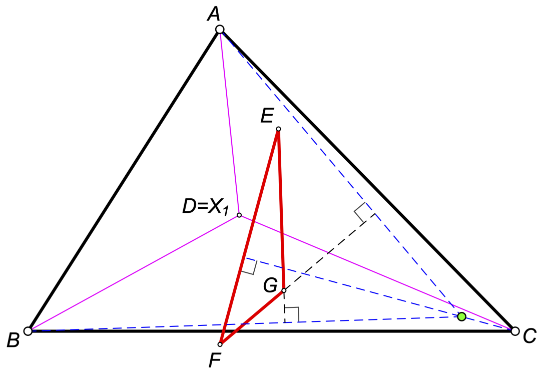


FIGURE 24. Central triangle using X_{157} points

Note that the X_{157}-point of a triangle is the symmedian point of its tangential triangle.

Open Question 1. *Triangles ABC and EFG are perspective when using centers X_n for $n = 1, 3, 13, 14, 15, 16, 19, 63, 326, 399, 920, 921$. Is there some pattern to these numbers?*

Theorem 4.4. *Let D be the incenter (X_1) of $\triangle ABC$. Let E, F, and G be the incenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are perspective (Figure 25). The perspector is X_{1127}.*

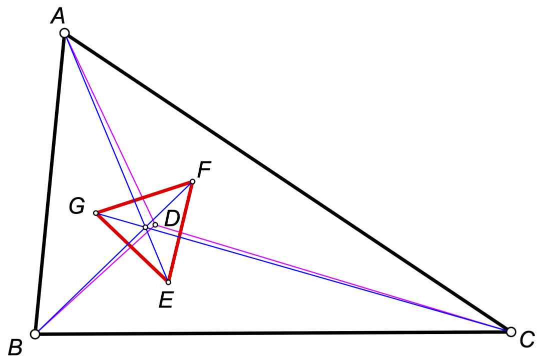


Figure 25. Central triangle using X_1 points

Theorem 4.5. *Let D be the incenter (X_1) of $\triangle ABC$. Let E, F, and G be the X_{13}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are perspective (Figure 26). The perspector is X_{66889}.*

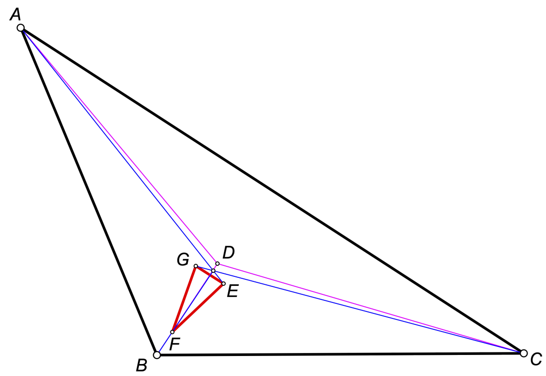


Figure 26. Central triangle using X_{13} points

Theorem 4.6. *Let D be the incenter (X_1) of $\triangle ABC$. Let E, F, and G be the X_{14}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are perspective. The perspector is X_{66890}.*

Theorem 4.7. *Let D be the incenter (X_1) of $\triangle ABC$. Let E, F, and G be the X_{15}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are perspective. The perspector is X_{39152}.*

Theorem 4.8. *Let D be the incenter (X_1) of $\triangle ABC$. Let E, F, and G be the X_{16}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are perspective. The perspector is X_{39153}.*

Open Question 2. *Are there geometric proofs for Theorems 4.4, 4.5, 4.6, 4.7, and 4.8?*

5. Results when $D = X_2$

Our computer study found a few additional results when D is the centroid of $\triangle ABC$. They are shown in the following table.

Central Triangles of $\triangle ABC$, $D = X_2$	
Relationship	**centers**
$\mathrm{m}(ABC) = \mathrm{m}(EFG)$	2
$\mathrm{eu}(ABC) = \mathrm{eu}(EFG)$	2
$\mathrm{euler}[EFG, ABC]$	13, 14
$ABC \sim EFG$	51
$\mathrm{ortho}[ABC, EFG]$	52, 143, 185, 373, 389, 568, 591

We highlight a few of these results.

Theorem 5.1. *Let D be the centroid (X_2) of $\triangle ABC$. Let E, F, and G be the centroids of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG have the same centroid (Figure 27).*

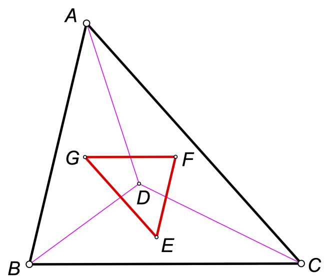


FIGURE 27. Central triangle using X_2 points

Proof. The two triangles are homothetic (with perspector D) by Theorem 3.2. The perspector represents corresponding points of both triangles. In this case, the perspector is the centroid of $\triangle ABC$, so D must also be the centroid of $\triangle EFG$. $\square$

Theorem 5.2. *Let D be the centroid (X_2) of $\triangle ABC$. Let E, F, and G be the centroids of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG have a common Euler line (Figure 28).*

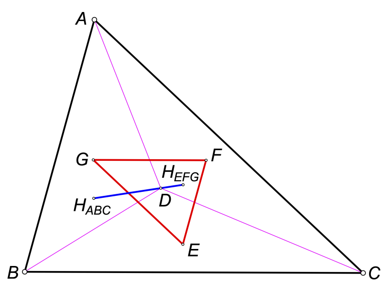


FIGURE 28. Central triangle using X_2 points

Proof. The two triangles are homothetic (with perspector D) by Theorem 3.2. Corresponding points are collinear with the perspector. So the orthocenter of $\triangle ABC$, the orthocenter of $\triangle EFG$, and D are collinear. Since D is both the centroid of $\triangle ABC$ and $\triangle EFG$, this means that the line joining the two orthocenters is the common Euler line of the two triangles. □

Theorem 5.3. *Let D be the centroid (X_2) of $\triangle ABC$. Let E, F, and G be the X_{51}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are inversely similar (Figure 29).*

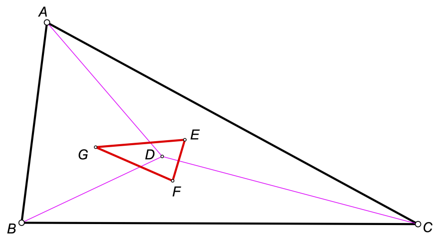


Figure 29. Central triangle using X_{51} points

Note that the X_{51}-point of a triangle is the centroid of its orthic triangle.

