Some Inequalities Associated with the Electrostatic Center of a Triangle

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Abstract

Let $P = X(5626)$ be the electrostatic center of a nondegenerate triangle ABC, with side lengths $a = BC$, $b = CA$, and $c = AB$. A characterization of Abraham and Kovač expresses the three sums $PB + PC$, $PC + PA$, and $PA + PB$ in terms of a single positive parameter. We use this characterization to derive several elementary inequalities involving distances naturally associated with P. In particular,

$$a < b \iff PA > PB, \qquad 0 < PA - PB < b - a.$$

We also obtain strict orderings for the perimeters and detours of the three subtriangles determined by P. Finally, we study the cevians through P. Each such cevian lies strictly between the corresponding median and internal angle bisector, and if $a < b < c$, then their lengths satisfy

$$AD > BE > CF.$$

A quantitative comparison with the incenter gives

$$0 < PA - PB < AI - BI < b - a$$

and

$$0 < AD - BE < \ell_a - \ell_b < AI - BI < b - a,$$

where I is the incenter and ℓ_a, ℓ_b are internal angle-bisector lengths.

1 Introduction

Let ABC be a nondegenerate triangle, with

$$a = BC, \qquad b = CA, \qquad c = AB,$$

and let $P = X(5626)$ be its electrostatic center. Abraham and Kovač [1] proved that P is an interior point of ABC and obtained several analytic and geometric characterizations of it. The point is listed as $X(5626)$ in Kimberling's *Encyclopedia of Triangle Centers* [2] and belongs to the broader theory of radial centers; see, for example, Sakata [3].

The characterization used here is the following. There is a unique number $q > 0$ such that

$$PB + PC = a\coth(qa), \tag{1}$$
$$PC + PA = b\coth(qb), \tag{2}$$
$$PA + PB = c\coth(qc). \tag{3}$$

Our purpose is to extract several elementary inequalities from these three relations.

We first compare the distances from P to the vertices. We then study the perimeters of the three subtriangles and the excesses of broken paths through P. The remaining sections concern the cevians through P: their locations relative to the medians and angle bisectors, their mutual ordering, and quantitative comparisons with familiar classical lengths.

We do not claim that every individual ordering property below is unique to the electrostatic center. Rather, the interest lies in the coherent family of inequalities that follows from the Abraham–Kovač characterization, often without requiring explicit coordinates for $X(5626)$.

2 A basic derivative estimate

For fixed $q > 0$, put
$$F(t) = t\coth(qt), \qquad t > 0.$$

Lemma 2.1. *For $t > 0$,*
$$0 < F'(t) < 1.$$

Proof. Put $u = qt$. Then
$$F'(t) = \coth u - u\,\mathrm{csch}^2\, u = \frac{\sinh 2u - 2u}{2\sinh^2 u} > 0,$$
because $\sinh 2u > 2u$. Also,
$$1 - F'(t) = \frac{2u - \left(1 - e^{-2u}\right)}{2\sinh^2 u} > 0,$$
because $e^{-x} > 1 - x$ for $x > 0$. □

In terms of F, equations (1)–(3) become
$$PB + PC = F(a), \qquad PC + PA = F(b), \qquad PA + PB = F(c).$$
Solving for the individual vertex distances gives
$$PA = \frac{F(b) + F(c) - F(a)}{2}, \qquad PB = \frac{F(c) + F(a) - F(b)}{2}, \qquad PC = \frac{F(a) + F(b) - F(c)}{2}.$$
In particular,
$$PA - PB = F(b) - F(a). \tag{4}$$

3 Vertex distances

Theorem 3.1. *For the electrostatic center P,*
$$a < b \iff PA > PB.$$
More precisely, if $a < b$, then
$$0 < PA - PB < b - a.$$

Proof. By (4),

$$PA - PB = F(b) - F(a).$$

The first assertion follows because F is strictly increasing.

