A Small Triangle Containing the Electrostatic Center

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Abstract

Let $P = X(5626)$ be the electrostatic center of a scalene triangle ABC, where $X(n)$ denotes the n-th named triangle center cataloged in Kimberling's *Encyclopedia of Triangle Centers*. Very little is known about the geometric location of P, and no central line containing it appears to be known. We localize P by defining

$$I = X(1), \qquad G = X(2), \qquad O = X(3), \qquad K = X(6), \qquad G_e = X(7),$$

$$M = X(5703) = IG \cap OG_e, \qquad T = X(5736) = GK \cap OG_e,$$

and proving

$$P \in \operatorname{int} \triangle GMT.$$

Our proof uses a one-parameter curve arising naturally from the equations of Abraham and Kovač for the electrostatic center. We also show that this entire curve lies inside $\triangle IGK$ and inside the sharper central triangle $\triangle IGX(5219) \subset \triangle IGN$, where $N = X(5)$ and $X(5219) = IN \cap GG_e$. The curve has limiting tangent GK at its centroid endpoint G; at its incenter endpoint I, the limiting tangent is the angle bisector from the vertex opposite the shortest side.

1 Introduction

We use Kimberling's notation $X(n)$ for the n-th named triangle center in the *Encyclopedia of Triangle Centers* (ETC) [4]. The electrostatic center is $X(5626)$. Abraham and Kovač [1] proved that it is the unique point in the interior of a triangle maximizing the inverse-distance potential of a uniformly charged triangular lamina. We denote this point by P. Finch [5] observed another physical interpretation: in a two-dimensional illumination model, in which brightness from a point source is inversely proportional to distance, maximizing the total brightness of a triangular region leads to exactly the same formulation. This should be distinguished from Shibata's inverse-square streetlight model, which leads to a different center.

Very little seems to be known about the geometric location of the electrostatic center. For many of the more extensively studied triangle centers, ETC records numerous geometric properties: central lines on which the point lies, circles or other curves passing through it, and relations with other familiar centers. The entry for $X(5626)$ contains no comparable list of incidences. In particular, we have been unable to find a central line that always contains P.

Numerical experiments do, however, show that P lies remarkably close to certain central lines. This suggests replacing the search for an exact incidence by a localization problem: can a small region, defined entirely by familiar triangle centers and central lines, be proved to contain P for every triangle? The purpose of this paper is to give such a localization.

The same point has been studied from two complementary geometric viewpoints. Nicollier [6] gave an elegant localization as the common point of three suitably related ellipses. He calls it the *gravitational center*; his definition uses the same inverse-distance potential and hence gives the same point $X(5626)$. Sakata [7] subsequently studied radial centers of triangles from a Euclidean-geometric point of view and, in particular, localized them in the minimal unfolded region of the triangle. These works, together with Abraham and Kovač, provide the principal background for the present paper.

Throughout, ABC is a nondegenerate triangle with

$$a = BC, \qquad b = CA, \qquad c = AB,$$

semiperimeter s, inradius r, circumradius R, and area Δ. We use the standard notation

$$I = X(1), \quad G = X(2), \quad O = X(3), \quad N = X(5), \quad K = X(6), \quad G_e = X(7),$$

where I is the incenter, G the centroid, O the circumcenter, N the nine-point center, K the symmedian point, and G_e the Gergonne point. For a scalene triangle, the lines IG, GK, and OG_e are distinct. Define

$$M = IG \cap OG_e, \qquad T = GK \cap OG_e. \tag{1}$$

ETC [4] identifies these points as

$$M = X(5703), \qquad T = X(5736).$$

Our main result is

$$\boxed{P \in \operatorname{int} \triangle GMT}.$$

Here and throughout, $\operatorname{int} \triangle XYZ$ denotes the interior of triangle XYZ. The restriction to scalene triangles is natural here: in the isosceles case, the relevant centers lie on the symmetry axis (the altitude to the base), and $\triangle GMT$ degenerates. Figure 1 illustrates the main result for a typical scalene triangle.

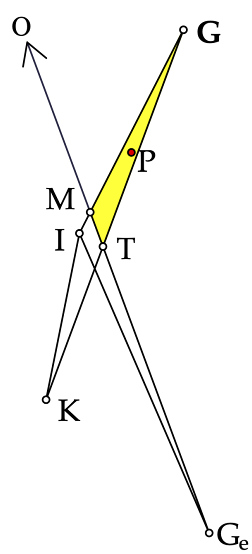

Figure 1: The electrostatic center P lies in the yellow area.

When we say that a point X lies on the G-side of a line ℓ, we mean that X and G lie in the same open half-plane determined by ℓ. Analogous phrases such as the M-side or the T-side have the corresponding meaning.

To prove that P lies inside $\triangle GMT$, it is enough to establish the following three side relations:

(i) P and T lie on the same side of $GM = IG$;

(ii) P and M lie on the same side of $GT = GK$;

(iii) P and G lie on the same side of $MT = OG_e$.

The first two relations will follow from a stronger localization result derived from the equations of Abraham and Kovač. The third relation is considerably more difficult and constitutes the main quantitative part of the proof.

The proof is organized accordingly. We begin by recalling the equations of Abraham and Kovač and use them to construct the Abraham–Kovač curve. After establishing an affine coordinate frame, we prove the global containment of that curve in $\triangle IGK$; this yields the first two side relations. We then prove the sharper whole-curve localization in $\triangle IGX(5219)$, from which the containment in $\triangle IGN$ follows immediately. We next determine the limiting tangents at both endpoints of the curve: GK at G, and at I the angle bisector from the vertex opposite the shortest side. Finally, we turn to the MT-side relation, derive a scalar equation that singles out the electrostatic center within the curve, and prove the required inequality by dividing the possible triangle shapes into several cases. Lengthy polynomial sign checks are collected in the appendix.

2 The Abraham–Kovač curve

We first recall the analytic characterization from which our one-parameter curve arises. If the triangular lamina carries uniform charge, its inverse-distance potential at an interior point X is, apart from an inessential constant,

$$V(X) = \iint_{ABC} \frac{dA(Y)}{|Y - X|}.$$

Abraham and Kovač proved that this function has the unique maximizer P mentioned above. A consequence of [1, Theorem 1] is that there is a number $q^* > 0$ such that

$$\begin{aligned} PB + PC &= a\coth(q^*a), \\ PC + PA &= b\coth(q^*b), \\ PA + PB &= c\coth(q^*c). \end{aligned} \tag{2}$$

To obtain a curve rather than a single point, we retain two difference equations and allow the parameter to vary. For $q > 0$, put

$$U = U(q) = a\coth(qa), \qquad V = V(q) = b\coth(qb), \qquad W = W(q) = c\coth(qc).$$

Define $P(q)$ to be the unique point X satisfying

$$XB^2 - XA^2 = (U - V)W, \qquad XC^2 - XA^2 = (U - W)V. \tag{3}$$

Each difference of squared distances is affine linear in X, so the two equations in (3) represent two nonparallel lines and determine a unique point $X = P(q)$.

We call the locus

$$\mathcal{C} = \{P(q) : q > 0\}$$

the *Abraham–Kovač curve* of ABC, since it arises naturally by allowing the parameter in the equations of Abraham and Kovač to vary. Subtracting equations in (2) and multiplying by the appropriate sums shows that the electrostatic center is the particular point

$$P = P(q^*)$$

on this curve. The remaining Abraham–Kovač condition, evaluated at the electrostatic parameter q^*, is

$$P(q^*)A = \frac{V(q^*) + W(q^*) - U(q^*)}{2}. \tag{4}$$

This condition singles out the electrostatic point from the one-parameter curve.

2.1 An affine normal frame

Put

$$\sigma = a^2 + b^2 + c^2.$$

For a scalene triangle, every point Z has unique affine coordinates (u, v) characterized by

$$\begin{aligned} ZB^2 - ZC^2 &= (c^2 - b^2)(u + va^2),\\ ZC^2 - ZA^2 &= (a^2 - c^2)(u + vb^2),\\ ZA^2 - ZB^2 &= (b^2 - a^2)(u + vc^2). \end{aligned} \tag{5}$$

The three equations are compatible because their left sides and their right sides each sum to zero; since the triangle is scalene, any two determine (u, v) uniquely.

