International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 7, 2022, pp. xx–yy
Received xx February 2022. Published on-line xx mmm 2022
web: http://www.journal-1.eu/

Inequalities involving Central Cevians

Stanley Rabinowitz
545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com/

Abstract. A cevian of a triangle is a line segment that extends from a vertex of the triangle to a point on the opposite side. A cevian that passes through a triangle center is called a central cevian. There are a number of inequalities known concerning central cevians. For example, if m_a, m_b, and m_c are the lengths of the medians of a triangle, then it is known that

$$27r^2 \leq m_a^2 + m_b^2 + m_c^2 \leq \frac{27}{4}R^2$$

where r is the inradius of the triangle and R is its circumradius. We use a computer to discover and prove similar inequalities for other central cevians. For example, if f_a, f_b, and f_c are the lengths of the Feuerbach cevians of a triangle, then

$$\frac{7}{8}s^2 \leq f_a^2 + f_b^2 + f_c^2 \leq \frac{64}{7}R^2$$

were s is the semiperimeter of the triangle.

Keywords. triangle centers, inequalities, computer-discovered mathematics, cevians, Mathematica.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

There are many notable points associated with a triangle, such as the incenter, centroid, circumcenter, and orthocenter, These are special cases of *triangle centers* as defined by Clark Kimberling in [3]. A *cevian* of a triangle is a line segment that extends from a vertex of the triangle to a point on the opposite side. A cevian that passes through a triangle center is called a *central cevian.* The cevian from vertex A is called the *A-cevian.* The other cevians are named similarly.

Let X_n denote the nth named triangle center as cataloged in the Encyclopedia of Triangle Centers [4]. Let $|PQ|$ denote the length of the line segment PQ.

The cevians through X_n will be named AA_n, BB_n, and CC_n as shown in Figure 1.

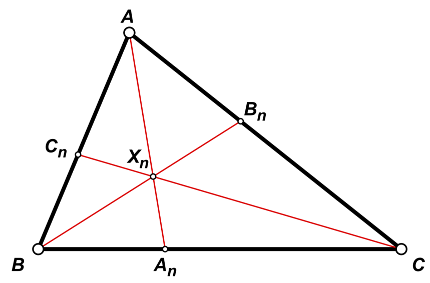

FIGURE 1. Cevians through X_n

2. THE DATA

We use barycentric coordinates in this study. The barycentric coordinates for triangle centers X_1 through X_{12} in terms of the sides of the triangle, a, b, and c, are shown in Table 1. Only the first barycentric coordinate is given, because if $f(a,b,c)$ is the first barycentric coordinate for a point P, then the barycentric coordinates for P are

$$\Big(f(a,b,c) : f(b,c,a) : f(c,a,b)\Big).$$

These were derived from [4].

TABLE 1. Barycentric coordinates for the first 12 centers

n	First barycentric coordinate for X_n
1	a
2	1
3	$a^2(a^2-b^2-c^2)$
4	$(a^2+b^2-c^2)(a^2-b^2+c^2)$
5	$c^4-a^2b^2+b^4-a^2c^2-2b^2c^2$
6	a^2
7	$(a+b-c)(a-b+c)$
8	$a-b-c$
9	$a(a-b-c)$
10	$b+c$
11	$(b-c)^2(-a+b+c)$
12	$(a+b-c)(a-b+c)(b+c)^2$

We will find inequalities that involve the squares of the lengths of central cevians and other elements of a triangle, as listed in Table 2.

TABLE 2. Elements of a triangle

symbol	**Description**
a,b,c	the sides of the triangle
K	the area of the triangle
r	the inradius of the triangle
R	the circumradius of the triangle
s	the semiperimeter of the triangle

To find the distance between two points, we used the following formula which comes from [2].

Proposition 1 (Distance Formula). *Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates. Denote $x = u_1 - u_2$, $y = v_1 - v_2$ and $z = w_1 - w_2$. Then the distance between P and Q is*

$$\sqrt{-a^2yz - b^zx - c^2xy}.$$

To find the length of a cevian of a triangle, we proceed as follows. Set up a barycentric coordinate system with $\triangle ABC$ as the reference triangle, so that $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, and $C = (0 : 0 : 1)$. Let P be an arbitrary point in the plane other than A. Let the barycentric coordinates for P be $(p : q : r)$. Let AP meet BC at A' (Figure 2). Then it is straightforward to show that the barycentric coordinates for A' are $(0 : q : r)$.

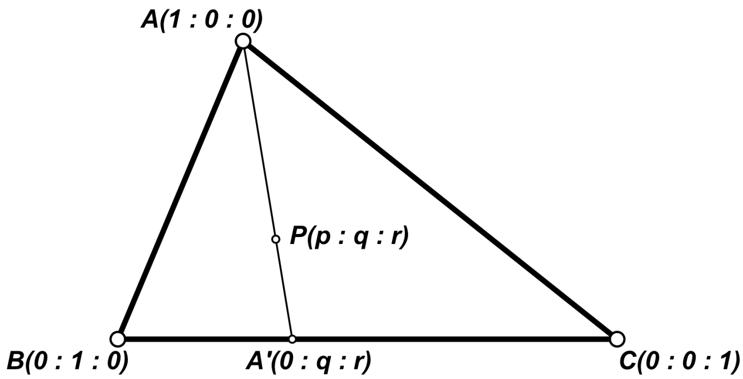

Figure 2. Barycentric Coordinates

Using Proposition 1, we get the following result.