If $a < b$, the mean value theorem gives

$$PA - PB = F'(\xi)(b - a)$$

for some $a < \xi < b$. Lemma 2.1 gives

$$0 < PA - PB < b - a. \qquad \square$$

Corollary 3.2. *If $a < b < c$, then*

$$PA > PB > PC.$$

Corollary 3.3.

$$PA = PB \iff a = b.$$

Equivalently,

$$P \text{ lies on the perpendicular bisector of } AB \iff CA = CB.$$

The same argument gives

$$0 < PA - PC < c - a$$

whenever $a < c$.

Remark 3.4. The reversal $a < b \iff PA > PB$ is not peculiar to the electrostatic center. It also occurs for several classical centers, including the incenter, centroid, and isodynamic points. The interest here is the direct derivation from the Abraham–Kovač equations together with the quantitative contraction $PA - PB < b - a$.

4 Subtriangle perimeters and detours

Define

$$p_a = PB + PC + a$$

and

$$\delta_a = PB + PC - a,$$

and define $p_b, p_c, \delta_b, \delta_c$ cyclically. Thus p_a is the perimeter of $\triangle PBC$, while δ_a, called the *detour*, is the excess of the broken path BPC over the side BC.

From (1),

$$p_a = a(\coth(qa) + 1) = \frac{2a}{1 - e^{-2qa}},$$

$$\delta_a = a(\coth(qa) - 1) = \frac{2a}{e^{2qa} - 1}.$$

Equivalently,

$$p_a = F(a) + a, \qquad \delta_a = F(a) - a.$$

Theorem 4.1. *If $a < b$, then*

$$p_a < p_b, \qquad \delta_a > \delta_b.$$

More precisely,

$$b - a < p_b - p_a < 2(b - a)$$

and

$$0 < \delta_a - \delta_b < b - a.$$

Proof. We have

$$p_b - p_a = (b-a) + F(b) - F(a)$$

and

$$\delta_a - \delta_b = (b-a) - \{F(b) - F(a)\}.$$

The result follows from

$$0 < F(b) - F(a) < b - a. \qquad \square$$

Thus, if $a < b < c$,

$$p_a < p_b < p_c$$

while

$$\delta_a > \delta_b > \delta_c.$$

Remark 4.2. For an arbitrary interior point Q, the identities

$$p_b(Q) - p_a(Q) = (b-a) + (QA - QB)$$

and

$$\delta_a(Q) - \delta_b(Q) = (b-a) - (QA - QB)$$

show that the quantitative inequalities in Theorem 4.1 are reformulations of the contraction inequality in Theorem 3.1.

The isoperimetric point $X(175)$ and the equal-detour point $X(176)$ are characterized by equality of the corresponding perimeter and detour triples, respectively; see [2]. For the electrostatic center, these triples are instead strictly and oppositely ordered in a scalene triangle.

5 The electrostatic cevians

Let

$$D = AP \cap BC, \qquad E = BP \cap CA, \qquad F = CP \cap AB.$$

The three electrostatic cevians AD, BE, and CF, which are concurrent at P, are shown in Figure 1.

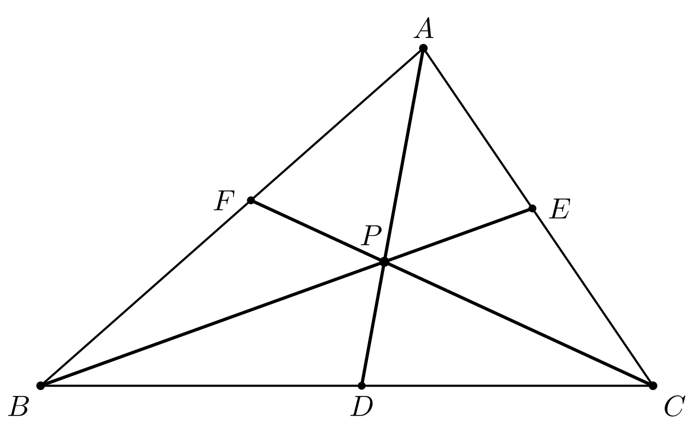

Figure 1: Electrostatic cevians.

Let M_a be the midpoint of BC, and let L_a be the foot of the internal angle bisector from A. Define M_b, M_c, L_b, L_c cyclically. Let

$$m_a = AM_a, \qquad \ell_a = AL_a$$

be the corresponding median and internal angle-bisector lengths.

