Three Ellipses Associated with the Electrostatic Center of a Triangle

Stanley Rabinowitz
545 Elm St Unit 1, Milford, New Hampshire 03055, USA
stan.rabinowitz@comcast.net

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Abstract

Let $P = X(5626)$ be the electrostatic center of a nondegenerate triangle ABC. Nicollier associated with P three ellipses through P, with focal pairs (B, C), (C, A), and (A, B). We study the pairwise second intersections of these ellipses. If Q_a is the second intersection of the ellipses with focal pairs (C, A) and (A, B), and $D_a = PQ_a \cap BC$, then for $b < c$ these points belong to a natural ordered configuration on the line BC:

$$M_a < X_a < D_a < U_a < J_a < T_a < C < S_a,$$

where $X_a = AP \cap BC$. Here M_a is the midpoint of BC, U_a is the internal similarity center of the circles centered at B, C through P, J_a is a vertex of the associated focal hyperbola, T_a is the incircle contact point, and S_a is the trace on BC of the tangent at P to the ellipse with foci B, C. We also prove that the three cevians AD_a, BD_b, CD_c are concurrent if and only if ABC is isosceles. In the scalene case the sign of their Ceva defect is determined by the ordering of the side lengths.

1 Introduction

Let ABC be a nondegenerate triangle, with

$$a = BC, \qquad b = CA, \qquad c = AB,$$

and let $P = X(5626)$ be its electrostatic center. Abraham and Kovač [1] introduced and studied this point as the unique point of maximal electrostatic potential of a triangular lamina carrying uniform surface charge. It is listed as $X(5626)$ in Kimberling's *Encyclopedia of Triangle Centers* [3].

Nicollier [4] gave a geometric localization of the same center using three ellipses through P, with focal pairs

$$(B, C), \qquad (C, A), \qquad (A, B).$$

More generally, triads of conics associated with an arbitrary point of a triangle have been studied by Garcia, Gheorghe, Moses, and Reznik [2]. Our interest here is in those consequences that use the special metric relations satisfied by the electrostatic center.

The characterization of Abraham and Kovač that we shall use is the following. There is a unique number $q > 0$ such that

$$PB + PC = a\coth(qa), \tag{1}$$
$$PC + PA = b\coth(qb), \tag{2}$$
$$PA + PB = c\coth(qc). \tag{3}$$

For brevity, put

$$F(t) = t\coth(qt), \qquad x = F(a), \quad y = F(b), \quad z = F(c). \tag{1.4}$$

Thus

$$x = PB + PC, \qquad y = PC + PA, \qquad z = PA + PB.$$

The first section below records only the basic ellipse formulas needed later. The main part of the paper begins with the pairwise second intersections of Nicollier's ellipses. From them we obtain a new point on each sideline, a nested configuration involving an associated hyperbola and the incircle, and finally a nonconcurrence theorem whose equality case is exactly the isosceles case.

2 Basic metric formulas

Let $\mathcal{E}_a$ be the ellipse through P having foci B, C, and define $\mathcal{E}_b, \mathcal{E}_c$ cyclically. Let M_a^* and N_a denote the full major-axis and minor-axis lengths of $\mathcal{E}_a$, and let e_a be its eccentricity. The superscript on M_a^* is used to avoid conflict with the midpoint M_a of BC, which will appear later.

From (1),

$$M_a^* = a\coth(qa), \qquad N_a = a\,\mathrm{csch}(qa), \qquad e_a = \tanh(qa), \tag{2.1}$$

with cyclic analogues. In particular,

$$\frac{\operatorname{artanh} e_a}{a} = \frac{\operatorname{artanh} e_b}{b} = \frac{\operatorname{artanh} e_c}{c} = q. \tag{2.2}$$

This is a convenient equivalent form of Nicollier's eccentricity characterization; we shall not use it as a separate result.