Proposition 2 (Cevian Length). *Let P be a point in the plane of $\triangle ABC$ with trilinear coordinates $(p : q : r)$. Let AP meet BC at A'. Then*

$$|AA'| = \frac{\sqrt{b^2r(q+r) + c^2q(q+r) - a^2qr}}{q+r}.$$

Using Proposition 2 and Table 1, we can find the length of the A-cevian that passes through the point X_n. Table 3 shows the lengths for n ranging from 1 to 12, where $K = \sqrt{s(s-a)(s-b)(s-c)}$ and $s = (a+b+c)/2$.

TABLE 3. Cevian lengths for the first 12 centers

n	Square of length of A-cevian passing through X_n
1	$bc\left(1-\frac{a^2}{(b+c)^2}\right)$
2	$\frac{1}{4}\left(2\left(b^2+c^2\right)-a^2\right)$
3	$-\frac{a^2b^2c^2\left(a^4-2a^2\left(b^2+c^2\right)+\left(b^2-c^2\right)^2\right)}{\left(\left(b^2-c^2\right)^2-a^2\left(b^2+c^2\right)\right)^2}$
4	$\frac{4K^2}{a^2}$
5	$\frac{16K^2\left(a^6-3a^4b^2-3a^4c^2+3a^2b^4+3a^2b^2c^2+3a^2c^4-b^6+b^4c^2+b^2c^4-c^6\right)}{\left(2a^4-3a^2b^2-3a^2c^2+b^4-2b^2c^2+c^4\right)^2}$
6	$\frac{b^2c^2\left(2\left(b^2+c^2\right)-a^2\right)}{\left(b^2+c^2\right)^2}$
7	$-\frac{a^3+a\left(-3b^2+2bc-3c^2\right)+2(b-c)^2(b+c)}{4a}$
8	$\frac{-a^3+a\left(3b^2-2bc+3c^2\right)+2(b-c)^2(b+c)}{4a}$
9	$-\frac{bc\left(a^4-2a^2\left(b^2+c^2\right)+(b-c)^4\right)}{\left((b-c)^2-a(b+c)\right)^2}$
10	$\frac{a^2(-(a+b))(a+c)+b^2(a+b)(2a+b+c)+c^2(a+c)(2a+b+c)}{(2a+b+c)^2}$
11	$\frac{(b^2x+c^2y)(x+y)-a^2xy}{(x+y)^2}$ where $x=(a-b)^2(a+b-c)$, $y=(a-c)^2(a-b+c)$
12	$\frac{xy\left(-a^2+b^2+c^2\right)+b^2y^2+c^2x^2}{(x+y)^2}$ where $x=(a+c)^2(a+b-c)$, $y=(a+b)^2(a-b+c)$

3. Main Results

Notation. The symbol S_n represents the sum of the squares of the lengths of the cevians of $\triangle ABC$ that pass through triangle center X_n. In other words,

$$S_n = |AA_n|^2 + |BB_n|^2 + |CC_n|^2.$$

For example, if $n = 2$, then the cevians are medians and $S_2 = m_a^2 + m_b^2 + m_c^2$.

Conventions. In this section, all inequalities listed are best possible.

The inequality $S_n \le k_0 f(a, b, c)$ is said to be *best possible* if there is no constant k with $k < k_0$ such that $S_n \le kf(a, b, c)$ is true for all triangles.

The inequality $k_0 f(a, b, c) \le S_n$ is said to be *best possible* if there is no constant k with $k > k_0$ such that $kf(a, b, c) \le S_n$ is true for all triangles.

If no upper bound is listed for S_n with respect to $f(a, b, c)$, this means that there is no constant k such that $S_n \le kf(a, b, c)$ is true for all triangles.

If no lower bound is listed for S_n with respect to $f(a, b, c)$, this means that there is no constant $k > 0$ such that $kf(a, b, c) \le S_n$ is true for all triangles.

Methodology. The best constants for all inequalities were found using Mathematica and Algorithm K from [8]. Since all computations were performed using exact symbolic algebra (as opposed to numerical approximations), these computer calculations constitute proofs that the inequalities are correct.

Theorem 1. *The following inequalities are true for all triangles.*

$$27r^2 \le S_1 \le \frac{27}{4}R^2$$
$$27r^2 \le S_2 \le \frac{27}{4}R^2$$
$$27r^2 \le S_3$$
$$27r^2 \le S_4 \le \frac{27}{4}R^2$$
$$27r^2 \le S_5 \le \frac{27}{4}R^2$$
$$27r^2 \le S_6 \le \frac{27}{4}R^2$$
$$27r^2 \le S_7 \le \frac{27}{4}R^2$$
$$27r^2 \le S_8 < 12R^2$$
$$27r^2 \le S_9 < \frac{68}{9}R^2$$
$$27r^2 \le S_{10} < \frac{68}{9}R^2$$
$$k_1 r^2 \le S_{11} \le \frac{64}{7}R^2$$
$$27r^2 \le S_{12} \le \frac{27}{4}R^2$$

where $k_1 \approx 30.91612615$ is the positive root of $x^3 - 32x^2 + 48x - 448$.

