International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 11, 2026 pp. xx–yy
web: http://www.journal-1.eu/
Received xx July 2026. Published on-line xx MMM 2026

Perspective Central Triangles Formed from a Triangle and a Transversal

Stanley Rabinowitz[a] and Ercole Suppa[b]
[a] 545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net[2]
web: http://www.StanleyRabinowitz.com/
[b] Via B. Croce 54, 64100 Teramo, Italia
e-mail: ercolesuppa@gmail.com
web: https://www.esuppa.it

Abstract. Let ℓ be a line not passing through any vertex of a triangle ABC and not parallel to any side. Line ℓ meets the sidelines BC, CA, AB of $\triangle ABC$ at points D, E, F, respectively. We consider three of the triangles that are formed: $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$. Placing a fixed triangle center (such as the incenter, centroid, or orthocenter) in each of these three triangles determines a *central triangle*. We investigate when the reference triangle and its central triangle are perspective, i.e., when the lines AD, BE, and CF are concurrent.

A computer search over the first 1000 centers in the Encyclopedia of Triangle Centers suggested numerous examples of concurrence. We give elementary geometric proofs for the circumcenter, orthocenter, and Clawson point, develop a general criterion for concurrence, and identify several operations, including isogonal and isotomic conjugation, that preserve this property.

Our main result is a complete characterization of the center functions whose associated cevians are concurrent for every transversal ℓ. This yields an explicit normal form for such centers. We also show that concurrence depends only on the direction of the transversal, and we investigate the special case in which the transversal is parallel to the Euler line.

Keywords. triangle centers, computer-discovered mathematics, Euclidean geometry, GeometricExplorer.

Mathematics Subject Classification (2020). 51M04, 51-08.

[2]Corresponding author

1. Introduction

Given a reference triangle ABC, suppose three associated triangles are constructed from it. If a fixed triangle center (such as the incenter, centroid, or orthocenter) is chosen in each of these three triangles, the resulting three points determine another triangle, called a *central triangle*.

One way to form three triangles is by picking an arbitrary point D in the plane of $\triangle ABC$, but not on any of its sidelines. Lines are then drawn from point D to the vertices of $\triangle ABC$. This determines the three *side triangles*, $\triangle DBC$, $\triangle DCA$, and $\triangle DAB$ as shown in Figure 1. Each side triangle has D as one vertex and two of the vertices of $\triangle ABC$ as its other two vertices.

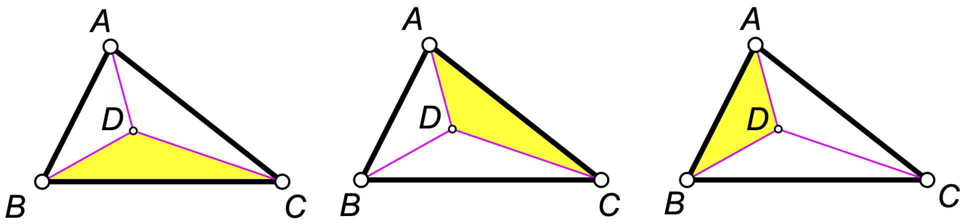

FIGURE 1. Three side triangles determined by D

Let X_n denote the n-th named triangle cataloged in the Encyclopedia of Triangle Centers, [4]. In [5], the center X_n (for n ranging from 1 to 1000) was placed in each of the three side triangles, forming a central triangle. We investigated the geometric relationship between this central triangle and the reference triangle. In particular, we determined when the two triangles were similar, homothetic, or perspective.

Another way to form three triangles is to start with a line ℓ not passing through any vertex of $\triangle ABC$ and not parallel to any side. Three points D, E, and F are determined by where line ℓ intersects the sidelines BC, CA, and AB; we call them the *pierce points* of ℓ. This creates three *corner triangles* associated with the line ℓ: $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ as shown in Figure 2. Each corner triangle is bounded by ℓ and two of the sidelines of $\triangle ABC$.

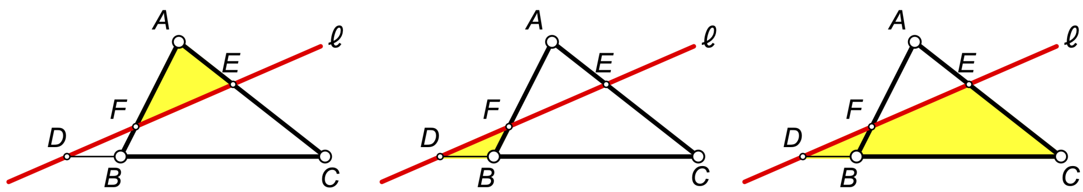

FIGURE 2. Three corner triangles determined by ℓ

In this paper, we will place triangle centers X_n (for n ranging from 1 to 1000) in each of the three corner triangles forming a central triangle associated with the line ℓ. We will determine when this central triangle is perspective with the reference triangle.

Figure 3 illustrates the construction using the centroid X_2. Here G, H, and I are the centroids of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$, respectively. The red triangle is the resulting central triangle. In this example the triangles are not perspective. That is, lines AG, BH, and CI are not concurrent.

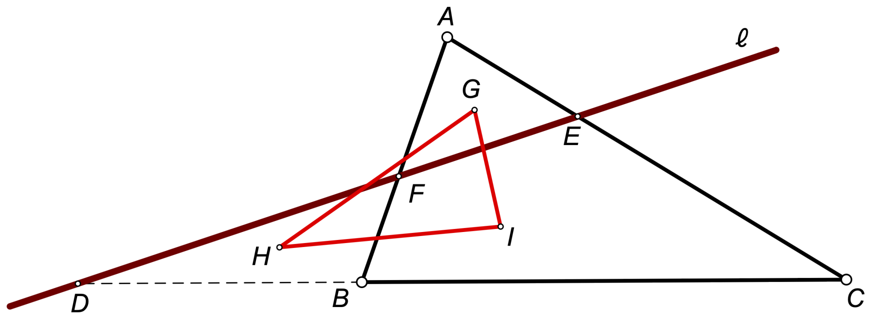

Figure 3. Central triangle determined by ℓ

We used a computer program called GeometricExplorer to compare the reference triangle with its central triangle. For an arbitrary triangle ABC and an arbitrary line ℓ, the program constructs the three corner triangles formed by ℓ and the triangle. It then places the center X_n in each of these corner triangles, for $1 \leq n \leq 1000$ (omitting centers at infinity). The program then determines whether the resulting central triangle is perspective with the reference triangle.

GeometricExplorer uses numerical coordinates (to 15 digits of precision) for locating all the points. Thus, a relationship found by this program does not constitute a proof that the result is correct, but gives us compelling evidence for the validity of the result.

The computer search suggested many perspectivities. The remainder of the paper establishes these results rigorously, using elementary geometry whenever possible, and barycentric coordinates otherwise.

We assume the reader is familiar with barycentric coordinates. For two centers P and Q, we write $P \equiv Q$ to mean that the centers are the same, so that their barycentric coordinates are proportional. If their center functions [6] are f and g, we will also write $f \equiv g$.

The paper proceeds as follows. Section 3 develops a general barycentric criterion for concurrence. Section 4 gives elementary geometric proofs for three representative centers: the circumcenter, the orthocenter, and the Clawson point. Sections 5 and 6 use these examples to build larger families of concurrent centers and to identify operations that preserve concurrence. Section 7 gives a complete characterization of the center functions that are concurrent for every admissible transversal, including an explicit trigonometric normal form. The remaining sections apply these results to the centers found by the computer search and then study the special case of transversals parallel to the Euler line.

2. Results

An *admissible transversal* is a line that does not pass through any vertex of $\triangle ABC$ and is not parallel to any side.

Theorem 2.1. *Let ℓ be an arbitrary admissible transversal to $\triangle ABC$. Three points D, E, and F are determined by where line ℓ intersects the sidelines BC, CA, and AB, respectively. Let the X_n-points of corner triangles AEF, BFD, and CDE be G, H, and I respectively. If*

$$n \in \{3, 4, 19, 24, 25, 48, 49, 63, 68, 69, 92, 93, 184, 186,$$
$$264, 265, 304, 305, 317, 328, 340, 378, 563, 847\},$$

then triangles ABC and GHI are perspective.

Figure 4 shows the case where $n = 19$. The red triangle is the central triangle. The green point is the *perspector* (where AG, BH, and CI meet).

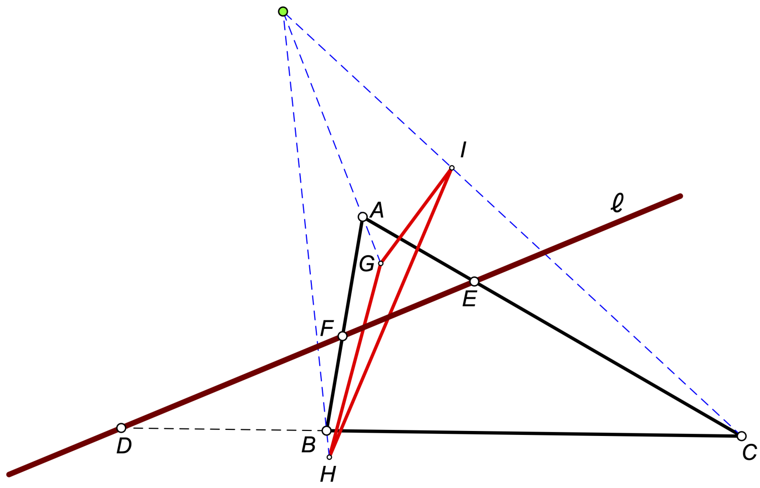

FIGURE 4. Central triangle formed by ℓ and X_{19} points

Remark 2.2. For $n \leq 1000$, our computations found no additional centers for which triangles ABC and GHI are always perspective.

3. PRELIMINARY REMARKS

Before proving Theorem 2.1, we establish a general criterion for deciding when a triangle center is concurrent.

Let ℓ be an admissible transversal of $\triangle ABC$, meeting the sidelines BC, CA, and AB at D, E, and F, respectively. The associated corner triangles are

$$T_A = AEF, \qquad T_B = BFD, \qquad T_C = CDE.$$

Let X be a triangle center determined by the barycentric center function f. For a triangle with side lengths $\lambda_1, \lambda_2, \lambda_3$, the barycentric coordinates of X are

$$\big(f(\lambda_1, \lambda_2, \lambda_3) : f(\lambda_2, \lambda_3, \lambda_1) : f(\lambda_3, \lambda_1, \lambda_2)\big), \tag{1}$$

where f is homogeneous and symmetric in its last two arguments.

Let G, H, and I denote the X-points of T_A, T_B, and T_C, respectively. Our goal is to determine when the cevians AG, BH, and CI are concurrent.

We say that a center *is concurrent for* ℓ if the three lines AG, BH, CI are concurrent. If the center X is concurrent for all lines ℓ, we simply say X is concurrent.

Throughout the paper, we use homogeneous barycentric coordinates with respect to $\triangle ABC$, writing points as $(x : y : z)$ and lines as $[u : v : w]$ as in [7]. Taking $\ell = [u : v : w]$, the admissibility of ℓ is equivalent to u, v, w being nonzero and pairwise distinct.

Using the formula for the intersection of two lines in barycentric coordinates [3, eq. (5)], we find that the coordinates for points D, E, and F are

$$D = (0 : w : -v), \qquad E = (-w : 0 : u), \qquad F = (v : -u : 0). \tag{2}$$

Lengths $\overline{PQ}$ measured along a fixed line are signed, while $|PQ|$ denotes the unsigned length. All angle equalities may be read as equalities of directed angles modulo 180°, which renders every statement independent of the position of ℓ relative to $\triangle ABC$.

