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Chains of Congruent Circles in a Right Triangle

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Abstract. Consider three congruent circles in a right triangle, each tangent to the hypotenuse, with consecutive circles tangent and the end circles touching the two legs. If a fourth circle near the right-angle vertex touches both legs and the middle circle, then it is known that the fourth circle must have the same radius as the original three. We generalize this phenomenon to longer chains of congruent circles.

Keywords. tangent circles, congruent circles, circle chain, right triangle, sangaku.

Mathematics Subject Classification (2020). 51M04.

1. Introduction

In a recent paper [1], the first author considered three congruent circles in a right triangle, each tangent to its hypotenuse, with consecutive circles tangent and the end circles touching the two legs. If a fourth circle near the right-angle vertex touches both legs and the middle circle, then it too must have the same radius. The proof is short and pleasant. The arrangement is shown in Figure 1.

What happens if we start with four circles instead of three? Or five? We shall see that both examples belong to the same pattern. The surprising point is that, however long the original chain may be, *one* suitably placed tangency determines the radius of the shorter chain and forces all its corresponding tangencies.

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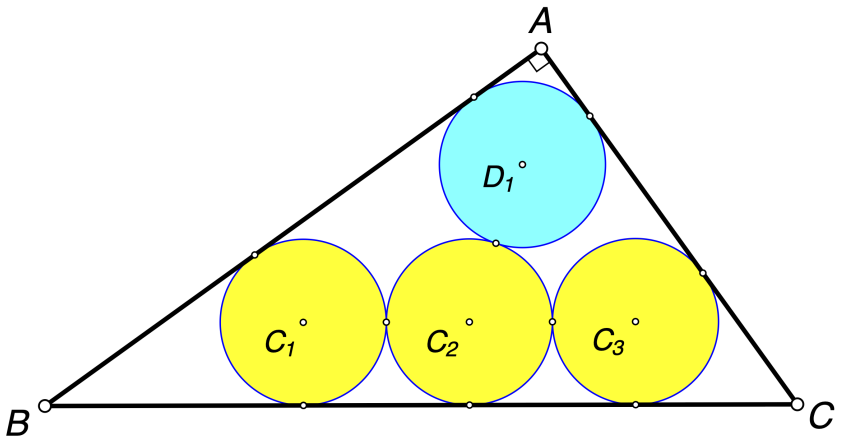


FIGURE 1. The original 3+1 configuration from [1]. All four circles are congruent.

The setting is related to the Japanese tradition of problems about tangent circles. For historical examples and earlier configurations, see Okumura [2], and the references in [1]. Our aim here is to show how the four-circle phenomenon extends to arbitrary chains.

2. CHAINS OF CIRCLES

We shall repeatedly use the same type of configuration, so it is convenient to introduce a definition.

Definition 2.1. *Let ABC be a right triangle with $\angle A = 90^\circ$ and hypotenuse BC. We say that circles $C_1, C_2, \ldots, C_n$ form a* chain of circles *in $\triangle ABC$ if the following conditions hold:*

- *all n circles have the same radius;*
- *all the circles lie inside triangle ABC;*
- *for each $i = 2, 3, \ldots, n$, the circle C_i is tangent to the preceding circle C_{i-1};*
- *the centers of $C_1, C_2, \ldots, C_n$ lie on a straight line parallel to the hypotenuse BC;*
- *C_1 is tangent to the leg AB;*
- *C_n is tangent to the leg AC.*

The common radius is called the radius *of the chain, and n is called the* length *of the chain.*

In our figures, all circles belonging to one chain have the same color, reminding us that they have the same radius.

Throughout, we name a circle by its center. Thus C_iD_j denotes the distance between the centers of circles C_i and D_j. If the two circles have radii r and s, they are externally tangent precisely when $C_iD_j = r + s$.

Figure 2 shows a chain of length 3.

3. THE 4 + 2 CONFIGURATION

We first increase the length of the original chain from three to four. Figure 3 shows the arrangement. The four circles of the longer chain touch the hypotenuse; the two circles of the shorter chain are congruent but do not have any prescribed radius.

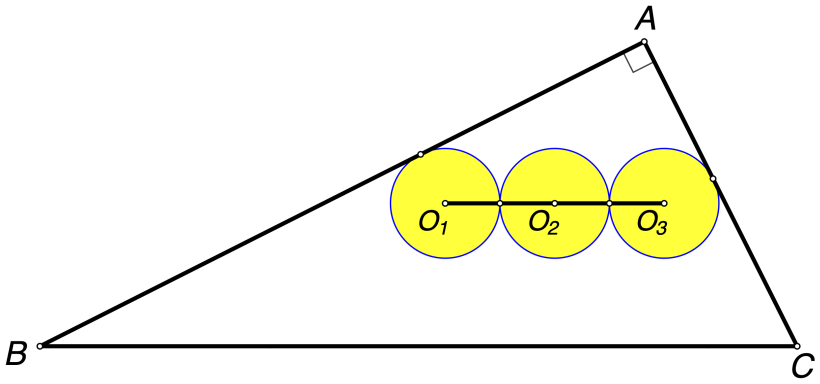


FIGURE 2. A chain of three circles in a right triangle.