Equality occurs when the triangle is equilateral, except in the following cases.
For $27r^2 \leq S_3$, $S_5 \leq \frac{27}{4}R^2$, and $27r^2 \leq S_5$, equality occurs when the sides of the triangle are proportional to 1, 1, and $\sqrt{3}$.
For $S_{11} \leq \frac{64}{7}R^2$, equality occurs when the sides of the triangle are proportional to 1, 1, and $2\sqrt{\frac{3}{7}}$.
For $k_1 r^2 \leq S_{11}$, equality occurs when the sides of the triangle are proportional to 1, 1, and the positive root of $7x^3 + 2x^2 + 4x - 8$.

Lemma 1. *Let A' be a point in the interior of side BC of $\triangle ABC$. Let $|AB| = c$, $|AC| = b$ and $|AA'| = x_a$ (Figure 3). Then*

$$h_a \leq x_a < \max(b, c)$$

where h_a is the length of the altitude from A.

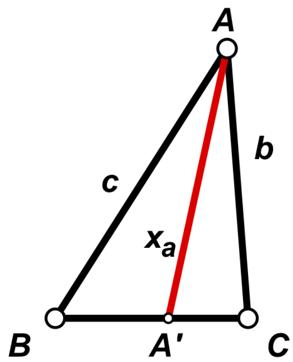

FIGURE 3. Cevian from A

Proof. Let H be the foot of the altitude from A (Figure 4). By the Pythagorean Theorem, it can be seen that the closer A' gets to H, the smaller x_a gets. The minimum value of x_a is h_a and the maximum value for x_a is the larger of b and c. □

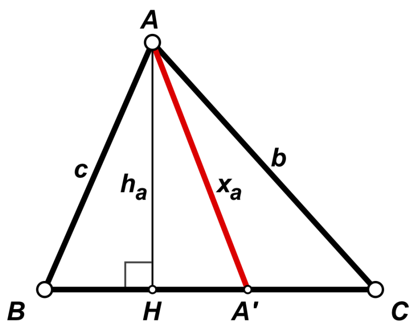

FIGURE 4. Cevian from A

Proposition 3. *Let P be a point inside $\triangle ABC$. Let x be the sum of the squares of the lengths of the cevians through P. Then*

$$27r^2 \leq x < 12R^2.$$

Proof. We will prove a more general result. Let x_a be the length of any interior cevian from vertex A of $\triangle ABC$. (An *interior cevian* meets the opposite side at an interior point of that side.) Define x_b and x_c similarlly. Note that the three cevians need not all pass through the same point P. Then we will show that

$$27r^2 \leq x_a^2 + x_b^2 + x_c^2 < 12R^2. \tag{1}$$

By Lemma 1, $x_a \geq h_a$. Similarly for x_b and x_c. Thus,

$$x_a^2 + x_b^2 + x_c^2 \geq h_a^2 + h_b^2 + h_c^2.$$

But

$$h_a^2 + h_b^2 + h_c^2 \geq 27r^2$$

from inequality $27r^2 \leq S_4$ of Theorem 1. This proves the left side of Equation (1). Without loss of generality, we can assume that $a \leq b \leq c$. By Lemma 1, we have $x_a < c$, $x_b < c$, and $x_c < b$. Thus

$$x_a^2 + x_b^2 + x_c^2 < b^2 + 2c^2. \tag{2}$$

The right side of Equation (1) will then be true if we can prove that $b^2+2c^2 < 12R^2$. This inequality is not homogeneous, so we cannot use the methods of [8]. Instead, we use the Simplify command in Mathematica. The formula for R in terms of a, b, and c is well known, namely

$$R = \frac{abc}{4K}$$

where K is the area of $\triangle ABC$. We thus issue the following Mathematica commands.

```
s = (a+b+c)/2;
K = Sqrt[s(s-a)(s-b)(s-c)];
R = a*b*c/(4K);
inequality = b^2+2c^2 < 12R^2;
triangCondition = a>0 && b>0 && c>0 && a+b>c && b+c>a && c+a>b;
Simplify[inequality, triangCondition]
```

Mathematica responds with True, indicating that the inequality is correct. Note that we did not need the condition $a \leq b \leq c$. This concludes the proof of the right side of Equation (1). □

The constants in Proposition 3 are best possible as can be seen by the inequality for S_8 in Theorem 1.

Continuing with Algorithm K, we get the following results.

