

Convex Polygons Formed by Triangle Centers

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Abstract. Let X_n denote the n th center in Kimberling’s *Encyclopedia of Triangle Centers*. We investigate sets of triangle centers that form a convex polygon for every scalene reference triangle. Among the first twenty centers $X_1, \dots, X_{20}$, exactly 38 six-element subsets form a strictly convex hexagon for every scalene triangle. No seven of these centers always form a convex heptagon. As a preliminary case, we also determine the convex polygons formed from the first ten centers: there are 87 universally convex quadrilaterals, 14 universally convex pentagons, and a unique universally convex hexagon, formed by $X_1, X_3, X_5, X_7, X_9, X_{10}$. We also give a higher-index example: adjoining X_{79} to this hexagon produces a universal convex heptagon.

The classification combines exact tests on a few specially chosen triangles with symbolic sign computations.

Keywords. triangle centers, convex polygons, convex hull, barycentric coordinates, computer-discovered mathematics.

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1. INTRODUCTION

A triangle has many distinguished associated points. Kimberling’s *Encyclopedia of Triangle Centers* [1] provides a standard indexing $X_1, X_2, X_3, \dots$ for thousands of such centers. Classical investigations of triangle centers commonly concern collinearities, concurrencies, loci, metric relations, or special configurations. Here we consider a different question:

Which finite collections of triangle centers form a convex polygon for every scalene triangle?

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For example, the four centers

$$X_1, X_2, X_7, X_6$$

always form a strictly convex quadrilateral when the reference triangle is scalene. The word *scalene* cannot be dropped: in an isosceles triangle, reflection in the axis of symmetry maps the triangle to itself, so it fixes every triangle center, and all triangle centers lie on that axis. This simple example led us to investigate the problem systematically.

The main result of the paper concerns the first twenty ETC centers. Among the $\binom{20}{6} = 38\,760$ six-element subsets of $X_1, \dots, X_{20}$, exactly 38 always form a strictly convex hexagon. Moreover, no seven of the first twenty centers always form a convex heptagon. Thus, six is the largest possible number of vertices for a universally convex polygon formed from these twenty centers. Higher-index centers can do better: Section 7.2 gives a universal convex heptagon obtained by adjoining X_{79} to the universal hexagon $X_5X_7X_1X_3X_9X_{10}$.

Before treating the twenty-center classification, we introduce the orientation method on two particularly transparent examples involving the first ten centers. The first is the quadrilateral just mentioned. The second is the unique universal hexagon among $X_1, \dots, X_{10}$,

$$X_5X_7X_1X_3X_9X_{10}.$$

After proving this result, we turn immediately to the complete twenty-center hexagon classification. The remaining pentagon and quadrilateral results for the first ten centers are collected later in the paper.

Throughout, a *convex n -gon* means a strictly convex polygon: all n points are distinct vertices of their convex hull, and no three consecutive vertices are collinear.

A search of the ETC and of related literature on configurations and locations of triangle centers [2, 3, 4, 5] did not reveal a previous systematic study of this convex-position question.

2. NOTATION AND ORIENTATION

Let ABC be a nondegenerate triangle with side lengths

$$a = BC, \quad b = CA, \quad c = AB,$$

and semiperimeter s . Put

$$x = s - a, \quad y = s - b, \quad z = s - c.$$

Then

$$a = y + z, \quad b = z + x, \quad c = x + y,$$

with $x, y, z > 0$. Also,

$$x - y = b - a, \quad x - z = c - a, \quad y - z = c - b.$$

Thus

$$V = (x - y)(x - z)(y - z)$$

is nonzero exactly when the reference triangle is scalene. When discussing signs, we may relabel the vertices and assume

$$a < b < c,$$

equivalently $x > y > z$. In that case $V > 0$. The other five possible side orders follow by relabeling.

We shall also use

$$p = x + y + z, \quad q = xy + yz + zx, \quad r = xyz.$$

Since $p = s$, Heron's formula gives

$$\Delta^2 = pr,$$

where Δ is the area of ABC . For centers involving $\sqrt{3}\Delta$, it is convenient to put

$$h = \sqrt{3pr} = \sqrt{3}\Delta.$$

We use the Conway notation

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}.$$

Thus $S_A = bc \cos A$, and cyclically.