Open Question 3. *Triangles ABC and EFG are orthologic when using centers X_n for $n = 52, 143, 185, 373, 389, 568, 591$. Is there some pattern to these numbers?*

Theorem 5.4. *Let D be the centroid (X_2) of $\triangle ABC$. Let E, F, and G be the orthocenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are Eulerologic (Figure 30) with $X_{5485} = \text{euler}[ABC, EFG]$. The Eulerologic center is the orthocenter of triangles AFG, BEG, and CEF.*

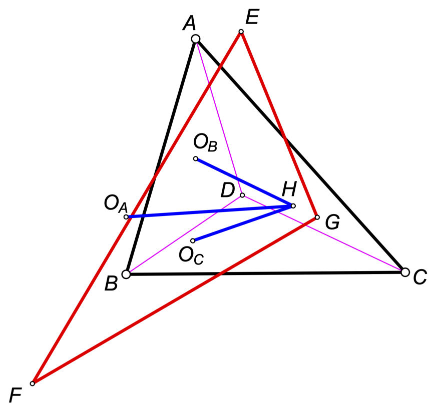


Figure 30. Central triangle using X_4 points. Euler lines in blue meet at H.

Theorem 5.5. *Let D be the centroid (X_2) of $\triangle ABC$. Let E, F, and G be the orthocenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are Eulerologic (Figure 31) with $X_2 = \text{euler}[EFG, ABC]$.*

Theorem 5.6. *Let D be the centroid (X_2) of $\triangle ABC$. Let E, F, and G be the circumcenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are Eulerologic (Figure 32) with $X_3 = \text{euler}[EFG, ABC]$.*

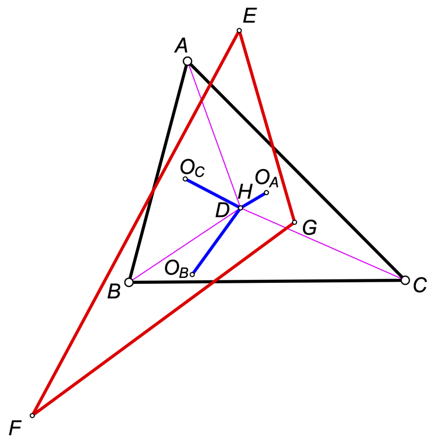


Figure 31. Central triangle using X_4 points. Euler lines in blue meet at D.

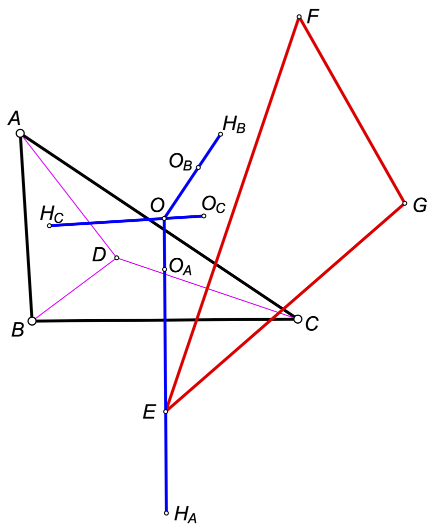


Figure 32. Central triangle using X_3 points. Euler lines in blue meet at O.

Theorem 5.7. *Let D be the centroid (X_2) of $\triangle ABC$. Let E, F, and G be the X_{13}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are Eulerologic (Figure 33).*

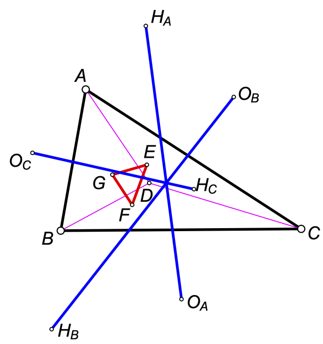


Figure 33. Central triangle using X_{13} points. Euler lines in blue.

The Eulerologic center, euler$[EFG, ABC]$ is not cataloged in [10] as of 2025.

Theorem 5.8. *Let D be the centroid (X_2) of $\triangle ABC$. Let E, F, and G be the X_{14}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are Eulerologic.*

The Eulerologic center, euler$[EFG, ABC]$ is not cataloged in [10] as of 2025.

Open Question 4. *Are there geometric proofs for Theorems 5.4, 5.5, 5.6, 5.7, and 5.8?*

6. Results when $D = X_3$

Our computer study found many additional results when D is the circumcenter of $\triangle ABC$. They are shown in the following table. We have omitted from the table any triangle center that lies on the line at infinity.

Note that by Lemma 5.13 of [22], the center X_n lies on the line at infinity for all isosceles triangles, but not for all triangles, for the following values of n: 351, 647, 649, 650, 652, 654, 656, 657, 659, 661, 663, 665, 667, 669, 676, 684, 686, 693, 764, 770, 798, 810, 822, 850, 875, 876, 878, 879, 881, 882, 884, 885, 887, 890, 905.

Central Triangles of $\triangle ABC$, $D = X_3$	
Relationship	**centers**
$[ABC] = 4[EFG]$	113, 114, 117–121, 126, 128, 129, 131–133, 138
$ABC \cong EFG$	148–150
cyclo$[ABC, EFG]$	148–150
cyclo$[EFG, ABC]$	148–150
eu(ABC) = eu(EFG)	$\mathbb{M}$, 2, 148–150, 290, 402, 620, 671, 903
X_{30} = euler$[ABC, EFG]$	$\mathbb{M}$
D = euler$[EFG, ABC]$	all
X_2 = homot$(-1:2)[ABC, EFG]$	$\mathbb{M}$
X_{631} = homot$(-1:4)[ABC, EFG]$	402, 620
X_{1656} = homot$(-2:3)[ABC, EFG]$	290, 671, 903
m(ABC) = m(EFG)	$\mathbb{M}$
n(ABC) = n(EFG)	148–150
ortho$[ABC, EFG]$	all
bilogic$[ABC, EFG]$	$\mathbb{M}$, 2, 3, 74, 98, 102–106, 111, 148–150, 290, 399, 402, 477, 620, 671, 675, 699, 701, 703, 705, 707, 709, 711, 713, 715, 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 745, 747, 753, 755, 759, 761, 767, 840–843, 903, 915, 917, 953, 972
X_5 = conic$[ABCEFG]$	148–150

The word "all" appearing in the previous table means that the relationship is true for all centers considered in this study for which $\triangle EFG$ is not degenerate.