If P has barycentric coordinates $(p : q : r)$, then the *normalized barycentric coordinates* of P, namely $(\frac{p}{p+q+r} : \frac{q}{p+q+r} : \frac{r}{p+q+r})$, are denoted by $\widehat{P}$. These have the property that the sum of the three component entries is 1.

Theorem 3.1 (Concurrence Condition)**.** *Order the corner triangles as $T_A = (A, E, F)$, $T_B = (B, F, D)$, $T_C = (C, D, E)$ and set*

$$\begin{aligned} s_A &= f(|FA|, |AE|, |EF|), & t_A &= f(|AE|, |EF|, |FA|), \\ s_B &= f(|DB|, |BF|, |FD|), & t_B &= f(|BF|, |FD|, |DB|), \\ s_C &= f(|EC|, |CD|, |DE|), & t_C &= f(|CD|, |DE|, |EC|), \end{aligned} \tag{3}$$

where f is a center function. Write $S = s_A s_B s_C$ and $T = t_A t_B t_C$. Then AG, BH, CI are concurrent if and only if $S + T = 0$.

Proof. We will first work in barycentric coordinates relative to the corner triangle $T_A = (A, E, F)$. By the definition of the center function, the X-point, G, has coordinates

$$G = (* : s_A : t_A),$$

where

$$s_A = f(|FA|, |AE|, |EF|), \qquad t_A = f(|AE|, |EF|, |FA|),$$

with the A-coordinate $*$ playing no further role. The cevian AG meets the side EF at

$$D_A = (0 : s_A : t_A).$$

The corresponding points for B and C are defined similarly.

By the barycentric change of coordinates rule, [3, eq. (10)], we find that the coordinates of D_A relative to $\triangle ABC$ are $D_A = s_A\widehat{E} + t_A\widehat{F}$, where $\widehat{E} = (-w, 0, u)/(u-w)$, and $\widehat{F} = (v, -u, 0)/(v-u)$.

A line through $A = (1 : 0 : 0)$ has the form $[0 : \mu : \nu]$ and passes through $(d_1 : d_2 : d_3)$ if and only if $\mu : \nu = d_3 : -d_2$. Reading the last two coordinates of D_A gives, after clearing the common factor u,

$$AG = \left[0 \;:\; \tfrac{s_A}{u-w} \;:\; \tfrac{t_A}{v-u}\right],$$

and cyclically

$$BH = \left[\tfrac{t_B}{w-v} \;:\; 0 \;:\; \tfrac{s_B}{v-u}\right], \qquad CI = \left[\tfrac{s_C}{w-v} \;:\; \tfrac{t_C}{u-w} \;:\; 0\right].$$

The three lines are concurrent precisely when the determinant of their line coordinates is zero [3, eq (6)]. Since each diagonal entry is zero, the determinant reduces to

$$\det\begin{pmatrix} 0 & \frac{s_A}{u-w} & \frac{t_A}{v-u} \\ \frac{t_B}{w-v} & 0 & \frac{s_B}{v-u} \\ \frac{s_C}{w-v} & \frac{t_C}{u-w} & 0 \end{pmatrix} = \frac{s_A s_B s_C + t_A t_B t_C}{(u-w)(v-u)(w-v)} = \frac{S+T}{(u-w)(v-u)(w-v)}.$$

The denominator is nonzero because ℓ is admissible. Hence the determinant vanishes exactly when $S+T=0$. □

Remark 3.2. Theorem 3.1 reduces the geometric problem of concurrence to the algebraic identity $S+T=0$. All subsequent proofs amount to showing that this identity holds for particular center functions.

4. Three geometric proofs

4.1. The circumcenter X_3.

Lemma 4.1. *Let PQR have circumcenter O. Then PO and the perpendicular from P to QR are isogonal with respect to $\angle QPR$ (Figure 5).*

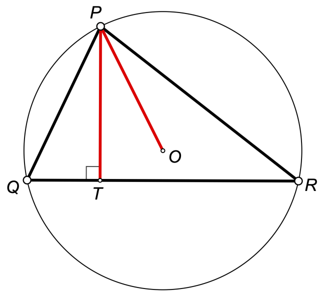

FIGURE 5. Altitude and circumradius of a triangle

Proof. Let the perpendicular from P meet QR at T. In the right triangle PTQ the angle at Q is $\angle PQR$, so $\angle(PT, PQ) = 90° - \angle PQR$. Since $OP = OR$, triangle OPR is isosceles. Hence

$$\angle(OP, PR) = \frac{1}{2}(180° - \angle POR) = 90° - \angle PQR,$$

because $\angle POR = 2\angle PQR$. Thus, $\angle(OP, PR) = \angle(QP, PT)$, and PO and PT are reflections in the bisector of $\angle QPR$. □

Since E lies on line CA and F on line AB, the sides of $\angle EAF$ are the lines CA, AB, so its bisector is the bisector of $\angle BAC$; and $EF \subset \ell$. Applying Lemma 4.1 to $T_A = AEF$, and cyclically, yields the following result.

Corollary 4.2. *The line AG is the isogonal in $\triangle ABC$ of the perpendicular to ℓ through A; likewise BH, CI are the isogonals of the perpendiculars to ℓ through B, C.*

Theorem 4.3. *Center X_3 is concurrent. The lines AG, BH, CI concur at a point on the circumcircle of $\triangle ABC$.*

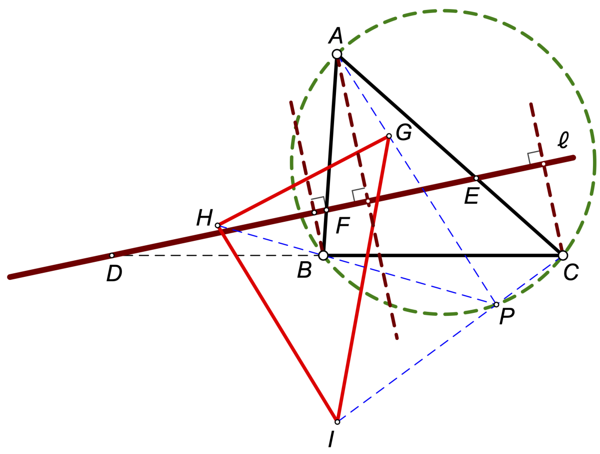

FIGURE 6. G, H, and I are circumcenters

Proof. The three perpendiculars to ℓ through A, B, C are parallel (Figure 6). Hence, the cevians $AP_\infty, BP_\infty, CP_\infty$ of the single point at infinity P_∞ in the direction orthogonal to ℓ, are also parallel. By Corollary 4.2, reflecting these in the bisectors at A, B, C produces AG, BH, CI. These reflected cevians therefore concur at the isogonal conjugate P_∞^*. Isogonal conjugation carries the line at infinity to the circumcircle [7, §6], so P_∞^* lies on the circumcircle of ABC, and AG, BH, CI concur there. □

4.2. **The orthocenter X_4.**

Assume here that $\triangle ABC$ has no right angle, so that the orthocenter of each corner triangle is distinct from its apex and its cevians are well defined.

Theorem 4.4. *Center X_4 is concurrent. The lines AG, BH, CI are mutually parallel. Consequently, $\triangle ABC$ and $\triangle GHI$ are perspective from the point at infinity orthogonal to ℓ.*

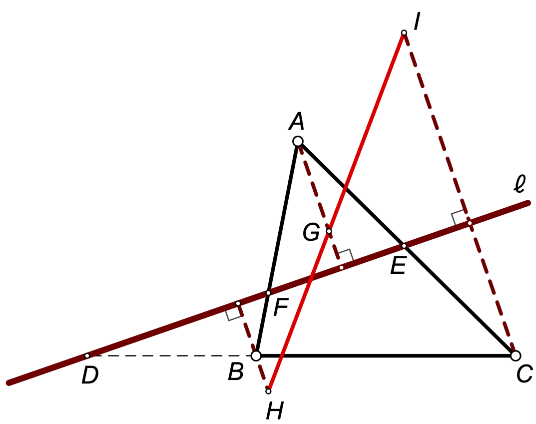

FIGURE 7. G, H, and I are orthocenters

Proof. Since G is the orthocenter of $\triangle AEF$, line AG is an altitude and therefore $AG \perp \ell$. Similarly, $BH \perp \ell$ and $CI \perp \ell$. Each of AG, BH, CI is perpendicular to the fixed line ℓ, so the three lines are parallel and meet at the infinite point P_∞ of the direction orthogonal to ℓ. Thus, $\triangle ABC$ and $\triangle GHI$ are perspective from P_∞ (Figure 7). □

Remark 4.5. Theorems 4.3 and 4.4 are closely related. For each corner triangle, the cevian to the circumcenter is the isogonal of the cevian to the orthocenter. Thus, the two perspectors are themselves isogonal conjugates: one lies on the circumcircle, while the other is the corresponding point at infinity.

4.3. **The Clawson point X_{19}.**

From [4], we find that the Clawson point has center function

$$f(x,y,z) = \frac{x}{y^2+z^2-x^2}. \tag{4}$$

Since D, E, F are collinear, Menelaus' theorem [1, §3.10] applies to $\triangle ABC$.

Lemma 4.6 (Product Identity)**.** $|AF|\,|BD|\,|CE| = |AE|\,|BF|\,|CD|$.

Proof. The unsigned Menelaus relation for the transversal ℓ is $\frac{|AF|}{|FB|}\cdot\frac{|BD|}{|DC|}\cdot\frac{|CE|}{|EA|} = 1$, i.e. $|AF|\,|BD|\,|CE| = |FB|\,|DC|\,|EA| = |BF|\,|CD|\,|AE|$. □

Lemma 4.7 (Signed Ratios)**.** $\dfrac{\overline{EA}}{\overline{EC}}\cdot\dfrac{\overline{FB}}{\overline{FA}}\cdot\dfrac{\overline{DC}}{\overline{DB}} = 1.$

Proof. The signed Menelaus relation is $\frac{\overline{AF}}{\overline{FB}}\cdot\frac{\overline{BD}}{\overline{DC}}\cdot\frac{\overline{CE}}{\overline{EA}} = -1$. Replacing $AF = -FA$, $BD = -DB$, $CE = -EC$ and inverting yields the asserted relation. □

Theorem 4.8. *Center X_{19} is concurrent.*

Proof. Unlike the proofs for X_3 and X_4, we have not found a purely synthetic argument for X_{19}. Instead, we apply the concurrence criterion of Theorem 3.1.

Substituting the center function (4) into the concurrence criterion (Theorem 3.1) gives

$$s_A = \frac{|FA|}{d_A},\quad t_A = \frac{|AE|}{d'_A},\qquad s_B = \frac{|DB|}{d_B},\quad t_B = \frac{|BF|}{d'_B},\qquad s_C = \frac{|EC|}{d_C},\quad t_C = \frac{|CD|}{d'_C}.$$

Hence

$$S+T = \frac{|FA|\,|DB|\,|EC|}{d_Ad_Bd_C} + \frac{|AE|\,|BF|\,|CD|}{d'_Ad'_Bd'_C}.$$

By Lemma 4.6 the two numerators are equal, so proving $S+T=0$ is equivalent to showing

$$d_Ad_Bd_C + d'_Ad'_Bd'_C = 0. \tag{5}$$

To prove this identity we express each denominator as a dot product.