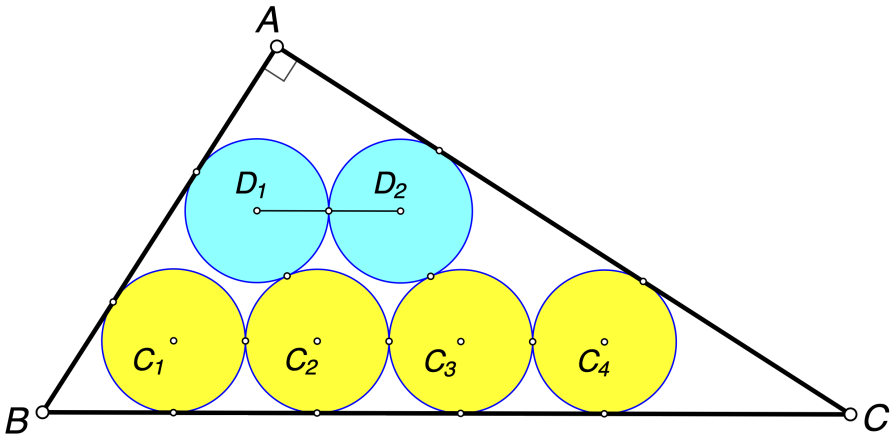


FIGURE 3. The $4+2$ configuration. If D_1 touches C_2, then all six circles are congruent and D_2 touches C_3.

Theorem 3.1. *Let C_1, C_2, C_3, C_4 be a chain of circles of radius r, all tangent to the hypotenuse of a right triangle. Let D_1, D_2 be another chain in the same triangle, of radius s. If D_1 is externally tangent to C_2, then $s = r$, and D_2 is externally tangent to C_3.*

We shall prove the equality of radii as part of a more general theorem in Section 5. For now, notice what happens once $s = r$. The segments D_1D_2 and C_2C_3 are parallel and each has length $2r$. Hence $D_1C_2C_3D_2$ is a parallelogram. Its side D_1C_2 has length $2r$ by the given tangency, so all its sides have length $2r$. Thus it is a rhombus and $D_2C_3 = 2r$, as claimed.

Figure 4 makes this rhombus particularly easy to see. It also illustrates an independent geometric observation: if the two chains are *known* to have the same radius, the two inter-chain tangencies follow without assuming either one. Indeed, let X be the intersection of the line through C_1 parallel to AB and the line through C_4 parallel to AC. The corresponding centers X, D_1, C_1 and X, D_2, C_4 are collinear. Since $D_1D_2 = 2r$ and $C_1C_4 = 6r$, similar triangles give

$$\frac{C_1D_1}{C_1X} = \frac{2}{3} = \frac{C_1C_3}{C_1C_4}.$$

Therefore $D_1C_3 \parallel AC$. Similarly, $D_2C_2 \parallel AB$. These are the perpendicular diagonals of a parallelogram, which must therefore be a rhombus.

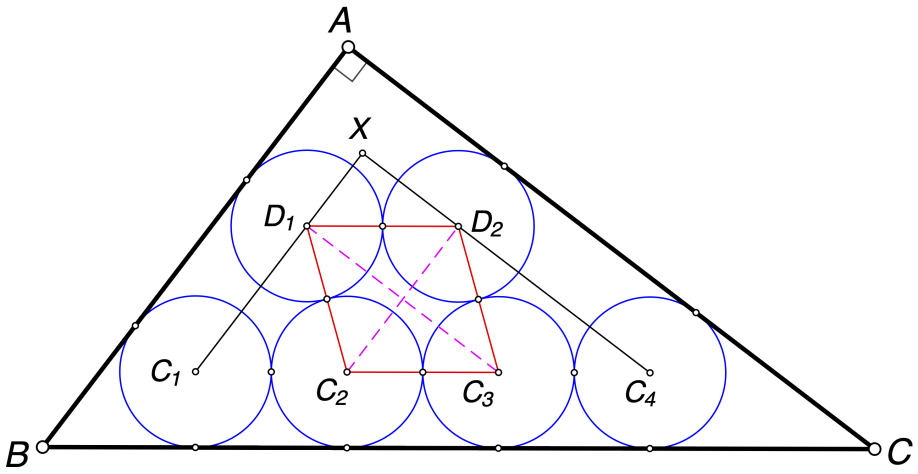


FIGURE 4. The auxiliary point X and the rhombus of centers in the $4+2$ configuration.