The symbol $\mathbb{M}$ denotes the set of X_n for which in an isosceles triangle, X_n coincides with the midpoint of the base. See Lemma 5.12 of [22]. The first few such n are 11, 115, 116, 122–125, 127, 130, 134–137, 139, 244-247, 338, 339, 865–868.

For the cases when triangles ABC and EFG are perspective, we list the perspectors in the following table.

Perspectors of perspective triangles ABC and EFG	
centers	**perspector**
11, 115, 116, 122–125, 127, 130, 134–137, 139, 244–247, 338, 339, 865–868	X_2
148–150	X_5
74, 98, 102–106, 111, 477, 675, 699, 701, 703, 705, 707, 709, 711, 713, 715, 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 745, 747, 753, 755, 759, 761, 767, 840–843, 915, 917, 953, 972	X_6
15	X_{15}†
16	X_{16}†
14	X_{17}†
13	X_{18}†
3	X_{54}
399	X_{74}
2	X_{140}
402, 620	X_{631}
290, 671, 903	X_{1656}
† only if $\triangle ABC$ is acute	

We highlight a few of these results.

Theorem 6.1. *Let D be the circumcenter of $\triangle ABC$. Let X be any triangle center. Let E, F, and G be the X-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then $DE \perp BC$, $DF \perp CA$, and $DG \perp AB$.*

Proof. When $D = X_3$, $AD = BD = CD$, and the three side triangles, $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, are isosceles with vertex D. In an isosceles triangle, all triangle centers lie on the perpendicular bisector of the base. Thus $DE \perp BC$, $DF \perp CA$, and $DG \perp AB$. □

Theorem 6.2. *Let D be the circumcenter of $\triangle ABC$. Let X be any triangle center. Let E, F, and G be the X-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are Eulerologic with $D = \mathrm{euler}[EFG, ABC]$.*

Proof. Since E is a triangle center of isosceles triangle BCD, it must lie on the perpendicular bisector of base BC (Figure 34). This perpendicular bisector passes through D.

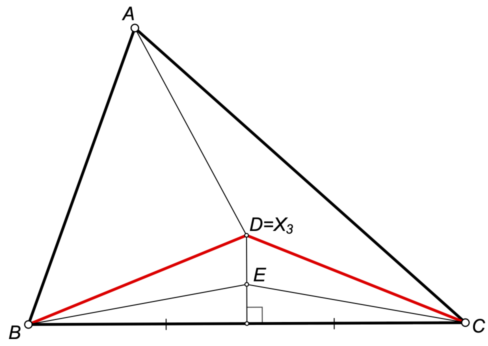


Figure 34.

Thus, the Euler line of $\triangle BCE$ is the perpendicular bisector of BC and passes through D. Similarly, the Euler lines of triangles CAF and ABG also pass through D. Hence, $D = \text{euler}[EFG, ABC]$. □

Theorem 6.3. *Let D be the circumcenter of $\triangle ABC$. Let X be any triangle center. Let E, F, and G be the X-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are orthologic with $D = \text{ortho}[EFG, ABC]$.*

Proof. The proof is the same as the proof of Theorem 6.2. We can conclude that $D = \text{ortho}[EFG, ABC]$ and thus, we also have $\text{ortho}[ABC, EFG]$. □

Note: This same proof shows that this result is true if E, F, and G are any points on the perpendicular bisectors of BC, CA, and AB, respectively.

Theorem 6.4. *Let D be the circumcenter of $\triangle ABC$. Let $X \in \mathbb{M}$. Let E, F, and G be the X-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then $\triangle ABC$ and $\triangle EFG$ are homothetic and have the same centroid.*

Proof. Let M be the midpoint of BC. By Lemma 5.12 of [22], E coincides with M. Similarly, F coincides with the midpoint of CA and G coincides with the midpoint of AB. Thus $\triangle ABC$ and $\triangle EFG$ are homothetic with perspector X_2 which is the centroid of both $\triangle ABC$ and $\triangle EFG$. □

Theorem 6.5. *Let D be the circumcenter of $\triangle ABC$. Let $X \in \mathbb{M}$. Then $X_{30} = \text{euler}[ABC, EFG]$ and the centroids of triangles ABC and EFG coincide.*

Proof. From the definition of $\mathbb{M}$, points E, F, and G are the midpoints of sides BC, CA, and AB, respectively. Thus $\triangle EFG$ is the medial triangle of $\triangle ABC$ and the results follow. □

We found a few additional results when $\triangle ABC$ is acute. They are shown in the following table.

Central Triangles of acute $\triangle ABC$, $D = X_3$	
Relationship	**centers**
$\text{m}(ABC) = \text{m}(EFG)$	13, 14
$[ABC] = [EFG]$	616, 617

Theorem 6.6. *Let D be the circumcenter of $\triangle ABC$. Let n be 13 or 14. Let E, F, and G be the X_n-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are perspective. If $\triangle ABC$ is acute, then triangles ABC and EFG have a common centroid and $\triangle EFG$ is equilateral.*

Figure 35 illustrates this theorem when $n = 14$.

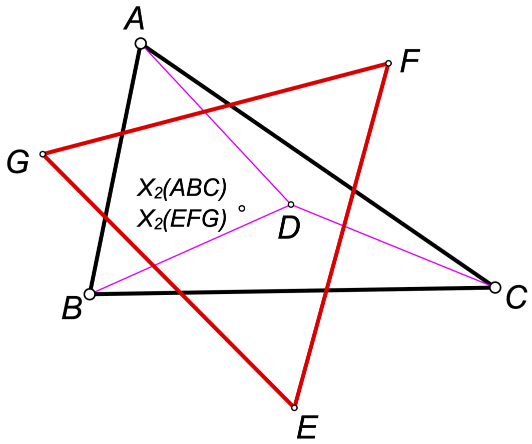


Figure 35. Central triangle using X_{14} points.