By the Law of Cosines, each d is a dot product and we have

$$\begin{aligned} d_A &= 2\,\overrightarrow{EA}\cdot\overrightarrow{EF}, & d'_A &= 2\,\overrightarrow{FA}\cdot\overrightarrow{FE},\\ d_B &= 2\,\overrightarrow{FB}\cdot\overrightarrow{FD}, & d'_B &= 2\,\overrightarrow{DB}\cdot\overrightarrow{DF},\\ d_C &= 2\,\overrightarrow{DC}\cdot\overrightarrow{DE}, & d'_C &= 2\,\overrightarrow{EC}\cdot\overrightarrow{ED}. \end{aligned}$$

Form the ratio

$$\frac{d_A d_B d_C}{d'_A d'_B d'_C}.$$

Grouping together the factors associated with the same vertex (E, F, and D), the common cosine factors cancel, leaving

$$\frac{d_A d_B d_C}{d'_A d'_B d'_C} = \underbrace{\frac{\overline{EA}}{\overline{EC}} \cdot \frac{\overline{FB}}{\overline{FA}} \cdot \frac{\overline{DC}}{\overline{DB}}}_{\text{side factors}} \cdot \underbrace{\frac{\overline{EF}}{\overline{ED}} \cdot \frac{\overline{FD}}{\overline{FE}} \cdot \frac{\overline{DE}}{\overline{DF}}}_{\ell \text{ factors}}.$$

Each ℓ factor equals -1 since $\overline{ED} = -\overline{DE}$, and $\overline{FE} = -\overline{EF}$, $\overline{DF} = -\overline{FD}$. Thus, the product of the three ℓ factors is -1. The side factors equal $+1$ by Lemma 4.7. Therefore the ratio is -1, which is (5). By Theorem 3.1, AG, BH, CI concur. $\square$

Remark 4.9. Section 5 will show that this phenomenon is not peculiar to X_{19}.

5. A parity criterion for concurrence

The proof of Theorem 4.8 suggests a much larger family of concurrent centers. The key observation is that the argument depends only on whether the exponent of the Conway symbol is odd or even. This leads to the following parity criterion.

Set $S_A = b^2 + c^2 - a^2$ (twice the Conway symbol), and S_B, S_C cyclically. For integers k and m, define the center function

$$f_{k,m}(x, y, z) = x^k\,(y^2 + z^2 - x^2)^m,$$

homogeneous in x, y, z and symmetric in y, z. Its center has barycentric coordinates

$$\left(a^k S_A^m : b^k S_B^m : c^k S_C^m\right).$$

Theorem 5.1. *For the center X with center function $f_{k,m}$, the lines AG, BH, CI are concurrent if and only if m is odd. In particular, the conclusion is independent of k.*

Proof. With the notation of Theorem 3.1 and the abbreviations $d_A, d'_A, \ldots$ of Theorem 4.8,

$$s_A = |FA|^k d_A^m, \quad t_A = |AE|^k (d'_A)^m,$$

and cyclically. Rather than compute $S+T$ directly, we compare the two products by forming the ratio S/T. We find that

$$\frac{S}{T} = \prod_V \frac{s_V}{t_V} = \left(\frac{|FA|\,|DB|\,|EC|}{|AE|\,|BF|\,|CD|}\right)^k \left(\frac{d_A d_B d_C}{d'_A d'_B d'_C}\right)^m.$$

The first factor equals 1 by Lemma 4.6, for every k. The second factor equals $(-1)^m$ by (5). Hence $S = (-1)^m T$, so $S + T = \big(1 + (-1)^m\big)T$, which vanishes for every admissible ℓ if and only if m is odd. The conclusion now follows immediately from Theorem 3.1. $\square$

Corollary 5.2. *Among Kimberling centers of index below 100, the following belong to the family $f_{k,m}$ with m odd, and are therefore concurrent.*

$$\begin{aligned}
&X_3 = a^2 S_A = f_{2,1}, && X_4 = S_A^{-1} = f_{0,-1}, && X_{19} = aS_A^{-1} = f_{1,-1},\\
&X_{25} = a^2 S_A^{-1} = f_{2,-1}, && X_{48} = a^3 S_A = f_{3,1}, && X_{63} = aS_A = f_{1,1},\\
&X_{69} = S_A = f_{0,1}.
\end{aligned}$$

No other Kimberling center of index below 100 belongs to this family.

Remark 5.3. Theorem 5.1 explains why the proof of Theorem 4.8 worked. The specific properties of the Clawson point play only a minor role; what matters is that the Conway symbol appears to an odd power. Thus, the theorem replaces one isolated concurrence result by an infinite family.

6. OPERATIONS THAT PRESERVE CONCURRENCE

Section 5 produced one infinite family of concurrent centers. We now show that several natural operations on barycentric center functions preserve concurrence. Starting from any concurrent center, these operations generate many others. In particular, they imply that concurrence is preserved under both isogonal and isotomic conjugation, and they will play a central role in the completeness theorem of Section 7.

Several simple modifications of a barycentric center function leave the concurrency locus $\{\,\ell : S+T=0\,\}$ of Theorem 3.1 unchanged. We collect them here. Together they generate, from a single concurrent center, large families of concurrent centers; they yield the classical conjugation invariances as a special case, and they underlie both the sufficient condition of Theorem 6.9 below and the completeness theorem of §7.

The simplest preserving operation is multiplication by a factor that depends only on the angle opposite the first coordinate.

Lemma 6.1 (Multiplier Lemma)**.** *Let X_0 be concurrent with center function f_0. If*

$$f(x,y,z)=f_0(x,y,z)\,\Phi\!\left(\frac{(y^2+z^2-x^2)^2}{4y^2z^2}\right),$$

where Φ is any one-variable function, then X is also concurrent.

The argument of Φ is $\cos^2 A$, where A is the angle opposite the first coordinate.

Proof. The multiplier is symmetric in y,z and homogeneous of degree 0, so f is a center function. Let s_V^0, t_V^0 be the quantities of Theorem 3.1 for f_0, with products S_0, T_0. Each s_V, t_V for f acquires the factor $\Phi(\cos^2\theta)$, where θ is the angle of the corner triangle T_V at the vertex opposite the leg placed first. Since that leg always lies opposite a vertex, these angles are the corner-triangle angles at D,E,F.

At each of D,E,F exactly two corner triangles meet; the two angles there are both the angle between ℓ and the sideline through that point, equal as undirected lines (vertical or supplementary, according to the configuration; cf. Lemma 7.2(ii)), so in every case they share the same $\cos^2$ term. Hence, the factor $\varphi_P := \Phi(\cos^2\theta_P)$ produced at $P\in\{D,E,F\}$ is common to the one s and the one t that involve P, giving

$$S=S_0\,\varphi_D\varphi_E\varphi_F,\qquad T=T_0\,\varphi_D\phi_E\varphi_F.$$

Therefore $S+T=(S_0+T_0)\,\varphi_D\varphi_E\varphi_F=0$, and Theorem 3.1 applies. □

The proof of Theorem 5.1 identifies one infinite family of concurrent centers. We now isolate several simple transformations of barycentric center functions that preserve concurrence. These transformations allow us to generate many new concurrent centers from any known one and will play a central role in the characterization theorem of Section 7.

Definition 6.2. *For a barycentric center function $f = f(a, b, c)$ write $S_A = b^2 + c^2 - a^2$ and $\cos^2 A = S_A^2/(4b^2c^2)$. The following operations are called* preserving operations *because they preserve concurrence.*

- (O1) isotomic $f \mapsto 1/f$, *the center function of the isotomic conjugate* [7]*, defined wherever $f \neq 0$;*
- (O2) isogonal $f \mapsto a^2/f$, *the center function of the isogonal conjugate* [7]*, defined wherever $f \neq 0$;*
- (O3) side-power $f \mapsto a^k f$ *and* co-side-power $f \mapsto (bc)^j f,\ \ k, j \in \mathbb{Z}$*;*
- (O4) Conway-even $f \mapsto S_A^{2m} f$ *and* $f \mapsto (S_B S_C)^{2n} f$*;*
- (O5) symmetric $f \mapsto \sigma(a, b, c)\, f$ *with σ symmetric;*
- (O6) angular multiplier $f \mapsto \Phi(\cos^2 A)\, f$ *for any one-variable function Φ.*

Lemma 6.3. *Each preserving operation in Definition 6.2 leaves the concurrency locus*

$$\{\ell : S + T = 0\}$$

of Theorem 3.1 unchanged.

Proof. Recall from Theorem 3.1 that

$$s_A = f(|FA|, |AE|, |EF|), \qquad t_A = f(|AE|, |EF|, |FA|),$$

with analogous definitions for B and C, and that

$$S = s_A s_B s_C, \qquad T = t_A t_B t_C.$$

Operation (O1): isotomic conjugation. Replacing f by $1/f$ sends

$$S \longmapsto \frac{1}{S}, \qquad T \longmapsto \frac{1}{T}.$$

Consequently,

$$S + T \longmapsto \frac{1}{S} + \frac{1}{T} = \frac{S + T}{ST},$$

which has the same zero locus wherever S and T are finite and nonzero.

Operation (O2): isogonal conjugation. Replacing f by a^2/f is simply the composition of isotomic conjugation with the side-power operation ($k = 2$). Since each preserves the concurrency locus, their composition does as well.

Operations (O3)–(O5). Each of these operations multiplies every s_V and every t_V by a factor depending only on the corresponding corner triangle.

For the side-power operation $f \mapsto a^k f$, the products acquire the factors

$$|FA|^k |DB|^k |EC|^k \quad \text{and} \quad |AE|^k |BF|^k |CD|^k,$$

which are equal by Lemma 4.6. Thus, both S and T are multiplied by the same nonzero quantity.

The co-side-power operation is similar. The ℓ-segment lengths cancel, and Lemma 4.6 again shows that S and T receive the same factor.

For the Conway-even case, note that $S_A^{2m} = 4^m (bc)^{2m} (\cos^2 A)^m$. In other words, operation (O4) is just co-side-power (O3) composed with the angular multiplier (O6) and a symmetric constant.

For the Conway-even, symmetric, and angular operations, each corner triangle contributes exactly the same multiplier to the corresponding s and t. Hence both S and T are multiplied by the same cyclic product.

In every case,
$$S+T \longmapsto \Lambda(S+T),$$
where $\Lambda \neq 0$, so the zero locus is unchanged.

Operation (O6): angular multiplication. This has already been proven as the Multiplier Lemma (Lemma 6.1). □

In other words, we have proven the following.

Theorem 6.4. *Let center X be concurrent for some admissible transversal ℓ. Suppose the center function for X is f. Let X' be the center whose center function is obtained from f by applying a preserving operation. Then X' is concurrent for ℓ.*

Corollary 6.5 (Conjugation invariance)**.** *Suppose X is concurrent for the admissible line ℓ and lies on no sideline of any corner triangle, so that $S \neq 0$ and $T \neq 0$. Then its isogonal conjugate X^* and its isotomic conjugate X^{-1} are also concurrent for ℓ.*

Proof. These are the operations (O1) and (O2) of Definition 6.2, shown to preserve the concurrency locus in Lemma 6.3; the hypothesis $S, T \neq 0$ keeps the conjugate center functions finite and nonzero. □

Remark 6.6. The monomial family of §5 is visibly closed under (O1) and (O2): isogonal sends $f_{k,m}$ to $f_{2-k,-m}$ and isotomic sends it to $f_{-k,-m}$, both of which again have m odd. From the geometric seeds $X_3 = f_{2,1}$, $X_4 = f_{0,-1}$, and $X_{19} = f_{1,-1}$ the whole family $\{f_{k,m} : m \text{ odd}\}$ is generated (using (O3) and (O4), the seeds X_3, X_4 give the even values of k, and X_{19} the odd). More generally, Corollary 6.5 together with Theorems 4.3, 4.4, and 4.8 shows that every center obtained from X_3, X_4, or X_{19} by repeated isogonal and isotomic conjugation is concurrent—for instance X_{63}, X_{69}, X_{264}, and X_{304}—and this applies equally to concurrent centers outside the monomial family.