The barycentric coordinates used in the computations are listed in Table 1. Only the first coordinate is shown; the second and third are obtained by cyclically permuting a, b, c .

n	name	first barycentric coordinate of X_n
1	incenter	a
2	centroid	1
3	circumcenter	$a^2 S_A$
4	orthocenter	$S_B S_C$
5	nine-point center	$a^2(b^2 + c^2) - (b^2 - c^2)^2$
6	symmedian point	a^2
7	Gergonne point	$(c + a - b)(a + b - c)$
8	Nagel point	$b + c - a$
9	Mittenpunkt	$a(b + c - a)$
10	Spieker center	$b + c$
11	Feuerbach point	$(b - c)^2(b + c - a)$
12	$\{X_1, X_5\}$ -harmonic conjugate of X_{11}	$(b + c)^2(c + a - b)(a + b - c)$
13	first Fermat point	$(\sqrt{3} S_B + 2\Delta)(\sqrt{3} S_C + 2\Delta)$
14	second Fermat point	$(\sqrt{3} S_B - 2\Delta)(\sqrt{3} S_C - 2\Delta)$
15	first isodynamic point	$a^2(\sqrt{3} S_A + 2\Delta)$
16	second isodynamic point	$a^2(\sqrt{3} S_A - 2\Delta)$
17	first Napoleon point	$(S_B + 2\sqrt{3}\Delta)(S_C + 2\sqrt{3}\Delta)$
18	second Napoleon point	$(S_B - 2\sqrt{3}\Delta)(S_C - 2\sqrt{3}\Delta)$
19	Clawson point	$a S_B S_C$
20	de Longchamps point	$3a^4 - 2a^2(b^2 + c^2) - (b^2 - c^2)^2$
79	isogonal conjugate of X_{35}	$(2S_B + ca)(2S_C + ab)$

TABLE 1. Barycentric coordinates used in this paper. The second and third coordinates are obtained by cyclically permuting (a, b, c) . These forms agree with the ETC up to a common nonzero factor, which may be negative.

If P, Q, R have homogeneous barycentric coordinates

$$P = (p_1 : p_2 : p_3), \quad Q = (q_1 : q_2 : q_3), \quad R = (r_1 : r_2 : r_3),$$

write

$$[P, Q, R] = \det \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}.$$

Let

$$N_P = p_1 + p_2 + p_3, \quad N_Q = q_1 + q_2 + q_3, \quad N_R = r_1 + r_2 + r_3.$$

Homogeneous barycentric coordinates are unchanged if all three coordinates are multiplied by the same nonzero number. To convert $(p_1 : p_2 : p_3)$ to ordinary affine barycentrics, one divides by $N_P = p_1 + p_2 + p_3$. Thus $N_P \neq 0$ is exactly the condition that the barycentric triple represents a finite point. If $N_P = 0$, the point

is at infinity. For three finite points P, Q, R , the affine orientation has the sign of

$$\frac{[P, Q, R]}{N_P N_Q N_R}.$$

Therefore, the signs of the coordinate sums must be included when homogeneous barycentrics are used. Put

$$T = x^2 + y^2 + z^2 + xy + yz + zx.$$

For the coordinate forms in Table 1, direct simplification gives

$$N_{X_{14}} = -4h(T - 2h), \quad \sqrt{3} N_{X_{16}} = -4h(T - 2h).$$

Since $T \geq 2q \geq 2h$, with equality only in the equilateral case, these sums are negative for a scalene triangle. The coordinate sum of X_{20} is also negative in this normalization. For X_{18} , the coordinate sum is a positive multiple of

$$10pr - hT,$$

which changes sign on the scalene region. Consequently, X_{18} lies at infinity for some scalene triangles. No candidate surviving the exact elimination in Section 4 contains X_{18} . All orientation computations below include the appropriate normalization signs.

For a proposed cyclic order $P_1, P_2, \dots, P_n$, we first check that the polygon is strictly convex for one exact scalene triangle. We then verify that every successive affine orientation $[P_i, P_{i+1}, P_{i+2}]$ has constant nonzero sign while the side lengths remain in one fixed order, such as $a < b < c$. There are six such regions, one for each ordering of a, b, c , and they differ only by relabeling the vertices of the reference triangle.