We also retain one elementary metric identity. Define

$$p_a = PB + PC + a$$

and

$$\delta_a = PB + PC - a,$$

and define the corresponding quantities cyclically. Thus p_a is the perimeter of $\triangle PBC$, while δ_a is the excess of the broken path BPC over the side BC. We call δ_a the *detour* from B to C through P.

Proposition 2.1. *For the ellipse $\mathcal{E}_a$,*

$$N_a^2 = p_a\delta_a.$$

Cyclically, the analogous identities hold for $\mathcal{E}_b$ and $\mathcal{E}_c$.

Proof. Since $M_a^* = PB + PC$,

$$p_a = M_a^* + a, \qquad \delta_a = M_a^* - a.$$

Hence

$$p_a\delta_a = (M_a^*)^2 - a^2 = N_a^2. \qquad \square$$

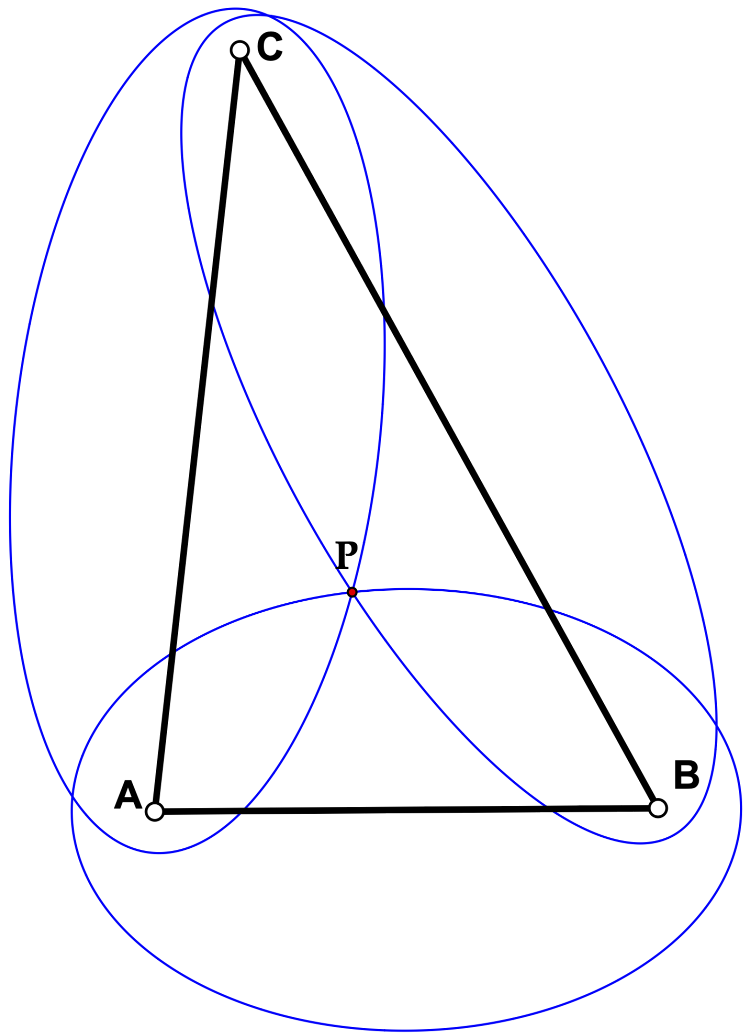

Figure 1: The three associated ellipses.

The following derivative estimate will be used repeatedly.

Lemma 2.2. *For $t > 0$,*

$$0 < F'(t) < 1.$$

Proof. Put $u = qt$. Then

$$F'(t) = \coth u - u \operatorname{csch}^2 u = \frac{\sinh 2u - 2u}{2\sinh^2 u} > 0.$$

Also,

$$1 - F'(t) = \frac{2u - (1 - e^{-2u})}{2\sinh^2 u} > 0.$$

□

Consequently, if $b < c$, then

$$0 < z - y < c - b, \tag{2.3}$$

and since $z - y = PB - PC$,

$$0 < PB - PC < c - b. \tag{2.4}$$

3 Second intersections

Let $Q_a \neq P$ be the second common point of $\mathcal{E}_b$ and $\mathcal{E}_c$, and put

$$D_a = PQ_a \cap BC.$$

Define Q_b, Q_c, D_b, D_c cyclically.