Remark 2.1. The map $Z \mapsto (u, v)$ is an affine bijection of the plane. Indeed, each left side in (5) is an affine function of Z, while fixing the first two distance differences gives two lines perpendicular to BC and CA, respectively; these lines are not parallel. Thus the map carries lines to lines, half-planes to half-planes, parallel lines to parallel lines, and affine combinations to affine combinations. In particular, all "same side" arguments below may be carried out in the (u, v)-frame.

In this frame

$$O = (0, 0), \qquad G = \left(\frac{1}{3}, 0\right), \qquad N = \left(\frac{1}{2}, 0\right), \qquad K = \left(0, \frac{2}{\sigma}\right).$$

If

$$d = (a + b)(b + c)(c + a),$$

then

$$I = \left(\frac{abc}{d}, \frac{a + b + c}{d}\right). \tag{6}$$

Write $I = (u_I, v_I)$. Comparing (3) with (5) gives

$$u + va^2 = \frac{U(W - V)}{c^2 - b^2}, \qquad u + vb^2 = \frac{V(U - W)}{a^2 - c^2}, \qquad u + vc^2 = \frac{W(V - U)}{b^2 - a^2}. \tag{7}$$

These formulas will be used repeatedly below.

3 The Abraham–Kovač curve lies inside $\triangle IGK$

The following global localization is one of the main results of the paper. Because $\mathcal{C} = \{P(q) : q > 0\}$ does not include its limiting endpoints, the containment is strict.

Theorem 3.1. *For every scalene triangle,*

$$\boxed{\mathcal{C} \subset \operatorname{int} \triangle IGK}.$$

Equivalently, for every $q > 0$,

$$P(q) \in \operatorname{int} \triangle IGK.$$

The proof requires one sign lemma for an affine decomposition of $P(q)$. We shall repeatedly use

$$\xi = s - a, \qquad \eta = s - b, \qquad \zeta = s - c. \tag{8}$$

Thus, $\xi + \eta + \zeta = s$.

For $q > 0$, write

$$P(q) = \alpha I + \beta G + \gamma N, \qquad \alpha + \beta + \gamma = 1. \tag{9}$$

The coefficients are unique because I, G, N are not collinear: G and N have second affine coordinate 0, whereas $v_I > 0$ by (6).

We use divided-difference notation. For a function f and distinct numbers $x_0, \dots, x_n$, define

$$f[x_0] = f(x_0), \qquad f[x_0, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0}.$$

In particular, $f[x, y] = (f(y) - f(x))/(y - x)$. For fixed $q > 0$, put

$$\Phi(t) = \frac{\sinh(2q\sqrt{t})}{\sqrt{t}}.$$

We first record the two signs that will be needed below.

Lemma 3.2. *For every scalene triangle and every $q > 0$, the coefficients in* (9) *satisfy*

$$\beta > 0, \qquad \gamma > 0.$$

Proof. Without loss of generality, assume that $a > b > c$. By (8), we then have $0 < \xi < \eta < \zeta < s$.

Since ABC is scalene, all arguments occurring below are distinct. Let

$$D_0 = (a - b)(a - c)(b - c)(a + b + c), \qquad S_q = \sinh(qa)\sinh(qb)\sinh(qc).$$

Product-to-sum identities applied to (7) give

$$\gamma = \frac{4\xi\eta\zeta(\zeta^2 - \eta^2)(\zeta^2 - \xi^2)(\eta^2 - \xi^2)}{D_0 S_q} \Phi[\zeta^2, \eta^2, \xi^2], \tag{10}$$

and

$$\beta = \frac{3}{2D_0 S_q} \left[B_1\{\Phi(\zeta^2) - \Phi(\eta^2)\} + B_3\{\Phi(\eta^2) - \Phi(\xi^2)\} \right]. \tag{11}$$

Here

$$B_1 = \zeta(\eta^2 - \xi^2)(\zeta^2 + \zeta\eta + \zeta\xi - 3\eta\xi),$$

$$B_3 = \xi(\zeta^2 - \eta^2)(3\zeta\eta - \zeta\xi - \eta\xi - \xi^2).$$

Now,

$$\zeta^2 + \zeta\eta + \zeta\xi - 3\eta\xi > 2\eta(\eta - \xi) > 0$$

and

$$3\zeta\eta - \zeta\xi - \eta\xi - \xi^2 > (\eta - \xi)(3\eta + \xi) > 0.$$

Thus, $B_1, B_3 > 0$. Since Φ is strictly increasing, (11) gives $\beta > 0$. Since Φ is strictly convex, the mean-value theorem for divided differences [3, p. 61, Eq. (44)] shows that the second divided difference in (10) is positive; hence, $\gamma > 0$. □

Proof of Theorem 3.1. Put

$$S_0 = s^2 + r^2 + 2Rr.$$

Using

$$abc = 4Rrs, \qquad (a+b)(b+c)(c+a) = 2s\,S_0,$$

formula (6) becomes

$$I = \left(\frac{2Rr}{S_0}, \frac{1}{S_0}\right).$$

The line GK has equation

$$3u + \frac{\sigma}{2}v = 1.$$

Define

$$L_{GK}(u, v) = 1 - 3u - \frac{\sigma}{2}v.$$

Since

$$\sigma = 2(s^2 - r^2 - 4Rr),$$

we obtain

$$L_{GK}(I) = \frac{2r^2}{S_0} > 0.$$

Without loss of generality, assume that $a > b > c$. With ξ, η, ζ as in (8), we have $0 < \xi < \eta < \zeta < s$. Let

$$D_1 = 2(a^2 - b^2)(a^2 - c^2)(b^2 - c^2) > 0, \qquad S_q = \sinh(qa)\sinh(qb)\sinh(qc).$$

Substitution of (7), followed by product-to-sum, gives

$$4D_1 S_q L_{GK}(P(q)) = C_0\,\Phi[s^2, \zeta^2, \eta^2, \xi^2], \tag{12}$$

where

$$C_0 = 8s\xi\eta\zeta(\eta^2 - \xi^2)(\zeta^2 - \eta^2)(\zeta^2 - \xi^2)(s + \xi)(s + \eta)(s + \zeta) > 0.$$

Now,

$$\Phi(t) = \sum_{n=0}^{\infty} \frac{(2q)^{2n+1}}{(2n+1)!} t^n,$$

so $\Phi'''(t) > 0$ for $t > 0$. By the mean-value theorem for divided differences [3, p. 61, Eq. (44)], the third divided difference in (12) is therefore positive, and $L_{GK}(P(q)) > 0$. Thus, $P(q)$ and I lie on the same side of GK.

It remains to check the other two sides. Since $\sigma - 2S_0 = -4r^2 - 12Rr < 0$, the line IK is not horizontal. It meets the u-axis at

$$u = \frac{R}{3R + r} < \frac{1}{3},$$

so G and N lie on the same side of IK. Also, if

$$L_{IG}(u, v) = v_I \left(u - \frac{1}{3}\right) - \left(u_I - \frac{1}{3}\right) v,$$

then

$$L_{IG}(N) = \frac{1}{6S_0} > 0, \qquad L_{IG}(K) = \frac{4r^2}{3\sigma S_0} > 0.$$

Thus, N and K lie on the same side of IG.

By Lemma 3.2, $\beta, \gamma > 0$. Since L_{IG} vanishes at I and G, the affine decomposition (9) gives

$$L_{IG}(P(q)) = \gamma L_{IG}(N) > 0.$$

Thus, $P(q)$ and K lie on the same side of IG. Likewise, let L_{IK} be an affine linear function that vanishes on IK and is positive at G and N. Then

$$L_{IK}(P(q)) = \beta L_{IK}(G) + \gamma L_{IK}(N) > 0,$$

so $P(q)$ and G lie on the same side of IK. Together with $L_{GK}(P(q)) > 0$, these are exactly the three side conditions for $P(q) \in \operatorname{int} \triangle IGK$. $\square$

3.1 Further containments of the Abraham–Kovač curve

The affine decomposition used above gives another whole-curve localization, sharper than the corresponding triangle involving the nine-point center. Let

$$Z = IN \cap GG_e.$$

The ETC [4] identifies this point as

$$Z = X(5219) = IX(5) \cap GX(7).$$

A direct simplification of its standard barycentrics gives

$$Z = (1 - \tau)I + \tau N, \qquad \tau = \frac{4r}{2R + 5r}.$$

Thus, $0 < \tau < 1$, so Z lies in the open segment IN.