Theorem 6.7. *Let D be the circumcenter of acute $\triangle ABC$. Let n be 616 or 617. Let E, F, and G be the X_n-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG have the same area.*

Figure 36 illustrates this theorem when $n = 616$.

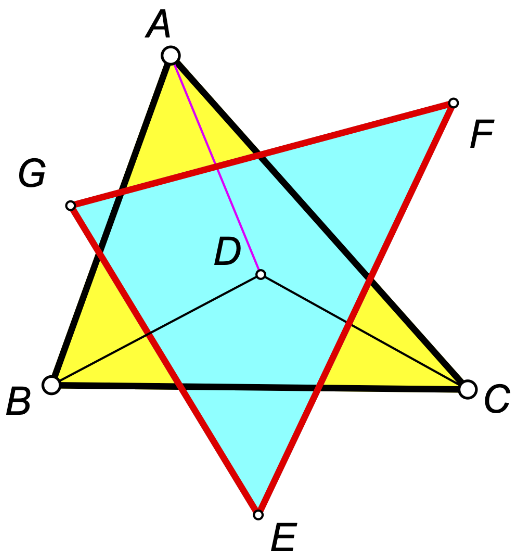


Figure 36. Central triangle using X_{616} points.

7. RESULTS WHEN $D = X_4$

Our computer study found many additional results when D is the orthocenter of $\triangle ABC$. They are shown in the following table.

Central Triangles of $\triangle ABC$, $D = X_4$	
Relationship	**centers**
$[ABC] = [EFG]$	684
$ABC \equiv EFG$	4
$ABC \cong EFG$	3, 930
$X_5 = \text{conic}[ABCEFG]$	3
$X_{128} = \text{conic}[ABCEFG]$	930
$X_{130} = \text{conic}[ABCEFG]$	184, 217 , 418
$X_{134} = \text{conic}[ABCEFG]$	571
$X_{137} = \text{conic}[ABCEFG]$	311, 327
$X_{139} = \text{conic}[ABCEFG]$	324, 467
$\text{conic}[ABCEFG]$	231
$X_5 = \text{homot}(n-m : 3m+n)[ABC, EFG]$	$\mathbb{S}$
$X_{128} = \text{homot}(-1 : 1)[ABC, EFG]$	930
$X_{51} = \text{homot}(-1 : 2)[ABC, EFG]$	389
$X_{143} = \text{homot}(-1 : 3)[ABC, EFG]$	568
$X_{51} = \text{homot}(-2 : 1)[ABC, EFG]$	185
$X_{27355} = \text{homot}(-2 : 9)[ABC, EFG]$	373
$\text{cyclo}[ABC, EFG]$	54, 95–97, 252
$\text{equi}[ABC, EFG]$	$\mathbb{A}$, 265
$\text{eu}(ABC) = \text{eu}(EFG)$	$\mathbb{S}$
$\text{euler}[ABC, EFG]$	3, 930
$\text{euler}[EFG, ABC]$	930
$\text{n}(ABC) = \text{n}(EFG)$	$\mathbb{S}$
$\text{bilogic}[ABC, EFG]$	157, 185, 373, 389, 428, 568, 930
$\text{para}[ABC, EFG]$	684
$\text{persp}[ABC, EFG]$	6, 21–29, 68, 69, 76, 93, 94, 143, 146, 155, 186, 195, 199, 217, 235, 237, 250, 254, 264, 265, 286, 297, 311, 314–317, 323-325, 340, 377–379, 383, 384, 387, 393, 394, 397–399, 401–431, 433, 436–455, 457, 458, 460–475, 562, 621, 622, 633, 634, 637, 638, 847, 851–868, 877, 933, 964

The symbol $\mathbb{A}$ denotes the set of all triangle centers that lie on the anticomplementary circle. The *anticomplementary circle* of a triangle is the circle through the anticomplements of the vertices. The *anticomplement* of a point P in a reference triangle ABC is a point P' satisfying the vector equation $\overrightarrow{P'G} = 2\overrightarrow{GP}$ where $G = X_2$ [32]. The first few centers that lie on the anticomplementary circle are X_n for $n = 146$–153, 3448.

For the cases when triangles ABC and EFG are perspective, we list the perspectors in the following table.

Perspectors of perspective triangles ABC and EFG	
centers	**perspector**
21–29, 186, 199, 235, 237, 297, 377–379, 383, 384, 401–431, 433, 436–455, 457, 458 460–475, 851–868, 964	X_5
389	X_{51}
68, 324, 847	X_{52}
6, 217, 387, 393, 397, 398	X_{53}
930	X_{128}
933	X_{137}
94, 143, 146, 265	X_{143}
93, 562	X_{1154}
195, 399	X_{1263}
69, 76, 264, 286, 311, 314–317, 340, 621, 622, 633, 634, 637, 638, 877	X_{5562}
16, 18	X_{6116}†
15, 17	X_{6117}†
155, 254	X_{8800}
250	X_{10216}
157	X_{27352}
323	X_{27353}
325	X_{27354}
394	X_{27356}
14, 62	X_{52670}†
13, 61	X_{52671}†
371, 372, 395, 396, 485, 486, 590, 615	not cataloged in [10] as of 2025†
† acute triangles only	

We found a few additional results if the triangle is acute. They are listed in the following table.

Central Triangles of acute $\triangle ABC$, $D = X_4$	
Relationship	**centers**
cyclo$[ABC, EFG]$	13–16

We highlight a few of these results.

Lemma 7.1. *The radius of the anticomplementary circle of $\triangle ABC$ is twice the radius of the circumcircle of $\triangle ABC$.*

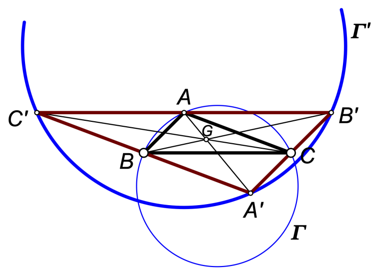


Figure 37. Anticomplementary circle of $\triangle ABC$

Proof. Let G be the centroid of $\triangle ABC$. Let A' be the anticomplement of A, so that $GA' = 2GA$. Similarly, for B' and C', the anticomplements of B and C, we have $GB' = 2GB$ and $GC' = 2GC$ (Figure 37).