The next lemma gives an explicit form of the common chord. Although the calculation is valid for an arbitrary interior point P, we state it in the notation already specialized to the electrostatic center because it will be used throughout the rest of the paper.

Lemma 3.1. *Take A as the origin. The common chord PQ_a has equation*

$$(zC - yB) \cdot X = \frac{yz(z-y) + zb^2 - yc^2}{2}. \tag{3.1}$$

Consequently,

$$\frac{BD_a}{D_aC} = \frac{n_a}{d_a}, \tag{3.2}$$

where

$$n_a = za^2 + (z-y)(yz - c^2), \tag{4}$$
$$d_a = ya^2 + (y-z)(yz - b^2). \tag{5}$$

Proof. For a variable point X, put $r = XA$. On $\mathcal{E}_c$,

$$r + XB = z.$$

Squaring gives

$$zr - B \cdot X = \frac{z^2 - c^2}{2}.$$

Similarly, on $\mathcal{E}_b$,

$$yr - C \cdot X = \frac{y^2 - b^2}{2}.$$

Eliminating r gives (3.1). Substituting

$$X = \frac{B + \rho C}{1 + \rho}, \qquad \rho = \frac{BD_a}{D_aC},$$

and using

$$2B \cdot C = b^2 + c^2 - a^2$$

gives (3.2). □

There is an associated hyperbola that will be useful. Since Q_a lies on both $\mathcal{E}_b$ and $\mathcal{E}_c$,

$$Q_aA + Q_aC = PA + PC, \qquad Q_aA + Q_aB = PA + PB.$$

Subtracting,

$$Q_aB - Q_aC = PB - PC. \tag{3.3}$$

Thus P and Q_a lie on the same hyperbola with foci B, C. This relation is part of the general theory of P-conic triads studied in [2]; here the special estimate (2.4) will place its vertex relative to classical points of BC.

For later use we record one sign identity. Define

$$\begin{aligned}\Phi = 2xyz &+ (a^2 - b^2 - c^2)x + (b^2 - c^2 - a^2)y \\ &+ (c^2 - a^2 - b^2)z.\end{aligned} \tag{3.4}$$

Lemma 3.2. *We have*

$$\Phi < 0.$$

Proof. Put

$$\alpha = PA, \qquad \beta = PB, \qquad \gamma = PC,$$

and

$$A_P = \angle BPC, \quad B_P = \angle CPA, \quad C_P = \angle APB.$$

Then

$$x = \beta + \gamma, \quad y = \gamma + \alpha, \quad z = \alpha + \beta.$$

Using the cosine law in the three triangles determined by P, a direct simplification gives

$$\Phi = 4\alpha\beta\gamma\big(1 + \cos A_P + \cos B_P + \cos C_P\big).$$

Since

$$A_P + B_P + C_P = 2\pi,$$

we have

$$1 + \cos A_P + \cos B_P + \cos C_P = -4\cos\frac{A_P}{2}\cos\frac{B_P}{2}\cos\frac{C_P}{2} < 0.$$

Hence $\Phi < 0$. □

4 An ordered configuration on a sideline

Assume throughout this section that

$$b < c.$$

Then (2.4) gives

$$\beta := PB > PC =: \gamma.$$

Let M_a be the midpoint of BC, and put

$$X_a = AP \cap BC.$$

Let U_a be the internal similarity center of the circles with centers B, C and radii β, γ; equivalently, PU_a is the internal angle bisector of $\angle BPC$. Thus

$$\frac{BU_a}{U_aC} = \frac{\beta}{\gamma}. \tag{4.1}$$

Let $\mathcal{H}_a$ be the hyperbola with foci B, C through P, and let J_a be the vertex of its branch containing P that lies on BC. Finally, let T_a be the incircle contact point on BC, and let t_a be the tangent at P to $\mathcal{E}_a$. Put

$$S_a = t_a \cap BC.$$

All order relations below are taken on the oriented line from B toward C.