Theorem 3.3. *For every scalene triangle and every $q > 0$,*

$$P(q) \in \operatorname{int} \triangle IGX(5219).$$

In particular,

$$X(5626) \in \operatorname{int} \triangle IGX(5219).$$

Proof. We verify the three side conditions for $\triangle IGZ$ directly. First, let L_{IG} be as in the proof of Theorem 3.1, chosen positive at N. Since $Z = (1-\tau)I + \tau N$ with $\tau > 0$,

$$L_{IG}(Z) = \tau L_{IG}(N) > 0.$$

By Lemma 3.2, $\gamma > 0$, and therefore

$$L_{IG}(P(q)) = \gamma L_{IG}(N) > 0.$$

Thus, $P(q)$ and Z lie on the same side of IG.

Next, $IZ = IN$. Let L_{IN} be an affine linear function vanishing on IN and positive at G. Since $I, N \in IN$, the decomposition (9) and Lemma 3.2 give

$$L_{IN}(P(q)) = \beta L_{IN}(G) > 0.$$

Thus, $P(q)$ and G lie on the same side of IZ.

It remains only to check the side $GZ = GG_e$. Without loss of generality, assume that $a > b > c$. With ξ, η, ζ as in (8), we have $0 < \xi < \eta < \zeta < s$.

For normalized barycentrics $T = (t_A, t_B, t_C)$, define

$$\ell(T) = \xi(\eta - \zeta)t_A + \eta(\zeta - \xi)t_B + \zeta(\xi - \eta)t_C.$$

Since $G = (1:1:1)$ and $G_e = (1/\xi : 1/\eta : 1/\zeta)$, the line $GZ = GG_e$ is $\ell = 0$. Moreover,

$$\ell(I) = -\frac{(\xi-\eta)(\xi-\zeta)(\eta-\zeta)}{2s} > 0.$$

Define

$$H_q(t) = \frac{\sinh(2qt)}{t}.$$

A direct elimination from (3), followed by product-to-sum, gives

$$\ell(P(q)) = -\frac{(\xi^2-\eta^2)(\xi^2-\zeta^2)(\eta^2-\zeta^2)}{8\sinh(qa)\sinh(qb)\sinh(qc)} H_q[\xi, \eta, \zeta, s].$$

Since

$$H_q(t) = \sum_{n=0}^{\infty} \frac{(2q)^{2n+1}}{(2n+1)!} t^{2n},$$

we have $H_q'''(t) > 0$ for $t > 0$. By the mean-value theorem for divided differences [3, p. 61, Eq. (44)], the third divided difference in the preceding display is positive. The product of the three differences of squares is negative, so $\ell(P(q)) > 0$. Thus, $P(q)$ and I lie on the same side of GZ. The three side conditions prove the theorem. □

The nine-point-center containment is now immediate, with no additional analytic argument.

Corollary 3.4. *For every scalene triangle and every $q > 0$,*

$$P(q) \in \operatorname{int} \triangle IGN.$$

Consequently,

$$X(5626) \in \operatorname{int} \triangle IGX(5).$$

Proof. Because $Z = X(5219)$ lies in the open segment IN,

$$\triangle IGZ \subset \triangle IGN.$$

The assertion follows from Theorem 3.3. $\square$

Thus,

$$\mathcal{C} \subset \operatorname{int} \triangle IGX(5219) \subset \operatorname{int} \triangle IGN,$$

while Theorem 3.1 gives a different whole-curve localization that is the one used in the proof of the small-triangle theorem.

3.2 Endpoint behavior and tangencies

The global containment theorem has useful local refinements at both endpoints. At the centroid they explain the appearance of the symmedian point K; at the incenter they single out the angle bisector opposite the shortest side.

Proposition 3.5. *Write $P(q) = (u(q), v(q))$ in the affine frame above. Then*

$$P(q) \longrightarrow G \quad (q \to 0^+), \qquad P(q) \longrightarrow I \quad (q \to \infty).$$

More precisely, as $q \to 0^+$,

$$u(q) = \frac{1}{3} - \frac{\sigma}{45} q^2 + O(q^4), \qquad v(q) = \frac{2}{15} q^2 + O(q^4). \tag{13}$$

Consequently, the Abraham–Kovač curve has limiting tangent GK at its centroid endpoint G.

Proof. For $q \to 0^+$, use

$$t \coth t = 1 + \frac{t^2}{3} - \frac{t^4}{45} + O(t^6).$$

Substitution in (7) gives

$$u + va^2 = \frac{1}{3} + \frac{q^2}{45}(6a^2 - \sigma) + O(q^4),$$

and cyclically. Comparing the constant and q^2 terms yields (13); in particular, $P(q) \to G$. Since

$$K - G = \left(-\frac{1}{3}, \frac{2}{\sigma} \right),$$

the leading displacement from G is

$$\left(-\frac{\sigma}{45}, \frac{2}{15} \right) q^2 = \frac{\sigma q^2}{15}(K - G).$$

Thus, the limiting tangent at G is the line GK.

As $q \to \infty$, we have $U \to a$, $V \to b$, and $W \to c$. Hence, (7) gives

$$u + va^2 \longrightarrow \frac{a}{b + c},$$

and cyclically. These are precisely the affine-frame equations for the incenter, so $P(q) \to I$. $\square$

Thus, GK is not merely a convenient boundary of the containing triangle: it is the tangent to the Abraham–Kovač curve at its centroid endpoint. Figure 2 illustrates both the global containment from Theorem 3.1 and this tangency.

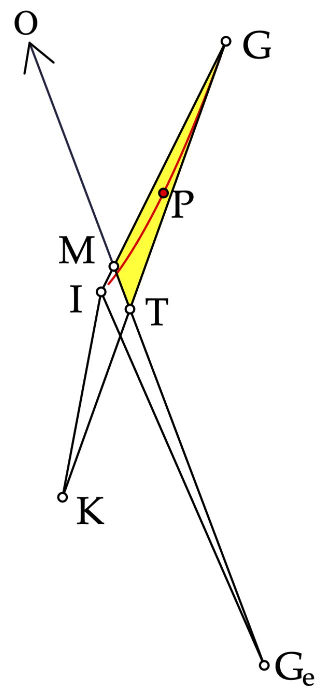

Figure 2: The Abraham–Kovač curve (red) passes through the electrostatic center P.

There is a complementary description at the incenter endpoint. Since the triangle is scalene, its shortest side is unique.

Proposition 3.6. *Suppose that c is the shortest side of ABC. Then, as $q \to \infty$,*

$$P(q) = I + \frac{2c}{a+b-c} e^{-2cq}(C - I) + o(e^{-2cq}). \tag{14}$$

Consequently, the Abraham–Kovač curve has limiting tangent CI at its incenter endpoint I. Cyclically, the limiting tangent at I is the angle bisector from the vertex opposite the shortest side.

Proof. Since $c < a, b$,

$$U = a + o(e^{-2cq}), \qquad V = b + o(e^{-2cq}), \qquad W = c + 2ce^{-2cq} + o(e^{-2cq}).$$

The defining equations (3) therefore give

$$\begin{aligned} P(q)B^2 - P(q)A^2 &= IB^2 - IA^2 + 2c(a-b)e^{-2cq} + o(e^{-2cq}), \\ P(q)C^2 - P(q)A^2 &= IC^2 - IA^2 - 2bc\, e^{-2cq} + o(e^{-2cq}). \end{aligned}$$

Here we used the limiting identities

$$IB^2 - IA^2 = c(a-b), \qquad IC^2 - IA^2 = b(a-c),$$

which follow by letting $q \to \infty$ in (3).

Both squared-distance differences are affine linear functions of the point. For the displacement from I toward C their changes are

$$(CB^2 - CA^2) - (IB^2 - IA^2) = (a-b)(a+b-c),$$
$$(CC^2 - CA^2) - (IC^2 - IA^2) = -b(a+b-c).$$

Thus the two affine coordinates of $P(q) - I$ agree, to leading order, with those of

$$\frac{2c}{a+b-c} e^{-2cq}(C - I).$$

Since the two distance-difference coordinates determine the point uniquely, (14) follows. Hence the limiting tangent is CI, the internal angle bisector at C. $\square$

Figure 3 displays the two endpoint tangencies in the case where c is the shortest side.