Thus, $\triangle A'B'C'$ is homothetic to $\triangle ABC$ with G being the center of the homothety and with homothetic ratio $2:1$. Let Γ be the circumcircle of $\triangle ABC$ and let Γ' be the circumcircle of $\triangle A'B'C'$. Corresponding linear elements of two homothetic figures are in the same ratio as their homothetic ratio. Thus, the radius of Γ' is twice the radius of Γ. □

Lemma 7.2. *The center of the anticomplementary circle of $\triangle ABC$ is the orthocenter of $\triangle ABC$.*

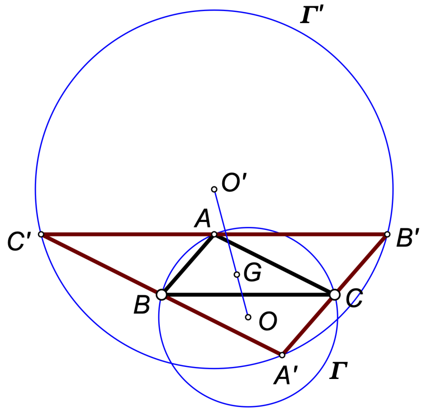


Figure 38. center of anticomplementary circle

Proof. The mapping that takes a point to its anticomplement maps the circumcircle of $\triangle ABC$ into the anticomplementary circle (the circumcircle of the anticomplementary triangle $A'B'C'$). The center of the circumcircle (the circumcenter O of $\triangle ABC$) therefore maps to the anticomplement of O (the circumcenter O' of triangle $A'B'C'$). But the anticomplement of O is H, the orthocenter of $\triangle ABC$ [11]. □

Lemma 7.3. *Let H be the orthocenter of $\triangle ABC$. Then the circumcircles of triangles ABC and HBC are congruent (Figure 39).*

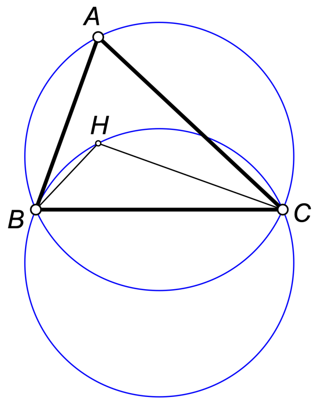


FIGURE 39. Congruent circumcircles

Proof. This is Theorem 180 in [1]. ☐

The following theorem is due to Francisco Javier García Capitàn.

Theorem 7.4. *Let R be the circumradius of $\triangle ABC$. Let H be the orthocenter of $\triangle ABC$. Then the anticomplementary circle of $\triangle HBC$ has center A and radius $2R$.*

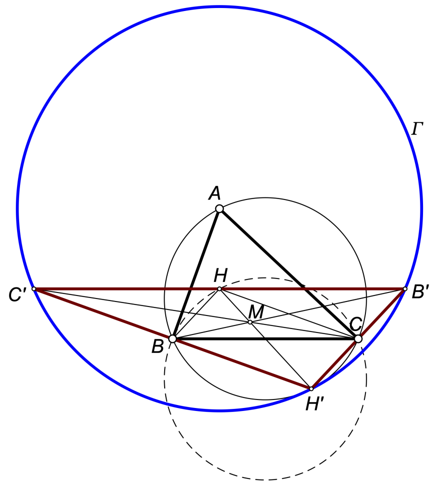


FIGURE 40. Blue circle has center A and radius $2R$

Proof. By Lemma 7.1, the radius R' of the anticomplementary circle Γ of $\triangle HBC$ is twice the radius of the the circumcircle of $\triangle HBC$. By Lemma 7.3, $R' = 2R$. By Lemma 7.2, the center of the anticomplementary circle Γ of $\triangle HBC$ is the orthocenter of $\triangle HBC$. Since A, B, C, and H form an orthocentric system [37], A is the orthocenter of $\triangle HBC$. Thus, A is the center of Γ. ☐

Theorem 7.5. *Let X be a triangle center that lies on the anticomplementary circle. Let D be the orthocenter of $\triangle ABC$. Let E, F, and G be the X-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are equilogic.*

Figure 41 illustrates this theorem using X_{146} points.

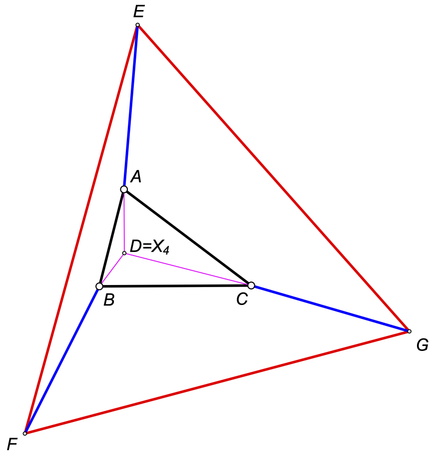


Figure 41. Central triangle using X_{146} points; blue lengths equal

Proof. Let R be the circumradius of $\triangle ABC$. Since E lies on the anticomplementary circle of $\triangle DBC$, $AE = 2R$ by Theorem 7.4. Similarly, $BF = 2R$ and $CG = 2R$. Thus $AE = BF = CG$ and the triangles are equilogic. □

Theorem 7.6. *Let D be the orthocenter (X_4) of $\triangle ABC$. Let E, F, and G be the X_{684}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG have the same area (Figure 42).*

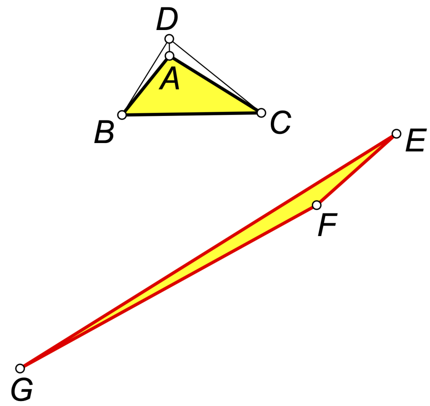


Figure 42. Central triangle using X_{684} points; yellow areas equal

Theorem 7.7. *Let D be the orthocenter of $\triangle ABC$. Let E, F, and G be the circumcenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are congruent Figure 43. They are also homothetic with N, the nine-point center of $\triangle ABC$ as the perspector. The six points A, B, C, E, F, and G lie on a conic with center N. The conic is not necessarily an ellipse.*