We must also exclude the possibility that the polygon changes from an ordinary convex polygon to a self-intersecting polygon while all the successive turns keep the same sign. For this we use the *turning number*. As one travels once around a polygon, the direction of travel turns through a total angle equal to an integer multiple of 360° ; that integer is the turning number. A convex polygon has turning number 1, whereas a star-shaped traversal such as a pentagram has a different turning number. As the reference triangle varies continuously, this integer cannot change unless a degeneracy occurs. Consequently, once the successive turns stay nonzero and the polygon is convex for one triangle in a fixed side-order region, the same cyclic order remains convex throughout that region.

It is not enough merely to check that all successive triples have the same orientation. Figure 1 illustrates why. As we travel along the polygon $ABCDEFGH$, the path makes a 90° counterclockwise turn at every vertex. Nevertheless, the polygon is self-intersecting and is not convex. Its total turning is 720° , so its turning number is 2, rather than 1 as for a convex polygon.

Each center in Table 1 has first coordinate $f(a, b, c)$ with $f(a, b, c) = f(a, c, b)$, and its other coordinates are obtained cyclically. Interchanging two side lengths therefore interchanges two coordinates of every center. In the determinant $[P, Q, R]$, this interchanges two columns and changes the sign of the determinant. Hence $[P, Q, R]$ vanishes whenever two side lengths are equal and is therefore divisible by

$$V = (x - y)(x - z)(y - z).$$

The real question is whether the remaining factor keeps a fixed sign for all $x, y, z > 0$.

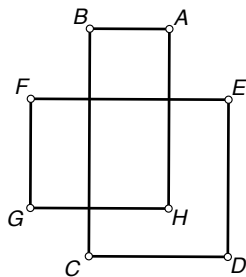


FIGURE 1. A self-intersecting polygon with a 90° counterclockwise turn at every vertex. Its turning number is 2, not 1.

3. TWO INTRODUCTORY EXAMPLES

3.1. A convex quadrilateral.

Theorem 3.1. *For every scalene triangle, the points*

$$X_1, X_2, X_7, X_6$$

in this order form a strictly convex quadrilateral.

Proof. Apart from positive common normalization factors,

$$\begin{aligned} [X_1, X_2, X_7] &= -V, \\ [X_2, X_7, X_6] &= -(x + y + z)V, \\ [X_7, X_6, X_1] &= -(xy + xz + yz)V, \\ [X_6, X_1, X_2] &= -V. \end{aligned}$$

Since

$$x + y + z > 0, \quad xy + xz + yz > 0,$$

all four successive affine turns have the same nonzero sign when the triangle is scalene. Under our convention $a < b < c$, we have $V > 0$, so all four turns are negative; that is, the displayed polygon is traversed in the sense opposite to that of triangle ABC . An exact check at any one scalene triangle gives the convex order displayed above. $\square$

3.2. The unique hexagon among the first ten centers. The first ten ETC centers are the incenter X_1 , centroid X_2 , circumcenter X_3 , orthocenter X_4 , nine-point center X_5 , symmedian point X_6 , Gergonne point X_7 , Nagel point X_8 , Mittenpunkt X_9 , and Spieker center X_{10} .

Theorem 3.2. *Among $X_1, \dots, X_{10}$, the unique six-element subset which forms a strictly convex hexagon for every scalene triangle is*

$$\{X_1, X_3, X_5, X_7, X_9, X_{10}\}.$$

One cyclic order, up to reversal and cyclic permutation, is

$$X_5, X_7, X_1, X_3, X_9, X_{10}.$$

Figure 2 shows this hexagon for a representative scalene triangle. The figure is illustrative only; Theorem 3.2 asserts that the same six centers are in convex position for every scalene reference triangle.

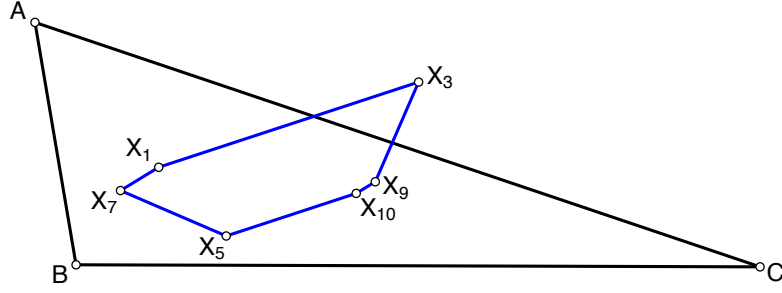


FIGURE 2. The universally convex hexagon $X_5X_7X_1X_3X_9X_{10}$ associated with a scalene triangle ABC .