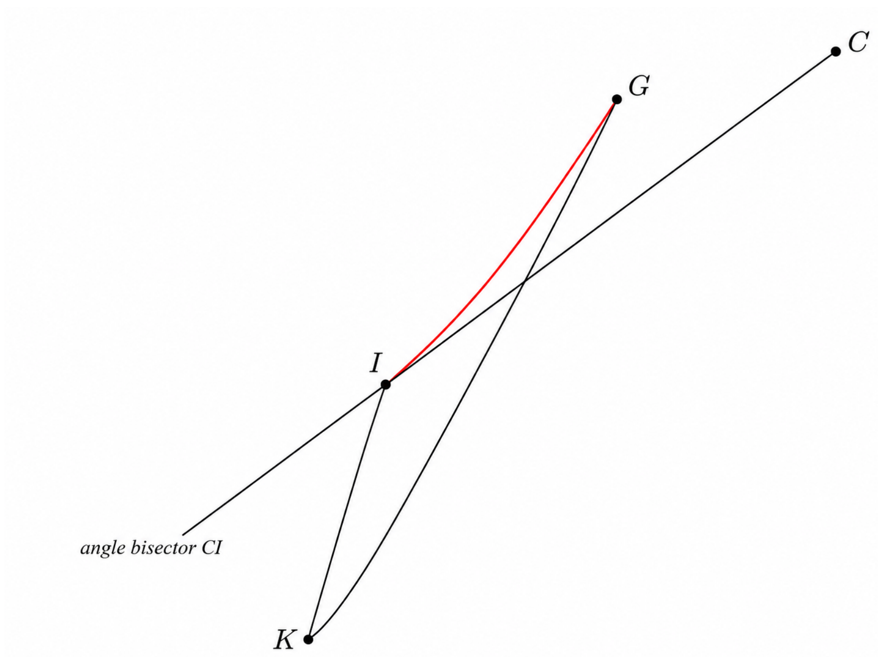

Figure 3: When c is the shortest side, the Abraham–Kovač curve has limiting tangents GK at G and CI at I.

4 The third side relation: the line MT

We now return to the three side relations listed in the Introduction. Theorem 3.1 will give the first two as soon as the positions of M and T on GI and GK are identified. The Gergonne point has barycentrics [4]

$$G_e = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c} \right).$$

Put

$$\lambda = 2R + r, \qquad \mu = s^2 + (4R + r)(2R + r).$$

A direct substitution in the affine frame gives the equation

$$OG_e : \qquad J(u,v) = 2\lambda u - r\mu v = 0.$$

At the centers needed below,

$$J(G) = \frac{2\lambda}{3} > 0, \qquad J(I) = -r < 0, \qquad J(N) = \lambda > 0, \qquad J(K) = -\frac{2r\mu}{\sigma} < 0. \tag{15}$$

Hence, OG_e cuts the interiors of both GI and GK. Its intersections are precisely the points M, T in (1); consequently,

$$\triangle GMT \subset \triangle IGK.$$

Because M lies in the open segment GI and T lies in the open segment GK, Theorem 3.1 now proves the first two side relations listed in the Introduction: P is on the T-side of GM and on the M-side of GT.

By Theorem 3.1, only the third side relation remains: we must show that P and G lie on the same side of $MT = OG_e$.

Retain the affine decomposition (9),

$$P(q) = \alpha(q)I + \beta(q)G + \gamma(q)N, \qquad \alpha + \beta + \gamma = 1.$$

Since $\beta, \gamma > 0$, (15) gives

$$\begin{aligned} J(P(q)) &= -r\alpha + \frac{2\lambda}{3}\beta + \lambda\gamma \\ &> -r\alpha + \frac{2\lambda}{3}(1-\alpha). \end{aligned}$$

Thus,

$$\alpha(q) < \frac{4R+2r}{4R+5r} \qquad \Longrightarrow \qquad J(P(q)) > 0. \tag{16}$$

It is enough to establish the preceding bound at $q = q^*$. A slightly stronger inequality turns out to be more convenient because it has a simple form in terms of the dimensionless shape parameter introduced below. We shall prove

$$\boxed{\alpha(q^*) < \frac{R}{R+2r}.}$$

The remainder of this section is devoted to that estimate. First, we derive a scalar equation determining q^*. Next, we obtain two bounds for the affine coefficient $\alpha(q)$. Finally, we combine them to prove the MT-side inequality.

4.1 Determining the electrostatic parameter

To express the shape of the triangle efficiently, use ξ, η, ζ as in (8). Then $s = \xi + \eta + \zeta$, and

$$a = \eta + \zeta, \qquad b = \zeta + \xi, \qquad c = \xi + \eta.$$

We shall use the dimensionless parameter

$$\kappa = \frac{4R}{r} = \frac{(\xi+\eta)(\eta+\zeta)(\zeta+\xi)}{\xi\eta\zeta}.$$

It is useful because the stronger target inequality becomes $\alpha(q^*) < \kappa/(\kappa+8)$. Euler's inequality $R \geq 2r$ gives $\kappa \geq 8$, with equality only for the equilateral triangle.

Define

$$\ell_A = \frac{V+W-U}{2}, \qquad \ell_B = \frac{W+U-V}{2}, \qquad \ell_C = \frac{U+V-W}{2}. \tag{17}$$

From (3),

$$P(q)A^2 - \ell_A^2 = P(q)B^2 - \ell_B^2 = P(q)C^2 - \ell_C^2.$$

Denote the common value by $E(q)$. Equation (4) says

$$E(q^*) = 0. \tag{18}$$

Theorem 4.1. *Define*

$$\mathcal{H}(q) = \cosh(2qs) - s\left(\frac{\cosh(2q\xi)}{\xi} + \frac{\cosh(2q\eta)}{\eta} + \frac{\cosh(2q\zeta)}{\zeta}\right). \tag{19}$$

Then

$$\boxed{\mathcal{H}(q^*) = \kappa.} \tag{20}$$

Moreover, q^* is the unique positive solution of (20)*, and*

$$\mathcal{H}'(q) = 8s\sinh(qa)\sinh(qb)\sinh(qc) > 0. \tag{21}$$

Proof. Consider the Cayley–Menger determinant (for background, see [2])

$$C_q = \det\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & \ell_A^2 \\ 1 & c^2 & 0 & a^2 & \ell_B^2 \\ 1 & b^2 & a^2 & 0 & \ell_C^2 \\ 1 & \ell_A^2 & \ell_B^2 & \ell_C^2 & 0 \end{pmatrix}.$$

A row-and-column reduction, or an ordinary Cartesian-coordinate calculation, gives

$$C_q = -32\Delta^2 E(q). \tag{22}$$

Now, put

$$x_\xi = e^{-2q\xi}, \qquad x_\eta = e^{-2q\eta}, \qquad x_\zeta = e^{-2q\zeta}.$$

Thus, for example,

$$U = (\eta+\zeta)\frac{1+x_\eta x_\zeta}{1-x_\eta x_\zeta},$$

and cyclically. Substituting (17) in the determinant and clearing the three denominators gives the exact factorization

$$\begin{aligned}(1-x_\xi x_\eta)^2(1-x_\xi x_\zeta)^2(1-x_\eta x_\zeta)^2 C_q \\ = -128\xi^2\eta^2\zeta^2 x_\xi^2 x_\eta^2 x_\zeta^2\big(\mathcal{H}(q)-\kappa\big)\big(\mathcal{H}(q)+\kappa\big).\end{aligned} \tag{23}$$

Since

$$1 - x_\xi x_\eta = 2e^{-qc}\sinh(qc)$$

and cyclically, while $a+b+c=2s$, equations (22)–(23), together with $\Delta = rs$ and $\xi\eta\zeta = r^2 s$, simplify to

$$\boxed{E(q) = \frac{r^2}{16\sinh^2(qa)\sinh^2(qb)\sinh^2(qc)}\big(\mathcal{H}(q)^2 - \kappa^2\big).} \tag{24}$$

Also,

$$\mathcal{H}(0) = 1 - s\left(\frac{1}{\xi} + \frac{1}{\eta} + \frac{1}{\zeta}\right) = -\kappa,$$

and the identity

$$\sinh(2qs) - \sinh(2q\xi) - \sinh(2q\eta) - \sinh(2q\zeta) = 4\sinh(qa)\sinh(qb)\sinh(qc)$$

gives (21). Thus, $\mathcal{H}(q) + \kappa > 0$ for $q > 0$. Also, $\mathcal{H}(q) \to +\infty$ as $q \to \infty$, since $s > \xi, \eta, \zeta$. Thus, $\mathcal{H}(q) = \kappa$ has exactly one positive solution. Equations (18) and (24) show that this solution is q^*. □

Integrating (21) from 0 to q^* yields the form that will be used in the thin-triangle estimates.