Open Question 5. *Is there a geometric proof for Theorem 7.7?*

Theorem 7.8. *Let D be the orthocenter of $\triangle ABC$. Let E, F, and G be the orthocenters of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG coincide Figure 44.*

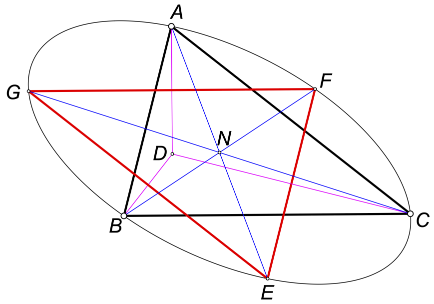


FIGURE 43. Central triangle using X_3 points

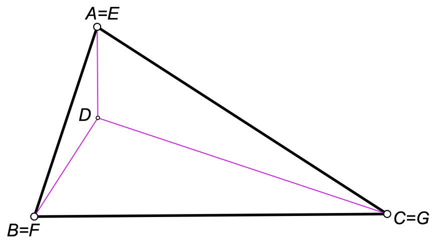


FIGURE 44. Central triangle using X_4 points

Proof. Since D is the orthocenter of $\triangle ABC$, the points A, B, C, and D form an orthocentric system, where each point is the orthocenter of the triangle formed by the other three points [37]. Thus, E coincides with A, F coincides with B, and G coincides with C. □

Theorem 7.9. *Let D be the orthocenter of $\triangle ABC$. Let E, F, and G be the X_{930}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are congruent (Figure 45). The triangles are also homothetic with perspector X_{128}. The triangles are also orthologic with $X_4 = \mathrm{ortho}[ABC, EFG]$ and $X_{930} = \mathrm{ortho}[EFG, ABC]$. The triangles are Eulerologic with $X_4 = \mathrm{euler}[ABC, EFG]$ and $X_{930} = \mathrm{euler}[EFG, ABC]$. The six points A, B, C, E, F, and G lie on a conic with center X_{128}.*

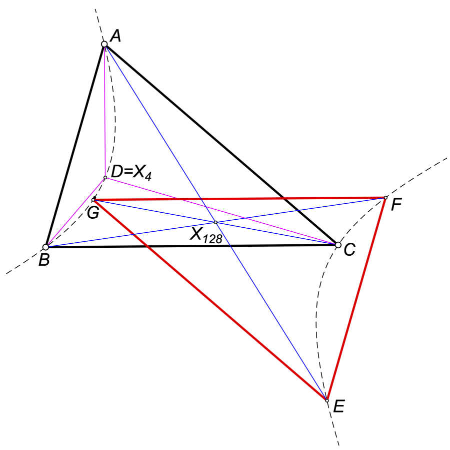


FIGURE 45. Central triangle using X_{930} points

Theorem 7.10. *Let D be the orthocenter of $\triangle ABC$. Let E, F, and G be the centroids of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are homothetic with perspector X_5 (Figure 46).*

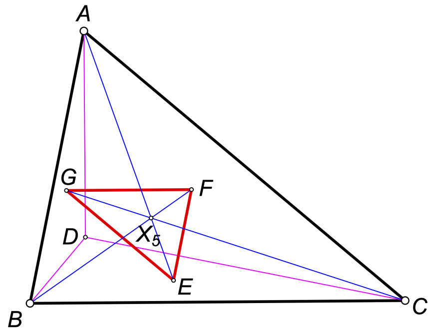


FIGURE 46. Central triangle using X_2 points

Theorem 7.11. *Let D be the orthocenter of $\triangle ABC$. Let E, F, and G be the symmedian points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles ABC and EFG are perspective with perspector X_{53} (Figure 47).*

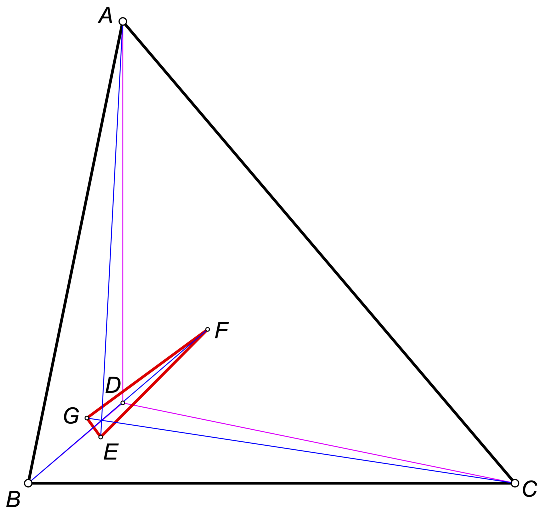


FIGURE 47. Central triangle using X_6 points

Theorem 7.12. *Let D be the orthocenter of $\triangle ABC$. Let $X \in \mathbb{S}$ with Shinagawa constants m and n. Let E, F, and G be the X-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG are homothetic with perspector X_5. The homothetic ratio is $n - m : 3m + n$.*

Theorem 7.13. *Let D be the orthocenter of $\triangle ABC$. Let $X \in \mathbb{S}$. Let E, F, and G be the X-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG have the same nine-point center.*

Proof. This is a consequence of Theorem 7.12. $\square$

Theorem 7.14. *Let D be the orthocenter of $\triangle ABC$. Let $X \in \mathbb{S}$. Let E, F, and G be the X-points of triangles BCD, CAD, and ABD, respectively. Then triangles ABC and EFG have the same Euler line.*

Proof. Since the triangles are homothetic (Theorem 7.12), their Euler lines are parallel. Since the triangles have the same nine-point center (Theorem 7.13), these Euler lines must coincide. □

8. RESULTS WHEN $D = X_5$

Our computer study found some additional results when D is the X_5 point of $\triangle ABC$. It is shown in the following table.