Proof. For the six successive triples in the displayed cyclic order, the orientation determinants have the common factor V . After positive numerical and normalization factors are removed, the remaining factors may be taken as

triple	factor apart from V
$X_5X_7X_1$	$-xyz(x+y+z)$
$X_7X_1X_3$	$-xyz(x+y+z)$
$X_1X_3X_9$	$-(x+y)(x+z)(y+z)(x+y+z)$
$X_3X_9X_{10}$	$-xyz(x+y+z)$
$X_9X_{10}X_5$	$-xyz(x+y+z)$
$X_{10}X_5X_7$	$-(x+y+z)Q,$

where

$$Q = x^2y + x^2z + xy^2 + xyz + xz^2 + y^2z + yz^2 > 0.$$

Thus all six successive affine turns have the same nonzero sign for every scalene triangle. Under $a < b < c$, all six turns are negative. The choice $x : y : z = 4 : 2 : 1$ is simply the triangle with side lengths $(a, b, c) = (3, 5, 6)$; an exact check for this triangle shows that the displayed order is strictly convex. Hence the same order remains convex throughout the scalene region.

An exhaustive check of the other six-element subsets of $X_1, \dots, X_{10}$ shows that none is universal. $\square$

4. UNIVERSAL CONVEX HEXAGONS AMONG $X_1, \dots, X_{20}$

We now turn to the main classification. There are

$$\binom{20}{6} = 38\,760$$

six-element subsets of the first twenty centers.

The proof has three stages. First, exact evaluation on five Heronian triangles eliminates 38,721 of the 38,760 possibilities, leaving 39 survivors. Second, one survivor is eliminated by an exact degeneration on the right-triangle locus. Third, the remaining 38 candidates are verified symbolically by showing that the relevant orientation factors keep a fixed sign. Thus the classification is not based on random sampling.

Computer code (Python/SymPy) reproducing the exact elimination of Section 4.1, every sign certificate of Sections 3–4 and Appendix A, and the classification of Section 5 is provided as supplementary material. The principal scripts are

`elimination.py`, `verify_signs.py`, `classify_first_ten.py`, and `verify_heptagon.py` (Theorem 7.1).

4.1. Exact elimination by Heronian test triangles. We used the following five scalene Heronian triangles:

(a, b, c)	Δ
(3, 25, 26)	36
(58, 65, 119)	924
(10, 17, 21)	84
(28, 85, 111)	504
(591, 884, 1465)	61530.

For a Heronian triangle the side lengths and area are integers. With the coordinates in Table 1, every coordinate then has the form $u + v\sqrt{3}$ with u, v rational. Its sign can therefore be decided exactly, with no floating-point tolerance. The test triangles were also chosen to sample a wide range of shapes; their largest angles are approximately $106^\circ, 151^\circ, 99^\circ, 155^\circ$, and 166° , respectively.

Starting with all 38,760 six-element subsets, the exact survivor counts after the five test triangles were

$$38\,760 \longrightarrow 330 \longrightarrow 68 \longrightarrow 54 \longrightarrow 51 \longrightarrow 39.$$

Lemma 4.1. *Every universally convex hexagon formed from $X_1, \dots, X_{20}$ is either one of the 38 sets listed in Theorem 4.3, or the set*

$$\{X_1, X_2, X_4, X_6, X_{16}, X_{19}\}$$

discussed in Remark 4.2.

Thus the five test triangles rigorously eliminate 38,721 possibilities.

4.2. The right-triangle exception. One of the 39 exact survivors is not universal.

Remark 4.2. The set

$$\{X_1, X_2, X_4, X_6, X_{16}, X_{19}\}$$

survives all five test triangles above, but it is not universal. If, for example, $\angle C = 90^\circ$, then

$$X_4 = X_{19} = C,$$

so only five distinct points remain. Moreover,

$$[X_2, X_{19}, X_4] = V(px - yz)(py - zx)(pz - xy),$$

where

$$px - yz = \frac{b^2 + c^2 - a^2}{2}, \quad py - zx = \frac{c^2 + a^2 - b^2}{2}, \quad pz - xy = \frac{a^2 + b^2 - c^2}{2}.$$

Thus one factor vanishes on each right-triangle locus.