Theorem 2. *The following inequalities are true for all triangles.*

$$\frac{8}{9}s^2 < S_1 \le s^2$$
$$s^2 \le S_2 < \frac{3}{2}s^2$$
$$S_4 \le s^2$$
$$S_5 \le \frac{33}{25}s^2$$
$$\frac{18}{25}s^2 < S_6 \le s^2$$
$$\frac{1}{2}s^2 < S_7 \le s^2$$
$$s^2 \le S_8 < 3s^2$$
$$s^2 \le S_9 < 2s^2$$
$$s^2 \le S_{10} < \frac{17}{9}s^2$$
$$\frac{7}{8}s^2 \le S_{11} \le 2s^2$$
$$\frac{1}{2}s^2 < S_{12} \le s^2$$

Equality occurs when the triangle is equilateral, except in the following cases.
For $S_5 \le \frac{33}{25}s^2$, equality occurs when the sides of the triangle are proportional to 1, 1, and $\sqrt{3}$.
For $\frac{7}{8}s^2 \le S_{11}$, equality occurs when the sides of the triangle are proportional to 1, 1, and $\frac{2}{7}$.

Proposition 4. *Let P be a point inside $\triangle ABC$. Let x be the sum of the squares of the lengths of the cevians through P. Then*

$$x < 3s^2.$$

Proof. This inequality follows from Equation (2) and the fact that the Mathematica code

```
inequality = b^2+2c^2 < 3s^2;
Simplify[inequality, triangCondition]
```

returns True. □

The constant "3" is best possible as can be seen from the inequality for S_8 in Theorem 2.

Lemma 2. *Let A' be a point in the interior of side BC of $\triangle ABC$. Let $|AB| = c$, $|AC| = b$ and $|AA'| = x_a$ (Figure 3). Then*

$$x_a \ge \min(b, c)\cos\frac{A}{2}.$$

Proof. In [5], it is shown that

$$x_a \geq (kb + k'c)\cos\frac{A}{2}$$

where $k = |BA'|/|A'C|$ and $k' = 1 - k$. The function $f(k) = kb + (1-k)c$ is a linear function of k over the interval $[0, 1]$. It takes on all values from $\min(b, c)$ to $\max(b, c)$. Therefore, when $b > 0$, $c > 0$, and $0 \leq k \leq 1$, we must have

$$kb + (1-k)c \geq \min(b, c).$$

The result now follows. □

Proposition 5. *Let P be a point inside $\triangle ABC$. Let x be the sum of the squares of the lengths of the cevians through P. Then*

$$\frac{1}{2}s^2 < x.$$

Proof. By Lemma 2, we have

$$x_a^2 \geq \min(b^2, c^2)\cos^2\frac{A}{2}.$$

From the half-angle formula for cosine,

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2},$$

and from the Law of Cosines,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

we see that

$$x_a^2 \geq \min(b^2, c^2)\frac{(b+c-a)(b+c+a)}{4bc}$$

with similar formulas for x_b and x_c.

Without loss of generality, assume that $a \leq b \leq c$. Then

$$x = x_a^2 + x_b^2 + x_c^2 \geq b^2\frac{(b+c-a)s}{2bc} + a^2\frac{(c+a-b)s}{2ca} + a^2\frac{(a+b-c)s}{2ab}. \tag{3}$$

Using this value for `x` in terms of a, b, and c, and the definitions of `s` and `triangCondition` from the proof of Proposition 3, we issue the following Mathematica commands.

```
inequality = x > (1/2)s^2;
Simplify[inequality, triangCondition]
```

The response of True proves the inequality. □

The proof shows that the result is true for any three internal cevians. They do not necessarily have to all pass through the same point P.

The constant "1/2" is best possible as can be seen from the inequality for S_7 in Theorem 2.

Related results can be found in Theorem 14.11 of [1, p. 124] and Theorem 7.22 of [6, p. 337].

Continuing with Algorithm K, we get the following results.

Theorem 3. *The following inequalities are true for all triangles.*

$$\begin{aligned}
\frac{27}{2}rR \leq S_1 &\leq \frac{3}{2}\sqrt{3}Rs \\
\frac{27}{2}rR \leq S_2 &< 3Rs \\
S_4 &\leq \frac{3}{2}\sqrt{3}Rs \\
S_5 &\leq \frac{66}{25}Rs \\
\frac{288}{25}rR < S_6 &\leq \frac{3}{2}\sqrt{3}Rs \\
8rR < S_7 &\leq \frac{3}{2}\sqrt{3}Rs \\
\frac{27}{2}rR \leq S_8 &< 6Rs \\
\frac{27}{2}rR \leq S_9 &< \frac{34}{9}Rs \\
\frac{27}{2}rR \leq S_{10} &< \frac{34}{9}Rs \\
k_2 rR \leq S_{11} &\leq k_3 Rs \\
8rR < S_{12} &\leq \frac{3}{2}\sqrt{3}Rs
\end{aligned}$$

where $k_2 \approx 14.12657721$ is the positive root of $2x^3 - 5x^2 - 256x - 1024$ and $k_3 \approx 3.737553924$ is the largest positive root of $4x^6 + 20117x^4 - 356864x^2 + 1048576$. Equality occurs when the triangle is equilateral, except for $S_5 \leq \frac{66}{25}Rs$, where equality occurs when the sides of the triangle are proportional to 1, 1, and $\sqrt{3}$.

