

Mapping a Point to the Center of Its Conic with Four Fixed Points

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Abstract. Fix four points that form a convex quadrilateral. Choose a fifth point, draw the conic through the five points, and replace the fifth point by the center of that conic. Repeating this operation gives a sequence of points. The sequence need not converge, even when the four fixed points lie on a circle. If it does converge to a finite point, however, that point must be one of the three intersections formed by pairs of opposite sides or diagonals. We prove convergence for several familiar starting points, including the vertex centroid, the area centroid, and the center of any ellipse tangent to all four sides. We also show that every interior starting point approaches the interior diagonal point exactly when the quadrilateral is a parallelogram, and we give a complete description for trapezoids, where the construction becomes Newton’s familiar method for solving a quadratic equation.

Keywords. conic, quadrilateral, iteration, centroid, Newton’s method.

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1. THE CONSTRUCTION

Let A, B, C, D be four fixed points in the plane, in order around a convex quadrilateral. Choose a point P_1 . The five points

$$A, B, C, D, P_1$$

usually determine one conic. Let P_2 be its center. Next, let P_3 be the center of the conic through A, B, C, D, P_2 , and continue in the same way. Our question is simple:

What happens to the sequence $P_1, P_2, P_3, \dots$?

The construction is not always possible. A parabola has no finite center, and neither does a pair of parallel lines. Also, if the fifth point repeats one of A, B, C, D , the five listed points need not determine one conic. We say that the construction *stops* in any of these cases.

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We do allow a pair of intersecting lines as a limiting conic. Its center is their intersection. This convention is natural here and makes the three special intersections below into fixed points of the construction.

Write

$$X = AC \cap BD,$$

$$Y = AB \cap CD,$$

$$Z = AD \cap BC.$$

The lines are extended when necessary. We call X, Y, Z the three *diagonal points*. For a convex quadrilateral, X is inside the quadrilateral; Y and Z are outside it or at infinity. For a nonparallelogram trapezoid, one of them is at infinity.

Figure 1 illustrates the first step of the construction and shows the three diagonal points.

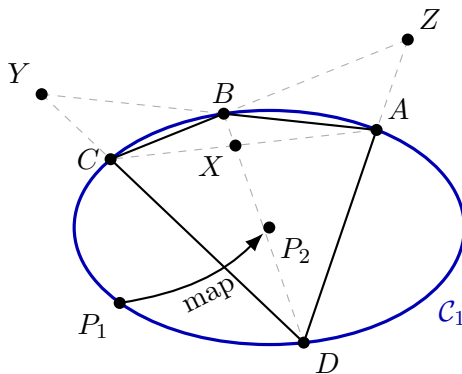


FIGURE 1. The conic through the four fixed vertices and P_1 has center P_2 ; thus the map takes P_1 to P_2 . Repeating the operation produces $P_1, P_2, P_3, \dots$. The three intersections X, Y, Z are also shown.

Proposition 1.1. *If P_1 , other than a vertex, is an ordinary point of either diagonal AC or BD , then $P_2 = X$. Similarly, a point on AB or CD gives $P_2 = Y$, and a point on AD or BC gives $P_2 = Z$.*

Proof. For example, the pair of lines $AC \cup BD$ is a degenerate conic and passes through the four vertices and through every point on either diagonal. Its center is $AC \cap BD = X$. The other cases are the same. $\square$

2. A COORDINATE FORM OF THE CONSTRUCTION

The construction is unchanged by translating, rotating, stretching, or shearing the entire figure. These operations carry conics to conics and carry their centers to their centers. We may therefore choose coordinates that make the algebra easier.