4. THE $5+3$ CONFIGURATION

Next, start with five circles along the hypotenuse and form a shorter chain of three circles. Again, the radius of the shorter chain is initially unknown.

Theorem 4.1. *Let C_1, C_2, C_3, C_4, C_5 be a chain of circles in a right triangle ABC, all tangent to the hypotenuse BC. Let D_1, D_2, D_3 be another chain in the same triangle. If the middle circle D_2 is externally tangent to C_3, then the radii of the two chains are equal; moreover, D_1 is tangent to C_2 and D_3 is tangent to C_4.*

The arrangement is shown in Figure 5. Like the $4+2$ result, it follows from the general theorem in the next section.

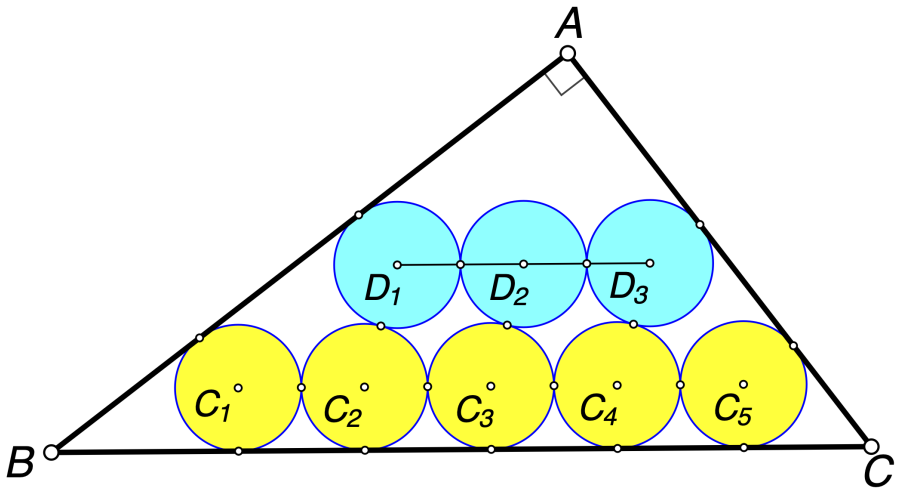


FIGURE 5. The $5+3$ configuration: one middle tangency forces two more.

There is no requirement that every pair of circles have disjoint interiors. As Figure 6 illustrates, circles from different chains may overlap. Only the tangency of D_2 with C_3 is given. The tangencies of D_1 with C_2 and D_3 with C_4 are conclusions of the theorem.

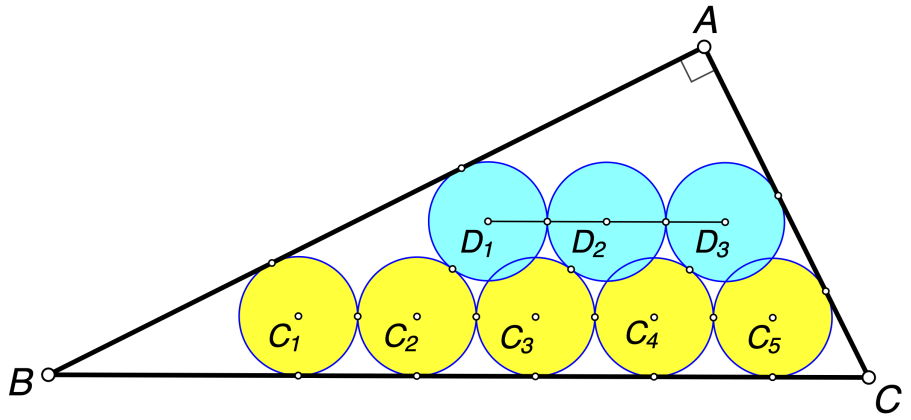


FIGURE 6. The 5+3 configuration with permitted overlaps between other pairs of circles.

5. LONGER CHAINS

The $4+2$ and $5+3$ configurations suggest a general pattern. Before proving the result for an arbitrary chain length, we give a geometric construction that explains why all the expected tangencies can occur. It will also be useful in the proof of the general theorem.

Theorem 5.1. *Let $n \geq 3$, and let $C_1, C_2, \ldots, C_n$ be a chain of n circles of radius r in a right triangle ABC. Then there exists a chain $D_1, D_2, \ldots, D_{n-2}$ of $n-2$ circles, also of radius r, whose line of centers is closer to A, such that for each i with $1 \leq i \leq n-2$, the circle D_i is externally tangent to C_{i+1}.*

Figure 7 illustrates the theorem in the case of chains of lengths 9 and 7.

