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# Inequalities for Circles in the Corners of Convex Polygons

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**Abstract.** Let a circle of radius $r$ be tangent to the sides of an angle, and let a second circle, nearer the vertex, be tangent to the same sides and to the first circle. We study sums of the radii of such corner circles at selected vertices of a triangle or convex polygon. We establish sharp two-sided bounds for two corners of a triangle, for all but one corner of a convex $n$-gon, and for all its corners.

**Keywords.** tangent circles, corner circles, convex polygons, geometric inequalities, sharp bounds.

**Mathematics Subject Classification (2020).** 51M16, 52A10.

## 1. Introduction

Let $ABC$ be a triangle. Place a circle of radius $r$ in the corner at $B$, tangent to $BA$ and $BC$, and another circle of the same radius in the corner at $C$, tangent to $CA$ and $CB$. In each corner, insert a smaller circle tangent to its two sides and to the corresponding radius-$r$ circle, as in Figure 1. Write $y$ and $z$ for their radii.

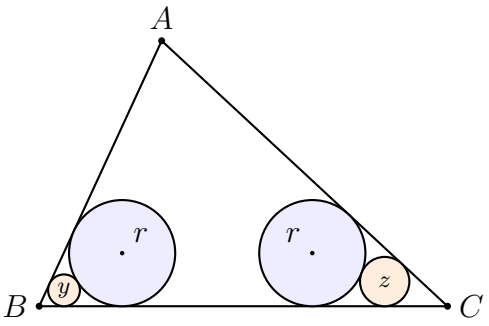



Figure 1. Two circles of radius $r$ at $B$ and $C$, with smaller corner circles of radii $y$ and $z$.

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Our starting observation is the sharp inequality

$$\boxed{y + z > (6 - 4\sqrt{2})r.} \tag{1}$$

The inequality is strict for every nondegenerate triangle, and its constant is best possible. We shall give a geometric proof, sharpen the bound when the angle at $A$ is prescribed, and obtain sharp upper bounds as well. The method will extend to selected corners of convex polygons and lead to a criterion for realizable pairs $(y, z)$. We also obtain a polygonal identity generalizing a familiar three-corner identity for a triangle.

The local relation between the two radii appears in solutions to GoGeometry Problem 454 [2]; a related three-corner triangle inequality was proposed in *Romanian Mathematical Magazine* [1]. Here our emphasis is on sharp sums of corner radii over convex polygons, especially when a vertex is omitted and its angle is fixed.

## 2. The corner-circle construction

For an angle $0 < \theta < \pi$, let a circle of radius $r > 0$ be tangent to both sides (Figure 2). A circle of radius $x$ lies nearer the vertex, is tangent to both sides, and is externally tangent to the radius-$r$ circle. We always mean this smaller, vertex-side circle. The two centers lie on the internal angle bisector. Their separation is $r + x$, while their distances from the vertex are $r/\sin(\theta/2)$ and $x/\sin(\theta/2)$. Thus

$$r + x = \frac{r}{\sin(\theta/2)} - \frac{x}{\sin(\theta/2)}.$$

Solving for $x/r$ and using a standard half-angle identity gives

$$\frac{x}{r} = \frac{1 - \sin(\theta/2)}{1 + \sin(\theta/2)} = \tan^2\left(\frac{\pi - \theta}{4}\right). \tag{2}$$

To verify the last equality, set $u = (\pi - \theta)/4$. Then $\sin(\theta/2) = \cos(2u)$, and $\cos(2u) = (1 - \tan^2 u)/(1 + \tan^2 u)$. The formula depends only on the angle, not on the position or scale of the initial circle. In a polygon, a radius-$r$ circle is constructed at each selected corner independently. The assertions do not require these circles to be mutually disjoint. If desired, the polygon can be enlarged until each circle lies inside it, without changing the corner radii.

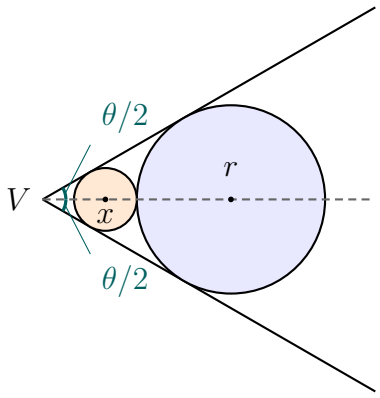



FIGURE 2. Two circles tangent to the sides of an angle and to each other. Their centers lie on the angle bisector.

## 3. A GEOMETRIC EQUAL-ANGLE PRINCIPLE

The proofs in the following sections will require inequalities for sums of squared tangents. We establish them here, beginning with a simple construction involving two rays. The basic idea is that replacing two unequal angles by their average decreases the sum of their squared tangents, whereas spreading the angles farther apart increases it.

**Lemma 3.1** (Two-ray symmetrization)**.** *For* $0 \le \alpha, \beta \le \pi/4$,

$$\tan^2 \alpha + \tan^2 \beta \ge 2 \tan^2 \left( \frac{\alpha + \beta}{2} \right), \tag{3}$$

*with equality if and only if* $\alpha = \beta$.