Theorem 5.1. *If $b \neq c$, then D lies strictly between M_a and L_a. Cyclically, the analogous statement holds at the other two vertices.*

Proof. It is enough to consider $b < c$. Let

$$P = (X : Y : Z)$$

be normalized barycentric coordinates, so $X + Y + Z = 1$. Since $D = AP \cap BC$,

$$\frac{BD}{DC} = \frac{Z}{Y}.$$

Thus it suffices to prove

$$Y < Z < \frac{c}{b} Y.$$

Put

$$u = F(a), \qquad v = F(b), \qquad w = F(c),$$

and let $\Delta = [ABC]$ denote the area of $\triangle ABC$. Take A as the origin. Since $P = YB + ZC$, the identities

$$PA^2 - PB^2 = (v - u)w, \qquad PA^2 - PC^2 = (w - u)v$$

give

$$2c^2 Y + (b^2 + c^2 - a^2)Z = c^2 + w(v - u)$$

and

$$(b^2 + c^2 - a^2)Y + 2b^2 Z = b^2 + v(w - u).$$

First we prove $Z < cY/b$. Eliminating Y, Z and simplifying with $u = a\coth(qa)$, $v = b\coth(qb)$, and $w = c\coth(qc)$ gives

$$\begin{aligned} 16\Delta^2(cY - bZ) = &\left((b+c)^2 - a^2\right) bc \left(\coth(qb)\coth(qc) - 1\right) \\ &\times \left[a\coth(qa)\tanh(q(c-b)) - (c-b)\right]. \end{aligned}$$

All factors outside the final brackets are positive. Since

$$0 < c - b < a$$

and $t \mapsto \tanh(qt)/t$ is strictly decreasing,

$$\frac{\tanh(q(c-b))}{c-b} > \frac{\tanh(qa)}{a}.$$

Thus $cY - bZ > 0$.

It remains to prove $Z > Y$. Put

$$\xi = q(c - b), \qquad \eta = qa, \qquad \zeta = q(b + c).$$

The triangle inequalities give

$$0 < \xi < \eta < \zeta.$$

For fixed $\eta > 0$, define

$$\Phi_\eta(t) = \frac{\sinh t}{t} \, \frac{t \coth t - \eta \coth \eta}{t^2 - \eta^2},$$

with the continuous value understood at $t = \eta$. A direct calculation gives

$$\Phi_\eta(t) = \frac{1}{t \sinh \eta} \int_0^1 \sinh(ts) \sinh(\eta s) \, ds.$$

For each $0 < s \leq 1$, the function $\sinh(ts)/t$ is strictly increasing in t. Hence $\Phi_\eta(t)$ is strictly increasing.

A second elimination gives

$$16\Delta^2(Z - Y) = \frac{\xi\zeta(\eta^2 - \xi^2)(\zeta^2 - \eta^2)}{2q^4 \sinh(qb)\sinh(qc)} \left(\Phi_\eta(\zeta) - \Phi_\eta(\xi)\right).$$

Every factor on the right is positive, so $Z > Y$. Therefore

$$1 < \frac{BD}{DC} < \frac{c}{b}, \qquad \square$$

which is equivalent to saying that D lies strictly between M_a and L_a.

Remark 5.2. The two boundary cases are familiar. For the centroid G, the cevian from A meets BC at M_a, whereas for the incenter I it meets BC at L_a. Thus the electrostatic cevian lies strictly between the corresponding centroid and incenter cevians.