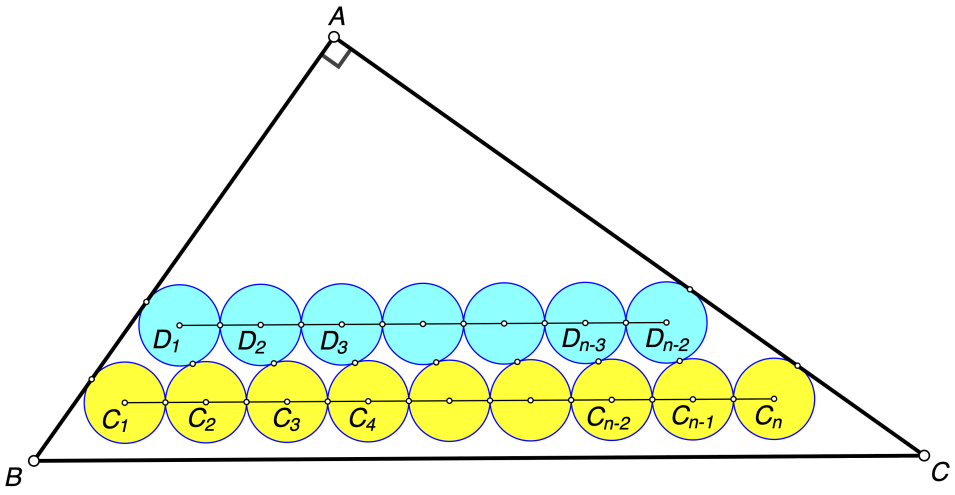


FIGURE 7. A chain of length n together with the inward chain of length $n-2$ from Theorem 5.1.

Proof. We give a direct construction.

Draw the ray C_1X parallel to BA. On this ray choose the point $D_1 \neq C_1$ such that $C_2D_1 = 2r$; equivalently, D_1 is the second intersection of the ray C_1X with the circle centered at C_2 of radius $2r$. Figure 8 shows this first step. Since circle C_1 is tangent to AB and $C_1D_1 \parallel AB$, the centers C_1 and D_1 are at the same

distance r from AB. Hence circle D_1 is also tangent to AB. Because $C_2D_1 = 2r$, it is also tangent to C_2.

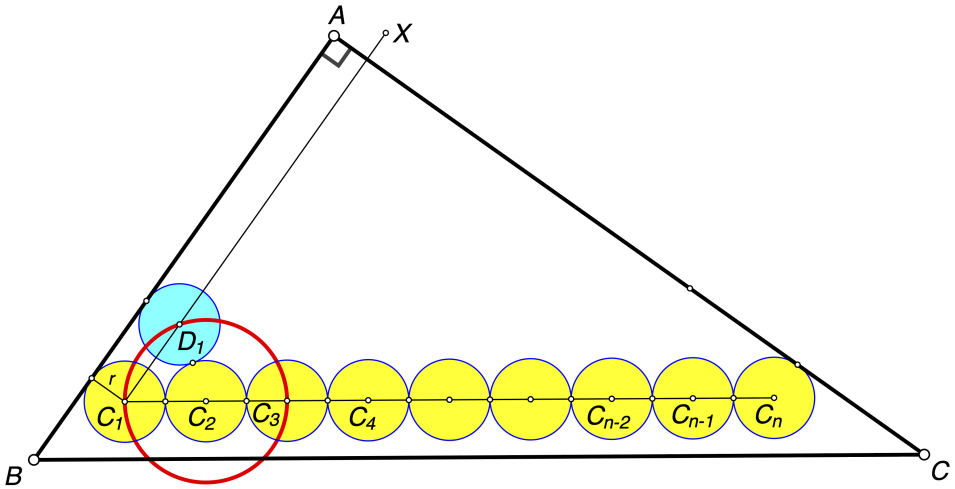


FIGURE 8. Constructing D_1: the ray C_1X is parallel to BA, and D_1 is chosen so that $C_2D_1 = 2r$.

Next translate circle D_1 parallel to C_1C_n through a distance $2r$ toward AC, obtaining a new center D_2. Then $D_1D_2 = 2r$ and $D_1D_2 \parallel C_2C_3$. Moreover, D_2 is the translate of D_1 by the same vector that carries C_2 to C_3, so the quadrilateral $D_1C_2C_3D_2$ is a parallelogram. Since $D_1C_2 = 2r = C_2C_3$, this parallelogram is in fact a rhombus. Therefore $D_2C_3 = 2r$, so circle D_2 is tangent to C_3. Figure 9 shows the first two steps of the construction.