Remark 6.7. Equivalently, if a concurrent center has first barycentric coordinate of the form $g(A)$, then the center whose first barycentric coordinate is $g(A)\, f(\cos^2 A)$ is also concurrent, for any function f of one variable.

Definition 6.8. *An expression that is of the form*

$$\cos A \cdot f(\cos^2 A) \tag{6}$$

or

$$\sin A \cos A \cdot f(\cos^2 A) \tag{7}$$

where f is a function of one variable is said to be in concurrent form*.*

The circumcenter and the Clawson point furnish one seed for each form. By Corollary 5.2, X_3 has barycentric center function $a^2 S_A$ and X_{19} has a/S_A; using $a \equiv \sin A$ and $S_A \equiv \csc A \cos A = \cot A$ modulo symmetric factors (the Law of Sines and $S_A = 2bc \cos A$), these reduce to

$$[X_3] \quad a^2 S_A \equiv \sin A \cos A, \qquad [X_{19}] \quad a/S_A \equiv \cos A \cdot \frac{1-\cos^2 A}{\cos^2 A}.$$

The center function for X_3 has been written in the concurrent form (7). The center function for X_{19} has been written in the concurrent form (6). Both were proven concurrent by elementary geometry (Theorems 4.3 and 4.8). Multiplying each by an arbitrary $f(\cos^2 A)$, as in Remark 6.7, fills out both forms, so we obtain the following, giving a broad sufficient condition for concurrence.

Theorem 6.9. *Let X be a center whose first barycentric coordinate is in concurrent form. Then X is concurrent.*

The next section shows that Theorem 6.9 is best possible. Every concurrent center is, up to a symmetric factor, of one of the forms described above.

7. Characterization of concurrent centers

Sections 5 and 6 produced broad families of centers that are concurrent for every admissible transversal. We now prove that these families are complete. The goal of this section is to characterize all center functions whose cevians are concurrent for every admissible transversal.

The main result is the Characterization Theorem (Theorem 7.3), followed by the explicit trigonometric normal form in Theorem 7.4.

7.1. **The shape function and the leg–swap ratio.** We now change the point of view. Instead of working directly with the side-length function $f(a, b, c)$, we describe a corner triangle by its angles.

A *shape* will mean an ordered angle triple

$$\sigma = (\alpha, \beta, \gamma), \qquad \alpha + \beta + \gamma = \pi, \qquad 0 < \alpha, \beta, \gamma < \pi.$$

The angles α, β, γ are ordered so that they are opposite the first, second, and third sides of the corner triangle corresponding to the arguments of the center function f, as shown in Figure 8. We shall call α and β the *base angles* of the shape, and γ the *apex angle*.

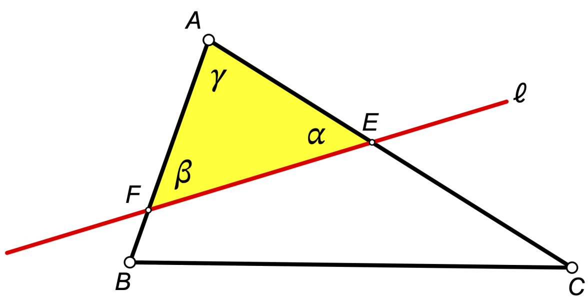

Figure 8. The ordered angle triple for the corner triangle $T_A = AEF$.

Since f is homogeneous, the value of

$$f(|FA|, |AE|, |EF|)$$

depends only on the shape of the triangle AEF, not on its size. By the Law of Sines, the side lengths of a triangle are proportional to the sines of the opposite angles. We therefore define the *shape function* associated with f by

$$F(\alpha, \beta, \gamma) := f(\sin \alpha, \sin \beta, \sin \gamma).$$

Since every center function is symmetric in its second and third arguments,

$$f(p,q,r) = f(p,r,q),$$

the shape function satisfies

$$F(\alpha,\beta,\gamma) = F(\alpha,\gamma,\beta). \tag{8}$$

Consider the corner triangle $T_A = AEF$. The side FA is opposite $\angle AEF$, the side AE is opposite $\angle AFE$, and the side EF is opposite $\angle EAF$. Thus the shape of T_A, in the order corresponding to

$$s_A = f(|FA|, |AE|, |EF|),$$

is

$$\sigma_A = (\angle AEF, \angle AFE, \angle EAF).$$

Similarly, let σ_B and σ_C denote the corresponding ordered shapes of the corner triangles T_B and T_C.

Using the homogeneity of f and the Law of Sines, we have

$$\begin{aligned}\frac{t_A}{s_A} &= \frac{f(|AE|, |EF|, |FA|)}{f(|FA|, |AE|, |EF|)} = \frac{f(2R\sin\beta, 2R\sin\gamma, 2R\sin\alpha)}{f(2R\sin\alpha, 2R\sin\beta, 2R\sin\gamma)}\\ &= \frac{f(\sin\beta,\sin\gamma,\sin\alpha)}{f(\sin\alpha,\sin\beta,\sin\gamma)} = \frac{F(\beta,\gamma,\alpha)}{F(\alpha,\beta,\gamma)}.\end{aligned}$$

Using the symmetry of F in its last two variables, this becomes

$$\frac{t_A}{s_A} = \frac{F(\beta,\alpha,\gamma)}{F(\alpha,\beta,\gamma)}.$$

This motivates the following definition. For any shape $\sigma = (\alpha,\beta,\gamma)$, define the *leg–swap ratio*

$$Q(\sigma) = Q(\alpha,\beta,\gamma) := \frac{F(\beta,\alpha,\gamma)}{F(\alpha,\beta,\gamma)}.$$

Thus, Q measures the effect of interchanging the two base angles while keeping the apex angle fixed.

Interchanging the two base angles twice returns to the original shape, so

$$Q(\beta,\alpha,\gamma) = Q(\alpha,\beta,\gamma)^{-1}.$$

For the three corner triangles, we therefore have

$$\frac{t_A}{s_A} = Q(\sigma_A), \qquad \frac{t_B}{s_B} = Q(\sigma_B), \qquad \frac{t_C}{s_C} = Q(\sigma_C).$$

The concurrence condition of Theorem 3.1 can now be written in terms of Q. Since

$$\frac{T}{S} = \frac{t_At_Bt_C}{s_As_Bs_C} = Q(\sigma_A)Q(\sigma_B)Q(\sigma_C),$$

the condition $S+T=0$ is equivalent, when $S \neq 0$, to

$$Q(\sigma_A)Q(\sigma_B)Q(\sigma_C) = -1. \tag{9}$$

Thus, the problem of characterizing concurrent centers has been reduced to understanding the single function Q.

7.2. **Geometry of the corner triangles.** The three corner triangles are not independent. Once the admissible transversal ℓ is chosen, the angles of one corner triangle determine those of the other two. The following lemma records the elementary geometry of the pierce points that will be used in the proof of the Characterization Theorem.

Throughout this subsection, ℓ is an admissible transversal meeting the sidelines BC, CA, and AB at the distinct points D, E, and F.

Definition 7.1. *A pierce point is* interior *if it lies in the interior of its side (for example, F is interior precisely when $A - F - B$), and* exterior *otherwise.*

Exactly one of the three pierce points lies between the other two on the line ℓ. We call this the median pierce point*; the other two are called* lateral pierce points.

Lemma 7.2 (Flip Lemma)**.** *Let ℓ be an admissible transversal.*

(1) *At each pierce point the two corner base angles are either equal or vertical angles or they are supplementary. They are supplementary precisely when the pierce point is interior and lateral, or exterior and median.*
(2) *Exactly one pierce point is supplementary; the other two are equal.*
(3) *Consequently, exactly one corner triangle has an external apex angle. It is the corner triangle that does not contain the supplementary pierce point. The other two corner triangles have internal apex angles.*
(4) *The number of interior pierce points is even. Hence either none or exactly two of the pierce points are interior.*

Proof. At each pierce point, the two adjacent corner triangles are either equal or determine a pair of vertical angles; or they are a pair of supplementary angles (Figure 9).

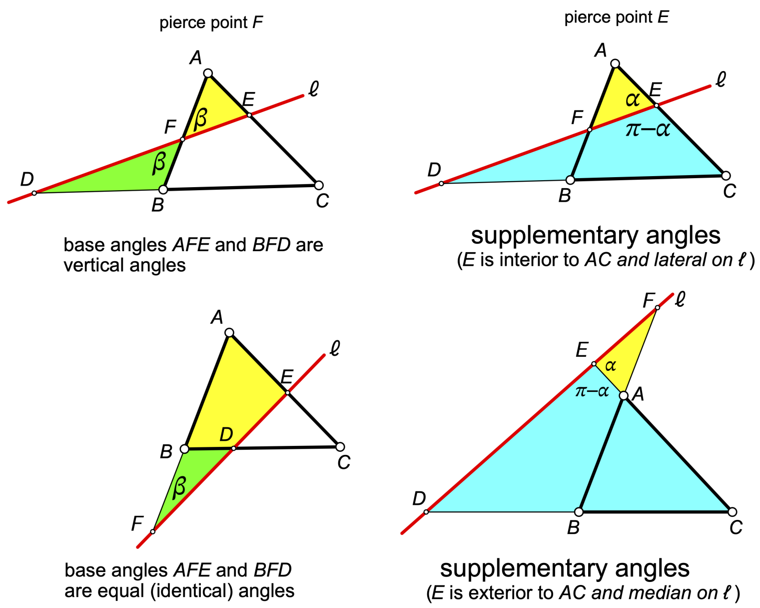

Figure 9. case (1)

The supplementary case occurs precisely when the pierce point is interior and lateral, or exterior and median. This proves (1).

We next show that exactly one pierce point is supplementary.

If none of the pierce points is interior, then all three are exterior (Figure 10, left). In this case, the unique median pierce point is exterior and median, so it is supplementary by (1), while the two lateral pierce points are exterior and lateral, hence equal.

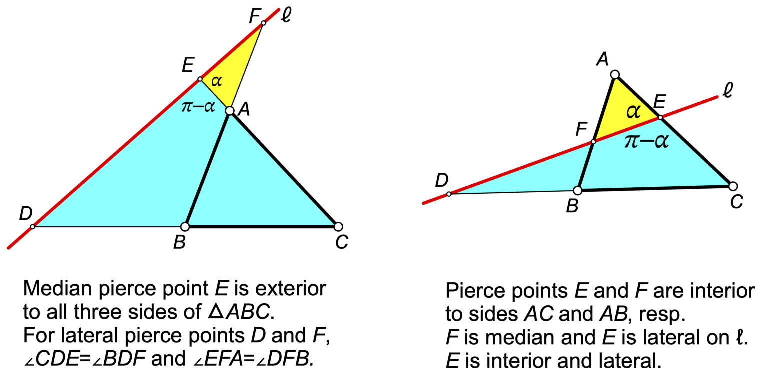

FIGURE 10. case (2)

If two pierce points are interior, then these two points are the endpoints of the segment $\ell \cap ABC$. The third pierce point is exterior and lies outside this segment (Figure 10, right). Hence the median pierce point is one of the two interior points. Of the two interior points, one is median and one is lateral. Thus exactly one pierce point is interior and lateral, and this is the unique supplementary point. This proves (2).