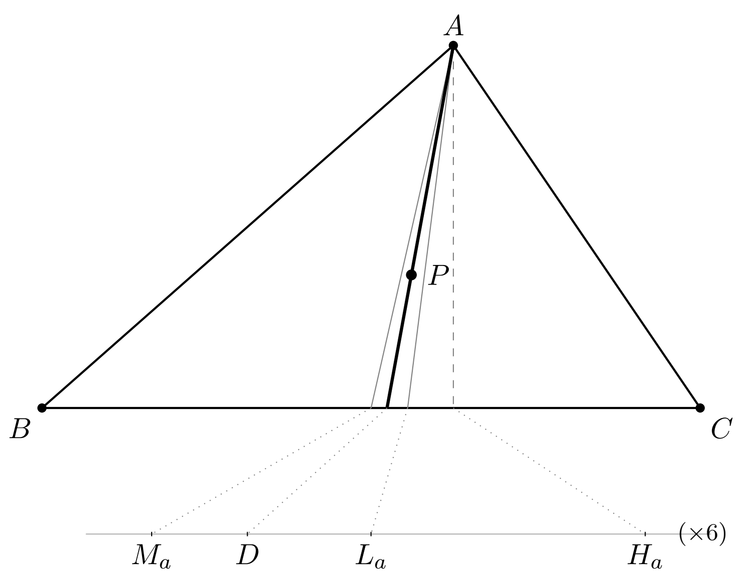

Figure 2: The electrostatic cevian AD lies between the median AM_a and the internal angle bisector AL_a; here $(a, b, c) = (6, 4, 5)$. The strip below magnifies BC near the four points, which occur in the order M_a, D, L_a, H_a.

For later use, note that if $a < b < c$ and $P = (X : Y : Z)$, the three cyclic instances of Theorem 5.1 are equivalently summarized by

$$X < Y < Z, \qquad \frac{X}{a} > \frac{Y}{b} > \frac{Z}{c}.$$

The next lemma is simply Stewart's theorem written in a convenient form.

Lemma 5.3. *For $p, r > 0$, define*

$$h_{p,r}(\lambda) = \frac{p^2\lambda + c^2}{1+\lambda} - \frac{r^2\lambda}{(1+\lambda)^2}, \qquad \lambda > 0.$$

Then

$$(1+\lambda)^3 h'_{p,r}(\lambda) = (p^2 + r^2 - c^2)\lambda + (p^2 - r^2 - c^2).$$

If $\sigma = BD/DC$ and $\tau = AF/FB$, then

$$AD^2 = h_{b,a}(\sigma), \qquad BE^2 = h_{a,b}(\sigma\tau),$$

and

$$CF^2 = \frac{a^2\tau + b^2}{1+\tau} - \frac{c^2\tau}{(1+\tau)^2}.$$

Proof. The derivative formula is immediate. The three length formulas follow from Stewart's theorem. By Ceva's theorem, we have

$$\frac{EA}{CE} = \sigma\tau. \qquad \square$$

Corollary 5.4. *If ABC is scalene, then*

$$\ell_a < AD < m_a, \qquad \ell_b < BE < m_b, \qquad \ell_c < CF < m_c.$$

Proof. Without loss of generality, assume $a < b < c$. By Theorem 5.1,

$$\sigma \in (1, c/b), \qquad \sigma\tau \in (1, c/a), \qquad \tau \in (1, b/a).$$

The endpoints are the parameters of the median and internal angle-bisector feet at the corresponding vertices.

By Lemma 5.3, the numerators of the three relevant derivatives are linear in the parameter. At the two endpoints of the first interval they are

$$2(b^2 - c^2) \quad \text{and} \quad \frac{(c-b)\big(a^2 - (b+c)^2\big)}{b},$$

at the two endpoints of the second they are

$$2(a^2 - c^2) \quad \text{and} \quad \frac{(a-c)(a+c-b)(a+b+c)}{a},$$

and at the two endpoints of the third they are

$$2(a^2 - b^2) \quad \text{and} \quad \frac{(a-b)\big((a+b)^2 - c^2\big)}{a}.$$

All six quantities are negative. Hence each squared cevian length is strictly decreasing between its median and angle-bisector endpoint, which gives

$$\ell_a < AD < m_a, \qquad \ell_b < BE < m_b, \qquad \ell_c < CF < m_c. \qquad \square$$

Corollary 5.5. *Let*

$$\rho_a = d(P, BC), \qquad \rho_b = d(P, CA), \qquad \rho_c = d(P, AB).$$

If $a < b$, then

$$1 < \frac{\rho_a}{\rho_b} < \frac{b}{a}.$$

Consequently, if $a < b < c$, then

$$\rho_a > \rho_b > \rho_c.$$

Proof. Apply Theorem 5.1 to the cevian from C. Since $a < b$,

$$1 < \frac{AF}{FB} < \frac{b}{a}.$$

If $\Delta_a = [PBC]$ and $\Delta_b = [PCA]$, then

$$\frac{AF}{FB} = \frac{\Delta_b}{\Delta_a} = \frac{b\rho_b}{a\rho_a}.$$

The stated inequality follows immediately. □

6 A comparison with the incenter

The cevian-location theorem also sharpens the vertex-distance estimate from Section 3.