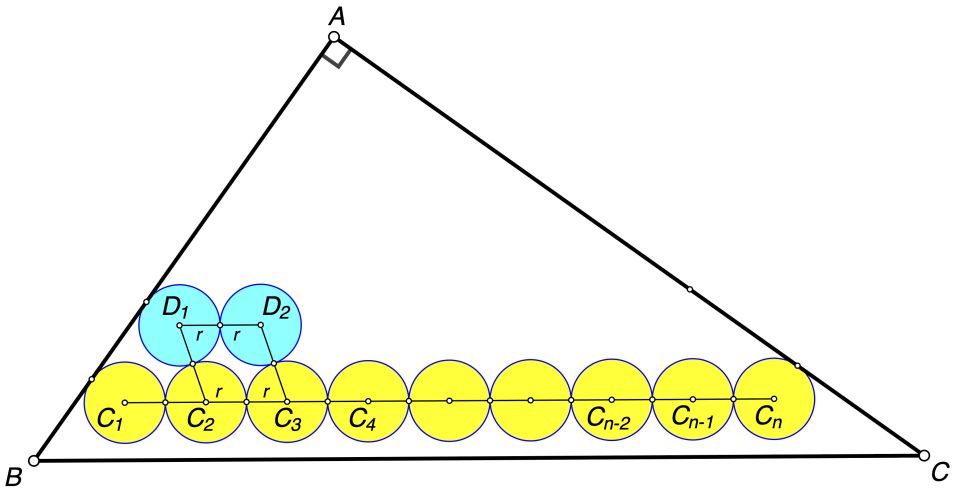


FIGURE 9. After D_1 is found, translating by one step parallel to BC gives D_2. The quadrilateral $D_1C_2C_3D_2$ is a rhombus.

Continuing in the same way, we obtain centers $D_3, D_4, \ldots, D_{n-2}$, each one obtained from the preceding center by the same translation that carries C_i to C_{i+1}. Thus the centers $D_1, D_2, \ldots, D_{n-2}$ lie on a line parallel to BC, consecutive D-circles are tangent, and each D_i is tangent to C_{i+1}. Figure 10 shows the general pattern and the geometry needed for the final tangency.

It remains to prove that the last circle D_{n-2} is tangent to AC. Here the construction circle from Figure 8 gives a particularly short argument.

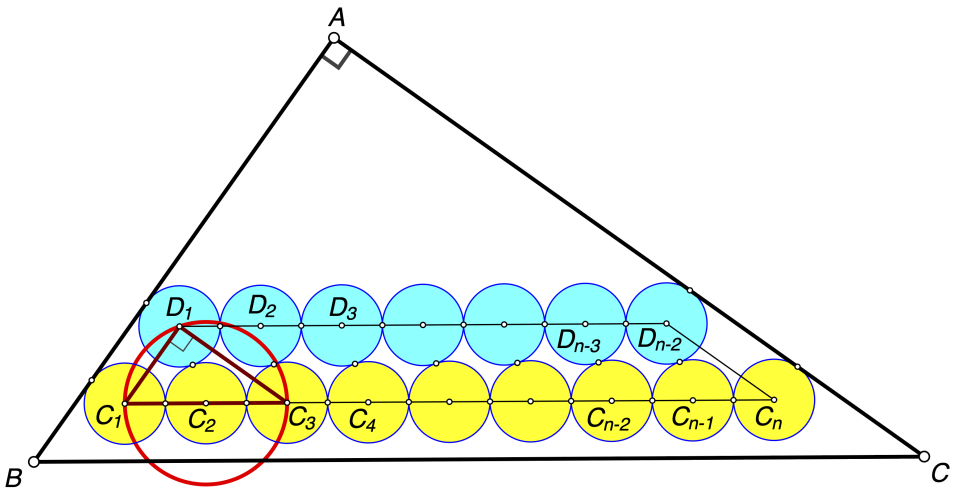


FIGURE 10. The completed construction. The red circle, with diameter C_1C_3, gives the right angle at D_1 used to prove the final tangency.

Since the three points C_1, C_2, C_3 are collinear and $C_1C_2 = C_2C_3 = 2r$, the segment C_1C_3 is a diameter of the circle centered at C_2 that we used to construct D_1. But D_1 lies on this circle. The right-angle marker in Figure 10 makes visible the conclusion of Thales' theorem (an angle in a semicircle is a right angle):

$$C_1D_1 \perp D_1C_3.$$

As $C_1D_1 \parallel AB$, we obtain

$$D_1C_3 \parallel AC.$$

Now D_{n-2} is obtained from D_1 by exactly the same succession of translations that carries C_3 to C_n. Hence

$$D_{n-2}C_n \parallel D_1C_3 \parallel AC.$$

(The assertion is immediate when $n = 3$; otherwise it follows from the parallelogram $D_1C_3C_nD_{n-2}$, as in Figure 10.) Since the circle C_n is tangent to AC and both circles have radius r, the circle D_{n-2} must also be tangent to AC.