Theorem 4.1. *If $b < c$, then*

$$M_a < X_a < D_a < U_a < J_a < T_a < C < S_a. \tag{4.2}$$

Cyclically, the analogous statements hold on the other sidelines.

Proof. We first locate X_a. Write the normalized barycentric coordinates of P as $(X : Y : Z)$. Since $X_a = AP \cap BC$,

$$\frac{BX_a}{X_aC} = \frac{Z}{Y}.$$

It is enough to prove $Z > Y$. Put

$$\xi = q(c-b), \qquad \eta = qa, \qquad \zeta = q(b+c).$$

The triangle inequalities give

$$0 < \xi < \eta < \zeta.$$

For fixed $\eta > 0$, define

$$\Psi_\eta(t) = \frac{\sinh t}{t} \, \frac{t\coth t - \eta \coth \eta}{t^2 - \eta^2},$$

with the continuous value at $t = \eta$. A direct calculation gives

$$\Psi_\eta(t) = \frac{1}{t \sinh \eta} \int_0^1 \sinh(ts) \sinh(\eta s)\, ds,$$

so Ψ_η is strictly increasing. Solving for the two barycentric coordinates Y, Z from the squared-distance equations for PB and PC, and simplifying with (1)–(3), gives

$$16[ABC]^2(Z - Y) = \frac{\xi\zeta(\eta^2 - \xi^2)(\zeta^2 - \eta^2)}{2q^4 \sinh(qb)\sinh(qc)} \big(\Psi_\eta(\zeta) - \Psi_\eta(\xi)\big) > 0.$$

Hence $M_a < X_a$.

We next locate D_a. Put

$$w = z - y = \beta - \gamma.$$

By (2.3),

$$0 < w < c - b < a. \tag{4.3}$$

In (4)–(5), we have $n_a > 0$ and $d_a > 0$. Indeed, if $yz \geq c^2$, then $n_a > 0$ is immediate. If $yz < c^2$, then

$$c^2 - yz < c(c-b),$$

so by (4.3),

$$w(c^2 - yz) < c(c-b)^2 < ca^2 < za^2.$$

Hence again $n_a > 0$.

For d_a, write $z = y + w$. Since P is interior,

$$z = PA + PB < a + b,$$

so $y - b < a - w$. Therefore

$$\begin{aligned} d_a &= y(a^2 - w^2) - w(y^2 - b^2) \\ &= y(a-w)(a+w) - w(y-b)(y+b) \\ &> (a-w)\big(y(a+w) - w(y+b)\big) \\ &= (a-w)(ay - bw) > 0. \end{aligned}$$

Thus D_a lies on the open segment BC.

Moreover,

$$n_a - d_a = (z - y)(a^2 - b^2 - c^2 + 2yz) > 0,$$

because

$$a^2 - b^2 - c^2 = -2bc\cos A$$

and $y > b, z > c$. Hence

$$\frac{BD_a}{D_aC} > 1,$$

so

$$M_a < D_a.$$

We now compare X_a with D_a. Let L denote the left side of (3.1) minus the right side, so that $L = 0$ is the line PQ_a. From (4)–(5),

$$2L(B) = -n_a < 0, \qquad 2L(C) = d_a > 0.$$

Since A, P, X_a are collinear and P lies between A and X_a, we have $L(X_a) < 0$ exactly when $L(A) > 0$. Taking A as the origin,