Theorem 6.1. *If $a < b < c$, then*

$$PA - PB < AI - BI.$$

Proof. Let $P = (X : Y : Z)$ be normalized barycentric coordinates. The cyclic instances of Theorem 5.1 give

$$bX > aY, \qquad cY > bZ.$$

The boundary lines $bX = aY$ and $cY = bZ$ are respectively the lines CI and AI. Hence P lies in the open cone at I generated by the ray from I toward A and the ray from I directly away from C.

Consider

$$d(Q) = QA - QB.$$

Since $a < b$, we have $AI > BI$, so the level curve

$$d(Q) = AI - BI$$

through I is the positive branch of a hyperbola with foci A, B. Choose Cartesian coordinates with the midpoint of AB as origin, with AB as the horizontal axis and the positive x-direction toward B. If $k = AI - BI$, then this branch has the form

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1, \qquad x > 0,$$

for some $\alpha, \beta > 0$, where $2\alpha = k$. Hence the region $d(Q) \geq d(I)$ is the epigraph

$$x \geq \alpha\sqrt{1 + \frac{y^2}{\beta^2}},$$

of a strictly convex function. Its tangent at I is therefore a supporting line for this region.

Put
$$\mathbf{u}_A = \frac{I-A}{AI}, \qquad \mathbf{u}_B = \frac{I-B}{BI}, \qquad \mathbf{u}_C = \frac{I-C}{CI}.$$
Let ∇ denote the gradient operator. Then
$$\nabla d(I) = \mathbf{u}_A - \mathbf{u}_B.$$
Since $\nabla d(I)$ points toward increasing values of d, the supporting half-plane containing the region $d(Q) \geq d(I)$ is
$$\nabla d(I) \cdot (Q - I) \geq 0.$$
Along the ray toward A,
$$\nabla d(I) \cdot (A - I) = -AI\big(1 - \cos\angle AIB\big) < 0.$$
Along the ray directly away from C,
$$\begin{aligned}\nabla d(I) \cdot (I - C) &= CI\big(\cos\angle AIC - \cos\angle BIC\big)\\ &= CI\left(\sin\frac{A}{2} - \sin\frac{B}{2}\right) < 0,\end{aligned}$$
because $A < B$. Since $P - I$ is a positive linear combination of the two generating directions $A - I$ and $I - C$, it follows that
$$\nabla d(I) \cdot (P - I) < 0.$$
Thus P lies strictly outside the supporting half-plane containing $d(Q) \geq d(I)$, and therefore
$$PA - PB < AI - BI.$$
□

7 Ordering the cevian lengths

Theorem 7.1. *If $a < b < c$, then*
$$AD > BE > CF.$$

Proof. Put
$$\sigma = \frac{BD}{DC}, \qquad \tau = \frac{AF}{FB}.$$
Theorem 5.1 gives
$$1 < \sigma < \frac{c}{b}, \qquad 1 < \tau < \frac{b}{a},$$
and hence
$$1 < \sigma \leq \sigma\tau < \frac{c}{a}.$$

By Lemma 5.3,
$$(1+\lambda)^3 h'_{a,b}(\lambda) = (a^2 + b^2 - c^2)\lambda + (a^2 - b^2 - c^2),$$
a linear function of λ. Its values at 1 and c/a are
$$2(a^2 - c^2) < 0$$

and

$$\frac{(a-c)(a+c-b)(a+b+c)}{a} < 0,$$

respectively. Thus $h_{a,b}$ is strictly decreasing on $[1, c/a]$. Since $\sigma < \sigma\tau$,

$$BE^2 = h_{a,b}(\sigma\tau) < h_{a,b}(\sigma) < AD^2,$$

where the final inequality follows from

$$AD^2 - h_{a,b}(\sigma) = \frac{(b^2-a^2)\sigma(\sigma+2)}{(1+\sigma)^2} > 0.$$

Hence $AD > BE$.