Move X to the origin and use the two diagonals as the coordinate axes of an affine, possibly oblique, coordinate system. Thus, the axes need not be perpendicular. After changing the scale along each axis, the four vertices can be written as

$$(1) \quad A = (a, 0), \quad B = (0, b), \quad C = (-1/a, 0), \quad D = (0, -1/b),$$

where $a, b > 0$. Put

$$u = \frac{a - a^{-1}}{2}, \quad v = \frac{b - b^{-1}}{2}.$$

A general conic equation contains the terms x^2, xy, y^2, x, y , and a constant. The coefficient of x^2 is nonzero unless the conic contains the entire x -axis, which happens only for the pair of diagonals $xy = 0$. Hence we may multiply the equation by a nonzero constant so that its x -terms are

$$(x - a)(x + 1/a) = x^2 - 2ux - 1.$$

The y -terms are a multiple of $(y - b)(y + 1/b) = y^2 - 2vy - 1$. Since the x -terms and the y -terms have the same constant term -1 , that multiple is 1. The only coefficient still free is the coefficient of xy . Writing it as $2t$, we find that all the ordinary conics through the four vertices have the form

$$(2) \quad \mathcal{C}_t: \quad x^2 + y^2 + 2txy - 2ux - 2vy - 1 = 0.$$

We call t the *conic parameter*. Equation (2) describes every conic in the family except the pair of diagonal lines $xy = 0$. We use $t = \infty$ to represent this remaining member. If $P = (x, y)$, then the conic through A, B, C, D, P has parameter

$$(3) \quad t(P) = \frac{1 + 2ux + 2vy - x^2 - y^2}{2xy},$$

provided $xy \neq 0$. The center of $\mathcal{C}_t$ is

$$(4) \quad C(t) = \left(\frac{u - tv}{1 - t^2}, \frac{v - tu}{1 - t^2} \right).$$

The values $t = 1$ and $t = -1$ do not give a unique finite center.

Substituting (4) into (3) gives the one-variable rule

$$(5) \quad F(t) = \frac{t^4 + 4uvt^3 - (3u^2 + 3v^2 + 2)t^2 + u^2 + v^2 + 1}{2(ut - v)(vt - u)}.$$

Thus, the original plane construction can be studied by repeatedly applying one ordinary fraction F .

The following difference will be useful:

$$(6) \quad F(t) - t = \frac{(t^2 - 1)(t^2 + 2uvt - 1 - u^2 - v^2)}{2(ut - v)(vt - u)}.$$

By reversing the coordinate axes if necessary, we may use $u, v > 0$ in the arguments below. Let

$$M = \max \left(\frac{u}{v}, \frac{v}{u} \right).$$

Suppose $t > M$. At $t = M$, the quadratic factor $t^2 + 2uvt - 1 - u^2 - v^2$ in (6) is nonnegative, and it increases for $t > 0$. Indeed, if $u \geq v$, then $M = u/v$ and its value at M is $((1 + v^2)(u^2 - v^2))/v^2 \geq 0$. The case $v \geq u$ is the same with u and v interchanged. All the remaining factors have the sign that makes $F(t)$ larger than t . Thus, the successive parameters increase. They cannot approach a finite limit, because such a limit would be a fixed value of F , while (6) shows that there is no fixed value greater than M . The parameters therefore increase without bound. Equation (4) then shows that their centers approach $(0, 0) = X$. We record this simple test.

Theorem 2.1 (A useful convergence test). *If at some stage the conic parameter satisfies*

$$t > \max \left(\frac{u}{v}, \frac{v}{u} \right),$$

then the subsequent points approach X .

3. THE ONLY POSSIBLE FINITE LIMITS

There is also a useful geometric fact hidden in formula (4). Direct substitution shows that every center $C(t)$ lies on the fixed conic

$$\mathcal{K} : \quad x^2 - y^2 - ux + vy = 0.$$

Thus, after the first step, the entire sequence lies on one fixed conic. Figure 2 shows an example where the points $P_1, P_2, P_3, P_4, P_5, P_6$, and P_7 lie on a hyperbola passing through the intersection of AC and BD . The original point P is not shown in this figure.