$$2L(A) = yc^2 - zb^2 - yz(z - y).$$

Put $u = qb$ and $v = qc$. Then $0 < u < v$, and after substituting $y = b\coth(qb)$, $z = c\coth(qc)$ and clearing positive factors, the last expression has the same sign as

$$\frac{u}{\sinh 2u} - \frac{v}{\sinh 2v}.$$

The function $t/\sinh 2t$ is strictly decreasing for $t > 0$. Thus $L(A) > 0$, so $L(X_a) < 0$, and hence

$$X_a < D_a.$$

Next compare D_a with U_a. Since

$$\beta = \frac{x + z - y}{2}, \qquad \gamma = \frac{x + y - z}{2},$$

a simplification using (4)–(5) gives

$$\gamma n_a - \beta d_a = \frac{z - y}{2}\,\Phi.$$

Lemma 3.2 and $z - y > 0$ show that this quantity is negative. Thus

$$\frac{BD_a}{D_aC} < \frac{\beta}{\gamma} = \frac{BU_a}{U_aC},$$

and therefore

$$D_a < U_a.$$

Since the transverse semi-axis of $\mathcal{H}_a$ is $w/2$,

$$BJ_a = \frac{a+w}{2}.$$

Also

$$BU_a = \frac{a\beta}{\beta+\gamma} = \frac{a}{2} + \frac{aw}{2(\beta+\gamma)}.$$

Because $\beta+\gamma > a$,

$$U_a < J_a.$$

The incircle contact point satisfies

$$BT_a = s - b = \frac{a+c-b}{2}.$$

By (4.3),

$$BJ_a = \frac{a+w}{2} < \frac{a+c-b}{2} = BT_a,$$

so $J_a < T_a$. The triangle inequality gives $T_a < C$.

Finally, the tangent t_a is the external angle bisector of $\angle BPC$. By the external angle-bisector theorem,

$$\frac{BS_a}{S_aC} = \frac{\beta}{\gamma} > 1$$

in unsigned lengths. Hence S_a lies beyond C. This completes the ordering (4.2). □

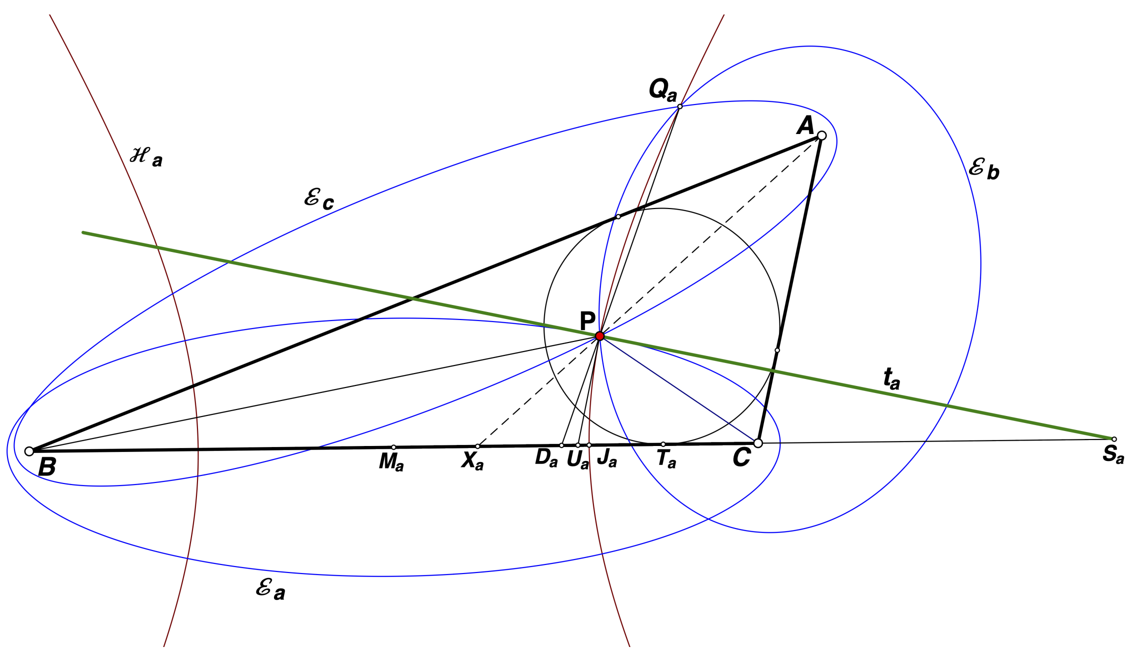

Figure 2: The ordered points on BC.