*Geometric proof.* Take a unit segment $OT$ and draw the perpendicular to $OT$ at $T$, as shown in Figure 3. Rays from $O$ making angles $\alpha$ and $\beta$ with $OT$ meet this perpendicular at $P$ and $Q$. If $\alpha = \beta$, the assertion is immediate. Otherwise assume $\alpha < \beta$, so $P$ lies between $T$ and $Q$. Let the bisector of $\angle POQ$ meet $PQ$ at $N$. The angle-bisector theorem and the Pythagorean theorem give

$$\frac{PN}{NQ} = \frac{OP}{OQ} < 1.$$

Thus $N$ lies strictly below the midpoint of $PQ$, and hence

$$TN < \frac{TP + TQ}{2}, \qquad 2TN^2 < \frac{(TP + TQ)^2}{2} \le TP^2 + TQ^2,$$

where the last step follows from $(TP - TQ)^2 \ge 0$. Since $OT = 1$, the three vertical lengths are respectively $\tan((\alpha + \beta)/2)$, $\tan \alpha$, and $\tan \beta$. This proves the strict form of (3) for unequal angles. Equality holds precisely when $P = Q$. □

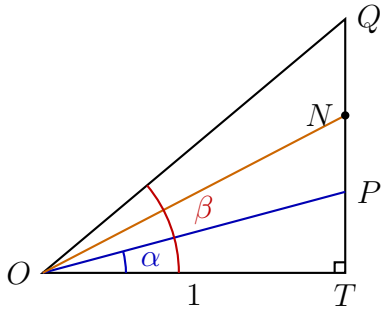



FIGURE 3. The two-ray symmetrization lemma. The marked angles are $\alpha = \angle POT$ and $\beta = \angle QOT$; $ON$ bisects $\angle POQ$. The angle-bisector theorem places $N$ below the midpoint of $PQ$.

**Lemma 3.2** (Strict convexity)**.** *The function* $f(t) = \tan^2 t$ *is strictly convex on* $[0, \pi/4]$.

*Proof.* Lemma 3.1 gives the strict midpoint inequality

$$f\left( \frac{a + b}{2} \right) < \frac{f(a) + f(b)}{2} \qquad (a \ne b).$$

Applying the midpoint inequality repeatedly gives the same inequality for points one quarter, three quarters, one eighth, and so on, of the way from $a$ to $b$. In

fact, after repeated halving it holds for every fraction $m/2^j$ of the way along the segment joining $a$ and $b$. Such fractions can be made arbitrarily close to any number between 0 and 1. Since $f(t) = \tan^2 t$ is continuous, it follows that

$$f((1-\lambda)a + \lambda b) \le (1-\lambda)f(a) + \lambda f(b) \qquad (0 \le \lambda \le 1).$$

This is the definition of convexity. The inequality is strict when $a \ne b$ and $0 < \lambda < 1$: otherwise the graph of a convex function would follow the straight chord between $(a, f(a))$ and $(b, f(b))$, contradicting the strict midpoint inequality. □

*A calculus check.* For readers familiar with derivatives, strict convexity also follows immediately from

$$f''(t) = 2\sec^2 t\,(1 + 3\tan^2 t) > 0.$$

**Lemma 3.3** (Equal-angle principle)**.** *If $t_1, \dots, t_k \in (0, \pi/4)$ and $t_1 + \cdots + t_k = T$, then*

$$\sum_{i=1}^{k} \tan^2 t_i \ge k \tan^2 \frac{T}{k}, \tag{4}$$

*with equality precisely when all $t_i = T/k$.*

*Proof.* By Lemma 3.2, $f(t) = \tan^2 t$ is strictly convex. Jensen's inequality states that the value of a convex function at the average of several numbers is no greater than the average of its values at those numbers. Here it says

$$\tan^2\left(\frac{t_1 + \cdots + t_k}{k}\right) \le \frac{\tan^2 t_1 + \cdots + \tan^2 t_k}{k},$$

which is exactly (4) after multiplying by $k$. Strict convexity shows that equality is possible only when all the angles are equal. □

**Lemma 3.4** (Angle-spreading principle)**.** *Let $k \ge 2$, $0 < t_i < \pi/4$, and*

$$T = t_1 + \cdots + t_k, \qquad \frac{\pi}{4} < T \le \frac{\pi}{2}.$$

*Then*

$$\sum_{i=1}^{k} \tan^2 t_i < 1 + \tan^2\left(T - \frac{\pi}{4}\right). \tag{5}$$

*The bound is best possible for each admissible $k$ and $T$.*

*Geometric proof.* Put $f(t) = \tan^2 t$ on $[0, \pi/4]$. By Lemma 3.2, $f$ is strictly convex, so spreading two arguments apart while preserving their sum strictly increases the sum of their $f$-values.