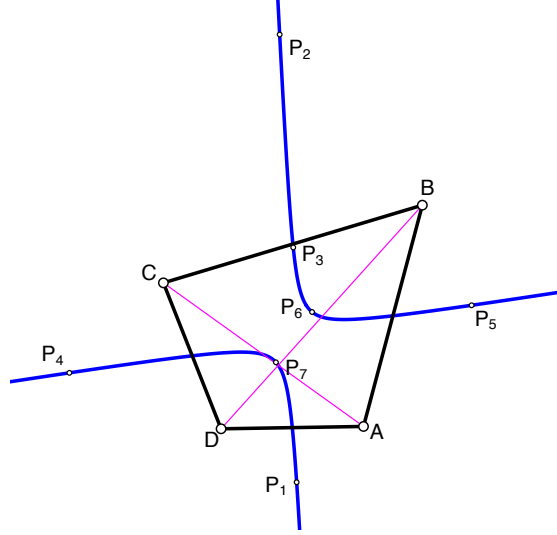


FIGURE 2. $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ lie on a hyperbola

Assume first that $|u| \neq |v|$. Then $\mathcal{K}$ is a genuine hyperbola. It passes through X, Y, Z and through the midpoints of the six segments joining pairs of vertices. Formula (4) gives a continuous one-to-one correspondence between the points of this hyperbola, including its two points at infinity, and the parameter values $t \in \mathbb{R} \cup \{\infty\}$. Consequently, if a sequence of centers converges, its parameter values also converge. When $|u| = |v|$, the curve $\mathcal{K}$ splits into two lines and all the centers lie on one of them. This is precisely the trapezoid case treated in Section 6.

The possible fixed values of the one-variable rule can also be found directly.

Proposition 3.1. *Assume $|u| \neq |v|$. The fixed values of F are*

$$\infty, \quad 1, \quad -1, \quad t_{\pm} = -uv \pm \sqrt{(1+u^2)(1+v^2)}.$$

The value ∞ represents $AC \cup BD$, whose center is X . The values t_+ and t_- represent $AB \cup CD$ and $AD \cup BC$, whose centers are Y and Z . At $t = 1$ or $t = -1$, the conic has no unique finite center, so the construction stops.

Moreover,

$$F'(t_+) = F'(t_-) = 0, \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau^2 F(1/\tau)} = 2uv.$$

Proof. Equation (6) gives the fixed values. Substitution in (2) identifies the three pairs of lines. Differentiating (5) and comparing its highest-degree terms give the last two formulas. $\square$

Theorem 3.2. *If the construction continues indefinitely and P_n approaches a finite point L , then*

$$L \in \{X, Y, Z\}.$$

Proof. For a nontrapezoid, convergence of the points on $\mathcal{K}$ gives a fixed value of the parameter. Proposition 3.1 lists all such values. The values 1 and -1 stop the construction, so a finite limiting point can only be the center of one of the three line pairs, namely X, Y , or Z . The nonparallelogram trapezoid case follows from the Newton formula in Section 6. For a parallelogram, every conic in (2) that has a unique center is centered at X . $\square$

The last formulas in Proposition 3.1 show that near any finite diagonal point, the error is approximately squared at every step. Thus, each finite diagonal point attracts all sufficiently nearby starting points. The trapezoid formula (11) below gives the same conclusion in that case.

This local attraction gives a simple answer to a natural question about all the points inside the quadrilateral. Here an interior point means a point strictly inside the quadrilateral, not on its boundary.

Theorem 3.3 (Interior-point theorem). *Every interior starting point approaches X if and only if $ABCD$ is a parallelogram. For a parallelogram, every interior starting point reaches X in one step.*

Proof. Suppose first that $ABCD$ is a parallelogram. Then $u = v = 0$ in the coordinates of Section 2. Formula (4) shows that every conic in the family having a unique center is centered at X . The two exceptional values $t = 1$ and $t = -1$ give pairs of parallel sides, and no strictly interior point lies on those sides. A point on either diagonal reaches X by Proposition 1.1. Thus, every interior starting point reaches X in one step.

Conversely, suppose that $ABCD$ is not a parallelogram. At least one of the two exterior diagonal points Y and Z is finite; call it W . Choose a point Q , other than a vertex, on one of the two opposite sides whose extended lines meet at W . The conic through A, B, C, D, Q is that pair of lines, so its center is W . Because Q is an interior point of a side, it lies on neither diagonal, so the conic center depends continuously on P near Q . Thus, as a point P inside the quadrilateral approaches Q , the center of the conic through A, B, C, D, P approaches W . Since every finite diagonal point attracts all sufficiently nearby points, all interior points P sufficiently close to Q produce sequences approaching W , rather than X . Therefore a nonparallelogram cannot have the stated property. $\square$

4. CONVERGENCE IS NOT AUTOMATIC

The four fixed vertices may lie on a circle and the sequence may still fail to converge.