Figure 2 shows the complete configuration used in Theorem 4.1. For an arbitrary interior point P with $PB > PC$, the conic and similarity-center construction already gives the subchain

$$M_a < D_a < U_a < J_a < C < S_a.$$

The electrostatic center supplies the additional comparisons involving X_a and T_a. In particular, $J_a < T_a$ is exactly the consequence of the strict contraction $PB - PC < c - b$.

5 The tangent traces

The tangent traces used in Theorem 4.1 have a classical interpretation. Define

$$S_a = t_a \cap BC, \qquad S_b = t_b \cap CA, \qquad S_c = t_c \cap AB,$$

where t_a, t_b, t_c are the tangents at P to $\mathcal{E}_a, \mathcal{E}_b, \mathcal{E}_c$, respectively.

Proposition 5.1. *The points S_a, S_b, S_c are collinear. In barycentric coordinates their common line is*

$$PA\,X + PB\,Y + PC\,Z = 0. \tag{5.1}$$

Proof. The tangent t_a is the external angle bisector of $\angle BPC$. Hence, in directed ratios,

$$\frac{BS_a}{S_aC} = -\frac{PB}{PC}.$$

Cyclically,

$$\frac{CS_b}{S_bA} = -\frac{PC}{PA}, \qquad \frac{AS_c}{S_cB} = -\frac{PA}{PB}.$$

Their product is -1, so Menelaus' theorem gives the collinearity. Moreover,

$$S_a = (0 : PC : -PB),$$

with cyclic analogues, and these three points satisfy (5.1). □

Remark 5.2. No electrostatic property of P is used in Proposition 5.1. The circles centered at A, B, C with radii PA, PB, PC all pass through P, and S_a, S_b, S_c are their pairwise external similarity centers. Thus their common line is the Monge line of these three circles. We include the proposition only because the points S_a, S_b, S_c occur naturally in the electrostatic ordering of Theorem 4.1.

6 A Ceva defect

The points D_a, D_b, D_c suggest asking whether the cevians

$$AD_a, \qquad BD_b, \qquad CD_c$$

are concurrent. They are not concurrent in a scalene triangle. More precisely, the sign of the Ceva defect is determined by the ordering of the side lengths.

We first need a one-variable lemma. Put

$$X(t) = t\coth(qt), \qquad \Lambda(t) = \frac{2t}{\sinh(2qt)}.$$

The second quantity is the latus-rectum length of the ellipse corresponding to the side length t.

Lemma 6.1. *As a function of X, the quantity Λ is strictly decreasing and strictly convex.*

Proof. The factor q only rescales both coordinates, so it is enough to take $q = 1$. Put

$$X = t\coth t, \qquad H = t\tanh t, \qquad \Lambda = X - H = \frac{2t}{\sinh 2t}.$$

The function $\Lambda(t)$ is strictly decreasing. Since $X'(t) > 0$, to prove convexity of Λ as a function of X, it is enough to show that

$$R(t) = \frac{H'(t)}{X'(t)}$$

is strictly decreasing. A simplification gives

$$R(t) = \tanh^2 t\,\frac{\sinh 2t + 2t}{\sinh 2t - 2t}.$$

Differentiating,

$$R'(t) = -\frac{4\tanh t\,G(2t)}{(2t - \sinh 2t)^2(\cosh 2t + 1)},$$

where

$$G(v) = 1 + v^2 + \frac{v}{2}\sinh 2v - \cosh 2v.$$

The power series of G is

$$G(v) = \sum_{m=3}^{\infty} \frac{2^{2m-1}(m-2)}{(2m)!}\,v^{2m} > 0 \qquad (v > 0).$$