Central Triangles of $\triangle ABC$, $D = X_5$	
Relationship	**centers**
$X_{1487} = \text{persp}[ABC, EFG]$	5
$X_4 = \text{persp}[ABC, EFG]$	382

Theorem 8.1. *Let D be the nine-point center (X_5) of $\triangle ABC$. Let E, F, and G be the nine-point centers of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are perspective with perspector X_{1487} (Figure 48).*

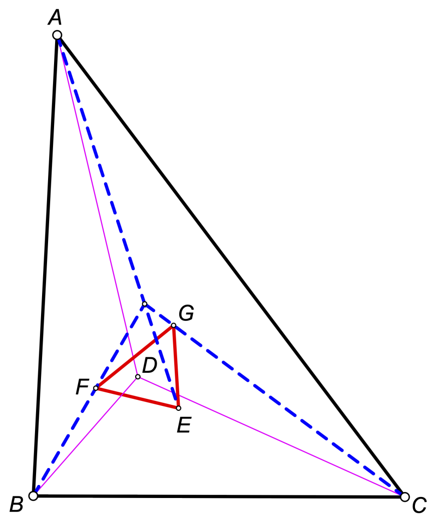


FIGURE 48. Central triangle using X_5 points

9. RESULTS WHEN $D = X_6$

Our computer study found one additional result when D is the X_6 point of $\triangle ABC$. It is shown in the following table.

Central Triangles of $\triangle ABC$, $D = X_6$	
Relationship	**centers**
$X_{251} = \text{persp}[ABC, EFG]$	6

Theorem 9.1. *Let D be the symmedian point (X_6) of $\triangle ABC$. Let E, F, and G be the symmedian points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are perspective with perspector X_{251} (Figure 49).*

Proof. The X_{251}-point is the Kosnita(X_6,X_6) point of $\triangle ABC$ [12], thus showing that triangles ABC and EFG are perspective. □

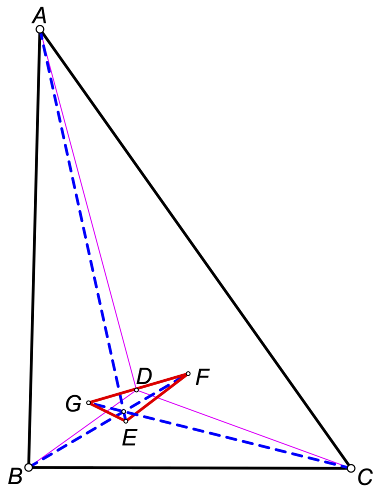


FIGURE 49. Central triangle using X_6 points

10. RESULTS WHEN $D = X_7$ THROUGH X_{11}

No new results were found when D is the X_7, X_8, X_9, X_{10}, or X_{11} point of $\triangle ABC$.

11. AREAS FOR FUTURE RESEARCH

There are many avenues for future investigation.

11.1. **Investigate other triangle centers.**

In our study, we only investigated triangle placing centers X_n for $n \leq 11$ in the side triangles. Extend this study to larger values of n.

11.2. **Ask about uniqueness.**

Find an entry in one of our tables where there is only one center giving a particular relationship. For example, for an arbitrary D, $[ABC] = [EFG]$ seems to be true only when $n = 4$. Is this because we only searched the first 1000 values of n? Expand the search and find other values of n for which the relationship is true or prove that the result is unique. For example, we can state the following.

Conjecture 1. *Let D be a point in the plane of $\triangle ABC$ (not on the boundary). Let X denote a triangle center. Let E, F, and G be the X-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then the areas of $\triangle ABC$ and $\triangle EFG$ are equal if and only if $X = X_4$.*

11.3. **Use notable points that are not triangle centers.**

There are other points associated with a triangle that are not triangle centers. Look for properties when some of these points are used.

11.4. **Place different centers in different side triangles.**

Investigate interesting results if we place different triangle centers in the three side triangles. For example, the following result was found by computer.

Theorem 11.1. *Let D be the circumcenter of $\triangle ABC$. Let E be the $X(617)$-point of $\triangle BCD$. Let F be the $X(617)$-point of $\triangle CAD$. Let G be the $X(616)$-point of $\triangle ABD$. Then $[ABC] = 3[EFG]$ (Figure 50).*

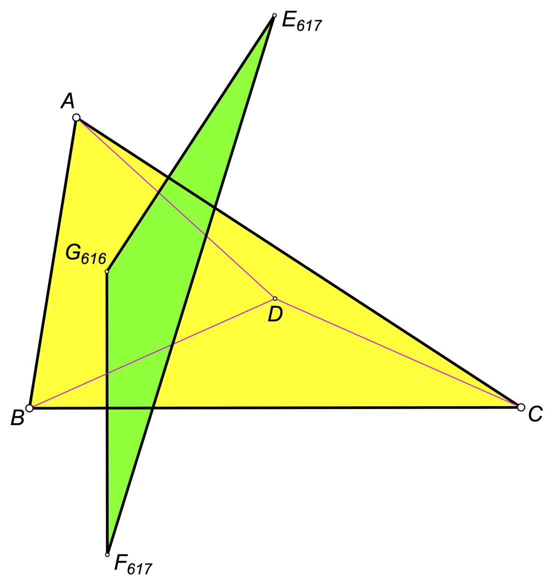


FIGURE 50. Central triangle using X_{616} and X_{617} points

11.5. **Determine properties of points of concurrence.**

Many of the results found conclude that three lines are concurrent. Investigate how the point of concurrence relates to $\triangle ABC$. For example, the following result was found by computer.

Theorem 11.2. *Let D be an arbitrary point in the plane of $\triangle ABC$ (not on the boundary). Let E, F, and G be the X_{13}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then triangles EFG and ABC are orthologic with the orthologic center being the X_{14}-point of $\triangle ABC$.*

The theorem remains true if the points X_{13} and X_{14} are interchanged.

Similarly, when the `Conic` relationship is found, Investigate how the center of the conic relates to $\triangle ABC$. For example, the following result was found by computer.