Example 4.1 (An exact two-cycle). Take

$$A = \left(0, \frac{-1 + \sqrt{13}}{2}\right), \quad B = (1, -2),$$

$$C = \left(0, \frac{-1 - \sqrt{13}}{2}\right), \quad D = (-1, -2).$$

All four vertices lie on

$$x^2 + y^2 + y - 3 = 0.$$

The ellipse

$$x^2 + y^2 - xy - 2x + y - 3 = 0$$

passes through the four vertices and through $(-1, 0)$; its center is $(1, 0)$. The ellipse

$$x^2 + y^2 + xy + 2x + y - 3 = 0$$

passes through the four vertices and through $(1, 0)$; its center is $(-1, 0)$. Hence

$$(-1, 0), (1, 0), (-1, 0), (1, 0), \dots$$

is a two-cycle.

The two ellipses and the alternating centers in this example are shown in Figure 3.

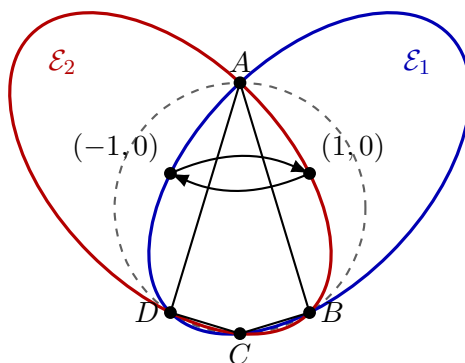


FIGURE 3. An exact two-cycle. The point $(-1, 0)$ lies on the blue ellipse, whose center is $(1, 0)$. The point $(1, 0)$ lies on the red ellipse, whose center is $(-1, 0)$. The dashed curve is the circumcircle of $ABCD$.

This example is a kite as well as a cyclic quadrilateral. Every convex kite has an incircle, so the example also shows a bicentric quadrilateral. Thus, none of these three shape conditions guarantees convergence for an arbitrary starting point.

4.1. A cyclic example starting at the circumcenter. Even a familiar starting point need not approach X . Consider the four points

$$\left(-\frac{4}{5}, -\frac{3}{5}\right), \quad \left(-\frac{3}{5}, -\frac{4}{5}\right), \quad (0, -1), \quad (1, 0),$$

which lie on the unit circle. Starting at the circumcenter $(0, 0)$, the first two new points are

$$\left(\frac{1}{2}, -\frac{1}{2}\right), \quad \left(-\frac{1}{5}, -\frac{6}{5}\right).$$

The first of these lies on the line CD . Proposition 1.1 therefore gives the second point immediately. It is the exterior diagonal point Y , where the sequence remains. For a cyclic kite, on the other hand, the circumcenter lies on the line of symmetry and reaches X in one step. Cyclically relabeling the four vertices in the example above makes the same exterior point the diagonal point Z . Thus, a circumcenter can lead to any of the three diagonal points.

5. FAMILIAR STARTING POINTS

Two different points are commonly called the centroid of a quadrilateral. Both work well in this problem. The following lemma will also handle centers of ellipses tangent to the four sides.

If $u = 0$ or $v = 0$, all the points considered in this section lie on one of the diagonals, so Proposition 1.1 sends them to X in one step. We may therefore assume, after reversing coordinate directions if necessary, that $u, v > 0$.

Lemma 5.1. *Let*

$$P = (\lambda u, \mu v),$$

where $\lambda, \mu > 0$ and

$$\lambda + 2\mu \leq 2, \quad 2\lambda + \mu \leq 2.$$

Then the sequence starting at P approaches X .