Starting with the given angles, choose any two strictly inside $(0, \pi/4)$, say $a \le b$, and set $h = \min\{a, \pi/4 - b\} > 0$. Replace $(a, b)$ by $(a - h, b + h)$. Their sum is preserved, and strict convexity gives

$$f(b + h) - f(b) > f(a) - f(a - h),$$

so the sum of squared tangents strictly increases. At least one of these two angles reaches 0 or $\pi/4$. Repeat the process whenever two angles remain strictly inside the interval. Each step reduces their number, so the process terminates after

finitely many steps with at most one angle in $(0, \pi/4)$. Since $\pi/4 < T \le \pi/2$, the final list, up to order, must be

$$\frac{\pi}{4}, \quad T - \frac{\pi}{4}, \quad 0, \ldots, 0.$$

Its sum of squared tangents is the right side of (5). Because all the original angles were strictly inside the interval, at least one strict increase occurred. Approximating the final list by positive angles below $\pi/4$ proves sharpness. □

## 4. Two corners of a triangle

Let $ABC$ be a nondegenerate triangle. Construct a circle of radius $r$ tangent to the two sides at $B$, and an independent circle of the same radius tangent to the two sides at $C$. At each corner construct the smaller circle tangent to those sides and to the corresponding radius-$r$ circle. Denote their radii by $y$ and $z$, respectively (Figure 1).

**Theorem 4.1.** *For every nondegenerate triangle,*

$$y + z > (6 - 4\sqrt{2})r. \tag{6}$$

*The constant is best possible. More precisely, for a fixed apex angle $A$, the sharp two-sided bounds are*

$$2r \tan^2\left(\frac{\pi + A}{8}\right) \le y + z < r \sec^2\left(\frac{A}{4}\right). \tag{7}$$

*Equality on the left holds exactly when $B = C$. The upper bound is strict and best possible.*

*Geometric proof.* By formula (2), let

$$\alpha = \frac{\pi - B}{4}, \qquad \beta = \frac{\pi - C}{4}, \qquad \frac{y+z}{r} = \tan^2\alpha + \tan^2\beta.$$

The angle sum of the triangle gives

$$\frac{\alpha + \beta}{2} = \frac{\pi + A}{8}.$$

Apply Lemma 3.1 to obtain the lower estimate in (7). To deduce the universal lower bound, put $\varphi = (\alpha + \beta)/2 = (\pi + A)/8$. Since $A > 0$, we have $\varphi > \pi/8$. Figure 4 illustrates why $\tan\varphi > \tan(\pi/8)$: draw a horizontal segment $OT$ of length 1, and let each ray from $O$ meet the perpendicular line through $T$. The larger the angle made by a ray with $OT$, the higher it meets that line. The two heights in the figure are $TN = \tan\varphi$ and $TN_0 = \tan(\pi/8)$. Bisecting the 45° angle of an isosceles right triangle gives $\tan(\pi/8) = \sqrt{2} - 1$. Consequently,

$$\frac{y+z}{r} \ge 2\tan^2\varphi > 2(\sqrt{2} - 1)^2 = 6 - 4\sqrt{2}.$$

The equality criterion in the geometric averaging step is $B = C$. Finally, as $A \to 0^+$ through isosceles triangles, the ratio in (6) approaches $6 - 4\sqrt{2}$. Thus the bound is strict but optimal.

For the upper bound, put $T = \alpha + \beta = (\pi + A)/4$. Lemma 3.4 gives

$$\frac{y+z}{r} < 1 + \tan^2\left(T - \frac{\pi}{4}\right) = \sec^2\frac{A}{4}.$$

For fixed $A$, letting $B \to 0^+$ and $C \to \pi - A$ gives $\alpha \to \pi/4$ and $\beta \to A/4$. The upper bound is therefore sharp and is never attained. □

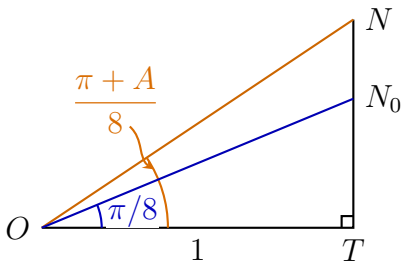



FIGURE 4. Comparison of two rays from $O$. The ray $ON_0$ (blue) makes angle $\pi/8$ with $OT$, while the ray $ON$ (orange) makes the larger angle $\varphi = (\pi + A)/8$. Because $OT = 1$, $TN_0 = \tan(\pi/8) = \sqrt{2} - 1$ and $TN = \tan\varphi > TN_0$.

*Remark* 4.2. Only the two corner angles are used. If $ABC$ is a right triangle with $A = \pi/2$, substituting this value into (7) gives $(\pi + A)/8 = 3\pi/16$ and $A/4 = \pi/8$. Thus the sharp bounds become

$$2r \tan^2 \frac{3\pi}{16} \le y + z < r \sec^2 \frac{\pi}{8}.$$

Equality on the left holds for $B = C = \pi/4$, the right isosceles triangle. The upper bound is approached, but not attained, as one of $B, C$ tends to zero.