The corner triangle with external apex angle is the one opposite the supplementary pierce point, meaning the corner triangle that does not contain that pierce point ($\triangle BFD$ in Figure 10). This proves (3).

Finally, a line not passing through a vertex can meet the interior of a triangle only in a segment. Therefore it enters and leaves the triangle through two sides, or else it does not enter the triangle at all. Hence the number of interior pierce points is either two or zero, proving (4). □

To prove the Characterization Theorem it is convenient to write the three corner shapes explicitly. Since the three possible branches differ only by cyclic permutation of A, B, and C, it suffices to consider one representative case.

Assume that the corner triangle opposite C, namely $T_C = CDE$, is the unique corner triangle with an external apex angle, and set

$$x = \angle BDF.$$

Then an angle chase gives

$$\angle CDE = x, \qquad \angle CED = C - x, \qquad \angle DCE = \pi - C.$$

Similarly for the other two corner triangles. Writing $C = \pi - A - B$, the three ordered shapes are

$$\begin{aligned} \sigma_A &= (C - x,\ B + x,\ A), \\ \sigma_B &= (\pi - B - x,\ x,\ B), \\ \sigma_C &= (x,\ C - x,\ \pi - C). \end{aligned} \tag{10}$$

Here the first two entries are the base angles of the shape and the third entry is the apex angle.

The remaining two branches are obtained by cyclically permuting $A \to B \to C$. Moreover, the parameters (A, B, x) vary independently over an open set. This local freedom will be essential in the proof of Theorem 7.3.

7.3. **The Characterization Theorem.** Sections 5 and 6 produced broad families of centers that are concurrent for every admissible transversal. We now prove that these families are complete.

The idea is that a universally concurrent center has only one essential non-symmetric part. Up to multiplication by a fully symmetric factor, its shape function depends only on the first angle of the ordered shape.

We shall use the following terminology. A function

$$M(\alpha, \beta, \gamma)$$

is called *fully symmetric* if it is unchanged by every permutation of α, β, γ.

A function ϕ on $(0, \pi)$ is said to be *odd about* $\pi/2$ if

$$\phi(\pi - t) = -\phi(t)$$

for all $t \in (0, \pi)$. Thus, ϕ takes opposite values at supplementary angles.

It is said to be *even about* $\pi/2$ if

$$\phi(\pi - t) = \phi(t)$$

for all $t \in (0, \pi)$.

Sections 5 and 6 produced broad families of centers that are concurrent for every admissible transversal. We now prove that these families are complete. The central result of this section shows that every universally concurrent center has a remarkably simple structure.

Theorem 7.3 (Characterization Theorem)**.** *A center function f is concurrent for every admissible transversal if and only if its shape function factors as*

$$F(\alpha, \beta, \gamma) = \phi(\alpha)\, M(\alpha, \beta, \gamma), \tag{11}$$

where M is fully symmetric and ϕ is odd about $\pi/2$.

Equivalently, the leg–swap ratio has the form

$$Q(\alpha, \beta, \gamma) = \frac{\phi(\beta)}{\phi(\alpha)}, \tag{12}$$

where ϕ is odd about $\pi/2$.

Before proving the theorem, we verify that the two formulations are equivalent.

Suppose first that

$$F(\alpha, \beta, \gamma) = \phi(\alpha)\, M(\alpha, \beta, \gamma),$$

where M is fully symmetric. Then

$$Q(\alpha, \beta, \gamma) = \frac{F(\beta, \alpha, \gamma)}{F(\alpha, \beta, \gamma)} = \frac{\phi(\beta) M(\beta, \alpha, \gamma)}{\phi(\alpha) M(\alpha, \beta, \gamma)} = \frac{\phi(\beta)}{\phi(\alpha)}.$$

Conversely, suppose that

$$Q(\alpha, \beta, \gamma) = \frac{\phi(\beta)}{\phi(\alpha)}.$$

Define

$$M(\alpha,\beta,\gamma)=\frac{F(\alpha,\beta,\gamma)}{\phi(\alpha)}.$$

Then

$$\frac{M(\beta,\alpha,\gamma)}{M(\alpha,\beta,\gamma)}=Q(\alpha,\beta,\gamma)\,\frac{\phi(\alpha)}{\phi(\beta)}=1.$$

Thus M is fully symmetric in its first two arguments. Since every center function is symmetric in its last two arguments, we also have

$$F(\alpha,\beta,\gamma)=F(\alpha,\gamma,\beta).$$

It follows that M is fully symmetric in its last two arguments as well. Hence M is fully symmetric.

The proof of the Characterization Theorem has two parts. The sufficiency is geometric: assuming the factorization

$$F(\alpha,\beta,\gamma)=\phi(\alpha)M(\alpha,\beta,\gamma),$$

we show that

$$Q(\sigma_A)Q(\sigma_B)Q(\sigma_C)=-1$$

for every admissible transversal. The necessity is analytic: starting from this identity for every admissible transversal, we determine the possible form of Q, and then recover the symmetric factor M.

Proof of sufficiency. Assume that

$$F(\alpha,\beta,\gamma)=\phi(\alpha)M(\alpha,\beta,\gamma),$$

where M is fully symmetric and ϕ is odd about $\pi/2$. Then

$$Q(\alpha,\beta,\gamma)=\frac{\phi(\beta)}{\phi(\alpha)}.$$

For the three corner triangles, write

$$\sigma_A=(\alpha_A,\beta_A,\gamma_A),\qquad \sigma_B=(\alpha_B,\beta_B,\gamma_B),\qquad \sigma_C=(\alpha_C,\beta_C,\gamma_C),$$

where the first two entries are the base angles of the corresponding shape. Then

$$Q(\sigma_A)Q(\sigma_B)Q(\sigma_C)=\frac{\phi(\beta_A)}{\phi(\alpha_A)}\frac{\phi(\beta_B)}{\phi(\alpha_B)}\frac{\phi(\beta_C)}{\phi(\alpha_C)}.$$

Writing these base angles explicitly gives

$$Q(\sigma_A)Q(\sigma_B)Q(\sigma_C)=\frac{\phi(\angle CED)}{\phi(\angle AEF)}\cdot\frac{\phi(\angle AFE)}{\phi(\angle BFD)}\cdot\frac{\phi(\angle BDF)}{\phi(\angle CDE)}.$$

Each factor compares the two base angles meeting at one pierce point: the first at E, the second at F, and the third at D.

By Lemma 7.2, at two pierce points the two angles are vertical, hence equal, so the corresponding factors are 1. At the remaining pierce point the two angles are supplementary, say θ and $\pi-\theta$. Since ϕ is odd about $\pi/2$,

$$\frac{\phi(\theta)}{\phi(\pi-\theta)}=-1.$$

Therefore

$$Q(\sigma_A)Q(\sigma_B)Q(\sigma_C)=-1.$$

By (9), this is equivalent to $S+T=0$. Hence, by Theorem 3.1, the cevians AG, BH, and CI are concurrent. □

Proof of necessity. Assume that the center is concurrent for every admissible transversal. We shall show that its shape function has the factorization described in the theorem.

We work on an open region where the relevant functions are nonzero. This is the only place where logarithms and signs are used; the resulting identities then extend to the zero set by continuity.

Step 1: The functional equation.

Write

$$q(u,v) = \log\left|Q(u,v,\pi-u-v)\right|.$$

Thus q is the logarithm of the absolute value of the leg–swap ratio, written as a function of the two base angles.

Taking absolute values in the concurrence condition

$$Q(\sigma_A)Q(\sigma_B)Q(\sigma_C) = -1$$

gives

$$q(\sigma_A)+q(\sigma_B)+q(\sigma_C)=0.$$

Now use the representative branch from (10):

$$\sigma_A = (C-x,\ B+x,\ A), \qquad \sigma_B = (\pi-B-x,\ x,\ B), \qquad \sigma_C = (x,\ C-x,\ \pi-C).$$

Introduce the variables

$$\lambda = C-x, \qquad \mu = x, \qquad \nu = \pi - B - x.$$

Then

$$A = \nu-\lambda, \qquad B = \pi-\nu-\mu, \qquad x=\mu,$$

and

$$B+x = \pi-\nu.$$

Thus the concurrence condition becomes

$$q(\lambda,\pi-\nu)+q(\nu,\mu)+q(\mu,\lambda)=0. \tag{13}$$

Step 2: Solving the functional equation.

Differentiate (13) with respect to μ. The first term is independent of μ. In the second term, μ occurs as the second argument of q; in the third term, μ occurs as the first argument of q. Hence

$$\frac{\partial q}{\partial v}(\nu,\mu)+\frac{\partial q}{\partial u}(\mu,\lambda)=0.$$

For fixed μ, the first term is independent of λ, while the second term is independent of ν. Since λ,μ,ν vary independently over an open set, each term must depend only on μ. In particular,

$$\frac{\partial q}{\partial v}(u,v)$$

depends only on v.

It follows that

$$q(u,v)=m(u)+K(v)$$

for some one-variable functions m and K.

On the other hand, interchanging the two base angles inverts Q, so

$$Q(v, u, \pi - u - v) = Q(u, v, \pi - u - v)^{-1}.$$

Therefore

$$q(v, u) = -q(u, v).$$

In particular $q(u, u) = 0$, and hence

$$m(u) + K(u) = 0.$$

Thus $m = -K$, and so

$$q(u, v) = K(v) - K(u).$$

Put

$$\Phi(t) = e^{K(t)}.$$

Then

$$|Q(\alpha, \beta, \gamma)| = \frac{\Phi(\beta)}{\Phi(\alpha)}. \tag{14}$$

We shall also need one consequence of (13). Setting $\mu = \lambda$ gives

$$q(\lambda, \pi - \nu) + q(\nu, \lambda) = 0.$$

Using $q(u, v) = K(v) - K(u)$, this becomes

$$K(\pi - \nu) - K(\nu) = 0.$$

Hence

$$\Phi(\pi - t) = \Phi(t).$$

Thus Φ is even about $\pi/2$.

Step 3: Determining the sign.

Equation (14) determines the absolute value of Q. It remains to determine its sign.

Write

$$Q(u, v, \pi - u - v) = \varepsilon(u, v)\frac{\Phi(v)}{\Phi(u)},$$

where

$$\varepsilon(u, v) \in \{1, -1\}.$$

Since interchanging u and v inverts Q, we have

$$\varepsilon(v, u) = \varepsilon(u, v).$$

Also,

$$Q(u, u, \pi - 2u) = 1,$$

so

$$\varepsilon(u, u) = 1.$$

Substituting the above expression for Q into the concurrence condition, and using the fact that the absolute values already multiply to 1, gives

$$\varepsilon(\lambda, \pi - \nu)\, \varepsilon(\nu, \mu)\, \varepsilon(\mu, \lambda) = -1. \tag{15}$$

Set $\mu = \lambda$ in (15). Since $\varepsilon(\lambda, \lambda) = 1$, we get

$$\varepsilon(\lambda, \pi - \nu)\, \varepsilon(\nu, \lambda) = -1.$$

Using the symmetry of ε, this becomes

$$\varepsilon(\lambda, \pi - \nu) = -\varepsilon(\lambda, \nu). \tag{16}$$

Substituting (16) back into (15) gives

$$\varepsilon(\lambda, \nu)\, \varepsilon(\nu, \mu)\, \varepsilon(\mu, \lambda) = 1.$$

Choose a fixed angle r in the interval under consideration, and define

$$\chi(t) = \varepsilon(t, r).$$

Taking $\mu = r$ in the last identity gives

$$\varepsilon(\lambda, \nu)\, \varepsilon(\nu, r)\, \varepsilon(r, \lambda) = 1.$$

Therefore

$$\varepsilon(\lambda, \nu) = \chi(\lambda)\chi(\nu).$$

Also, putting $\lambda = r$ in (16) gives

$$\chi(\pi - \nu) = -\chi(\nu).$$

Thus χ is odd about $\pi/2$.