Thus $R'(t) < 0$, and the required convexity follows. □

For the three points D_a, D_b, D_c, write

$$r_a = \frac{BD_a}{D_aC}, \qquad r_b = \frac{CD_b}{D_bA}, \qquad r_c = \frac{AD_c}{D_cB}.$$

Theorem 6.2. *If ABC is scalene, then*

$$(r_a r_b r_c - 1)(a - b)(b - c)(c - a) < 0. \tag{6.1}$$

Consequently,

$$AD_a, \quad BD_b, \quad CD_c$$

are concurrent if and only if ABC is isosceles.

Proof. By Lemma 3.1,

$$r_a = \frac{n_a}{d_a},$$

where n_a, d_a are given by (4)–(5); define n_b, d_b, n_c, d_c cyclically. Theorem 4.1 and its cyclic forms show that all these quantities have the signs required for the usual positive Ceva ratios.

Set

$$\Delta = n_a n_b n_c - d_a d_b d_c.$$

A direct expansion factors as

$$\Delta = \Phi\Theta, \tag{6.2}$$

where Φ is the quantity in (3.4), and

$$\Theta = a^2b^2z(y-x) + a^2c^2y(x-z) + b^2c^2x(z-y) + xyz(x-y)(x-z)(y-z). \tag{6.3}$$

By Lemma 3.2, $\Phi < 0$.

It remains to determine the sign of Θ. Assume first that

$$a < b < c.$$

Then

$$x < y < z.$$

Put

$$h_a = \frac{a^2}{x} = a\tanh(qa), \quad h_b = \frac{b^2}{y} = b\tanh(qb), \quad h_c = \frac{c^2}{z} = c\tanh(qc).$$

Also put

$$\lambda_a = x - h_a = \frac{2a}{\sinh(2qa)},$$

with cyclic definitions of λ_b, λ_c. From (6.3),

$$\frac{\Theta}{xyz} = \begin{vmatrix} 1 & 1 & 1 \\ h_a & h_b & h_c \\ a^2 & b^2 & c^2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}. \tag{6.4}$$

Let

$$p = y - x, \qquad r = z - y,$$

and

$$u = \lambda_a - \lambda_b, \qquad v = \lambda_b - \lambda_c.$$

Then $p, r, u, v > 0$, and

$$h_b - h_a = p + u, \qquad h_c - h_b = r + v.$$

By Lemma 6.1,

$$\frac{u}{p} > \frac{v}{r}. \tag{6.5}$$

Expanding (6.4) in these increments gives

$$\frac{\Theta}{xyz} = h_a(ru - pv) + 2pru + prv + r^2u + ru^2 + ruv. \tag{6.6}$$

Every term on the right is positive by (6.5). Hence

$$\Theta > 0$$

when $a < b < c$. Since Θ changes sign when two of the paired variables $(a, x), (b, y), (c, z)$ are interchanged, it follows for every scalene triangle that

$$\operatorname{sgn}\Theta = \operatorname{sgn}\big((a-b)(b-c)(c-a)\big).$$

Together with $\Phi < 0$, this proves (6.1).

For a scalene triangle, (6.1) gives $r_ar_br_c \neq 1$, so Ceva's theorem rules out concurrence. If ABC is isosceles, the construction is symmetric about the corresponding symmetry axis; the two remaining cevians are mirror images and meet on the third. Thus the three cevians are concurrent. This proves the final assertion. $\square$

Remark 6.3. For example, if $a < b < c$, then (6.1) gives

$$r_a r_b r_c < 1.$$

Thus the failure of concurrence is not arbitrary: its orientation is fixed by the ordering of the side lengths. Figure 3 illustrates the scalene case.