**Proposition 4.3** (Admissible pairs of corner radii)**.** *Given $r > 0$ and $0 < y, z < r$, there exists a nondegenerate triangle whose two-corner construction has these radii if and only if*

$$\boxed{\sqrt{ry} + \sqrt{rz} + \sqrt{yz} > r.} \tag{8}$$

*Proof.* Set $u = \sqrt{y/r}$ and $v = \sqrt{z/r}$. Formula (2), applied at $B$ and $C$, gives

$$u = \tan\frac{\pi - B}{4}, \qquad v = \tan\frac{\pi - C}{4}.$$

Since the two angles lie between 0 and $\pi/4$, taking inverse tangents gives

$$\alpha = \arctan u = \frac{\pi - B}{4}, \qquad \beta = \arctan v = \frac{\pi - C}{4}.$$

Thus $\alpha + \beta = (\pi + A)/4 > \pi/4$. Since $0 < u, v < 1$, the tangent addition formula gives

$$\frac{u + v}{1 - uv} > 1, \qquad \text{or equivalently} \qquad u + v + uv > 1,$$

which is (8).

Conversely, suppose (8) holds and define

$$B = \pi - 4\arctan u, \qquad C = \pi - 4\arctan v.$$

These angles are positive, and $u + v + uv > 1$ implies $B + C < \pi$. Hence $A = \pi - B - C > 0$, so $A, B, C$ are the angles of a nondegenerate triangle.

Construct circles of radius $r$ at $B$ and $C$. At $B$, formula (2) gives the smaller radius

$$r\tan^2\left(\frac{\pi - B}{4}\right) = r\tan^2(\arctan u) = ru^2 = y.$$

Similarly, the smaller radius at $C$ is $rv^2 = z$. The triangle may be scaled to accommodate the circles. □

## 5. Selected corners of a convex polygon

Let $n \geq 3$, and let $P_1 \cdots P_n$ be a strictly convex $n$-gon. Write $\theta_i$ for its interior angles and $\delta_i = \pi - \theta_i$ for its exterior turning angles. Then $0 < \delta_i < \pi$ and

$$\delta_1 + \cdots + \delta_n = 2\pi. \tag{9}$$

At each selected vertex construct a circle of radius $r$ and its smaller corner circle of radius $x_i$. Formula (2) becomes

$$x_i = r\tan^2\frac{\delta_i}{4}. \tag{10}$$

Figure 5 shows the construction at all five vertices of a convex pentagon.

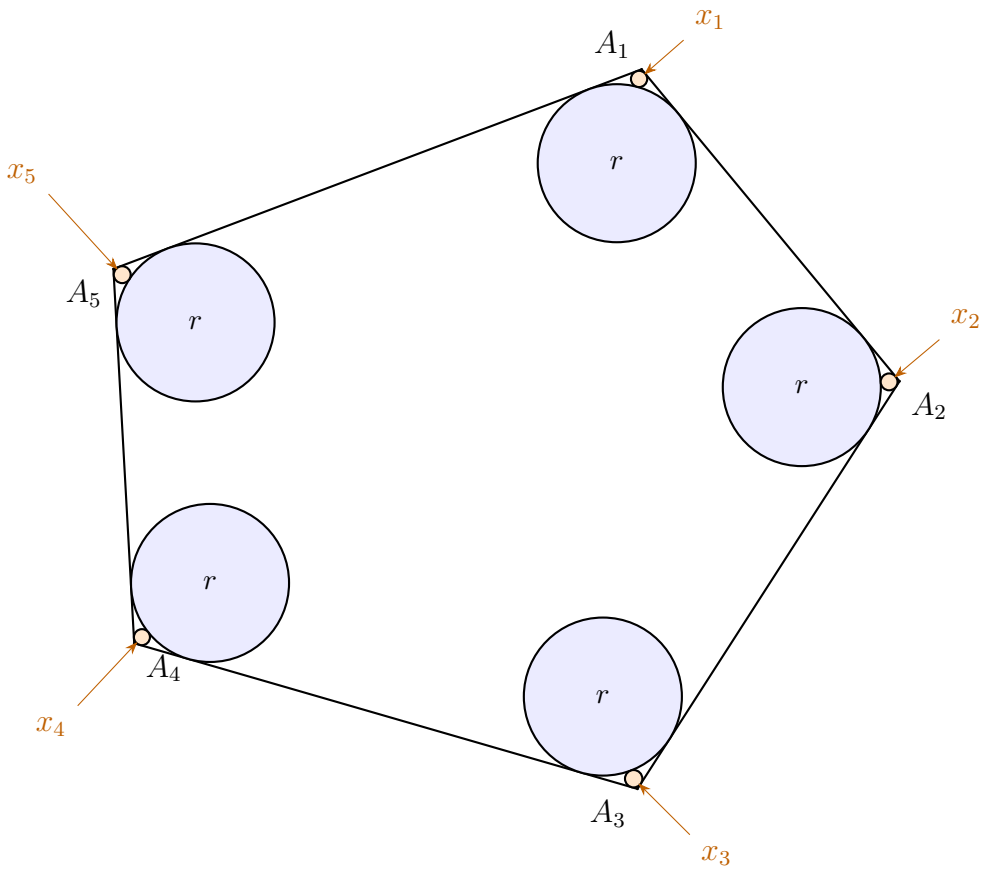



FIGURE 5. The all-corners construction for a convex pentagon. Each large circle has radius $r$; the smaller corner circles have radii $x_1, \ldots, x_5$.