Corollary 4.2.

$$\boxed{s\int_0^{q^*} \sinh(at)\sinh(bt)\sinh(ct)\,dt = \frac{R}{r} = \frac{\kappa}{4}.} \tag{25}$$

4.2 Bounding the position of P

We now seek bounds on the coefficient $\alpha(q)$, since the reduction above shows that controlling this single coefficient will place $P(q^*)$ on the correct side of MT. In the decomposition (9), the points G and N have second coordinate zero, so $\alpha = v/v_I$. Since $v_I = 1/S_0$, we have

$$\alpha(q) = S_0 v(q).$$

For later use, set

$$h(t) = \frac{\tanh(q\sqrt{t})}{\sqrt{t}}.$$

A direct simplification of (7) gives the algebraic identity

$$v = \frac{h[a^2, b^2, c^2]}{h(a^2)h(b^2)h(c^2)}. \tag{26}$$

All inequalities below are scale invariant, so from now on we normalize

$$s = 1.$$

Put

$$e_2 = \xi\eta + \eta\zeta + \zeta\xi, \qquad e_3 = \xi\eta\zeta.$$

Then

$$e_3 = \frac{e_2}{\kappa+1}, \qquad S_0 = 1 + \frac{e_2 + e_3}{2} = 1 + \frac{e_2(\kappa+2)}{2(\kappa+1)}.$$

It is convenient to use the positive product

$$\mathcal{D}(q) = 4\sinh(qa)\sinh(qb)\sinh(qc) > 0. \tag{27}$$

The product-to-sum identity already used in Theorem 4.1 gives

$$\mathcal{D}(q) = \sinh(2q) - \sinh(2q\xi) - \sinh(2q\eta) - \sinh(2q\zeta).$$

Set

$$\mathcal{N}(q) = \alpha(q)\mathcal{D}(q).$$

We first need a simple inequality for complete homogeneous symmetric polynomials. Recall that

$$h_n(x_1, \ldots, x_m) = \sum_{i_1+\cdots+i_m=n} x_1^{i_1} \cdots x_m^{i_m},$$

where the sum is over nonnegative integers $i_1, \ldots, i_m$. Let

$$\mathsf{h}_n = h_n(1, \xi^2, \eta^2, \zeta^2), \qquad \mathsf{h}_n = 0 \quad (n < 0).$$

Lemma 4.3. *For $n \geq 1$,*

$$\boxed{\mathsf{h}_n \geq \frac{n+1}{n}(1 - e_2)\mathsf{h}_{n-1},}$$

with equality if and only if $n = 1$.

Proof. Let

$$g(t) = \sum_{n \geq 0} \mathsf{h}_n t^n = \frac{1}{(1-t)(1-\xi^2 t)(1-\eta^2 t)(1-\zeta^2 t)}.$$

Since

$$1 + \xi^2 + \eta^2 + \zeta^2 = 2(1 - e_2),$$

the generating function of

$$f_n = n\mathsf{h}_n - (n+1)(1 - e_2)\mathsf{h}_{n-1}$$

is

$$\begin{aligned}\sum_{n \geq 1} f_n t^n &= t\{(1 - (1 - e_2)t)g'(t) - 2(1 - e_2)g(t)\} \\ &= t^2 g(t) \sum_{u \in \{1,\xi,\eta,\zeta\}} \frac{u^2(u^2 - 1 + e_2)}{1 - u^2 t}.\end{aligned}$$

Here, $u^2 < 1 - e_2$ for $u \in \{\xi, \eta, \zeta\}$; for example,

$$1 - e_2 - \zeta^2 = \xi + \eta - \xi\eta > 0,$$

and the other two cases are cyclic. For $m \geq 1$, define

$$M_m = \sum_{u \in \{1,\xi,\eta,\zeta\}} u^{2m}(u^2 - 1 + e_2).$$

Then

$$M_m \geq e_2 - \sum_{u \in \{\xi,\eta,\zeta\}} u^2(1 - e_2 - u^2) = 4e_3 > 0.$$

The displayed generating function begins with t^2, so $f_1 = 0$; all subsequent coefficients are positive. This proves the lemma. $\square$

A direct expansion of (26), followed by product-to-sum, gives the following coefficient formulas.

Lemma 4.4. *Write*

$$\mathcal{D}(q) = \sum_{m\geq 1} d_m \frac{(2q)^{2m+1}}{(2m+1)!}, \qquad \mathcal{N}(q) = \sum_{m\geq 2} n_m \frac{(2q)^{2m+1}}{(2m+1)!}.$$

Then

$$\sum_{m\geq 1} d_m t^m = \frac{abc\, t\{3-(1-e_2)t+e_3t^2\}}{(1-t)(1-\xi^2 t)(1-\eta^2 t)(1-\zeta^2 t)}, \tag{28}$$

$$\sum_{m\geq 2} n_m t^m = \frac{2abc\, S_0 t^2}{(1-t)(1-\xi^2 t)(1-\eta^2 t)(1-\zeta^2 t)}. \tag{29}$$

Equivalently,

$$d_m = abc\{3\mathsf{h}_{m-1} - (1-e_2)\mathsf{h}_{m-2} + e_3\mathsf{h}_{m-3}\},$$
$$n_m = 2abc\, S_0 \mathsf{h}_{m-2}.$$

Proposition 4.5 (A Padé-type bound)**.** *For every $q>0$,*

$$\boxed{\alpha(q) < \frac{2S_0q^2}{15+S_0q^2}.} \tag{30}$$

Proof. Consider

$$\Omega(q) = 2S_0q^2\mathcal{D}(q) - (15+S_0q^2)\mathcal{N}(q).$$

The coefficient corresponding to $m=2$ is zero. For $m=n+2$, $n\geq 1$, after division by a positive factor the coefficient is

$$c_n\mathsf{h}_n - (1-e_2+S_0)\mathsf{h}_{n-1} + e_3\mathsf{h}_{n-2}, \qquad c_n = 3 - \frac{30}{(n+2)(2n+5)}. \tag{31}$$

Now,

$$c_n\frac{n+1}{n} = 3 + \frac{3(2n-1)}{(n+2)(2n+5)} > 3.$$

Moreover,

$$S_0 < 2(1-e_2),$$

because this is equivalent to $5e_2+e_3<2$, while $e_2\leq 1/3$ and $e_3\leq 1/27$. By Lemma 4.3,

$$c_n\mathsf{h}_n \geq c_n\frac{n+1}{n}(1-e_2)\mathsf{h}_{n-1} > 3(1-e_2)\mathsf{h}_{n-1} > (1-e_2+S_0)\mathsf{h}_{n-1}.$$

The strict inequality holds also for $n=1$, where $\mathsf{h}_1 = 2(1-e_2)\mathsf{h}_0$. Since $e_3\mathsf{h}_{n-2}\geq 0$, every coefficient in (31) is strictly positive. Hence, $\Omega(q)>0$, and using $\mathcal{N}=\alpha\mathcal{D}$ and $\mathcal{D}>0$ gives (30). ◻

The second estimate comes directly from the scalar equation.

Proposition 4.6. *For $q > 0$,*

$$\frac{\mathcal{H}(q) + \kappa}{2\kappa} > e_3 q^4 \left[1 + \frac{2(1-e_2)}{9} q^2 + \frac{(1-e_2)^2 + e_3}{45} q^4\right]. \tag{32}$$

At $q = q^*$ the left side equals 1.