The six points are nevertheless in convex position for every *non-right* scalene triangle. Within a fixed side-order region, the cyclic order is

$$X_6 X_1 X_2 X_4 X_{19} X_{16}$$

on the acute part and

$$X_6 X_1 X_2 X_{19} X_4 X_{16}$$

on the obtuse part. The exact sign verification is given in Appendix A.1.

4.3. Symbolic verification of the remaining 38. The remaining 38 survivors were analyzed symbolically. For each candidate, a cyclic order was determined at an exact scalene triangle and the six successive affine orientations were evaluated exactly. After homogeneous coordinate factors were cleared, the cofactors were reduced to expressions in x, y, z, p, q, r, h , with $h^2 = 3pr$.

By a *sign certificate* we mean an exact algebraic argument proving that a cofactor is always positive or always negative for all allowed positive x, y, z . A certificate may be a factorization into obviously positive factors, an elementary inequality, or a substitution that turns the cofactor into a polynomial with nonnegative coefficients.

Two elementary inequalities are used repeatedly. First,

$$q^2 \geq 3pr, \quad \text{equivalently } q \geq h.$$

Indeed, with $u = xy, v = yz, w = zx$,

$$(u + v + w)^2 \geq 3(uv + vw + wu),$$

which follows from

$$(u - v)^2 + (v - w)^2 + (w - u)^2 \geq 0.$$

Second,

$$r \leq \frac{p^3}{27},$$

by AM–GM applied to x, y, z .

The factor $T - 2h$, which governs the coordinate sums of X_{14} and X_{16} in Section 2, is positive. Here

$$T = x^2 + y^2 + z^2 + xy + yz + zx.$$

Since $x^2 + y^2 + z^2 \geq q$ and $q \geq h$, we have $T \geq 2q \geq 2h$, with equality only in the equilateral case. Hence $T - 2h > 0$ for every scalene triangle.

Another recurring factor is

$$F = 3p^3q - 3p^2r - 2h(p^3 + r).$$

By homogeneity set $p = 1$. Then $h^2 = 3r$, so $3r = h^2$; also $q \geq h$ and $r \leq 1/27$, so $h \leq 1/3$. Thus

$$\begin{aligned} F &\geq 3h - 3r - 2h(1 + r) \\ &= h - 2hr - h^2 \\ &= h(1 - 2r - h) \\ &\geq h \left(1 - \frac{2}{27} - \frac{1}{3} \right) > 0. \end{aligned}$$

Several remaining cofactors vanish at the equilateral point. For these we choose one of the six possible orderings of x, y, z . For example, if $x \geq y \geq z > 0$, write

$$x = w + v + u, \quad y = w + v, \quad z = w, \quad u, v, w > 0.$$

This records the successive positive differences. In the required cases, the transformed polynomial has nonnegative coefficients and is not identically zero, hence is positive. The other five orderings are treated similarly. The remaining technical sign certificates, including the cases involving expressions of the form $A + Bh$, are summarized in Appendix A.2.

Theorem 4.3. *Exactly 38 six-element subsets of*

$$\{X_1, \dots, X_{20}\}$$

form a strictly convex hexagon for every scalene triangle.

For compactness, we abbreviate

$$\{X_{i_1}, X_{i_2}, \dots, X_{i_k}\}$$

by writing the indices with spaces between them. The 38 universal hexagons are

1 3 5 7 9 10	2 3 9 10 11 14	2 9 10 11 14 20
3 5 6 9 10 13	3 5 6 9 10 14	3 5 7 9 10 11
3 5 9 10 13 14	3 5 9 10 13 15	3 5 9 10 14 16
3 5 9 10 15 17	3 6 7 9 10 14	3 6 9 10 12 14
3 6 9 10 14 17	5 6 9 10 11 13	5 6 9 10 13 15
5 6 9 10 13 20	5 6 9 10 14 20	5 7 9 10 11 20
5 9 10 13 14 20	5 9 10 13 15 20	5 9 10 14 16 20
5 9 10 15 17 20	6 7 9 10 14 20	6 9 10 12 14 20
6 9 10 14 17 20	2 3 6 7 12 14	2 4 8 9 12 13
2 4 8 9 12 16	2 5 8 9 12 13	2 5 9 10 12 13
2 5 9 10 12 16	2 6 7 12 14 20	3 6 7 9 12 14
3 6 7 12 14 20	3 12 13 14 16 20	4 7 8 10 12 16
5 7 8 10 12 13	6 7 9 12 14 20.	