For the second inequality, $h_{a,b}(\sigma\tau)$ is decreasing in σ, so

$$BE^2 > h_{a,b}\left(\frac{c\tau}{b}\right).$$

A direct computation gives

$$h_{a,b}\left(\frac{c\tau}{b}\right) - CF^2 = \frac{(c-b)N(\tau)}{(b+c\tau)^2(1+\tau)^2},$$

where

$$\begin{aligned} N(\tau) = {} & c(a^2+b^2+2bc+c^2)\tau^3 \\ & + (b+c)(a^2+4bc)\tau^2 \\ & + b(a^2+b^2+4bc+c^2)\tau \\ & + b^2(b+c). \end{aligned}$$

Every coefficient of N is positive, so $BE > CF$. □

Remark 7.2. This reverse ordering of central cevian lengths is not unique to the electrostatic center. For example, when $a < b < c$, the medians and the internal angle bisectors also satisfy

$$m_a > m_b > m_c, \qquad \ell_a > \ell_b > \ell_c.$$

Theorem 7.1 shows that the electrostatic cevians share this familiar behavior. The proof itself uses only their locations between the corresponding medians and angle bisectors.

8 A quantitative cevian inequality

Theorem 8.1. *If $a < b < c$, then*

$$0 < AD - BE < \ell_a - \ell_b.$$

Proof. The first inequality follows from Theorem 7.1. We prove the upper bound.

Put

$$r = \frac{BD}{DC}, \qquad t = \frac{AF}{FB}.$$

By Theorem 5.1,

$$1 < r < \frac{c}{b}, \qquad 1 < t < \frac{b}{a}.$$

Ceva's theorem gives

$$\frac{CE}{EA} = \frac{1}{rt} > \frac{a}{br}.$$

Let

$$d_a = L_a D, \qquad d_b = L_b E.$$

At vertex A, L_a lies between D and the altitude foot H_a; at vertex B, L_b lies between E and H_b. Thus d_a, d_b are positive distances measured from the angle-bisector foot toward the corresponding cevian foot. The angle-bisector theorem gives

$$d_a = \frac{ac}{b+c} - \frac{ar}{1+r} = \frac{a(c-br)}{(b+c)(1+r)}.$$

Also,

$$CE > \frac{ab}{a+br}, \qquad CL_b = \frac{ab}{a+c},$$

and hence

$$d_b > \frac{ab(c-br)}{(a+br)(a+c)}.$$

The difference between the right side of the last inequality and d_a is

$$\frac{a(b-a)(c-br)(a+br+b+c)}{(a+c)(a+br)(b+c)(1+r)} > 0.$$

Thus $d_b > d_a$.

Let

$$u_a = L_a H_a, \qquad u_b = L_b H_b,$$

both taken as positive lengths. Writing $S = a+b+c$, direct calculation gives

$$u_a = \frac{(c-b)(b+c-a)S}{2a(b+c)}$$

and

$$u_b = \frac{(c-a)(a+c-b)S}{2b(a+c)}.$$

If

$$h_a = \frac{2\Delta}{a}, \qquad h_b = \frac{2\Delta}{b}$$

are the altitudes of ABC, then

$$\frac{u_b}{h_b} - \frac{u_a}{h_a} = \frac{ab(b-a)S}{2\Delta(a+c)(b+c)} > 0.$$

Also $h_a > h_b$.