Proof. From (19) and $s = 1$,

$$\mathcal{H}(q) + \kappa = \sum_{m \geq 2} \frac{(2q)^{2m}}{(2m)!} \left(1 - \xi^{2m-1} - \eta^{2m-1} - \zeta^{2m-1}\right). \tag{33}$$

The $m = 1$ term vanishes because $\xi + \eta + \zeta = 1$, which explains why the sum begins at $m = 2$. Every displayed coefficient is positive. Newton's identities give

$$\begin{aligned}
1 - (\xi^3 + \eta^3 + \zeta^3) &= 3(e_2 - e_3), \\
1 - (\xi^5 + \eta^5 + \zeta^5) &= 5(e_2 - e_3)(1 - e_2), \\
1 - (\xi^7 + \eta^7 + \zeta^7) &= 7(e_2 - e_3)\{(1 - e_2)^2 + e_3\}.
\end{aligned}$$

Since $e_2 - e_3 = \kappa e_3$, the first three nonzero terms of (33) give (32). The last statement follows from Theorem 4.1. $\square$

We shall also need a lower expansion for $1 - \alpha$. Let $\widetilde{h}_n = h_n(\xi^2, \eta^2, \zeta^2)$, with $\widetilde{h}_n = 0$ for $n < 0$.

Lemma 4.7. *If*

$$\mathcal{F}(q) = \mathcal{D}(q) - \mathcal{N}(q) = (1 - \alpha(q))\mathcal{D}(q),$$

then

$$\boxed{\mathcal{F}(q) = abc \sum_{m \geq 1} \left(3\widetilde{h}_{m-1} - e_3 \widetilde{h}_{m-2}\right) \frac{(2q)^{2m+1}}{(2m+1)!}.} \tag{34}$$

Proof. Subtract (29) from (28) and use $S_0 = 1 + (e_2 + e_3)/2$; the factor $1 - t$ cancels. Coefficient extraction gives (34). $\square$

4.3 The MT-side inequality

We now combine the scalar equation for q^* with the preceding estimates for α. The goal is precisely the stronger inequality $\alpha(q^*) < \kappa/(\kappa + 8)$, which will imply the required MT-side relation. The two large-κ cases use the same short-interval estimate, so we record it once.

Lemma 4.8. *Assume $s = 1$ and $q^* \geq t_0 + 1/2$. If*

$$\prod_{x \in \{a,b,c\}} (1 - e^{-2xt}) \geq \mu \qquad (t \geq t_0),$$

then

$$e^{2q^*} \leq \frac{4\kappa}{\mu(1 - e^{-1})}. \tag{35}$$

Proof. Since

$$8\sinh(at)\sinh(bt)\sinh(ct) = e^{2t}\prod_{x\in\{a,b,c\}}(1-e^{-2xt}),$$

we integrate only over $[q^*-1/2, q^*]$ in (25). This gives

$$2\kappa \geq \frac{\mu}{2}e^{2q^*}(1-e^{-1}),$$

which is (35). $\square$

Theorem 4.9. *For the electrostatic parameter q^* of every scalene triangle,*

$$\boxed{\alpha(q^*) < \frac{\kappa}{\kappa+8} = \frac{R}{R+2r}.} \tag{36}$$

Proof. Continue with $s=1$. Without loss of generality, assume that

$$0 < \xi \leq \eta \leq \zeta < 1.$$

Set

$$L_* = \frac{15\kappa}{S_0(\kappa+16)}$$

and

$$p_\kappa(e_2) = e_3 L_*^2\left[1 + \frac{2(1-e_2)}{9}L_* + \frac{(1-e_2)^2+e_3}{45}L_*^2\right], \qquad e_3 = \frac{e_2}{\kappa+1}. \tag{37}$$

Proposition 4.5 shows that it is enough to prove

$$(q^*)^2 < L_*, \tag{38}$$

because equality in the right side of (30) with $\alpha = \kappa/(\kappa+8)$ occurs precisely at $q^2 = L_*$. By Proposition 4.6, the implication

$$p_\kappa(e_2) \geq 1 \quad\Longrightarrow\quad (q^*)^2 < L_* \tag{39}$$

is immediate, since the truncated series is strictly increasing in q^2.

We now divide the proof into three cases. In the moderate range we combine the Padé bound of Proposition 4.5 with the truncated scalar-series bound of Proposition 4.6; the remaining polynomial checks are collected in the appendix. For large κ with $\zeta \leq 3/4$, the short-window estimate above controls q^* and a monotonicity argument for $f(q,\rho)$ supplies the needed lower bound for $1-\alpha$. When $\zeta \geq 3/4$, the triangle is thin, and we retain the small-side factor in $\mathcal{D}$ to obtain a sharper estimate adapted to that regime.

1. The moderate range $8 < \kappa \leq 48$. We first note the elementary bound

$$e_2 \geq \frac{4}{\kappa+4}.$$

Indeed, since $e_3 = e_2/(\kappa+1)$, this is equivalent to $e_2^2 + 3e_2e_3 - 4e_3 \geq 0$, and

$$e_2^2 + 3e_2e_3 - 4e_3 = e_3\sum_{\text{cyc}}\xi^2\left(\frac{\eta}{\zeta}+\frac{\zeta}{\eta}-2\right) \geq 0.$$

Lemma A.1 in the appendix shows that $p_\kappa(e_2)$ is increasing in e_2 for $8 \le \kappa \le 110$. Hence,

$$p_\kappa(e_2) \ge p_\kappa\left(\frac{4}{\kappa+4}\right) > 1 \qquad (8 \le \kappa \le 48),$$

where the last inequality is Lemma A.2. Thus, (38) follows from (39).

2. The range $\kappa > 48$ with $\zeta \le 3/4$. First, $\zeta > 12/25$. Suppose instead that $\zeta \le 12/25$. Since ζ is the largest of ξ, η, ζ and $\xi + \eta + \zeta = 1$, we have $\zeta \ge 1/3$. For fixed $\xi + \eta = 1 - \zeta$, the function

$$x \longmapsto \frac{1}{x} + \frac{1}{1-\zeta-x}$$

is convex on the admissible interval $1 - 2\zeta \le x \le \zeta$; hence its maximum occurs at an endpoint. Therefore,

$$\kappa + 1 = \frac{1}{\xi} + \frac{1}{\eta} + \frac{1}{\zeta} \le \frac{1}{1-2\zeta} + \frac{2}{\zeta} \le 25 + 6 = 31 < 49,$$

contradicting $\kappa > 48$. Hence,

$$e_2 > \zeta(1-\zeta) \ge \frac{3}{16}.$$

If $48 < \kappa \le 110$, Lemmas A.1 and A.3 give

$$p_\kappa(e_2) > p_\kappa(3/16) \ge p_{110}(3/16) > 1,$$

so again (38) follows.

It remains to treat $\kappa \ge 110$. Since $\mathcal{H}(q) < \cosh(2q)$ and $\cosh(16/3) < 110$, Theorem 4.1 gives

$$q^* > \frac{8}{3}. \tag{40}$$

Put

$$\rho = \sqrt{\eta\zeta}, \qquad \rho_0 = \sqrt{\frac{3}{32}}.$$

From $\eta \ge (1-\zeta)/2$ and $12/25 < \zeta \le 3/4$,

$$\rho \ge \rho_0.$$

Moreover, $\eta \le \zeta$ gives

$$\rho = \sqrt{\eta\zeta} \le \zeta \le \frac{3}{4},$$

so $\rho \in [\rho_0, 3/4]$. Also,

$$abc = (1-\xi)(1-\eta)(1-\zeta) = (1-\zeta)(\zeta + \xi\eta) \ge \frac{3}{16}. \tag{41}$$

In Lemma 4.7, $\widetilde{h}_{m-1} \ge \zeta^2 \widetilde{h}_{m-2}$ and

$$3\xi\eta \le \frac{3(1-\zeta)^2}{4} \le \zeta,$$

so

$$3\widetilde{h}_{m-1} - e_3\widetilde{h}_{m-2} \ge \frac{8}{3}\widetilde{h}_{m-1}.$$

The m monomials involving only η^2, ζ^2, together with AM–GM, give

$$\widetilde{h}_{m-1} \geq m(\eta\zeta)^{m-1} = m\rho^{2m-2}.$$

Summing (34) therefore yields, with $u = 2q\rho$,

$$\mathcal{F}(q) \geq \frac{8abc}{3} \frac{u\cosh u - \sinh u}{2\rho^3}.$$

From (27), $\mathcal{D}(q) = 4\sinh(qa)\sinh(qb)\sinh(qc) < e^{2q}/2$. Define

$$f(q,\rho) = e^{-2q}\frac{u\cosh u - \sinh u}{2\rho^3}, \qquad u = 2q\rho.$$

Together with (41), we obtain

$$1 - \alpha(q) > f(q,\rho).$$

For fixed q, $f(q,\rho)$ increases with ρ by its positive power series. For $q \geq 8/3$ and $\rho_0 \leq \rho \leq 3/4$, it decreases with q; after differentiation the latter statement reduces to

$$\rho < \coth(2q\rho) - \frac{1}{2q\rho},$$

which follows from $\coth t > 1$ and $16\rho(1-\rho) \geq 3$.