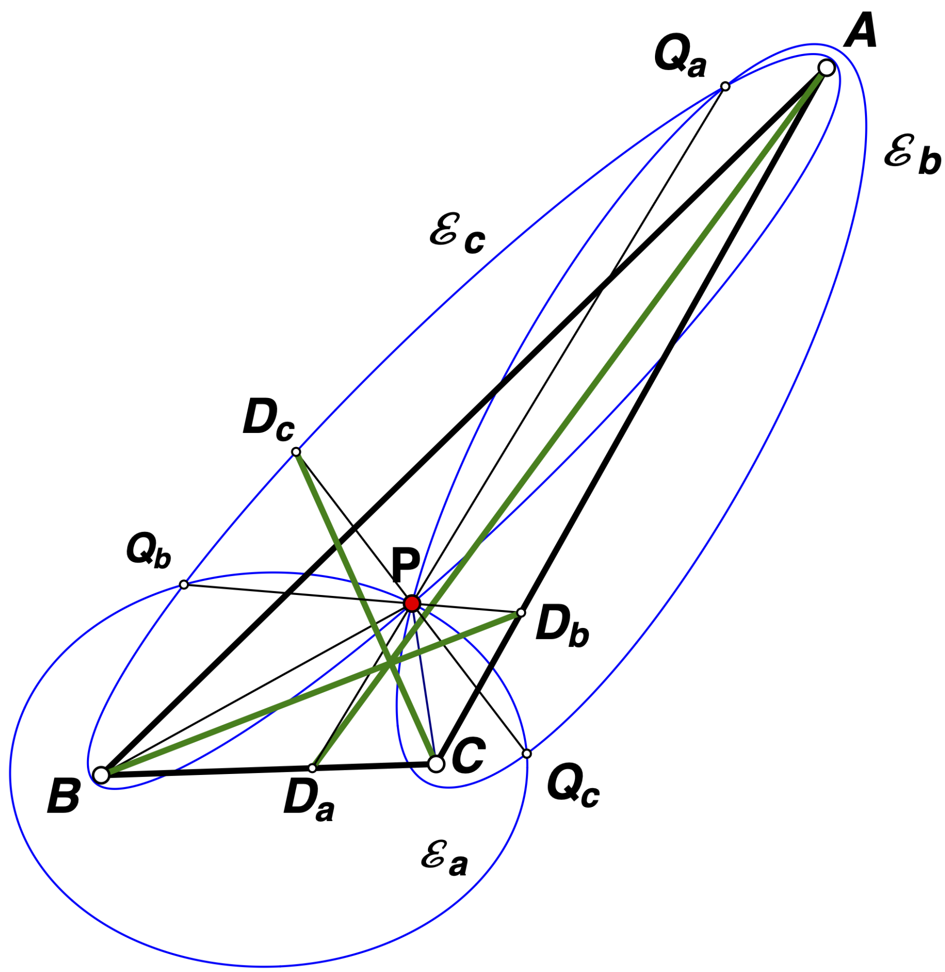

Figure 3: The three green lines are not concurrent.

7 Concluding remarks

The elementary axis formulas (2.1) are useful background, but the main geometry of the three associated ellipses appears when their pairwise intersections are considered. The second point Q_a determines a chord whose trace D_a on BC is trapped among several natural points. In the case $b < c$,

$$M_a < X_a < D_a < U_a < J_a < T_a < C < S_a.$$

The comparison $J_a < T_a$ is a direct geometric consequence of the strict electrostatic contraction

$$0 < PB - PC < c - b.$$

The three tangent traces belong to a classical Monge line, while the three new cevians through D_a, D_b, D_c are concurrent precisely in the isosceles case. Thus the second-intersection configuration produces properties that depend essentially on the Abraham–Kovač characterization, rather than only on the general theory of focal ellipses through an arbitrary point.

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