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A Quartic

Problem 69-5, *by* H. HOLLOWAY AND M. S. KLAMKIN (Ford Scientific Laboratory).

Solve the quartic equation

$$x^4 + (1-b)x^3 + (1-3a-b+3a^2+3ab+b^2)x^2 + b(2-3a-2b)x + b^2 = 0.$$

Comment by STANLEY RABINOWITZ (Westford, Massachusetts).

In the published solution [1], the quartic was factored as the product of two quadratic polynomials having complex coefficients. Since its zeros occur in complex conjugate pairs, the quartic must also factor as the product of two quadratics with real coefficients. The editor [M.S.K.] requested that I find these two factors.

The four roots are known. In [1], Holloway and Klamkin pointed out that the equation can be written $(x^2+\alpha x+b\omega)(x^2+\bar{\alpha}x+b\bar{\omega})=0$, where $\omega=(1+i\sqrt{3})/2$, and

$$\alpha = (1-b)\omega - ia\sqrt{3} = \frac{(1-b)+i(1-2a-b)\sqrt{3}}{2}.$$

Thus the roots of the equation are $z_1, \overline{z_1}, z_2, \overline{z_2}$, where

$$z_1 = \frac{-\alpha+\sqrt{\alpha^2-4b\omega}}{2} \quad \text{and} \quad z_2 = \frac{-\alpha-\sqrt{\alpha^2-4b\omega}}{2}.$$

Note that $\alpha^2-4b\omega = (p+iq\sqrt{3})/2$, where

$$p = -(1-b)^2+6a(1-b)-6a^2-4b \quad \text{and} \quad q = (1-b)^2-2a(1-b)-4b.$$

We need to determine $z_1+\overline{z_1}$, $z_2+\overline{z_2}$, $z_1\overline{z_1}$ and $z_2\overline{z_2}$ in terms of a and b. The required complex square root is

$$\sqrt{\frac{p+iq\sqrt{3}}{2}} = \frac{1}{2}\left[\sqrt{\sqrt{p^2+3q^2}+p} + i\,\operatorname{sgn} q\sqrt{\sqrt{p^2+3q^2}-p}\right].$$

(See, for example, [2, p. 95].) Thus

$$z_1+\overline{z_1} = \frac{1}{2}\left(b-1+\sqrt{\sqrt{p^2+3q^2}+p}\right) \quad \text{and} \quad z_2+\overline{z_2} = \frac{1}{2}\left(b-1-\sqrt{\sqrt{p^2+3q^2}+p}\right).$$

Also

$$z_1\overline{z_1} = \frac{1}{16}\left[\left(b - 1 + \sqrt{\sqrt{p^2+3q^2}+p}\right)^2 + \left((2a+b-1)\sqrt{3} + \operatorname{sgn} q\sqrt{\sqrt{p^2+3q^2}-p}\right)^2\right]$$

and

$$z_2\overline{z_2} = \frac{1}{16}\left[\left(b - 1 - \sqrt{\sqrt{p^2+3q^2}+p}\right)^2 + \left((2a+b-1)\sqrt{3} - \operatorname{sgn} q\sqrt{\sqrt{p^2+3q^2}-p}\right)^2\right].$$

Thus the desired factors are

$$x^2 - \frac{1}{2}\left(b - 1 + \sqrt{\sqrt{p^2+3q^2}+p}\right)x + (P_1 + P_2 + P_3)$$

and

$$x^2 - \frac{1}{2}\left(b - 1 - \sqrt{\sqrt{p^2+3q^2}+p}\right)x + (P_1 - P_2 - P_3),$$

where

$$P_1 = \frac{(b-1)^2 + 3(2a+b-1)^2 + 2\sqrt{p^2+3q^2}}{16},$$

$$P_2 = \frac{(b-1)\sqrt{\sqrt{p^2+3q^2}+p}}{8},$$

$$P_3 = \operatorname{sgn} q\,\frac{(2a+b-1)\sqrt{3\left(\sqrt{p^2+3q^2}-p\right)}}{8}.$$

REFERENCES

[1] H. HOLLOWAY AND M. S. KLAMKIN, *Solution of Problem* 69-5, SIAM Rev., 12 (1970), pp. 471–473; reprinted in M. S. KLAMKIN, *Problems in Applied Mathematics, Selections from SIAM Review*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1990, pp. 529–530.

[2] A. MOSTOWSKI AND M. STARK, *Introduction to Higher Algebra*, Pergamon Press, New York, 1964.