Proposition 6. *Let P be a point inside $\triangle ABC$. Let x be the sum of the squares of the lengths of the cevians through P. Then*

$$8rR < x < 6Rs.$$

Proof. The right side of the inequality follows from Equation (2) and the fact that the Mathematica code

```
inequality = b^2+2c^2 < 6R*s;
Simplify[inequality, triangCondition]
```

returns True.

The left side of the inequality follows from the fact that the Mathematica code

```
r = K/s;
inequality = x > 8rR;
Simplify[inequality, triangCondition]
```

returns True, where `x` is given by Equation (3). □

The constants in Proposition 6 are best possible as can be seen from the inequalities for S_7 and S_8 in Theorem 3.

Continuing with Algorithm K, we get the following results.

Theorem 4. *The following inequalities are true for all triangles.*

$$3\sqrt{3}K \le S_1$$
$$3\sqrt{3}K \le S_2$$
$$3\sqrt{3}K \le S_6$$
$$3\sqrt{3}K \le S_7$$
$$3\sqrt{3}K \le S_8$$
$$3\sqrt{3}K \le S_9$$
$$3\sqrt{3}K \le S_{10}$$
$$4\sqrt{2}K \le S_{11}$$
$$3\sqrt{3}K \le S_{12}$$

Equality occurs when the triangle is equilateral, except for $4\sqrt{2}K \le S_{11}$, where equality occurs when the sides of the triangle are proportional to 1, 1, and $\frac{2}{3}$.

Proposition 7. *Let x_a be the length of an internal A-cevian in $\triangle ABC$. Define x_b and x_c similarly. (The three cevians need not concur.) Let $x = x_a^2 + x_b^2 + x_c^2$. Then*

$$x \ge k_7 K$$

where $k_7 \approx 4.319536403$ is the positive real root of $x^{26} + 279x^{24} + 26353x^{22} + 1287331x^{20} + 29550479x^{18} - 84430591x^{16} - 19873132241x^{14} - 177553607339x^{12} + 3995469783904x^{10} - 20956237447808x^8 + 24820097419264x^6 - 17358828744704x^4 + 5114979942400x^2 - 1274019840000$ and is the best possible constant.

Proof. The following Mathematica code proves this result.

```
s = (a+b+c)/2;
K = Sqrt[s(s-a)(s-b)(s-c)];
expression = x/K;
Minimize[expression, triangCondition, {a,b,c}]
```

where `x` is given by Equation (3). □

If the 3 cevians concur, then Theorem 4 would suggest that $x_a^2+x_b^2+x_c^2 \geq 3\sqrt{3}K$. However, this is not the case. Figure 5 shows an example where