Step 4: Recovering ϕ and M.

Define

$$\phi(t) = \chi(t)\Phi(t).$$

Since Φ is even about $\pi/2$ and χ is odd about $\pi/2$, the function ϕ is odd about $\pi/2$.

Moreover,

$$Q(\alpha, \beta, \gamma) = \varepsilon(\alpha, \beta)\frac{\Phi(\beta)}{\Phi(\alpha)} = \chi(\alpha)\chi(\beta)\frac{\Phi(\beta)}{\Phi(\alpha)} = \frac{\phi(\beta)}{\phi(\alpha)}.$$

By the equivalence proved before the theorem, this implies that

$$F(\alpha, \beta, \gamma) = \phi(\alpha)M(\alpha, \beta, \gamma),$$

where M is fully symmetric.

This proves the necessity and completes the proof of the Characterization Theorem. □

7.4. **The complete family of concurrent centers.** Theorems 6.9 and 7.3 together give an explicit description of every center function whose cevians are concurrent for every admissible transversal.

Theorem 7.4 (The Trigonometric Normal Form)**.** *A center function is concurrent for every admissible transversal if and only if, up to multiplication by a symmetric factor,*

$$f(a, b, c) = \cos A\, \Phi(\cos^2 A) + \sin A \cos A\, \Psi(\cos^2 A),$$

where A is the angle opposite the first argument and Φ, Ψ are functions of one variable.

Thus, the two concurrent forms of Theorem 6.9 are not merely sufficient; together they exhaust all centers concurrent for every admissible transversal.

Proof. By the Characterization Theorem, a concurrent center has shape function

$$F(\alpha, \beta, \gamma) = \phi(\alpha)M(\alpha, \beta, \gamma),$$

where M is fully symmetric and ϕ is odd about $\pi/2$, i.e. $\phi(\pi - t) = -\phi(t)$. Since the symmetric factor M may be absorbed into the phrase "up to multiplication by a symmetric factor," it remains to describe the possible form of the one-variable factor $\phi(A)$.

Assume first that the center is concurrent. Then

$$\phi(\pi - A) = -\phi(A).$$

Away from $A = \pi/2$, define

$$h(A) = \frac{\phi(A)}{\cos A}.$$

Since

$$\cos(\pi - A) = -\cos A,$$

we have

$$h(\pi - A) = \frac{\phi(\pi - A)}{\cos(\pi - A)} = \frac{-\phi(A)}{-\cos A} = h(A).$$

Thus, h is even about $\pi/2$.

We now write h as a function of $\sin A$. For $0 < s < 1$, define

$$H(s) = h(\arcsin s)$$

and assume that $H(s)$ extends continuously to $s = 1$. If $0 < A < \pi$ and $s = \sin A$, then either

$$\arcsin(\sin A) = A$$

or

$$\arcsin(\sin A) = \pi - A.$$

In the second case, $h(\pi - A) = h(A)$. Therefore, in all cases,

$$h(A) = H(\sin A).$$

Hence

$$\phi(A) = \cos A \, H(\sin A).$$

It remains to rewrite $H(\sin A)$ in the stated form. Put $s = \sin A$. Since $0 < A < \pi$, we have $s > 0$.

Decompose H into its even and odd parts. That is, write $H(s) = H_e(s) + H_o(s)$, where H_e is even and H_o is odd. Since H_e is even, it depends only on s^2, so

$$H_e(s) = P(s^2)$$

for some one-variable function P. Since H_o is odd, $H_o(s)/s$ is even, so

$$H_o(s) = sQ(s^2)$$

for some one-variable function Q. Hence

$$H(s) = P(s^2) + sQ(s^2).$$

Since

$$s^2 = \sin^2 A = 1 - \cos^2 A,$$

we may rewrite this as

$$H(\sin A) = \Phi(\cos^2 A) + \sin A \, \Psi(\cos^2 A)$$

for suitable one-variable functions Φ and Ψ. Note that Φ and Ψ are arbitrary functions. Therefore

$$\phi(A) = \cos A\,\Phi(\cos^2 A) + \sin A \cos A\,\Psi(\cos^2 A),$$

as claimed. This proves the necessity.

Conversely, suppose that, up to multiplication by a symmetric factor, f has the form

$$\cos A\,\Phi(\cos^2 A) + \sin A \cos A\,\Psi(\cos^2 A).$$

When A is replaced by $\pi - A$, the factor $\cos A$ changes sign, while $\sin A$ and $\cos^2 A$ remain unchanged. Therefore, the whole expression changes sign:

$$\phi(\pi - A) = -\phi(A).$$

Thus, ϕ is odd about $\pi/2$. By the Characterization Theorem, the center is concurrent for every admissible transversal. □

8. Proof of Theorem 2.1

The characterization obtained in Section 7 now allows us to explain every concurrent center found in the original computer search.

We are now at a point where we can give a proof of Theorem 2.1.

Proof. The table below lists, for each center in Theorem 2.1, its barycentric center function. For these centers, it is a pure function of the angle A. Such centers are known as *major centers*. For each center, the table lists

(1) the ETC barycentric center function,

(2) an equivalent concurrent form, and

(3) the preserving operations used to obtain it.

Column 2 records the center function obtained from [4]. Since ETC often lists only trilinear coordinates, we sometimes take the first trilinear coordinate and multiply by a to convert it to a barycentric coordinate. Whenever this leaves a factor of a or of bc, we remove it by means of

$$a \equiv \sin A, \qquad bc \equiv \csc A.$$

Each replacement alters the center function only by a factor symmetric in a, b, c, and a symmetric factor multiplies all three barycentric coordinates equally; it therefore leaves the center unchanged, and preserves the concurrency locus by Lemma 6.3. Indeed, by the Law of Sines, $a = 2R \sin A$, where the circumradius R is symmetric, so a and $\sin A$ differ by the symmetric factor $2R$; and from the area formula $2K = bc \sin A$, we get $bc = 2K \csc A$, where the area K is symmetric, so bc and $\csc A$ differ by the symmetric factor $2K$. After these reductions every entry of column 2 is a function of A alone.

Column 3 lists an equivalent center function in concurrent form, obtained from column 2 by standard trigonometric identities together with the discarding of nonzero constant factors. Each column-3 entry is $\cos A$ or $\sin A \cos A$ times a function of $\cos^2 A$, hence is in concurrent form. By Theorem 6.9, the corresponding center is concurrent. As columns 2 and 3 define the same center, every center listed in Theorem 2.1 is concurrent. □

Barycentric Center Functions		
center	**ETC form**	**concurrent form**
X_3	$\sin 2A$	$\sin A\cos A$
X_4	$\tan A$	$\sin A\cos A/\cos^2 A$
X_{19}	$\sin A\tan A$	$(\cos A)(1-\cos^2 A)/\cos^2 A$
X_{24}	$\tan A\cos 2A$	$(\sin A\cos A)(2\cos^2 A-1)/\cos^2 A$
X_{25}	$\sin^2 A\tan A$	$(\sin A\cos A)(1-\cos^2 A)/\cos^2 A$
X_{48}	$\sin A\sin 2A$	$(\cos A)(1-\cos^2 A)$
X_{49}	$\sin A\cos 3A$	$(\sin A\cos A)(4\cos^2 A-3)$
X_{63}	$\cos A$	$\cos A$
X_{68}	$\tan 2A$	$(\sin A\cos A)/(2\cos^2 A-1)$
X_{69}	$\cot A$	$(\sin A\cos A)/(1-\cos^2 A)$
X_{92}	$\sec A$	$(\cos A)/\cos^2 A$
X_{93}	$\sin A\sec 3A$	$(\sin A\cos A)/\big((\cos^2 A)(4\cos^2 A-3)\big)$
X_{184}	$\sin^3 A\cos A$	$(\sin A\cos A)(1-\cos^2 A)$
X_{186}	$\sin A\sin 3A\csc 2A$	$(\sin A\cos A)(4\cos^2 A-1)/\cos^2 A$
X_{264}	$\csc 2A$	$(\sin A\cos A)/\big((1-\cos^2 A)(\cos^2 A)\big)$
X_{265}	$\sin A\sin 2A\csc 3A$	$(\sin A\cos A)/(4\cos^2 A-1)$
X_{304}	$\cos A\csc^2 A$	$(\cos A)/(1-\cos^2 A)$
X_{305}	$\csc^3 A\cos A$	$(\sin A\cos A)/(1-\cos^2 A)^2$
X_{317}	$\cot 2A$	$(\sin A\cos A)(2\cos^2 A-1)/\big((1-\cos^2 A)(\cos^2 A)\big)$
X_{328}	$\cos A\csc 3A$	$(\sin A\cos A)/\big((1-\cos^2 A)(4\cos^2 A-1)\big)$
X_{340}	$\sec A\sin 3A\csc^2 A$	$(\sin A\cos A)(4\cos^2 A-1)/\big((1-\cos^2 A)\cos^2 A\big)$
X_{378}	$\tan A+\sin 2A$	$(\sin A\cos A)(1+2\cos^2 A)/\cos^2 A$
X_{563}	$\sin A\sin 4A$	$(\cos A)(1-\cos^2 A)(2\cos^2 A-1)$
X_{847}	$\tan A\sec 2A$	$(\sin A\cos A)/\big((\cos^2 A)(2\cos^2 A-1)\big)$

Thus, the computer search is completely explained by the characterization developed in Sections 6 and 7.

9. Parallel Transversals

So far we have treated each admissible transversal independently. The next result shows that only its *direction* matters. As an admissible transversal slides parallel to itself, the cevians AG, BH, and CI remain fixed. The reason is that each corner triangle changes only by a homothety about its corresponding vertex.

Lemma 9.1. *Let ℓ_0 and ℓ be admissible transversals of $\triangle ABC$ with $\ell \parallel \ell_0$, and let X be any triangle center. If G_0, H_0, I_0 and G, H, I are the X-points of the corner triangles of ℓ_0 and ℓ respectively, then, as lines,*

$$AG = AG_0, \qquad BH = BH_0, \qquad CI = CI_0.$$

Proof. Consider the corner triangle $T_A = AEF$ of ℓ, with $E = \ell \cap CA$ and $F = \ell \cap AB$, alongside $T_A^0 = AE_0F_0$ of ℓ_0. The sidelines CA and AB meet at A, and the parallels ℓ_0, ℓ cut them; by the intercept theorem [1] the two signed cut ratios coincide,

$$\frac{\overline{AE}}{\overline{AE_0}} = \frac{\overline{AF}}{\overline{AF_0}} =: \lambda_A,$$

and $\lambda_A \neq 0$ since ℓ passes through no vertex. Hence $T_A = h_A(T_A^0)$, where h_A is the homothety of center A and ratio λ_A, which fixes A and carries $E_0 \mapsto E$ on CA and $F_0 \mapsto F$ on AB.