Finally, the new circles lie inside triangle ABC. Indeed, they are obtained by a translation of the first $n-2$ circles parallel to BA, toward A, so they move away from the hypotenuse BC. Their first circle touches AB, their last circle touches AC, and the intermediate circles lie between these two legs. Their center line is therefore closer to A than the original one. This completes the construction. □

We now turn from constructing such a shorter chain to the converse question. Suppose its radius is not known in advance. How much information is needed to show that it agrees with the radius of the longer chain? The answer is: only one corresponding tangency, provided the shorter chain lies closer to A.

Theorem 5.2. *Let $n \geq 3$. Suppose $C_1, C_2, \ldots, C_n$ is a chain of circles of radius r in a right triangle ABC. Let $D_1, D_2, \ldots, D_{n-2}$ be another chain in the same triangle, of radius s, with its line of centers* closer to A *than the line of centers of the first chain. That is, the second center line lies between A and the first center line.*

If D_j is externally tangent to C_{j+1} for any one *index j, where $1 \leq j \leq n-2$, then*

$$\boxed{s = r,}$$

and all the corresponding pairs are externally tangent:

$$\boxed{D_iC_{i+1} = 2r \qquad (1 \le i \le n-2).} \tag{1}$$

Furthermore, the quadrilaterals

$$D_iC_{i+1}C_{i+2}D_{i+1} \qquad (1 \le i \le n-3)$$

are rhombi.

Proof. By Theorem 5.1, there is a chain $P_1, \ldots, P_{n-2}$ of radius r nearer to A, with $P_iC_{i+1} = 2r$ for every i. We shall show that the given tangency forces the unknown radius s to equal r.

Fix the index j in the statement of the theorem. The construction in Theorem 5.1 gives

$$C_jP_j \parallel AB, \qquad P_jC_{j+2} \parallel AC.$$

Hence $\angle C_jP_jC_{j+2} = 90°$. Let ω be the circle with diameter C_jC_{j+2}. Its center is C_{j+1} and its radius is $2r$. By Thales' theorem, P_j lies on ω.

Consider the ray AP_j. Its continuation beyond P_j lies inside the right angle $C_jP_jC_{j+2}$, so it meets the segment C_jC_{j+2} at a point T. Being between the endpoints of a diameter, T lies strictly inside ω. On the other hand, A lies outside ω, since $\angle C_jAC_{j+2} < 90°$. Thus the ray enters the circle at P_j: points between A and P_j lie outside ω, while points between P_j and T lie inside it.

Figure 11 illustrates the construction. The center C_{j+1} of ω is not marked in the figure; it is usually distinct from T.

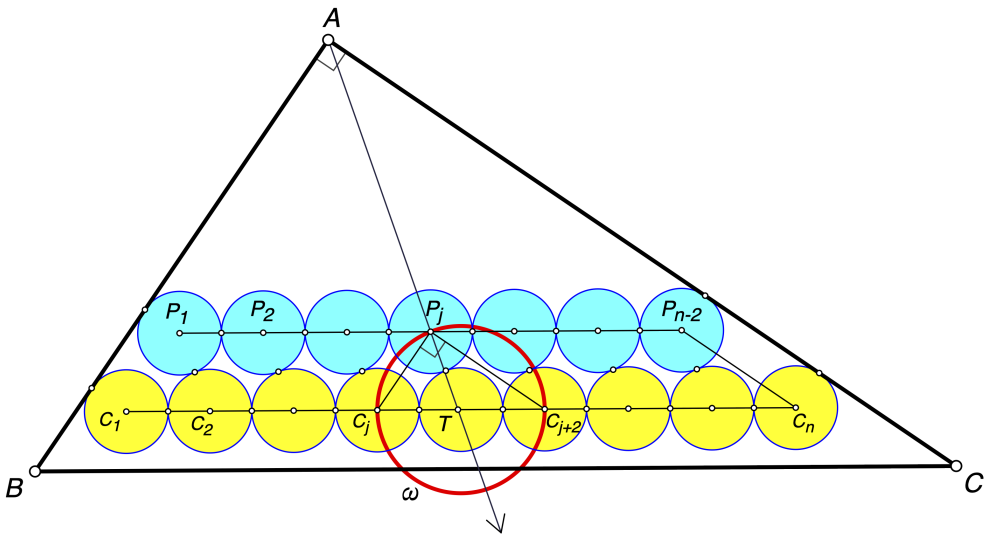


FIGURE 11. The red circle ω has diameter C_jC_{j+2}, and the ray AP_j meets that diameter at T.