The first entry is the unique universal hexagon already obtained among $X_1, \dots, X_{10}$.

4.4. No universal heptagons. We use the classical four-point principle: a finite planar set is in convex position if and only if every four of its points are in convex position. Consequently, a set of centers is universally in convex position if and only if each of its four-element subsets is; in particular, every subset of a universally convex set is universally convex.

Theorem 4.4. *No seven centers among $X_1, \dots, X_{20}$ form a strictly convex heptagon for every scalene triangle.*

Proof. If a seven-element set were universally in convex position, then each of its seven six-element subsets would also be universally convex. A direct combinatorial check of the

$$\binom{20}{7} = 77\,520$$

seven-element subsets shows that no 7-set has all seven of its 6-subsets in the list of Theorem 4.3; in fact, no 7-set has more than three of its 6-subsets in that list. $\square$

Corollary 4.5. *There is no universally convex n -gon formed from $X_1, \dots, X_{20}$ for any $n \geq 7$. Thus six is the largest possible number of vertices.*

5. SMALLER CONVEX POLYGONS AMONG $X_1, \dots, X_{10}$

We now return to the first ten centers and record the smaller universally convex polygons. This material is secondary to the twenty-center hexagon classification, but it gives a more complete picture of the low-index centers.

5.1. **Pentagons.** There are exactly 14 universally convex pentagons:

13478 13479 134710
 13578 13579 135710
 135910 137910 14789
 15789 157910 356910
 357910 367910.

Six of these are 5-subsets of the hexagon in Theorem 3.2. If inherited cases are omitted, the eight additional pentagons are

13478 13479 134710 13578
 14789 15789 356910 367910.

5.2. **Quadrilaterals.** Of the

$$\binom{10}{4} = 210$$

four-element subsets of the first ten centers, exactly 87 form a strictly convex quadrilateral for every scalene triangle.

Every 4-subset of a universal pentagon or hexagon is automatically universal. After all inherited cases are removed, there remain exactly 46 additional universal convex quadrilaterals:

1237 1239 1246 1247 1256 1257
 1267 1456 1457 1468 14610 1568
 15610 1678 16710 2367 2369 23910
 2468 24610 2478 24710 2489 2568
 25610 2578 25710 2589 25910 2678
 26710 3468 3469 34610 3568 3678
 4589 45810 4689 46810 47810 5689
 56810 57810 6789 67810

Thus, the first-ten-center classification can be summarized nonredundantly as

1 universal hexagon, 8 additional pentagons,
 46 additional quadrilaterals.

6. ISOSCELES TRIANGLES

As noted in the Introduction, in an isosceles triangle every triangle center lies on the symmetry axis, so all the polygons considered here collapse to a line.

Algebraically, this is reflected in the factor

$$V = (x - y)(x - z)(y - z),$$

which vanishes whenever two side lengths are equal. The right-triangle exception in Remark 4.2 illustrates a different phenomenon: additional shape loci can occur when the cofactor multiplying V vanishes.

7. AREAS FOR FUTURE RESEARCH

There are several natural avenues for further investigation.

7.1. Investigate higher-index triangle centers. The complete classification in this paper stops at X_{20} , but an exploratory search among higher-index centers already shows that new phenomena occur. The first example we found is the universal convex heptagon in the next subsection. It would be interesting to extend the systematic classification to $X_1, \dots, X_N$ for larger N and to study how the number and structure of universal convex polygons change.

A computational search can be organized efficiently: first use exact test triangles to eliminate most candidate subsets, and then apply symbolic orientation tests only to the survivors.

7.2. A universal convex heptagon. The universal hexagon from Theorem 3.2 can be enlarged by using a higher-index center.

Theorem 7.1. *For every scalene triangle, the seven centers*

$$X_5, X_{79}, X_7, X_1, X_3, X_9, X_{10}$$

in this cyclic order form a strictly convex heptagon.