Barycentric coordinates are preserved by every affine map, and a homothety is affine; so the point with prescribed barycentric coordinates in T_A^0 is carried by h_A to the point with the *same* coordinates in T_A. The X-point is defined by these barycentric coordinates (1), so $G = h_A(G_0)$. As h_A has center A, the points A, G_0, G are collinear; that is, $AG = AG_0$. The arguments at B and C are identical, with homotheties h_B, h_C of centers B, C. $\square$

Theorem 9.2. *If a triangle center X is concurrent for an admissible transversal ℓ_0, then X is concurrent for every admissible transversal parallel to ℓ_0.*

Proof. By Lemma 9.1, for any admissible $\ell \parallel \ell_0$ the cevians AG, BH, CI are the very same three lines as for ℓ_0. If they concur for ℓ_0, they concur for ℓ. $\square$

Corollary 9.3. *If X is concurrent for an admissible transversal ℓ_0 with perspector P, then X is concurrent for every admissible transversal parallel to ℓ_0, and the perspector is the same point P.*

Proof. The three cevians are unchanged as lines (Lemma 9.1), so their common point is unchanged. $\square$

Thus each parallel class of admissible transversals determines a fixed triple of cevians and, when they are concurrent, a fixed perspector.

10. The Perspector Formula

When a center X is concurrent for a fixed admissible transversal ℓ, we can compute the coordinates of the perspector explicitly.

Theorem 10.1 (Perspector Formula)**.** *Let X be concurrent for the admissible transversal*

$$\ell = [u : v : w],$$

and let f be the barycentric center function of X. Define

$$\begin{aligned} s_A &= f(|FA|, |AE|, |EF|), & t_A &= f(|AE|, |EF|, |FA|), \\ s_B &= f(|DB|, |BF|, |FD|), & t_B &= f(|BF|, |FD|, |DB|), \\ s_C &= f(|EC|, |CD|, |DE|), & t_C &= f(|CD|, |DE|, |EC|). \end{aligned}$$

Then the perspector of $\triangle ABC$ and $\triangle GHI$ is

$$P = \big(s_A s_B(w - v) : t_A t_B(u - w) : s_A t_B(u - v)\big).$$

Proof. From the proof of Theorem 3.1,

$$AG = \left[0 : \frac{s_A}{u - w} : \frac{t_A}{v - u}\right], \qquad BH = \left[\frac{t_B}{w - v} : 0 : \frac{s_B}{v - u}\right].$$

Their intersection is the cross product of these two line-coordinate vectors. After clearing the common factor

$$(u - w)(v - u)(w - v),$$

we obtain

$$P = \big(s_A s_B(w - v) : t_A t_B(u - w) : s_A t_B(u - v)\big).$$

It remains only to check that this point also lies on CI. Since

$$CI = \left[\frac{s_C}{w-v} : \frac{t_C}{u-w} : 0\right],$$

substitution gives

$$\frac{s_C}{w-v}\, s_A s_B(w-v) + \frac{t_C}{u-w}\, t_A t_B(u-w) = s_A s_B s_C + t_A t_B t_C = S + T.$$

This is zero by Theorem 3.1, because X is concurrent for ℓ. Hence P lies on all three cevians, and is the perspector. □

11. Expressions for the side lengths

It is useful to have expressions for the lengths of the sides of the corner triangles. They are given by the following theorem.

Theorem 11.1. *Let $\ell = [u : v : w]$ be admissible. Then*

$$\begin{aligned}
&FA = c\left|\frac{u}{u-v}\right|, \quad AE = b\left|\frac{u}{u-w}\right|, \quad EF = Q\left|\frac{u}{(u-v)(u-w)}\right|,\\
&DB = a\left|\frac{v}{v-w}\right|, \quad BF = c\left|\frac{v}{v-u}\right|, \quad FD = Q\left|\frac{v}{(v-w)(v-u)}\right|,\\
&EC = b\left|\frac{w}{w-u}\right|, \quad CD = a\left|\frac{w}{w-v}\right|, \quad DE = Q\left|\frac{w}{(w-u)(w-v)}\right|,
\end{aligned} \tag{17}$$

where $Q = a^2u^2 + b^2v^2 + c^2w^2 - S_A vw - S_B wu - S_C uv$,
$S_A = b^2 + c^2 - a^2$, $S_B = c^2 + a^2 - b^2$, and $S_C = a^2 + b^2 - c^2$.

Proof. The pierce points are $D = (0 : w : -v)$, $E = (-w : 0 : u)$, $F = (v : -u : 0)$. The result then follows from direct computation with the barycentric distance formula $|UV|^2 = -(a^2yz + b^2zx + c^2xy)$ applied to the normalized displacement $(x, y, z) = U - V$. □

Note that the squares of the side lengths are all rational functions of a, b, c, u, v, w.

Combining the Perspector Formula with (17) gives the coordinates of the perspector whenever the center function and the equation of the transversal are known.

12. The Euler Line

In earlier sections, we have determined when a center is concurrent for all admissible transversals. We can also ask when a center is concurrent for some specific transversal ℓ. We have the following.

Theorem 12.1. *There is no admissible transversal ℓ such that X_n is concurrent for ℓ when $n \in \{1, 2, 6\}$.*

Proof. Suppose $\ell = [u, v, w]$. The condition for X_n to be concurrent for ℓ is given by Theorem 3.1, namely $S + T = 0$, where $S = s_A s_B s_C$ and $T = t_A t_B t_C$. We can compute S and T from (3) and (17).

For X_1, $f(a, b, c) = a$ and we find

$$S + T = \frac{2abc|u| \cdot |v| \cdot |w|}{|u-v| \cdot |v-w| \cdot |w-u|}.$$

For an admissible transversal, $u \neq 0$, $v \neq 0$, $w \neq 0$, $u \neq v$, $v \neq w$, and $w \neq u$. Thus, each term is positive, so $S + T$ cannot equal 0.

For X_2, $f(a, b, c) = 1$ and we find $S + T = 2$ which is not 0.

For X_6, $f(a, b, c) = a^2$ and we find

$$S + T = \frac{2a^2b^2c^2u^2v^2w^2}{(u - v)^2(v - w)^2(w - u)^2}.$$

Again, each term is positive, so $S + T$ cannot equal 0. □

The situation is different for X_3 and X_4.

By Theorem 2.1, we know that X_3 and X_4 are concurrent for *all* admissible lines. For $n > 4$, we can ask if X_n is concurrent for *some* admissible line. We find for $n = 99$, that that although X_{99} is not concurrent for all admissible lines, it *is* concurrent for the Euler line of $\triangle ABC$.

In fact, we have the following theorem (discovered empirically).

Theorem 12.2 (Centers concurrent for the Euler line)**.** *Let ℓ be the Euler line of $\triangle ABC$. Three points D, E, and F are determined by the points where line ℓ intersects the sidelines BC, CA, and AB, respectively. Let the X_n-points of triangles AEF, BFD, and CDE be G, H, and I respectively. Assume that ℓ is an admissible transversal. If*

$$n \in \{99, 107, 110, 125, 131, 136, 163, 250, 339, 403, 476, 661, 662,$$
$$669, 670, 798, 799, 822, 823, 850, 925, 930, 974\},$$

then triangles ABC and GHI are perspective. That is, X_n is concurrent for the Euler line.

Remark 12.3. The values of n listed in Theorem 12.2 are ones that are in addition to the values of n in the general case (Theorem 2.1), which are also concurrent for the Euler line.

Before proving Theorem 12.2, we give some preliminary results.

Since the Euler line is a fixed transversal, the perspector will be a fixed triangle center. The equation of the Euler line is given in [7, §11.4.4] as

$$S_A(S_B - S_C)x + S_B(S_C - S_A)y + S_C(S_A - S_B)z = 0.$$

For a center that is concurrent for the Euler line, we can then use the Perspector Formula (Theorem 10.1) to find the coordinates for the perspector. The perspectors for various such centers are listed in the following table.

For centers concurrent for the Euler line that are not in the table, the perspectors are not cataloged in [4] as of January, 2026.

Perspectors for Centers Concurrent for the Euler Line	
center	**perspector**
X_3	X_{110}
X_4	X_{523}
X_{24}	X_{15453}
X_{68}	X_{7471}
X_{69}	X_{3233}
X_{93}	X_{18039}
X_{99}	X_{16163}
X_{107}	X_{125}
X_{110}	X_3
X_{125}	X_{107}
X_{265}	X_{476}
X_{476}	X_{265}
X_{523}	X_4
X_{850}	X_{34334}

By Theorem 9.2, concurrence depends only on the direction of the transversal, so each center listed in Theorem 12.2 is concurrent not merely for the Euler line but for *every* admissible transversal parallel to it; and by Corollary 9.3 the perspector is, in each case, the fixed center tabulated above.

Examining the perspectors in the above table suggests that for the Euler line, center X has perspector Y if and only if center Y has perspector X. We will prove this as Theorem 12.8 below.

The key is the following well-known property of the Euler Line [2].

Lemma 12.4. *If the Euler line of non-equilateral triangle ABC meets CA at E and meets AB at F (Figure 11), and if $\angle A \notin \{60°, 120°\}$, then triangle AEF is not equilateral and its Euler line is parallel to BC.*

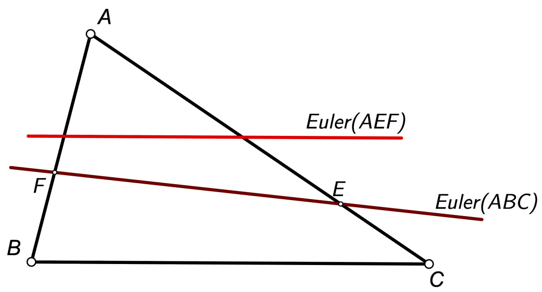

FIGURE 11. red line is parallel to BC

Remark 12.5. When angle A is 60° or 120°, the conclusion fails for a picturesque reason: there the corner triangle cut off by the Euler line is *equilateral*, so it has no Euler line at all.

Remark 12.6. This lemma propagates to the whole pencil of lines parallel to the Euler line because parallel transversals cut off corner triangles similar to one another (they are images of each other under homotheties centered at the vertex).

Remark 12.7. Similarly, if $\angle B \notin \{60^\circ, 120^\circ\}$ the Euler line of BFD is parallel to CA, and if $\angle C \notin \{60^\circ, 120^\circ\}$ the Euler line of CDE is parallel to AB.

Theorem 12.8 (The Reciprocity Theorem)**.** *Suppose no angle of $\triangle ABC$ equals 60° or 120° and the Euler line of $\triangle ABC$ is an admissible transversal. Let X be a triangle center that is concurrent for the Euler line, and let Y be its perspector. Then Y is concurrent for the Euler line of $\triangle ABC$ with perspector X.*

Simply put, for the Euler line, center X has perspector Y if and only if center Y has perspector X.

Proof. We will prove the result for any line parallel to the Euler line. Fix an admissible transversal ℓ parallel to the Euler line, with pierce points D, E, F, and consider the complete quadrilateral formed by the four lines BC, CA, AB, ℓ (see Figure 12). Its four triangles are ABC, AEF, BFD, and CDE. Each omits exactly one of the four lines, and the omitted line is a transversal of it. By Lemma 12.4 (and the remarks following), the configuration is *self-reciprocal*: each of the three corner triangles sees its omitted line as a transversal parallel to its own Euler line, namely

$$BC \parallel \text{Euler}(AEF), \qquad CA \parallel \text{Euler}(BFD), \qquad AB \parallel \text{Euler}(CDE),$$

and these transversals are admissible for the corner triangles (their pierce points are D, B, C on the sidelines of AEF, and cyclically, none of which is a vertex).