All chains of length $n-2$ between the same two legs, with their center lines parallel to BC, are homothetic about A. Consequently, D_j lies on the ray AP_j, with

$$\frac{AD_j}{AP_j} = \frac{s}{r}.$$

If $s < r$, then D_j is between A and P_j, outside ω. Therefore

$$D_jC_{j+1} > 2r > r + s,$$

so the two circles are separated.

If $s > r$, then D_j lies beyond P_j. Since its center line is closer to A than the line C_jC_{j+2}, it lies before T. Thus D_j is inside ω, and

$$D_jC_{j+1} < 2r < r + s,$$

so the two circles overlap. Neither case permits external tangency. Hence $s = r$.

The shorter chain is therefore the constructed chain: $D_i = P_i$ for every i. This proves (1). Finally, D_iD_{i+1} and $C_{i+1}C_{i+2}$ are parallel and both have length $2r$. Consequently, $D_iC_{i+1}C_{i+2}D_{i+1}$ is a parallelogram with all sides of length $2r$, hence a rhombus. □

The proof also tells us what happens when the radii differ: corresponding circles are separated if $s < r$ and overlap if $s > r$. Tangency occurs exactly when $s = r$.

For $n = 3$ this is the original $3 + 1$ theorem. The cases $n = 4$ and $n = 5$ give Theorems 3.1 and 4.1, respectively. In those cases the shorter chain is automatically closer to A: enlarge the equal-radius shorter chain of Theorem 5.1 about A until it touches BC. Its radius then exceeds r, so even this outermost shorter chain has its center line closer to A than the original chain, which touches BC. The same theorem applies to every greater length, odd or even, and also to chains that do not touch the hypotenuse.

6. Further chains toward the right-angle vertex

Theorem 5.1 can be applied repeatedly, adding chains toward the right-angle vertex A and producing lengths

$$n,\ n-2,\ n-4,\ \ldots,$$

until one circle remains if n is odd or two circles remain if n is even. In every step, the next line of centers lies closer to A than the preceding one.

Together with Theorem 5.2, this gives a converse: if a shorter chain is drawn on a line of centers closer to A and one corresponding pair is externally tangent, then the shorter chain must be the one produced by this construction. In particular, once one corresponding tangency is prescribed at each inward step, all the circles in all the chains are forced to have the same radius.

Figure 12 illustrates the $5 + 3 + 1$ configuration. The circle E_1 is tangent to D_2, and all nine circles are congruent.

Figure 13 gives a second example, with three chains of lengths 6, 4, and 2. The four-circle chain is placed between the original six circles and the pair E_1, E_2 nearest A. Every circle has the same radius, and each circle in a shorter chain is tangent to its corresponding circle in the preceding chain.

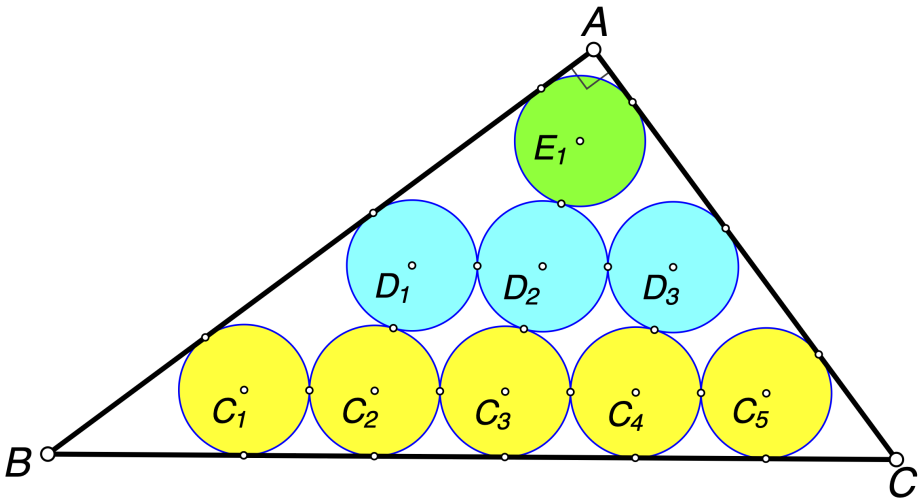


FIGURE 12. Three successive chains of lengths 5, 3, and 1.

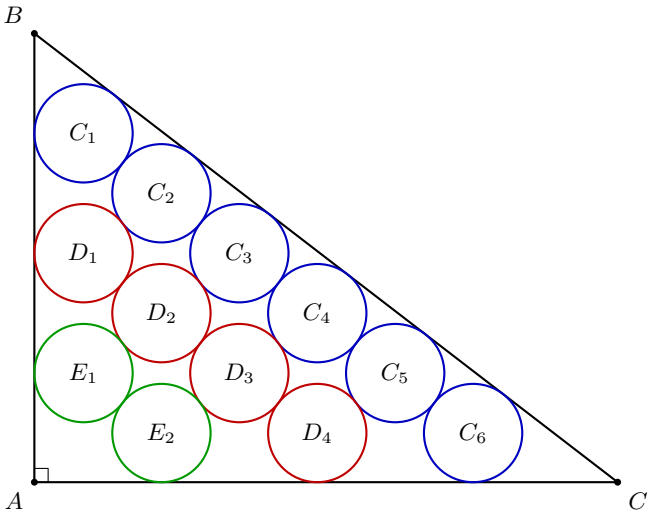


FIGURE 13. Three successive chains of lengths 6, 4, and 2.