**Theorem 5.1** (All corners)**.** *For $n \geq 3$, if all $n$ vertices are selected, then*

$$\boxed{nr \tan^2 \frac{\pi}{2n} \leq \sum_{i=1}^{n} x_i < 2r.} \tag{11}$$

*Equality on the left holds if and only if the polygon is equiangular. The upper bound is strict and best possible.*

*Geometric proof.* By (9), the quarter-angles $\delta_i/4$ sum to $\pi/2$. Apply Lemma 3.3 and multiply by $r$. Equality in the lower bound requires $\delta_i = 2\pi/n$ for every $i$. Such a polygon exists (for example, a regular $n$-gon).

For the upper bound, apply Lemma 3.4 with $T = \pi/2$. In the limiting configuration, two quarter-angles approach $\pi/4$ and the rest approach zero. This configuration can be approximated by strictly convex $n$-gons: choose positive exterior angles less than $\pi$ whose first two values tend to $\pi$, while the others tend to zero, and whose sum is $2\pi$. These angle lists are realizable, for example, by drawing successive tangents to a circle. Therefore $2r$ is the unattained supremum. □

**Theorem 5.2** (All but one corner; prescribed omitted angle)**.** *Let $n \geq 3$. Suppose the vertex $P_n$ is omitted and its interior angle is $\theta \in (0, \pi)$. Then*

$$\boxed{(n-1)r \tan^2\left(\frac{\pi + \theta}{4(n-1)}\right) \leq \sum_{i=1}^{n-1} x_i < r \sec^2 \frac{\theta}{4}.} \tag{12}$$

*Both bounds are sharp. Equality on the left is attained precisely when the remaining $n-1$ interior angles are equal, each being*

$$\frac{(n-2)\pi - \theta}{n-1}. \tag{13}$$

*Consequently, without prescribing the omitted angle,*

$$\boxed{(n-1)r \tan^2 \frac{\pi}{4(n-1)} < \sum_{i=1}^{n-1} x_i < 2r}. \tag{14}$$

*Both strict bounds are best possible.*

*Geometric proof.* The exterior angle at $P_n$ is $\pi - \theta$. By (9), the selected exterior angles sum to $2\pi - (\pi - \theta) = \pi + \theta$. Apply Lemma 3.3 to their quarter-angles. Equality means all selected exterior angles are $(\pi + \theta)/(n-1)$, giving (13).

The equality angle lists are realizable by strictly convex polygons: all exterior turns lie between 0 and $\pi$ and sum to $2\pi$; for instance, one may choose successive support lines tangent to a circle with these prescribed changes of direction. Thus the lower bound in (12) is attained.

For the upper bound, the selected quarter-angles sum to $T = (\pi+\theta)/4$. Lemma 3.4 yields

$$\sum_{i=1}^{n-1} \frac{x_i}{r} < 1 + \tan^2\Big(T - \frac{\pi}{4}\Big) = \sec^2 \frac{\theta}{4}.$$

The limiting selected exterior angles are $(\pi, \theta, 0, \ldots, 0)$, while the omitted exterior angle remains $\pi - \theta$. Approximating these limits by strictly convex angle lists proves sharpness; the upper bound is never attained.

Because $\theta > 0$, the lower bound in (12) is strictly greater than the lower bound in (14). Letting $\theta \to 0^+$ in the equality configurations of (12) shows that the universal lower bound is optimal. The upper bound follows immediately from Theorem 5.1, because the omitted radius is positive. It is sharp as $\theta \to \pi^-$ with two selected exterior angles tending to $\pi$ and all the other exterior angles tending to zero. □

The fixed-omitted-angle inequality (12) unifies the sharp two-corner triangle result and the $(n-1)$-corner polygon result. We have not located earlier statements of the sharp polygonal bounds in Theorems 5.1 and 5.2, although equivalent trigonometric inequalities may exist in other formulations.

## 6. Three corners of a triangle

We first consider the classical configuration in which the radius-$r$ circle is the incircle of $ABC$. At each vertex construct a smaller circle tangent to the incircle and to the two sides through that vertex. Denote their radii at $A, B, C$ by $x, y, z$, respectively. The case $n = 3$ of Theorem 5.1 reads

$$x + y + z \geq r, \tag{15}$$

with equality for an equilateral triangle. This inequality is not new: it was posed by Marian Ursărescu as Problem X.78 in the *Romanian Mathematical Magazine*, no. 27 (2020), p. 67, for acute triangles [1]. Although posed for acute triangles, the inequality holds for every triangle, either as the case $n = 3$ of Theorem 5.1 or directly from the identity (16) below. Figure 6 illustrates the equality case.