$$\frac{x_a^2+x_b^2+x_c^2}{K} \approx 4.95030 < 3\sqrt{3}.$$

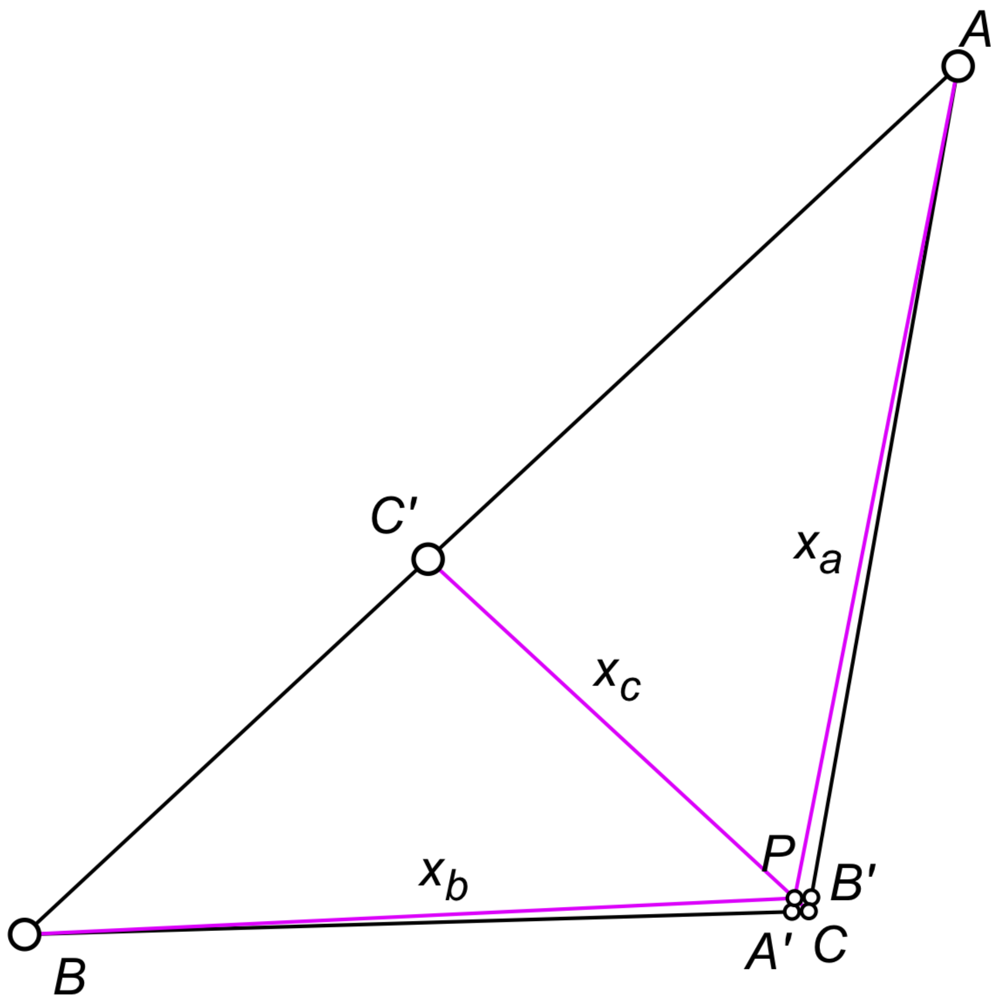

FIGURE 5. Three concurrent cevians with $(x_a^2+x_b^2+x_c^2) < 3\sqrt{3}K$.

In this figure, found using Geometer's Sketchpad, $a = 3$, $b = 105/32$, $c = 39/8$, $|B'C| \approx 2.10706$, $|AC'| \approx 2.76794$, $|BA'| \approx 2.93549$, $|CA'| \approx 0.06451$, $|AB'| \approx 3.22726$, $|CB'| \approx 0.05399$, $x_a \approx 3.29496$, $x_b \approx 3.01143$, $x_c \approx 1.98276$, and $(x_a^2+x_b^2+x_c^2)/K \approx 4.95030$.

Noting that $4.95030^2 \approx 24.5$ suggests the following conjecture.

Conjecture 1. *Let P be a point inside $\triangle ABC$. Let x be the sum of the squares of the lengths of the cevians through P. Then*

$$x \geq \frac{7}{2}\sqrt{2}K.$$

4. Related Results

In this section, the inequalities are all best possible, however, we omit inequalities involving S_n for n =3, 5, 11, or 12. The compute power available to us was insufficient to find many of the inequalities involving these central cevians. The results were found using Algorithm K.

Theorem 5. *The following inequalities are true for all triangles.*

$$\frac{1}{2}(a^2+b^2+c^2) < S_1 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$S_2 = \frac{3}{4}(a^2+b^2+c^2)$$
$$S_4 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$\frac{12}{25}(a^2+b^2+c^2) < S_6 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$\frac{1}{3}(a^2+b^2+c^2) < S_7 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$\frac{3}{4}(a^2+b^2+c^2) \le S_8 < \frac{3}{2}(a^2+b^2+c^2)$$
$$\frac{3}{4}(a^2+b^2+c^2) \le S_9 < \frac{4}{3}(a^2+b^2+c^2)$$
$$\frac{3}{4}(a^2+b^2+c^2) \le S_{10} < \frac{17}{18}(a^2+b^2+c^2)$$

Equality occurs when the triangle is equilateral.

Proposition 8. *Let P be a point inside $\triangle ABC$. Let x be the sum of the squares of the lengths of the cevians through P. Then*

$$\frac{1}{3}(a^2+b^2+c^2) < x < \frac{1}{3}(3+\sqrt{3})(a^2+b^2+c^2).$$

The proof uses Mathematica in the same way as in the proof of Proposition 6 and is omitted.

The constant "1/3" on the left side of the inequality in Proposition 8 is best possible as can be seen from the inequality for S_7 in Theorem 5. To show that the constant "$\frac{1}{3}(3+\sqrt{3})$" on the right side is best possible, we recall Equation (2), and then issue the following Mathematica command.

```
expression = (b^2+2c^2)/(a^2+b^2+c^2);
Maximize[expression, triangCondition, {a,b,c}]
```

Mathematica returns the maximum $\frac{1}{3}(3+\sqrt{3})$ and states that the maximum occurs for the degenerate triangle with sides 1, $1+\sqrt{3}$, and $2+\sqrt{3}$.

Continuing with Algorithm K, we get the following results.

Theorem 6. *The following inequalities are true for all triangles.*

$$\begin{aligned}
\frac{32}{45}(ab+bc+ca) &< S_1 < ab+bc+ca \\
\frac{3}{4}(ab+bc+ca) &\le S_2 < \frac{3}{2}(ab+bc+ca) \\
S_4 &< ab+bc+ca \\
\frac{72}{125}(ab+bc+ca) &< S_6 < ab+bc+ca \\
\frac{2}{5}(ab+bc+ca) &< S_7 < ab+bc+ca \\
\frac{3}{4}(ab+bc+ca) &\le S_8 < 3(ab+bc+ca) \\
\frac{3}{4}(ab+bc+ca) &\le S_9 < \frac{17}{9}(ab+bc+ca) \\
\frac{3}{4}(ab+bc+ca) &\le S_{10} < \frac{17}{9}(ab+bc+ca)
\end{aligned}$$

Equality occurs when the triangle is equilateral.

Proposition 9. *Let P be a point inside $\triangle ABC$. Let x be the sum of the squares of the lengths of the cevians through P. Then*

$$\frac{2}{5}(ab+bc+ca) < x < 3(ab+bc+ca).$$

The proof uses Mathematica in the same way as in the proof of Proposition 6 and is omitted.