Proof. Substitution in (3) gives

$$t(P) - \frac{u}{v} = \frac{1 + \lambda(2 - \lambda - 2\mu)u^2 + \mu(2 - \mu)v^2}{2\lambda\mu uv} > 0$$

and

$$t(P) - \frac{v}{u} = \frac{1 + \lambda(2 - \lambda)u^2 + \mu(2 - \mu - 2\lambda)v^2}{2\lambda\mu uv} > 0.$$

The result follows from Theorem 2.1. □

5.1. The vertex centroid. Let

$$G = \frac{A + B + C + D}{4}.$$

This is obtained by placing equal masses at the four vertices. It is also the intersection of the two lines joining the midpoints of opposite sides.

In the coordinates (1),

$$G = \left(\frac{u}{2}, \frac{v}{2}\right).$$

Lemma 5.1, with $\lambda = \mu = 1/2$, proves the following result.

Corollary 5.2 (Vertex-centroid theorem). *Starting at the vertex centroid always gives*

$$P_n \longrightarrow X.$$

5.2. The area centroid. Now regard the whole interior of $ABCD$ as a thin sheet of uniform density. Let G_A be its center of mass. Dividing the quadrilateral into four triangles with common vertex X gives

$$G_A = \left(\frac{2u}{3}, \frac{2v}{3} \right)$$

in the coordinates (1). Apply Lemma 5.1 with $\lambda = \mu = 2/3$.

Corollary 5.3 (Area-centroid theorem). *Starting at the area centroid always gives*

$$P_n \longrightarrow X.$$

The relation $G_A = X + \frac{4}{3}(G - X)$ shows, in particular, that X, G , and G_A lie on one line.

5.3. Centers of ellipses tangent to the four sides. A classical theorem of Newton says that the centers of all ellipses tangent to the four sides lie on the open segment joining the midpoints of the two diagonals [4, 5]. For background on the classical geometry used here, see [1, 2, 3]. In our coordinates, those two midpoints are

$$(u, 0) \quad \text{and} \quad (0, v).$$

Therefore the center of such an ellipse has the form

$$S = ((1-s)u, sv), \quad 0 < s < 1.$$

Lemma 5.1 applies with $\lambda = 1-s$ and $\mu = s$.

Corollary 5.4 (Inscribed-ellipse theorem). *If P_1 is the center of any ellipse tangent to all four sides, then*

$$P_n \longrightarrow X.$$

When $s = 1/2$, the point S is exactly the average G of the four vertices. Thus, the vertex centroid is the midpoint of Newton's segment.

A quadrilateral with an incircle is called *tangential*. Its incenter is the center of one of these ellipses—in this case, a circle.

Corollary 5.5. *Starting at the incenter of a tangential quadrilateral always gives $P_n \rightarrow X$.*

6. A COMPLETE RESULT FOR TRAPEZOIDS

In the coordinates of Section 2, the condition $|u| = |v|$ means that one pair of opposite sides is parallel. In this case, the rule (5) simplifies from a fourth-degree fraction to a quadratic one. The following coordinates make that simplification especially clear.

Every convex trapezoid can be changed by a stretch and shear into

$$(7) \quad A = (-1, 0), \quad B = (1, 0), \quad C = (r, 1), \quad D = (-r, 1),$$

where $r > 0$. Here $Y = AB \cap CD$ is the point at infinity in the direction of the parallel bases. The two finite diagonal points are

$$(8) \quad X = \left(0, \frac{1}{1+r} \right), \quad Z = \left(0, \frac{1}{1-r} \right).$$

The second expression is omitted when $r = 1$. That is the parallelogram case. From now on assume $r \neq 1$.

Except for the pair of parallel bases, the conics through the four vertices have equations of the form

$$(9) \quad x^2 + qy^2 + (1 - r^2 - q)y - 1 = 0.$$

Their centers lie on the y -axis. Thus, after the first step, write

$$P_n = (0, z_n).$$

Put $k = 1 - r^2$. Direct calculation from (9) gives

$$(10) \quad z_{n+1} = \frac{1 - kz_n^2}{2(1 - kz_n)}.$$

Formula (10) is Newton's method for solving

$$kz^2 - 2z + 1 = 0.$$

The two roots are exactly the two numbers in (8). If

$$\alpha = \frac{1}{1+r}, \quad \beta = \frac{1}{1-r},$$

then

$$(11) \quad \frac{z_{n+1} - \alpha}{z_{n+1} - \beta} = \left(\frac{z_n - \alpha}{z_n - \beta} \right)^2.$$

This identity completely describes the convergence.