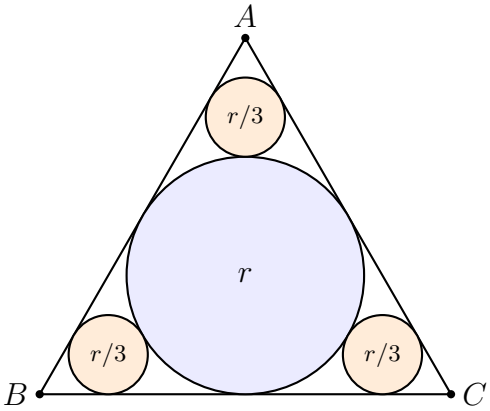



FIGURE 6. The equilateral equality case of $x + y + z \geq r$. Each of the three corner circles has radius $r/3$.

An earlier-known geometric identity supplies another proof when the same radius-$r$ circle is the incircle of $ABC$:

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = r. \tag{16}$$

This is explicitly stated as GoGeometry Problem 454 by Antonio Gutierrez [2]. It also appears as Problem 5374 in [3], together with its published solutions. Figure 7 illustrates the identity.

Subtracting the left side of (16) from $x + y + z$ gives

$$x + y + z - r = \frac{1}{2}\left[(\sqrt{x} - \sqrt{y})^2 + (\sqrt{y} - \sqrt{z})^2 + (\sqrt{z} - \sqrt{x})^2\right] \geq 0.$$

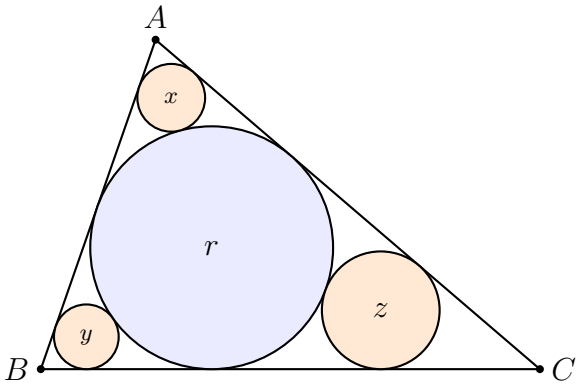



FIGURE 7. Three smaller circles tangent to the common incircle of a scalene triangle. Their radii $x, y, z$ satisfy $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = r$.

We include (15) only for completeness and do not claim originality for it or for (16). Indeed, the admissibility condition (8) can also be obtained from (16) by eliminating the third radius $x$.

The three larger circles need not be the incircle. In Figure 8, a separate circle of radius $r$ is placed at each vertex of a scalene triangle; the associated smaller circles again have radii $x, y, z$.

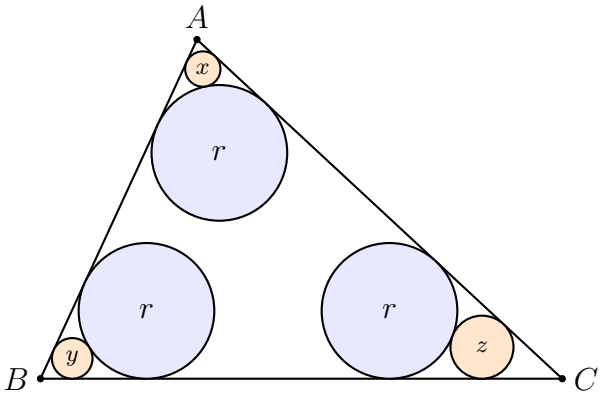



FIGURE 8. Three congruent circles of radius $r$, one at each corner of a scalene triangle, with vertex-side tangent circles of radii $x, y, z$. Their radii satisfy $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = r$.

**Proposition 6.1** (Three congruent initial circles)**.** *For any triangle $ABC$, construct at each vertex a circle of radius $r$, tangent to the two sides through that vertex. Let $x, y, z$ be the radii of the smaller circles tangent to the corresponding radius-$r$ circles and to the same two sides. Then*

$$\boxed{\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = r.} \tag{17}$$

*Moreover, $x + y + z \geq r$, with equality if and only if $ABC$ is equilateral.*

*Proof.* Let $\rho$ be the inradius of $ABC$, and let $x', y', z'$ be the smaller corner-circle radii when the common initial circle is the incircle. Formula (2) says that, at a given corner, the smaller radius divided by the larger radius depends only on the

angle of that corner. For example, the two constructions at $A$ give

$$\frac{x}{r} = \tan^2 \frac{\pi - A}{4} = \frac{x'}{\rho}.$$

The same argument at $B$ and $C$ gives

$$(x', y', z') = \frac{\rho}{r}(x, y, z).$$

Applying the known identity (16) to $x', y', z'$ and cancelling $\rho/r$ proves (17). The inequality and its equality case now follow from the sum-of-squares identity displayed immediately after (16) and the strict dependence of each corner-radius ratio on the vertex angle. □