Figure 3 shows the heptagon for a representative scalene triangle.

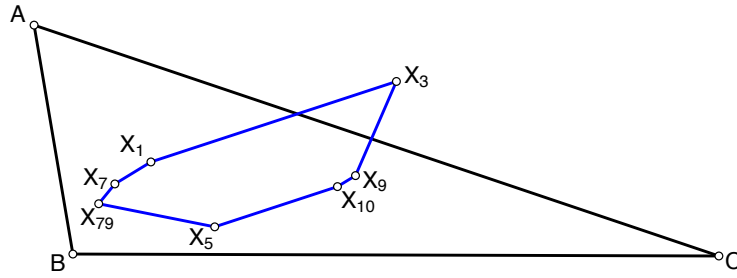


FIGURE 3. A universal convex heptagon obtained by adjoining X_{79} to $X_5X_7X_1X_3X_9X_{10}$.

Proof. Let

$$\Sigma = x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2.$$

Using the coordinates of X_{79} in Table 1, a direct symbolic calculation gives the positive coordinate sum

$$N_{X_{79}} = 2p(3\Sigma + 14xyz) > 0.$$

For the seven successive triples in the displayed cyclic order, after positive common normalization factors are removed, the orientation determinants are

$$\begin{aligned} [X_5, X_{79}, X_7] &= -16xyz pV(7\Sigma + 22xyz), \\ [X_{79}, X_7, X_1] &= -16xyz pV, \\ [X_7, X_1, X_3] &= -16xyz pV, \\ [X_1, X_3, X_9] &= -8pV(x+y)(x+z)(y+z), \\ [X_3, X_9, X_{10}] &= -16xyz pV, \\ [X_9, X_{10}, X_5] &= -16xyz pV, \\ [X_{10}, X_5, X_{79}] &= -8p^3V(3\Sigma - 2xyz). \end{aligned}$$

All factors other than V are positive. For the final factor, AM–GM applied to the six terms in Σ gives $\Sigma \geq 6xyz$, and hence

$$3\Sigma - 2xyz \geq 16xyz > 0.$$

Thus all seven successive affine turns have the same nonzero sign for every scalene triangle. An exact check for one triangle, for example $(a, b, c) = (13, 14, 15)$, shows that the displayed order is strictly convex. The turning-number argument of Section 2 then proves convexity throughout each fixed side-order region, and relabeling covers all scalene triangles. $\square$

This raises the next natural question: can a universal convex octagon be formed from triangle centers? A search obtained by adjoining one of $X_{80}, \dots, X_{100}$ to the heptagon of Theorem 7.1 found no such example, but a systematic search among higher-index centers remains open.

7.3. Classify pentagons and quadrilaterals among the first twenty centers. For $X_1, \dots, X_{10}$, we determined all universal convex quadrilaterals and pentagons. In the present paper, the twenty-center extension was restricted to hexagons. A complete classification of universal convex pentagons and quadrilaterals among $X_1, \dots, X_{20}$ would provide a fuller picture of the underlying order structure.

Because there are many more small subsets than hexagons, a useful goal would be to find a compact representation in terms of maximal convex families rather than simply listing every subset.

7.4. Explain the structure of the 38 hexagons. The list in Theorem 4.3 is explicit but not conceptual. Look for geometric or combinatorial principles that explain why these 38 sets occur and the others do not.

For example, one might form a hypergraph whose vertices are $X_1, \dots, X_{20}$ and whose hyperedges are the 38 universal hexagons. Its intersections, frequently occurring pairs or triples, and symmetries may point toward a shorter description of the classification.

7.5. Study exceptional shape loci. The set

$$\{X_1, X_2, X_4, X_6, X_{16}, X_{19}\}$$

is convex for every non-right scalene triangle but degenerates on the right-triangle locus. Similar phenomena can occur when an orientation cofactor has zeros away from the isosceles locus.

A systematic study could classify triples of centers whose orientation changes as the shape of the reference triangle varies and determine the corresponding algebraic curves in triangle-shape space. Such loci may have independent geometric interpretations.

7.6. Seek geometric proofs. The present results are primarily analytic and computational. The orientation-determinant method proves universality efficiently, but it does not explain geometrically why particular collections of centers surround one another in a fixed order.