The constants in the inequality in Proposition 9 are best possible as can be seen from the inequalities for S_7 and S_8 in Theorem 6.

Continuing with Algorithm K, we get the following results.

Theorem 7. *The following inequalities are true for all triangles.*

$$S_1 \le S_2 \qquad S_1 \le \frac{100}{81}S_6 \qquad S_1 \le \frac{16}{9}S_7 \qquad S_1 \le S_8 \qquad S_1 \le S_9 \qquad S_1 \le S_{10}$$

$$S_4 \le S_1 \qquad S_4 \le S_2 \qquad S_4 \le S_6 \qquad S_4 \le S_7 \qquad S_4 \le S_8 \qquad S_4 \le S_9 \qquad S_4 \le S_{10}$$

$$S_7 \le S_1 \qquad S_7 \le S_2 \qquad S_7 \le k_4 S_6 \qquad S_7 \le S_8 \qquad S_7 \le S_9 \qquad S_7 \le S_{10}$$

$$S_9 \le \frac{9}{4}S_1 \qquad S_9 \le \frac{16}{9}S_2 \qquad S_9 \le \frac{25}{9}S_6 \qquad S_9 \le 4S_7 \qquad S_9 \le S_8 \qquad S_9 \le \frac{25}{16}S_{10}$$

$$S_2 \le \frac{3}{2}S_1 \qquad S_2 \le \frac{25}{16}S_6 \qquad S_2 \le \frac{9}{4}S_7 \qquad S_2 \le S_8 \qquad S_2 \le S_9 \qquad S_2 \le S_{10}$$

$$S_6 \le S_1 \qquad S_6 \le S_2 \qquad S_6 \le \frac{36}{25}S_7 \qquad S_6 \le S_8 \qquad S_6 \le S_9 \qquad S_6 \le S_{10}$$

$$S_8 \le 3S_1 \qquad S_8 \le 2S_2 \qquad S_8 \le 3S_6 \qquad S_8 \le 4S_7 \qquad S_8 \le \frac{27}{17}S_9 \qquad S_8 \le \frac{27}{17}S_{10}$$

$$S_{10} \le \frac{17}{9}S_1 \qquad S_{10} \le \frac{34}{27}S_2 \qquad S_{10} \le \frac{17}{9}S_6 \qquad S_{10} \le \frac{64}{25}S_7 \qquad S_{10} \le S_8 \qquad S_{10} \le S_9$$

where $k_4 \approx 1.017624086$ is the smallest positive root of $25947x^7 + 653697x^6 - 49857885x^5 + 128952193x^4 - 112076935x^3 + 32426283x^2 - 3014327x + 2963603$.

We did not check for the conditions when equality occurs.

Corollary 8. *For all triangles,*

$$27r^2 \le S_4 \le S_6 \le S_1 \le S_2 \le S_{10} \le S_9 \le S_8 \le 12R^2.$$

Related results can be found in [7].

5. Acute Triangles

Algorithm K in [8] allows us to search for inequalities that are true for all acute triangles. We get the following results. We did not check for the conditions when equality occurs.

Theorem 9. *The inequalities given by Theorem 1 are best possible when the triangles are restricted to acute triangles. In addition, the following inequality is true for all acute triangles.*

$$S_3 \le 9R^2$$

Theorem 10. *The following inequalities are true for all acute triangles.*

$$\left(75 - 28\sqrt{7}\right)s^2 \le S_1 \le s^2$$
$$s^2 \le S_2 \le \frac{3}{2}s^2$$
$$1s^2 \le S_3 \le \frac{17}{9}s^2$$
$$(15 - 10\sqrt{2})s^2 \le S_4 \le s^2$$
$$\frac{49}{9}(3 - 2\sqrt{2})s^2 \le S_5 \le \frac{33}{25}s^2$$
$$k_5 s^2 \le S_6 \le s^2$$
$$\frac{147 - 19\sqrt{57}}{4}s^2 \le S_7 \le s^2$$
$$s^2 \le S_8 \le 3s^2$$
$$s^2 \le S_9 \le 2s^2$$
$$s^2 \le S_{10} \le \frac{17}{9}s^2$$
$$\frac{7}{8}s^2 \le S_{11} \le (27 - 18\sqrt{2})s^2$$
$$\frac{8043 - 5330\sqrt{2}}{529}s^2 \le S_{12} \le s^2$$

where $k_5 \approx 0.8742445769$ is the positive root of
$5000x^6 + 32241x^5 + 215799x^4 - 164970x^3 - 239166x^2 + 258633x - 77841$.

Constants in blue are those that differ from the corresponding constant in Theorem 2.