7. EXTENDING A CHAIN OUTWARD

We can also move in the opposite direction, away from the right-angle vertex. Starting with n circles tangent to BC, we seek a chain of $n+2$ circles on a parallel line farther from A. The key is to regard the new row as lying along the hypotenuse of a *larger* right triangle.

Theorem 7.1. *Let $C_1, \ldots, C_n$, where $n \geq 1$, be a chain of circles of radius r in a right triangle ABC, all tangent to BC. Construct a second chain $B_1, \ldots, B_{n+2}$ of radius s, within angle BAC and outward of BC, with consecutive circles tangent, its line of centers parallel to BC, and its first and last circles tangent to the rays AB and AC, respectively.*

Let $B'C'$ be the common tangent to the circles of this new chain on the side away from A, with B' and C' on the rays AB and AC beyond B and C. Thus $AB'C'$ is a larger right triangle containing the original configuration.

If C_j is externally tangent to B_{j+1} for any one index $1 \leq j \leq n$, then $s = r$, and all n corresponding pairs are externally tangent:

$$\boxed{C_iB_{i+1} = 2r \qquad (1 \leq i \leq n).}$$

Moreover, for $1 \leq i < n$ the quadrilateral $C_iB_{i+1}B_{i+2}C_{i+1}$ is a rhombus.

Proof. The circles $B_1, \ldots, B_{n+2}$ form a chain along the hypotenuse $B'C'$ of the larger right triangle $AB'C'$. The original circles $C_1, \ldots, C_n$ form a shorter chain inside this larger triangle: their centers lie on a line parallel to $B'C'$, closer to A than the line of centers of the B-chain, and the end circles still touch the appropriate legs. The given tangency $C_jB_{j+1} = r+s$ is precisely one corresponding tangency between these two chains.

Apply Theorem 5.2 in triangle $AB'C'$, taking the B-chain as the longer chain and the C-chain as the shorter one. The theorem immediately yields $s = r$ and $C_iB_{i+1} = 2r$ for every i. The stated rhombi follow as well. □

The case $n = 1$ starts with the incircle of ABC and adds an outward chain of three circles. Thus the theorem includes outward extensions beginning with a single circle.

Figure 14 illustrates the simplest outward extension beyond our original result: a chain of four circles along BC is extended to a chain of six circles along $B'C'$. The original hypotenuse BC is dashed to distinguish it from the hypotenuse of the larger triangle. In this figure the triangle is intentionally chosen to be non-isosceles.

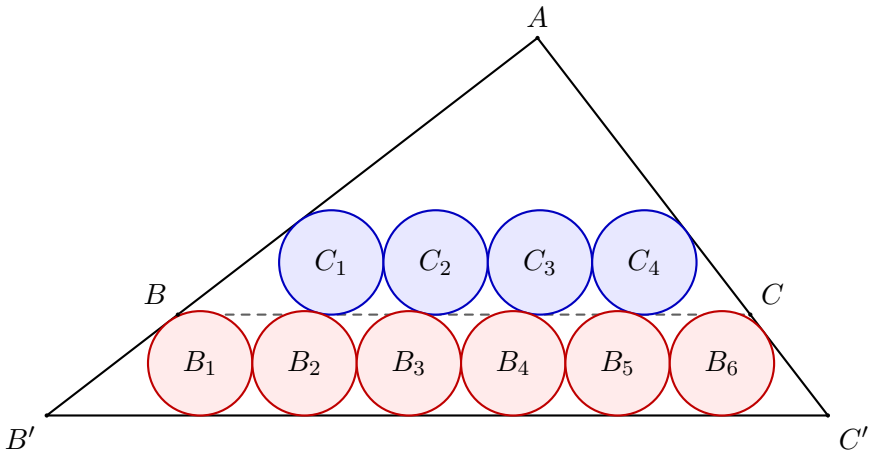


FIGURE 14. An outward extension $4 \to 6$. The original hypotenuse BC is dashed. All ten circles have the same radius, and each C_i is tangent to B_{i+1}.

The same idea works for further outward rows, of lengths $n + 4, n + 6, \ldots$: at each stage, use the common tangent of the new outer row as the hypotenuse of an enlarged right triangle.

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