Now

$$AD = \sqrt{h_a^2 + (u_a+d_a)^2}, \qquad \ell_a = \sqrt{h_a^2 + u_a^2},$$

and

$$BE = \sqrt{h_b^2 + (u_b+d_b)^2}, \qquad \ell_b = \sqrt{h_b^2 + u_b^2}.$$

Hence

$$AD - \ell_a = \int_0^{d_a} \frac{u_a + s}{\sqrt{h_a^2 + (u_a+s)^2}}\, ds,$$

while

$$BE - \ell_b = \int_0^{d_b} \frac{u_b + s}{\sqrt{h_b^2 + (u_b + s)^2}}\, ds.$$

For $s \geq 0$,

$$\frac{u_b + s}{h_b} > \frac{u_a + s}{h_a}.$$

Since $z/\sqrt{1+z^2}$ is strictly increasing and $d_b > d_a$,

$$BE - \ell_b > AD - \ell_a.$$

Therefore

$$AD - BE < \ell_a - \ell_b. \qquad \square$$

The remaining comparisons are classical inequalities of the original triangle.

Lemma 8.2. *If $a < b < c$, then*

$$\ell_a - \ell_b < AI - BI < b - a.$$

Proof. Let $s = (a + b + c)/2$. Standard half-angle formulas give

$$AI^2 = \frac{bc(s-a)}{s}, \qquad BI^2 = \frac{ca(s-b)}{s},$$

and

$$\ell_a = \frac{2s}{b+c} AI, \qquad \ell_b = \frac{2s}{a+c} BI.$$

Therefore

$$(AI - BI) - (\ell_a - \ell_b) = \frac{b}{a+c} BI - \frac{a}{b+c} AI.$$

Both terms are positive, so it suffices to compare their squares. Their difference is

$$\left(\frac{bBI}{a+c}\right)^2 - \left(\frac{aAI}{b+c}\right)^2 = \frac{abc(a-b)(a^2 - ab + b^2 - c^2)}{(a+c)^2(b+c)^2}.$$

Since $a < b < c$,

$$a - b < 0$$

and

$$a^2 - ab + b^2 - c^2 = a(a-b) + (b^2 - c^2) < 0.$$

Hence

$$\ell_a - \ell_b < AI - BI.$$

Finally,

$$AI^2 - BI^2 = c(b-a),$$

so

$$AI - BI = \frac{c(b-a)}{AI + BI}.$$

In triangle AIB,

$$AI + BI > AB = c,$$

and therefore

$$AI - BI < b - a. \qquad \square$$

Combining the preceding results gives two useful chains.

Corollary 8.3. *If $a < b < c$, then*

$$0 < PA - PB < AI - BI < b - a$$

and

$$0 < AD - BE < \ell_a - \ell_b < AI - BI < b - a.$$

Proof. The first chain follows from Theorems 3.1 and 6.1 together with Lemma 8.2. The second follows from Theorem 8.1 and the same lemma. □

Remark 8.4. The hypothesis $a < b < c$ in Theorem 8.1 is essential. For example, for

$$(a, b, c) = (3, 4, 2)$$

one finds

$$AD - BE = 1.07902\ldots, \qquad \ell_a - \ell_b = 0.97980\ldots, \qquad b - a = 1,$$

so both

$$AD - BE < \ell_a - \ell_b \qquad \text{and} \qquad AD - BE < b - a$$

fail.

9 Concluding remarks

The Abraham–Kovač characterization (1)–(3) gives a direct route to a collection of elementary inequalities associated with the electrostatic center.

For a scalene triangle with $a < b < c$, the vertex distances satisfy

$$PA > PB > PC,$$

with the quantitative comparison

$$0 < PA - PB < AI - BI < b - a.$$

The subtriangle perimeters have the same ordering as the side lengths, while the corresponding detours have the reverse ordering:

$$p_a < p_b < p_c, \qquad \delta_a > \delta_b > \delta_c.$$

Each electrostatic cevian lies strictly between the corresponding median and internal angle bisector, and the three cevian lengths satisfy

$$AD > BE > CF.$$

Finally,

$$0 < AD - BE < \ell_a - \ell_b < AI - BI < b - a.$$

These results require no explicit closed form for the barycentric coordinates of $X(5626)$. They arise instead from the three distance-sum equations of Abraham and Kovač, together with elementary monotonicity and Euclidean geometry.

Acknowledgment

The author gratefully acknowledges substantial assistance from ChatGPT, an AI system developed by OpenAI, in exploring conjectures, carrying out symbolic and numerical checks, developing proofs, and organizing and editing the presentation.

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