Theorem 11. *The following inequalities are true for all acute triangles.*

$$\frac{27}{2}rR \le S_1 \le \frac{3}{2}\sqrt{3}Rs$$
$$\frac{27}{2}rR \le S_2 \le 3Rs$$
$$\frac{27}{2}rR \le S_3 \le \frac{34}{9}Rs$$
$$(5+5\sqrt{2})rR \le S_4 \le \frac{3}{2}\sqrt{3}Rs$$
$$\frac{49+49\sqrt{2}}{9}rR \le S_5 \le \frac{66}{25}Rs$$
$$\frac{49+49\sqrt{2}}{9}rR \le S_6 \le \frac{3}{2}\sqrt{3}Rs$$
$$(3+7\sqrt{2})rR \le S_7 \le \frac{3}{2}\sqrt{3}Rs$$
$$\frac{27}{2}rR \le S_8 \le 6Rs$$
$$\frac{27}{2}rR \le S_9 \le \frac{34}{9}Rs$$
$$\frac{27}{2}rR \le S_{10} \le \frac{34}{9}Rs$$
$$k_2 rR \le S_{11} \le (9\sqrt{2}-9)Rs$$
$$\frac{3001+2905\sqrt{2}}{529}rR \le S_{12} \le \frac{3}{2}\sqrt{3}Rs$$

where $k_2 \approx 14.12657721$ is the positive root of $2x^3 - 5x^2 - 256x - 1024$.

Constants in blue are those that differ from the corresponding constant in Theorem 3.

Theorem 12. *The inequalities given by Theorem 4 are best possible when the triangles are restricted to acute triangles. In addition, the following inequality is true for all acute triangles.*

$$5K \le S_4$$

Theorem 13. *The following inequalities are true for all acute triangles.*

$$\frac{1}{2}(a^2+b^2+c^2) \le S_1 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$S_2 = \frac{3}{4}(a^2+b^2+c^2)$$
$$\frac{1}{2}(a^2+b^2+c^2) \le S_4 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$\frac{1}{2}(a^2+b^2+c^2) \le S_6 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$\frac{1}{2}(a^2+b^2+c^2) \le S_7 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$\frac{3}{4}(a^2+b^2+c^2) \le S_8 \le \frac{3}{2}(a^2+b^2+c^2)$$
$$\frac{3}{4}(a^2+b^2+c^2) \le S_9 \le \frac{17}{18}(a^2+b^2+c^2)$$
$$\frac{3}{4}(a^2+b^2+c^2) \le S_{10} \le \frac{17}{18}(a^2+b^2+c^2)$$

Constants in blue are those that differ from the corresponding constant in Theorem 5.

Theorem 14. *The following inequalities are true for all acute triangles.*

$$\frac{6\sqrt{2}-7}{2}(ab+bc+ca) \le S_1 \le ab+bc+ca$$
$$\frac{3}{4}(ab+bc+ca) \le S_2 \le \frac{3}{2}(ab+bc+ca)$$
$$\frac{10\sqrt{2}-5}{14}(ab+bc+ca) \le S_4 \le ab+bc+ca$$
$$k_6 \le S_6 \le ab+bc+ca$$
$$\frac{26\sqrt{2}-27}{14}(ab+bc+ca) \le S_7 \le ab+bc+ca$$
$$\frac{3}{4}(ab+bc+ca) \le S_8 \le 3(ab+bc+ca)$$
$$\frac{3}{4}(ab+bc+ca) \le S_9 \le \frac{17}{9}(ab+bc+ca)$$
$$\frac{3}{4}(ab+bc+ca) \le S_{10} \le \frac{17}{9}(ab+bc+ca)$$

where $k_6 \approx 0.7067084379$ is the positive root of $512000x^7+831488x^6+519424x^5-82176x^4+1093104x^3-2084400x^2+946647x-233523$.

Constants in blue are those that differ from the corresponding constant in Theorem 6.

References

[1] Oene Bottema, R. Ž. Djordjević, R. R. Janic, Dragoslav S. Mitrinović, and P. M. Vasic, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.

[2] Sava Grozdev and Deko Dekov, *Barycentric Coordinates: Formula Sheet*, International Journal of Computer Discovered Mathematics, **1**(2016)75–82.
http://www.journal-1.eu/2016-2/Grozdev-Dekov-Barycentric-Coordinates-pp.75-82.pdf

[3] Clark Kimberling, *Central Points and Central Lines in the Plane of a Triangle*, Mathematics Magazine, **67**(1994)163–187.
https://www.jstor.org/stable/2690608

[4] Clark Kimberling, *Encyclopedia of Triangle Centers.*
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[5] Martin Lukarevski and Dan Ştefan Marinescu, *An inequality for cevians and applications*, Elemente der Mathematik, **75**(2020)166–171.
http://doi.org/10.4171/EM/418

[6] Dragoslav S. Mitrinović, J. Pečarić, and Vladimir Volenec, *Recent advances in geometric inequalities.* Kluwer Academic Publishers, Dordrecht, The Netherlands, 1989.
https://books.google.com/books?id=_OvGErBOQK4C

[7] Stanley Rabinowitz. *Inequalities Involving Gergonne and Nagel Cevians*, International Journal of Computer Discovered Mathematics, **6**(2021)78–83.
http://www.journal-1.eu/2021/Stanley%20Rabinowitz.%20Inequalities%20Involving%20Gergonne%20and%20Nagel%20Cevians,%20pp.%2078-83..pdf

[8] Stanley Rabinowitz. *A Computer Algorithm for Proving Symmetric Homogeneous Triangle Inequalities*, submitted to the International Journal of Computer Discovered Mathematics in January, 2021.