Theorem 11.3. *Let D be the Gergonne point (X_7) of $\triangle ABC$. Let E, F, and G be the orthocenters (X_4-points) of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then the six points A, B, C, E, F, and G lie on a conic and the center of the conic (O in Figure 51) is the Feuerbach point (X_{11}) of $\triangle ABC$.*

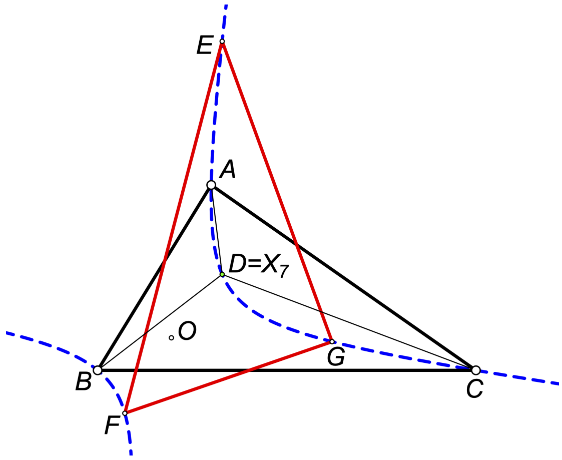


Figure 51. Central triangle using X_4 points

11.6. **Look for interesting properties of equilateral triangles.**

Find additional properties when $\triangle ABC$ is equilateral. For example, the following result was found by computer.

Theorem 11.4. *Let D be an arbitrary point in the plane of equilateral triangle ABC (not on the boundary). Let E, F, and G be the X_{18}-points of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then the points D, E, F, and G lie on a straight line (Figure 52).*

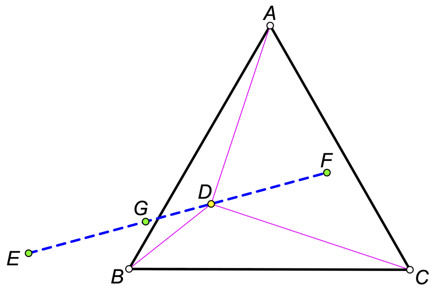


Figure 52. Central "triangle" using X_{18} points

11.7. **Investigate acute triangles.**

In our study, ABC could be any triangle. Do we get any new results if we restrict $\triangle ABC$ to be acute?

11.8. **Investigate points inside $\triangle ABC$.**

In our study, point D could be any point in the plane. Do we get any new results if we restrict D to be inside $\triangle ABC$ (other than the ones listed in Theorem 3.22)?

11.9. Investigate points outside $\triangle ABC$.

In our study, point D could be any point in the plane. Do we get any new results if we restrict D to be outside $\triangle ABC$? For example, the following result seems to be true.

Conjecture 2. *Let D be any point outside $\triangle ABC$ such that D and A are on opposite sides of BC. Let E be the X_{14}-point of $\triangle BCD$. Let F be the X_{13}-point of $\triangle CAD$. Let G be the X_{13}-point of $\triangle ABD$. Then triangles ABC and EFG are cyclologic and $X_{13} = \text{cyclo}[EFG, ABC]$.*

11.10. Look for different triangle centers.

In our study, we looked for relationships like the reference triangle and the central triangle have the same orthocenter, incenter, symmedian point, centroid, or nine-point center. Check for other centers such as the Nagel points or the Feuerbach points.

Extend the study to look for examples where one type of center of the reference triangle coincides with a different type of center of the central triangle. For example, Dung [4] found the following result.

Theorem 11.5. *Let D be the incenter of $\triangle ABC$. Let E, F, and G be the nine-point centers of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then the circumcenter of $\triangle EFG$ coincides with the nine-point center of $\triangle ABC$.*

11.11. Look for other properties of the central triangle.

Instead of comparing the central triangle with the reference triangle, look for interesting properties of the central triangle. For example, Suppa [29] found the following result.

Theorem 11.6. *Let D be the incenter of $\triangle ABC$. Let E, F, and G be the nine-point centers of $\triangle BCD$, $\triangle CAD$, and $\triangle ABD$, respectively. Then the circumcircle of $\triangle EFG$ passes through the Spieker center of $\triangle ABC$.*

11.12. Look for missing properties.

There were some properties that we checked for, but never found any pairs of triangles that satisfied these properties. For example, we never found a case where triangles ABC and EFG had the same perimeter but were not congruent. Does such a case ever occur or can it be proved that no such case is possible? Look for such cases. Also look for cases where triangles ABC and EFG do not coincide but have a common incircle. Look for cases where triangles ABC and EFG do not coincide but have a common inconic. Look for cases where triangles ABC and EFG have a common circumconic that is a rectangular hyperbola.

11.13. Find degenerate triangles.

Find instances where $\triangle EFG$ is degenerate, i.e. points E, F, and G coincide or are collinear. For example, when $D = X_1$, $\triangle EFG$ is degenerate when using center X_n for n =54, 95, 96, 97, 252. How does the common point relate to $\triangle ABC$?

11.14. **Check for bilogic triangles.**

If $D = X_1, X_2, X_5, X_6, X_7, X_8, X_9, X_{10}, X_{11}$, is it possible for triangles ABC and EFG to be bilogic? Our study found no such examples.

11.15. **Use other points for E, F, and G.**

Instead of choosing E to be a triangle center of $\triangle BCD$, find other ways to choose E. For example, pick E so that quadrilateral $BDCE$ is a parallelogram, kite, or a bicentric quadrilateral.

For example, the following result is easy to prove.

Theorem 11.7. *Let D be a point inside $\triangle ABC$. Let E be the point so that $BDCE$ is a convex bicentric quadrilateral. Define F and G correspondingly. Then triangles ABC and EFG are cyclologic (Figure 53).*

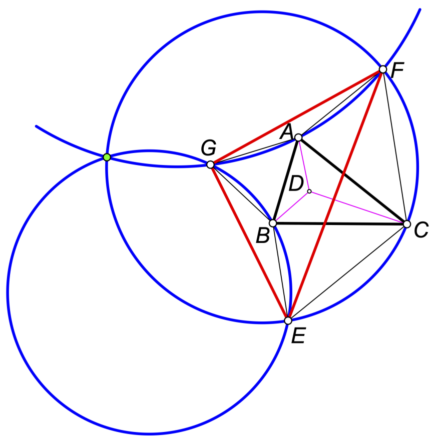


Figure 53. Central triangle using bicentric quadrilaterals

11.16. **How are these central triangles related to known triangles?**

Investigate various central triangles to see if they are named triangles in [10] or related to known triangles in some way. For example, if $D = X_1$ and E, F, G are the orthocenters of the side triangles, then central triangle EFG turns out to be the Garcia-reflection triangle.

11.17. **Work in 3-space.**

If point D is moved off the plane of $\triangle ABC$, then a tetrahedron is formed and the side triangles become faces of the tetrahedron. Investigate how the central triangle is related to the reference triangle.

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