It would be valuable to find synthetic proofs for selected cases, especially the hexagon

$$X_5X_7X_1X_3X_9X_{10}.$$

For example, one could look for familiar geometric constructions or separation properties showing directly that each vertex lies on the correct side of the lines through neighboring vertices, or for affine relations among the six centers that determine their cyclic order. Such an argument would explain the convexity geometrically rather than by checking determinant signs.

7.7. Develop general sign criteria for center functions. For the symmetric center coordinates used in this paper, the Vandermonde factor in an orientation determinant is automatic, as explained in Section 2. The essential question is whether the remaining cofactor has a fixed sign for positive x, y, z .

It would be useful to find general hypotheses on three center functions that guarantee such a fixed sign. A theorem of this type could replace many individual symbolic verifications and might extend the present results to infinite families of triangle centers.

APPENDIX A. ADDITIONAL SIGN CERTIFICATES

A.1. The non-right hexagon in Remark 4.2. For the set

$$\{X_1, X_2, X_4, X_6, X_{16}, X_{19}\},$$

consider the twenty orientations of triples chosen from the six points. Exactly the four triples containing both X_4 and X_{19} ,

$$(X_1, X_4, X_{19}), \quad (X_2, X_4, X_{19}), \quad (X_4, X_6, X_{19}), \quad (X_4, X_{16}, X_{19}),$$

contain the factor

$$(px - yz)(py - zx)(pz - xy)$$

to the first power. The other sixteen cofactors have fixed nonzero sign.

One of these sixteen, arising from $[X_2, X_{16}, X_{19}]$, has the form $A + Bh$ with $B > 0$ and A indefinite. Since $h \leq q$,

$$h = \frac{3pr}{h} \geq \frac{3pr}{q},$$

and therefore

$$A + Bh \geq \frac{qA + 3prB}{q}.$$

To verify $qA + 3prB > 0$, suppose $x \geq y \geq z > 0$ and write

$$x = w + v + u, \quad y = w + v, \quad z = w, \quad u, v, w > 0.$$

The transformed polynomial has nonnegative coefficients and is not identically zero. The other five orderings are treated in the same way. Thus, within each fixed side-order region, every triple orientation is constant on the acute part and on the obtuse part, each of which is connected. Exact checks in the two parts give the cyclic orders stated in Remark 4.2.

A.2. Cofactors of the form $A + Bh$. The sign computations for the 38 universal hexagons reduce to several standard types. The most technical are cofactors of the form $A + Bh$, where A and B are polynomials in p, q, r . They are handled in one of four ways:

- (1) A and B have the same fixed sign.
- (2) A has a fixed sign and

$$A^2 - 3prB^2 > 0.$$

Since $h^2 = 3pr$, this gives $|A| > |B|h$, so $A + Bh$ has the sign of A .

- (3) B has a fixed sign and

$$A^2 - 3prB^2 < 0.$$

Then $|B|h > |A|$, so $A + Bh$ has the sign of B .

- (4) $A > 0$ and $A + Bq > 0$. If $B \geq 0$, then $A + Bh \geq A > 0$. If $B < 0$, then $h \leq q$ gives

$$A + Bh \geq A + Bq > 0.$$

The inequality $A + Bq > 0$ is certified by the positive-coefficient substitution described in Section 4.3.

Together with direct positive factorizations, the two worked inequalities in Section 4.3, and the ordered-variable substitutions, these cases give exact fixed-sign certificates for all successive orientations required in Theorem 4.3.

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REFERENCES

- [1] Clark Kimberling, *Encyclopedia of Triangle Centers*, University of Evansville.
<https://faculty.evansville.edu/ck6/encyclopedia/>
- [2] Clark Kimberling, *Triangle Centers and Central Triangles*, *Congressus Numerantium* **129** (1998), 1–295.
- [3] Lawrence S. Evans, “Some Configurations of Triangle Centers,” *Forum Geometricorum* **3** (2003), 49–56.
- [4] Mowaffaq Hajja, “Triangle centres: some questions in Euclidean geometry,” *International Journal of Mathematical Education in Science and Technology* **32** (2001), 21–36.
- [5] Christopher J. Bradley and Geoffrey C. Smith, “The Locations of Triangle Centres,” *Forum Geometricorum* **6** (2006), 57–70.