## 7. A generalization of the three-corner identity

The identity (16) extends to all convex polygons. The initial circles may be a common incircle when the polygon is tangential, or separate circles of the same radius at its vertices. The general identity follows directly from the classical cosine-of-a-sum formula; its interest lies in the geometric interpretation of the terms as products of corner-circle radii. Let

$$u_i = \sqrt{\frac{x_i}{r}}, \qquad e_m(u_1, \ldots, u_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} u_{i_1} \cdots u_{i_m}.$$

**Theorem 7.1** (Alternating corner-radius identity)**.** *For every strictly convex $n$-gon, $n \ge 3$,*

$$\boxed{\sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^{j+1} e_{2j}(u_1, \ldots, u_n) = 1.} \tag{18}$$

*Equivalently, in terms of the corner radii,*

$$\sum_{i<j} \sqrt{x_i x_j} - \frac{1}{r} \sum_{i<j<k<l} \sqrt{x_i x_j x_k x_l} + \frac{1}{r^2} \sum_{i_1 < \cdots < i_6} \sqrt{x_{i_1} \cdots x_{i_6}} - \cdots = r.$$

*The last term involves products of $2\lfloor n/2 \rfloor$ radii.*

*Proof.* Put $t_i = (\pi - \theta_i)/4$, where $\theta_i$ is the interior angle at the $i$th vertex. By (10), $u_i = \tan t_i$, and by (9), $\sum_i t_i = \pi/2$. Expanding the cosine of this sum gives

$$0 = \cos\Big(\sum_i t_i\Big) = \Big(\prod_i \cos t_i\Big)\big(1 - e_2(u) + e_4(u) - e_6(u) + \cdots\big).$$

All the factors $\cos t_i$ are positive. Dividing by their product proves (18). □

For $n = 3$, this is also Proposition 6.1. The first higher-order case has an especially simple geometric interpretation. Figure 9 shows four congruent initial circles at the vertices of a convex quadrilateral.

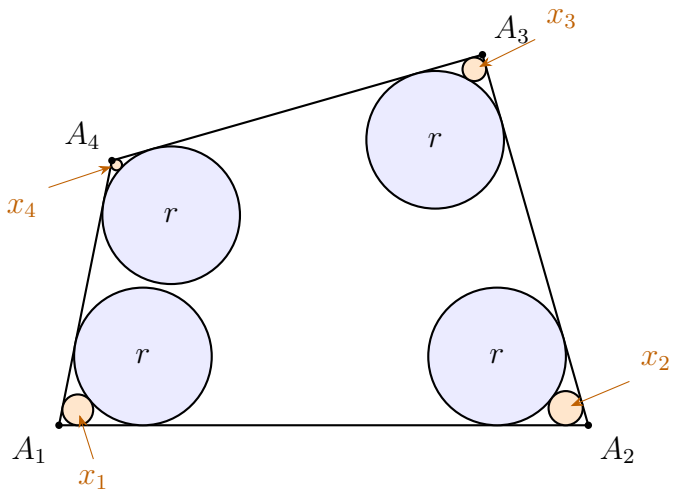



FIGURE 9. Four congruent circles of radius $r$ at the vertices of a convex quadrilateral, and four smaller tangent circles of radii $x_1, x_2, x_3, x_4$. Their radii satisfy $\displaystyle\sum_{1\le i<j\le 4}\sqrt{x_i x_j} = r + \frac{\sqrt{x_1x_2x_3x_4}}{r}$.

**Corollary 7.2** (Four congruent initial circles)**.** *For any strictly convex quadrilateral, construct one circle of radius $r$ at each vertex, tangent to the two incident sides. Let $x_1, x_2, x_3, x_4$ be the corresponding smaller corner-circle radii. Then*

$$\boxed{\sum_{1\le i<j\le 4}\sqrt{x_i x_j} = r + \frac{\sqrt{x_1x_2x_3x_4}}{r}.} \tag{19}$$

*Proof.* For $n = 4$, identity (18) reads $e_2(u_1,u_2,u_3,u_4) - e_4(u_1,u_2,u_3,u_4) = 1$. Substituting $u_i = \sqrt{x_i/r}$ and multiplying by $r$ gives (19). □

The sum of the four corner radii also obeys the sharp bounds

$$4r\tan^2\frac{\pi}{8} \le x_1 + x_2 + x_3 + x_4 < 2r, \tag{20}$$

by Theorem 5.1. Equality on the left holds precisely for a rectangle.

For five initial circles, the corresponding identity is

$$\sum_{i<j}\sqrt{x_i x_j} = r + \frac{1}{r}\sum_{i<j<k<l}\sqrt{x_i x_j x_k x_l}.$$

**Corollary 7.3.** *For every strictly convex $n$-gon with $n \ge 4$,*

$$\boxed{\sum_{i<j}\sqrt{x_i x_j} > r.} \tag{21}$$

*The constant is best possible, but equality is not attained.*

*Proof.* For $n = 4$ and $n = 5$, Theorem 7.1 immediately gives $e_2 = 1 + e_4 > 1$. We give an argument that works for all $n \ge 4$.