For an original starting point $P_1 = (x, y)$, define

$$\begin{aligned} N(x, y) &= 1 - x^2 - (1 - r^2)y, \\ H(x, y) &= (1 + r^2)N(x, y) + (1 - r^2)^2y(y - 1). \end{aligned}$$

These expressions have a direct meaning. For a starting point $P_1 = (x, y)$, with $y \neq 0, 1$, substitute P_1 into (9) and solve for q . This gives

$$q = \frac{N(x, y)}{y(y - 1)}.$$

Thus, $N = 0$ gives a parabola, while $y = 0$ or $y = 1$ gives the pair of parallel bases. A calculation of the first center gives

$$z_2 - \frac{1}{k} = -\frac{H(x, y)}{2kN(x, y)}.$$

The number $1/k$ is halfway between the two roots in (8). It is also the value at which the next Newton step has zero denominator. Consequently the sign of NH determines which root is approached.

Whenever the construction continues indefinitely,

$\begin{aligned} N(x, y)H(x, y) &> 0 &\implies P_n \rightarrow X, \\ N(x, y)H(x, y) &< 0 &\implies P_n \rightarrow Z. \end{aligned}$
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Completing the square gives $H = 0$ in the form

$$(1 + r^2)x^2 - k^2 \left(y - \frac{1}{k} \right)^2 = r^2.$$

This is a hyperbola centered at $(0, 1/k)$ whose lobes open in the x -direction. Both separating curves belong to the family (9): $N = 0$ is its unique parabola, obtained when $q = 0$, and $H = 0$ is the member with $q = -k^2/(1 + r^2)$. The center of this hyperbola is the midpoint $1/k$ of the two roots. Both curves pass through all four vertices, and they meet nowhere else.

Geometrically, the Z -regions consist of the points that are inside both curves, together with the points that are outside both curves. Here “inside the parabola” means on its convex side, that is, $N > 0$ (below the parabola when $r < 1$ and above it when $r > 1$). To be “inside the hyperbola” means to be inside either of its two lobes, that is, $H < 0$.

The construction stops, and the starting point must be excluded, if P_1 lies on $N = 0$, $H = 0$, or either base $y = 0$ or $y = 1$, or if some later z_n equals $1/k$, 0 , or 1 . The last two values are the midpoints of the two bases. At either midpoint the next conic is the pair of parallel bases, even though formula (10), considered merely as an algebraic formula, still produces the value $1/2$.

Theorem 6.1 (Trapezoid theorem). *For every nonparallelogram trapezoid, and as long as the geometric construction remains defined, its points follow Newton’s method for a quadratic equation after the first step. Except when the construction stops, the sequence approaches one of the two finite diagonal points.*

7. SUMMARY AND CLOSING REMARKS

The main results for familiar starting points are collected in Table 1.

TABLE 1. Results for familiar starting points.

Starting point	General result
Point on one of the two diagonals	Reaches X in one step
Arbitrary interior point	Approaches X for every interior point if and only if $ABCD$ is a parallelogram
Vertex centroid	Always approaches X
Area centroid	Always approaches X
Center of an ellipse tangent to all four sides	Always approaches X
Incenter, when an incircle exists	Always approaches X
Circumcenter, when a circumcircle exists	May approach X , Y , or Z

The map studied here begins with a simple geometric construction but has several different behaviors. A sequence can approach any one of the three diagonal points, can stop because a parabola or a pair of parallel lines appears, or can repeat periodically. Nevertheless, the vertex centroid, the area centroid, and centers of ellipses tangent to all four sides give dependable convergence to the interior diagonal point. For trapezoids the iteration has the especially simple form of Newton’s

method for a quadratic equation. Parallelograms are exactly the quadrilaterals for which every interior starting point approaches the interior diagonal point.

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