Now consider the center Y in the corner triangle AEF. By Corollary 9.3, the Y-point of AEF is the perspector formed by taking X-points in the corner triangles formed by triangle AEF and *any* admissible transversal parallel to the Euler line of $\triangle AEF$. By the self-reciprocity just established, we may take that transversal to be the line BC. The transversal BC meets the sidelines EF, FA, AE of $\triangle AEF$ at D, B, C respectively, so the corner triangles of the pair $(\triangle AEF, BC)$ at the vertices A, E, F are

$$ABC, \qquad CDE, \qquad BFD,$$

and the corresponding corner centers (using X-points) are $X = X_{ABC}$ itself, X_{CDE}, and X_{BFD} (Figure 12). In particular, the cevian issuing from A is the line AX, and the perspector, which is the Y-point of $\triangle AEF$, lies on it. Hence, when Y-points are taken in the corner triangles formed by $\triangle ABC$ and ℓ, the line joining A to the Y-point of $\triangle AEF$ is the line AX and passes through X.

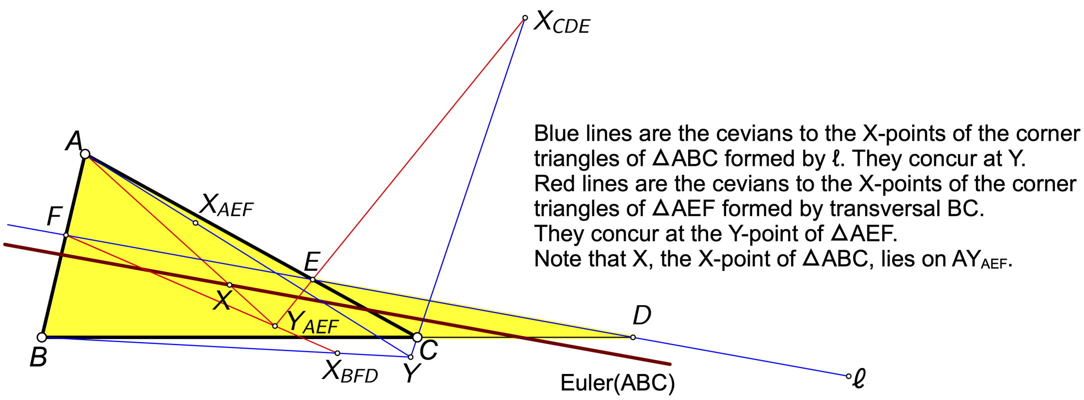

Figure 12. construction of Y-point of $\triangle AEF$

The same argument applied to the pairs $(\triangle BFD, CA)$ and $(\triangle CDE, AB)$. Their corner triangles at B and at C are again ABC. This shows that the cevians from B and from C (when using Y-points in the corner triangles) are the lines BX and CX (Figure 13).

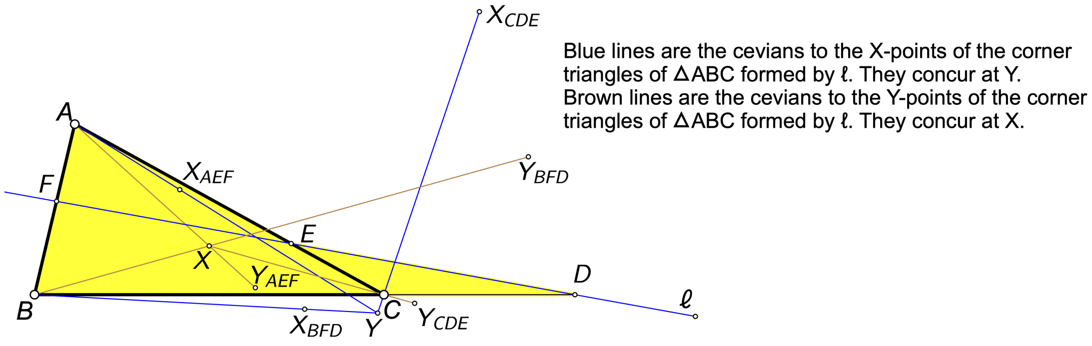

FIGURE 13. construction of X and Y points of $\triangle ABC$

The three cevians to the Y-points of the corner triangles of $\triangle ABC$ therefore concur at X, showing that the center Y is concurrent for ℓ, with perspector X. $\square$

Proposition 12.9. *The Steiner point, X_{99}, is concurrent for the Euler line.*

Proof. The Steiner point has barycentric center function $f(x,y,z) = 1/(y^2 - z^2)$. Put

$$p = |AE|^2 - |EF|^2, \qquad p' = |EF|^2 - |FA|^2,$$

and let q, q', r, r' be the cyclic analogues. By Theorem 3.1, $s_A = 1/p$, $t_A = 1/p'$, and so on, whence

$$S + T = \frac{p'q'r' + pqr}{pqr\, p'q'r'},$$

so AG, BH, CI concur if and only if $pqr + p'q'r' = 0$.

The trace points are $D = (0 : w : -v)$, $E = (-w : 0 : u)$, $F = (v : -u : 0)$. Using the barycentric distance formula, one computes

$$p = \frac{-u^2 m_A}{(u-v)^2(u-w)}, \qquad p' = \frac{u^2 m'_A}{(u-v)(u-w)^2},$$

where

$$m_A = (a^2 - b^2)(u - v) + c^2(v - w), \qquad m'_A = (a^2 - c^2)(u - w) + b^2(w - v),$$

the remaining quantities following by the cyclic substitution $(a, b, c, u, v, w) \mapsto (b, c, a, v, w, u)$. Multiplying and using $(u - w)(v - u)(w - v) = -(u - v)(v - w)(w - u)$, we have

$$pqr = \frac{u^2v^2w^2\, m_A m_B m_C}{\Delta^3}, \qquad p'q'r' = \frac{u^2v^2w^2\, m'_A m'_B m'_C}{\Delta^3},$$

where $\Delta = (u - v)(v - w)(w - u)$.

As u, v, w are nonzero and distinct, concurrency is equivalent to

$$\nu := m_A m_B m_C + m'_A m'_B m'_C = 0.$$

The cubic ν factors as $\nu = -L\,Q$ with L linear and Q quadratic in (u, v, w). The coefficient vector of L is

$$\big(2a^4 - a^2b^2 - a^2c^2 - b^4 + 2b^2c^2 - c^4 \ : :\big) \ = \ X_4 - X_3,$$

the orthocenter minus the circumcenter, taken in the common normalization in which each has coordinate sum $16K^2$ (with K the area of ABC). Its coordinates sum to zero, so $X_4 - X_3$ is the point at infinity of the line X_3X_4, namely of the Euler line. Hence $L = 0$ expresses exactly that ℓ passes through the infinite point of the Euler line, i.e. that ℓ is parallel to it. By hypothesis this holds, so $\nu = 0$; therefore $S + T = 0$, and AG, BH, CI concur. □

If X is a concurrent center, by *the orbit of* X we mean the set of all centers that can be obtained from X by successively applying the preserving operations (O1)-(O6) of Definition 6.2.

Theorem 12.10. *Every center in the orbit of X_{99} is concurrent for the Euler line.*

Proof. Since X_{99} is concurrent for the Euler line, the result follows from Theorem 6.4. □

Theorem 12.11. *The center X_n is concurrent for the Euler line for $n \in \{107, 110, 163, 661, 662, 669, 670, 798, 799, 822, 823, 850, 930\}$.*

Proof. From [4], we find that the center function for X_{99}, the Steiner point, is $(a^2 - b^2)(c^2 - a^2)$ and that the isogonal conjugate of X_{99} is X_{512}. The following identities therefore show that each X_n is concurrent by Theorem 12.10.

$$\begin{array}{lll}
X_{512} = a^2/X_{99}, & X_{523} = 1/X_{99} & \\
X_{107} = S_A^{-2}X_{99}, & X_{110} = a^2X_{99}, & X_{163} = a^3X_{99}, \\
X_{662} = aX_{99}, & X_{669} = a^4X_{512}, & X_{670} = (bc)^2X_{99}, \\
X_{661} = a^2/X_{662}, & X_{799} = bc\,X_{99}, & X_{822} = a^3S_A^2X_{512}, \\
X_{823} = bc(S_BS_C)^2X_{99}, & X_{850} = (bc)^2X_{512}, & X_{930} = (bc)^{-2}\left(4\cos^2\!A - 3\right)^{-1}X_{99}, \\
X_{798} = a^3/X_{99}, & & X_{925} = (bc)^{-2}\left(4\cos^2\!A - 2\right)^{-1}X_{99}.
\end{array}$$

Note: For X_{930}, $a^4+b^4+c^4-2a^2b^2-2a^2c^2-b^2c^2 = S_A^2-3b^2c^2 = (bc)^2(4\cos^2 A-3)$, so dividing by it is the co-side-power operation (O3) with $j = -2$ followed by the angular multiplier (O6) with $\Phi(t) = 1/(4t - 3)$; both preserve concurrence, so X_{930} lies in the orbit of X_{99}. A similar remark holds for X_{925}. □

Theorem 12.12. *The center X_n is concurrent for the Euler line for $n \in \{125, 250, 339, 476\}$.*

Proof. Since X_{107} is concurrent for the Euler line with perspector X_{125}, by the Reciprocity Theorem (Theorem 12.8), X_{125} must be concurrent for the Euler line. Similarly, X_{265} is concurrent for all lines (Theorem 2.1), and for the Euler line, the perspector is X_{476}. Hence X_{476} must be concurrent for the Euler line. Examining their center functions from [4], we find

$$X_{250} = a^2/X_{125}, \quad X_{339} = 1/X_{250},$$

so X_{250} and X_{339} are concurrent for the Euler line by Theorem 6.4. □

Theorem 12.13. *The center X_{131} is concurrent for the Euler line.*

Proof. By Theorem 3.1, concurrence for a given transversal is equivalent to the identity $S + T = 0$. Substituting the ETC center function of X_{131}, the side-length expressions of Theorem 11.1, and the coordinates of the Euler line reduces $S + T = 0$ to a polynomial identity in a, b, c. We have verified this identity with Mathematica. The intermediate polynomials are large and unilluminating, so we omit them. □

Theorem 12.14. *The center X_n is concurrent for the Euler line for $n \in \{136, 403, 974\}$.*

Concurrence for these centers is verified by the same computer-algebra reduction as in Theorem 12.13, and the details are omitted.

We leave the reader with something to ponder on.

Conjecture 1. *A center on the circumcircle is concurrent for the Euler line if and only if it lies in the orbit of X_{99}.*

Acknowledgments

The authors gratefully acknowledge the Anthropic AI program Claude (Opus 4.8 and Fable 5) for substantial assistance in proving the results of this paper. We also thank OpenAI's ChatGPT (version 5.5) for clarifying and simplifying much of the exposition.

References

[1] Nathan Altshiller-Court, *College Geometry*, 2nd edition. Barnes & Noble, Inc. NY: 1952.
[2] Alexander Bogomolny, *Iterations on Euler Lines* at Cut The Knot, 2018.
https://www.cut-the-knot.org/Curriculum/Geometry/EulerToEuler.shtml
[3] Sava Grozdev and Deko Dekov, *Barycentric Coordinates: Formula Sheet*, International Journal of Computer Discovered Mathematics, **1**(2016)75–82.
http://www.journal-1.eu/2016-2/Grozdev-Dekov-Barycentric-Coordinates-pp.75-82.pdf
[4] Clark Kimberling, *Encyclopedia of Triangle Centers*, 2025.
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html
[5] Stanley Rabinowitz and Ercole Suppa, *Relationships between a Central Triangle and its Reference Triangle*, submitted to International Journal of Computer Discovered Mathematics. **11**(2026)60–103.
url to be supplied
[6] Eric W. Weisstein, *Triangle Center Function.* From MathWorld–A Wolfram Resource.
https://mathworld.wolfram.com/TriangleCenterFunction.html
[7] Paul Yiu, Introduction to the Geometry of the Triangle, version 13 (April 2013).
https://users.math.uoc.gr/~pamfilos/Yiu.pdf