We show that, for positive angles $t_1, \dots, t_m$ summing to $\pi/2$, the quantity

$$F(t_1,\dots,t_m) = \sum_{i<j}\tan t_i \tan t_j$$

strictly increases if one angle is split into two positive angles, provided at least two other angles remain. Write the two new tangents as $a = \tan s$ and $b = \tan t$, and put $w = \tan(s+t) = (a+b)/(1-ab)$. If $S$ is the sum of the tangents of the remaining angles, the increase is

$$ab + (a+b)S - wS = ab(1 - wS).$$

There are at least two remaining angles, and their sum is $\pi/2 - s - t$. The tangent of a sum of at least two positive angles (whose sum is below $\pi/2$) is strictly greater than the sum of their tangents. Hence

$$S < \tan\Big(\frac{\pi}{2} - s - t\Big) = \frac{1}{w},$$

so the increase is positive.

Starting with the $n$ quarter-exterior angles, repeatedly merge two angles until three remain. Each merge strictly decreases $F$. The remaining three angles $p, q, s$ are positive and sum to $\pi/2$. Expanding $\cos(p+q+s) = 0$ and dividing by the positive product $\cos p \cos q \cos s$ gives

$$\tan p \tan q + \tan q \tan s + \tan s \tan p = 1.$$

This calculation does not require the merged angles to be quarter-exterior angles of a triangle. For the original quarter-exterior angles, formula (2) gives $\sqrt{x_i/r} = \tan t_i$. Therefore $\sum_{i<j} \sqrt{x_i x_j} = rF(t_1, \ldots, t_n) > r$, proving (21). For sharpness, let $n-3$ quarter-exterior angles tend to zero while the remaining three tend to $\pi/6$; the corresponding convex polygons exist. □

## 8. Conclusion

The sharp bounds and identities established above are as follows.

(1) *Two corners of a triangle:*

$$y + z > (6 - 4\sqrt{2})r, \qquad 2r \tan^2 \frac{\pi + A}{8} \le y + z < r \sec^2 \frac{A}{4}.$$

Equality in the second lower bound holds exactly when $B = C$.

(2) *Right triangles ($A = \pi/2$):*

$$2r \tan^2 \frac{3\pi}{16} \le y + z < r \sec^2 \frac{\pi}{8},$$

with equality on the left exactly when $B = C = \pi/4$.

(3) *Admissible triangle radii:* for $0 < y, z < r$, such a triangle exists precisely when

$$\sqrt{ry} + \sqrt{rz} + \sqrt{yz} > r.$$

(4) *Three corners of a triangle:* whether the initial radius-$r$ circle is the common incircle or three congruent radius-$r$ circles are placed at the vertices, the smaller radii satisfy

$$x + y + z \ge r, \qquad \sqrt{xy} + \sqrt{yz} + \sqrt{zx} = r.$$

Equality in the first inequality holds exactly for an equilateral triangle.

(5) *Four separate radius-$r$ circles in a convex quadrilateral:*

$$\sum_{1\le i<j\le 4}\sqrt{x_i x_j} = r + \frac{\sqrt{x_1x_2x_3x_4}}{r}.$$

Also,

$$4r\tan^2\frac{\pi}{8} \le x_1+x_2+x_3+x_4 < 2r,$$

with lower equality precisely for rectangles.

(6) *All corners of a convex $n$-gon ($n \ge 3$):*

$$nr\tan^2\frac{\pi}{2n} \le \sum_{i=1}^{n} x_i < 2r.$$

Equality on the left holds exactly for equiangular polygons.

(7) *All but one corner:* if the omitted interior angle is $\theta$, then

$$(n-1)r\tan^2\frac{\pi+\theta}{4(n-1)} \le \sum_{i=1}^{n-1} x_i < r\sec^2\frac{\theta}{4},$$

with lower equality exactly when the selected angles are equal. Without fixing $\theta$,

$$(n-1)r\tan^2\frac{\pi}{4(n-1)} < \sum_{i=1}^{n-1} x_i < 2r.$$

(8) *All corner radii, $n \ge 3$:* the exact alternating identity (18) holds; for $n \ge 4$ it also yields the sharp strict inequality

$$\sum_{i<j}\sqrt{x_i x_j} > r.$$

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## References

[1] M. Ursărescu, Problem X.78, *Romanian Mathematical Magazine*, no. 27 (Winter 2020), p. 67.
https://www.ssmrmh.ro/wp-content/uploads/2020/02/RMM-27-FINAL_compressed.pdf

[2] A. Gutierrez, *Online Geometry Problem 454: Triangle, Incircle, Inradius, Tangent, Circle, Radius, Radii*, GoGeometry.
https://gogeometry.com/education/p454_triangle_incircle_inradius_tangent_circle_radii.htm

[3] R. Izard, Problem 5374, *School Science and Mathematics*, Problems section (December 2015); solutions by B. Salgueiro Fanego and T. Shimizu (March 2016).
https://www.ssma.org/assets/docs/Dec-2015.pdf
https://www.ssma.org/assets/docs/march-2016.pdf