How to Find the Square Root of a Complex Number

Stanley Rabinowitz
12 Vine Brook Road
Westford, Massachusetts 01886 USA

It is known that every polynomial with complex coefficients has a complex root. This is called "The Fundamental Theorem of Algebra". In particular, the equation

$$z^2 = c$$

where c is a complex number, always has a solution. In other words, every complex number has a square root. We could write this square root as $\sqrt{c}$. But – it would be nice to find an explicit representation for that square root in the form $p + qi$ where p and q are real numbers. It is the purpose of this note to show how to actually find the square root of a given complex number. This method is not new (see for example page 95 of Mostowski and Stark [1]) but appears to be little-known.

Let us start with the complex number

$$c = a + bi$$

where a and b are real ($b \neq 0$) and attempt to find an explicit representation for its square root. Of course, every complex number (other than 0) will have two square roots. If w is one square root, then the other one will be $-w$. We will find the one whose real part is non-negative.

Let us assume that a square root of c is $p + qi$ where p and q are real. Then we have

$$(p + qi)^2 = a + bi.$$

Equating the real and imaginary parts gives us the two equations

$$p^2 - q^2 = a \tag{1}$$
$$2pq = b. \tag{2}$$

We must have $p \neq 0$ since $b \neq 0$. Solving equation (2) for q gives

$$q = \frac{b}{2p} \tag{3}$$

and we can substitute this value for q into equation (1) to get

$$p^2 - \left(\frac{b}{2p}\right)^2 = a$$

Reprinted from Mathematics and Informatics Quarterly, 3(1993)54–56

or

$$4p^4 - 4ap^2 - b^2 = 0.$$

This is a quadratic in p^2, so we can solve for p^2 using the quadratic formula. We get (taking just the positive solution):

$$p^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$

so that

$$p = \frac{1}{\sqrt{2}} \sqrt{a + \sqrt{a^2 + b^2}} \; .$$

From equation (3), we find

$$\begin{aligned} q = \frac{b}{2p} &= \frac{b}{\frac{2}{\sqrt{2}} \sqrt{\sqrt{a^2 + b^2} + a}} \\ &= \frac{b}{\frac{2}{\sqrt{2}} \sqrt{\sqrt{a^2 + b^2} + a}} \cdot \frac{\sqrt{\sqrt{a^2 + b^2} - a}}{\sqrt{\sqrt{a^2 + b^2} - a}} \\ &= \frac{b}{\sqrt{2}} \frac{\sqrt{\sqrt{a^2 + b^2} - a}}{\sqrt{(a^2 + b^2) - a^2}} \\ &= \frac{b}{\sqrt{2}} \frac{\sqrt{\sqrt{a^2 + b^2} - a}}{\sqrt{b^2}} = \frac{b}{\sqrt{2}} \frac{\sqrt{\sqrt{a^2 + b^2} - a}}{|b|} \\ &= \frac{\operatorname{sgn} b}{\sqrt{2}} \sqrt{\sqrt{a^2 + b^2} - a} \; . \end{aligned}$$

Note that $\sqrt{b^2} = |b|$, so that $b/|b| = \operatorname{sgn}(b)$, the sign of b (defined to be +1 if $b > 0$ and -1 if $b < 0$).

Thus we have our answer:

Theorem 1. If a and b are real ($b \neq 0$), then

$$\sqrt{a + bi} = p + qi$$

where p and q are real and are given by

$$p = \frac{1}{\sqrt{2}} \sqrt{\sqrt{a^2 + b^2} + a}$$

and

$$q = \frac{\operatorname{sgn} b}{\sqrt{2}} \sqrt{\sqrt{a^2 + b^2} - a} \; .$$

In practice, square roots of complex numbers are more easily found by first converting to polar form and then using DeMoivre's Theorem. Any complex number $a+bi$ can be written as

$$r(\cos\theta + i\sin\theta)$$

where

$$r = \sqrt{a^2+b^2}, \quad \cos\theta = \frac{a}{r}, \quad \text{and} \quad \sin\theta = \frac{b}{r} \tag{4}$$

DeMoivre's Theorem states that if n is any positive real number, then

$$(a+bi)^n = r^n(\cos n\theta + i\sin n\theta).$$

In particular, if $n = 1/2$, we have

$$\sqrt{a+bi} = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right). \tag{5}$$

This gives us a straightforward way to calculate $\sqrt{a+bi}$.

This method also gives us an alternate proof of Theorem 1. If we apply the half-angle formulae

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

and

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

to equation (5), we get

$$\sqrt{a+bi} = \sqrt{r}\left(\sqrt{\frac{1+\cos\theta}{2}} \pm i\sqrt{\frac{1-\cos\theta}{2}}\right)$$

where we have arbitrarily chosen the "+" sign for the first radical. Using the value for $\cos\theta$ from equation (4), we get

$$\begin{aligned}\sqrt{a+bi} &= \sqrt{r}\left(\sqrt{\frac{1+a/r}{2}} \pm i\sqrt{\frac{1-a/r}{2}}\right)\\ &= \sqrt{\frac{r+a}{2}} \pm i\sqrt{\frac{r-a}{2}}\\ &= \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} \pm i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\end{aligned}$$

which is equivalent to Theorem 1. As before, the "±" sign should be chosen to be the same as the sign of b.

We sometimes need to find the square root of an expression of the form $s+\sqrt{-d}$ where s and d are real numbers and $d > 0$. We can use Theorem 1 to get an explicit formula for this square root which is of the form $p+qi$ where p and q are real. Since $s+\sqrt{-d} = s+i\sqrt{d}$, we can let $a = s$ and $b = \sqrt{d}$ in Theorem 1, to get the result:

Theorem 2. If s and d are real with $d > 0$, then

$$\sqrt{s+\sqrt{-d}} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{s^2+d}+s} + i\frac{1}{\sqrt{2}}\sqrt{\sqrt{s^2+d}-s} \quad .$$

Reference

[1] A. Mostowski and M. Stark, *Introduction to Higher Algebra.* Pergamon Press. New York: 1964.