We next bound q^* from above. By (40), every $t \in [q^* - 1/2, q^*]$ is at least $13/6$. The two larger sides are at least $\zeta > 12/25$, and the smallest side $1 - \zeta$ is at least $1/4$. Hence,

$$\prod_{x \in \{a,b,c\}} (1 - e^{-2xt}) \geq (1 - e^{-52/25})^2(1 - e^{-13/12}).$$

Lemma 4.8 gives

$$e^{2q^*} \leq c_1\kappa, \qquad c_1 = \frac{4}{(1-e^{-1})(1-e^{-52/25})^2(1-e^{-13/12})}.$$

Define

$$q_{\max}(\kappa) = \frac{1}{2}\log(c_1\kappa).$$

Thus, $q^* \leq q_{\max}(\kappa)$. Combining the monotonicities of f gives

$$1 - \alpha(q^*) > f(q_{\max}(\kappa), \rho_0).$$

At $\kappa = 110$,

$$\frac{f(q_{\max}(110), \rho_0)}{8/118} = 1.0685097\ldots > 1.$$

This ratio increases with κ. Indeed, if $g(u) = u\cosh u - \sinh u$, logarithmic differentiation gives, after multiplication by κ,

$$-\frac{8}{\kappa+8} + \rho_0\frac{u\sinh u}{g(u)} > 0,$$

because $u\sinh u > g(u)$ and $\rho_0 > 8/118$. Therefore,

$$1 - \alpha(q^*) > \frac{8}{\kappa+8},$$

which is equivalent to (36). This completes the second case.

3. The range $\kappa > 48$ with $\zeta \geq 3/4$. Put

$$w = 1 - \zeta = \xi + \eta \leq \frac{1}{4}.$$

Since $\cosh(9/2) < 48$, we have

$$q^* > \frac{9}{4}. \tag{42}$$

In (34),

$$\frac{e_3}{\zeta^2} = \frac{\xi\eta}{\zeta} \leq \frac{w^2}{4\zeta} \leq \frac{1}{48},$$

so

$$\mathcal{F}(q) \geq \frac{143}{48}\frac{abc}{\zeta^3}\{\sinh(2q\zeta) - 2q\zeta\}.$$

This time retain the small-side factor in $\mathcal{D}$: By (27),

$$4\sinh(qa)\sinh(qb) < e^{q(a+b)} = e^{q(2-w)},$$

so

$$\mathcal{D}(q) < e^{q(2-w)}\sinh(qw) = \frac{1}{2}e^{2q}(1 - e^{-2wq}). \tag{43}$$

The function $k(u) = e^{-u}(\sinh u - u)$ is increasing for $u > 0$. Since $abc = w(\zeta + \xi\eta) \geq w\zeta$, equations (42)–(43) give

$$1 - \alpha(q^*) > d_0 \frac{w}{(1-w)^2}\frac{1}{e^{2wq^*} - 1}, \tag{44}$$

where

$$d_0 = \frac{143}{24}e^{-27/8}\left(\sinh\frac{27}{8} - \frac{27}{8}\right) = 2.2875734\ldots.$$

For $u > 0$,

$$\sinh u \geq \frac{ue^u}{1+2u},$$

because $e^{2u} \geq 1 + 2u$. Since $a + b + c = 2$, AM–GM gives

$$(1 + 2at)(1 + 2bt)(1 + 2ct) \leq \left(1 + \frac{4t}{3}\right)^3.$$

For $t \geq 7/4$,

$$\frac{t}{1 + 4t/3} \geq \frac{21}{40}.$$

For $t \in [q^* - 1/2, q^*]$ we have $t \geq 7/4$. The preceding estimates imply

$$\prod_{x\in\{a,b,c\}}(1 - e^{-2xt}) \geq 8abc\left(\frac{21}{40}\right)^3 \geq 6w\left(\frac{21}{40}\right)^3.$$

Lemma 4.8 therefore gives

$$e^{2q^*} \leq \frac{\kappa}{3d_1 w}, \qquad d_1 = \left(\frac{21}{40}\right)^3\frac{1 - e^{-1}}{2},$$

Put

$$a_0 = \frac{1}{3d_1} = 7.2883784\ldots.$$

Then (44) becomes

$$1 - \alpha(q^*) > d_0 \frac{w}{(1-w)^2} \frac{1}{(a_0\kappa/w)^w - 1}.$$

Define the ratio of the right side to $8/(\kappa+8)$ by

$$\mathcal{B}(\kappa, w) = \frac{\kappa+8}{8} d_0 \frac{w}{(1-w)^2} \frac{1}{(a_0\kappa/w)^w - 1}.$$

For fixed $w \le 1/4$, this is increasing in κ. Indeed, if $L_0 = \log(a_0\kappa/w)$ and $Y = wL_0$, logarithmic differentiation gives

$$\kappa \frac{\partial}{\partial\kappa} \log \mathcal{B} = \frac{\kappa}{\kappa+8} - \frac{w}{1-e^{-Y}} > 0,$$

because

$$\frac{w}{1-e^{-Y}} < w + \frac{1}{L_0} < \frac{1}{4} + \frac{1}{7} < \frac{6}{7} \le \frac{\kappa}{\kappa+8}.$$

Here, $L_0 > 7$ follows from $\kappa > 48$ and $w \le 1/4$.

If $1/12 \le w \le 1/4$, it is therefore enough to use $\kappa = 48$. Both $w/(1-w)^2$ and $(48a_0/w)^w - 1$ are increasing in w on this interval; for the latter, differentiate $w\log(48a_0/w)$ and use $\log(48a_0/w) > 7$. Taking the left endpoint in the numerator and the right endpoint in the denominator on each of the following subintervals gives:

interval for w	lower bound for $\mathcal{B}(48, w)$
$[1/12, 7/60]$	1.02805
$[7/60, 3/20]$	1.08823
$[3/20, 11/60]$	1.11024
$[11/60, 13/60]$	1.11282
$[13/60, 1/4]$	1.10515

Thus, $\mathcal{B} > 1$ in this range.

Finally, suppose $0 < w < 1/12$. From

$$\kappa + 1 = \frac{1}{\xi} + \frac{1}{\eta} + \frac{1}{\zeta} \ge \frac{4}{w} + \frac{1}{1-w}$$

we obtain a useful lower bound. Define

$$\kappa_m(w) = \frac{(2-w)^2}{w(1-w)}.$$

Then the preceding inequality gives $\kappa \ge \kappa_m(w)$. Since $\mathcal{B}$ increases with κ, it is enough to consider $b(w) = \mathcal{B}(\kappa_m(w), w)$. This simplifies to

$$b(w) = \frac{d_0}{8} \frac{4 + 4w - 7w^2}{(1-w)^3} \frac{1}{e^{Y(w)} - 1},$$

where

$$Y(w) = w \log \frac{a_0(2-w)^2}{w^2(1-w)}.$$

For $0 < w \le 1/12$,

$$\frac{d}{dw} \log \frac{4 + 4w - 7w^2}{(1-w)^3} - 4 = \frac{w(-28w^2 + 37w - 6)}{4 - 11w^2 + 7w^3} < 0.$$

Indeed, the denominator is positive on this interval, while $-28w^2 + 37w - 6 < 0$ for

$$0 < w < \frac{37 - \sqrt{697}}{56} = 0.1892\dots.$$

Also,

$$Y'(w) = \log \frac{a_0(2-w)^2}{w^2(1-w)} - \frac{2w}{2-w} - 2 + \frac{w}{1-w} > 6,$$

the last bound following already from its value at $w = 1/12$ in the logarithmic term. Hence,

$$(\log b)' < 4 - \frac{Y'}{1 - e^{-Y}} < 0.$$

Therefore,

$$b(w) > b(1/12) = 1.5836395\ldots > 1.$$

Thus, $\mathcal{B} > 1$ also in this last range. The third case is complete, and (36) now follows in all cases. $\square$

Corollary 4.10. *The electrostatic center P and the centroid G lie in the same open half-plane determined by the central line $X(12)X(35)$.*

Proof. Direct simplification of the standard ETC barycentrics [4] gives

$$X(12) = \frac{R}{R+2r} I + \frac{2r}{R+2r} N, \qquad X(35) = \frac{R}{R+2r} I + \frac{2r}{R+2r} O.$$

Hence, $X(12)X(35)$ is parallel to the Euler line OGN. In the affine frame of Section 2, both $X(12)$ and $X(35)$ have second coordinate

$$\frac{R}{R+2r} v_I,$$

whereas G has second coordinate 0 and $P = P(q^*)$ has second coordinate $\alpha(q^*)v_I$. Since $v_I > 0$, Theorem 4.9 shows that P and G lie on the same side of $X(12)X(35)$. $\square$

5 Completion of the proof

We can now finish the geometric argument.

Theorem 5.1. *Let ABC be scalene and let $P = X(5626)$ be its electrostatic center. With*

$$M = IG \cap OG_e, \qquad T = GK \cap OG_e,$$

one has

$$\boxed{P \in \operatorname{int} \triangle GMT.}$$

Proof. By Theorem 3.1, P is already on the correct side of IG and GK. Theorem 4.9 gives

$$\alpha(q^*) < \frac{R}{R+2r} < \frac{4R+2r}{4R+5r}.$$

The second inequality is immediate after cross multiplication. By (16), $J(P) > 0$, so P and G lie on the same side of $MT = OG_e$. These are exactly the three side conditions for $P \in \operatorname{int} \triangle GMT$. $\square$

Thus, although no central line containing the electrostatic center is currently known to us, its location is constrained by the small central triangle GMT. Along the way we obtained several whole-curve localizations: the Abraham–Kovač curve lies in $\triangle IGK$ and also in $\triangle IGX(5219) \subset \triangle IGN$. The quantitative bound used for the final side of GMT also yields Corollary 4.10, placing the electrostatic center and the centroid on the same side of the central line $X(12)X(35)$. The endpoint analysis also shows that the Abraham–Kovač curve is tangent to GK at G and, at I, to the angle bisector from the vertex opposite the shortest side. These additional containments and tangencies may be useful in studying other geometric properties of $X(5626)$.

A Algebraic checks for the moderate range

For completeness, we record the elementary polynomial checks used in the proof of Theorem 4.9. They involve only the function $p_\kappa(e_2)$ in (37).

Lemma A.1. *For*

$$8 \le \kappa \le 110, \qquad 0 < e_2 \le \frac{1}{3},$$

the function $p_\kappa(e_2)$ is strictly increasing in e_2.

Proof. Clear the positive denominator in $\partial p_\kappa/\partial e_2$, put

$$u = \kappa - 8, \qquad v = \frac{1}{3} - e_2,$$

and discard an overall positive factor. The resulting polynomial is

$$c_0(u) + vc_1(u) + v^2c_2(u) + v^3c_3(u),$$

where

$$\begin{aligned}
c_0(u) &= (u+8)^2(u+9)\big[-85u^5 + 22328u^4 + 1025176u^3 \\
&\qquad\qquad + 16232416u^2 + 110957568u + 279631872\big], \\
c_1(u) &= 9(u+8)^3(u+9)(1559u^4 + 65690u^3 + 1015968u^2 \\
&\qquad\qquad + 6865968u + 17159040), \\
c_2(u) &= 27(u+8)^2(u+9)(471u^5 + 18964u^4 + 298584u^3 \\
&\qquad\qquad + 2274896u^2 + 8236800u + 10920960), \\
c_3(u) &= 27(u+8)^2(u+9)(u+10)(43u^4 + 1224u^3 + 13088u^2 \\
&\qquad\qquad + 65280u + 138240).
\end{aligned}$$

The last three are positive. Here, $0 \le u \le 102$, and the only negative term in c_0 is controlled by

$$-85u^5 + 22328u^4 = u^4(22328 - 85u) > 0.$$

Thus, the derivative is positive. $\square$

Lemma A.2. *For* $8 \le \kappa \le 48$,

$$p_\kappa\left(\frac{4}{\kappa+4}\right) > 1.$$

Proof. Define

$$g_0(\kappa) = p_\kappa\left(\frac{4}{\kappa+4}\right).$$

Direct simplification gives

$$\begin{aligned} g_0(\kappa) = &\frac{1200\kappa^2(\kappa+1)(\kappa+4)}{(\kappa+16)^4(\kappa^2+7\kappa+8)^4} \\ &\times \big(7\kappa^6 + 102\kappa^5 + 953\kappa^4 + 5027\kappa^3 + 15596\kappa^2 + 23040\kappa + 12288\big). \end{aligned}$$

Put $u = \kappa - 8$, so $0 \le u \le 40$. The numerator of $g_0(\kappa) - 1$ is

$$\begin{aligned} &-u^{12} - 188u^{11} - 7574u^{10} + 25300u^9 + 10405871u^8 \\ &\quad + 371930144u^7 + 7165872592u^6 + 87815164544u^5 \\ &\quad + 713827668224u^4 + 3820748464128u^3 \\ &\quad + 12796459646976u^2 + 23709367664640u + 17517687865344. \end{aligned}$$

The negative part satisfies

$$u^{12} + 188u^{11} + 7574u^{10} \le 16694u^{10}.$$

The displayed positive terms from u^9 through u^4 alone exceed $16778u^{10}$ for $0 < u \le 40$. Hence, the numerator is positive; at $u = 0$ it is plainly positive as well. $\square$

Lemma A.3. *For* $48 \le \kappa \le 110$,

$$p_\kappa(3/16) \ge p_{110}(3/16) > 1.$$

Proof. Differentiation gives

$$\frac{d}{d\kappa}p_\kappa(3/16) = -\frac{14400\kappa Q_7(\kappa)}{(\kappa+16)^5(35\kappa+38)^5},$$

where

$$\begin{aligned} Q_7(\kappa) = {}&802025\kappa^7 - 17573060\kappa^6 - 306422140\kappa^5 - 1785807856\kappa^4 \\ &- 4991290880\kappa^3 - 7108570624\kappa^2 - 4963848192\kappa - 1348534272. \end{aligned}$$

Now, $Q_7(\kappa)/\kappa^5$ is increasing for $\kappa \ge 48$, because its derivative has positive numerator

$$\begin{aligned} &2\big(802025\kappa^7 - 8786530\kappa^6 + 892903928\kappa^4 + 4991290880\kappa^3 \\ &\quad + 10662855936\kappa^2 + 9927696384\kappa + 3371335680\big), \end{aligned}$$

and $802025\kappa - 8786530 > 0$. Also,

$$\frac{Q_7(48)}{48^5} = \frac{426708436075}{648} > 0.$$

Thus, $Q_7 > 0$ and $p_\kappa(3/16)$ decreases. Finally,

$$p_{110}(3/16) = \frac{24553223694360625}{19288556575029504} = 1.2729425\ldots > 1.$$

$\square$

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References

[1] Hrvoje Abraham and Vjekoslav Kovač, *From electrostatic potentials to yet another triangle center*, Forum Geometricorum **15** (2015), 73–89. arXiv:1312.3176.

[2] Leonard Mascot Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1953.

[3] Carl de Boor, *Divided Differences*, Surveys in Approximation Theory **1** (2005), 46–69.

[4] Clark Kimberling, *Encyclopedia of Triangle Centers*, University of Evansville, `https://faculty.evansville.edu/ck6/encyclopedia/`.

[5] Steven R. Finch, *In Limbo: Three Triangle Centers*, arXiv:1406.0836, 2014.

[6] Grégoire Nicollier, *A simple dynamic localization of the gravitational center of a triangle*, Forum Geometricorum **15** (2015), 263–265.

[7] Shigehiro Sakata, *Euclidean geometric description of radial centers of a triangle*, Colloquium Mathematicum **170** (2022), no. 2, 275–288. doi:10.4064/cm8493-12-2021.