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Catalog of Properties of the Ellipse – Part 1

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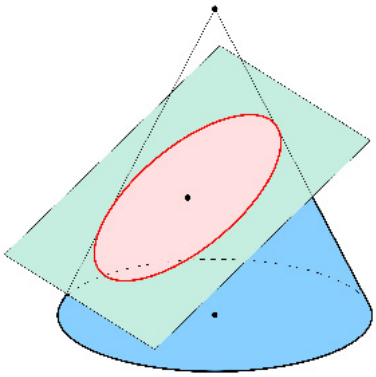
Abstract. This paper surveys known results about the ellipse and gives additional properties that were discovered by computer.

Keywords. ellipse, conic, computer-discovered mathematics, GeometricExplorer.

Mathematics Subject Classification (2020). 51M04, 51-08.

Introduction

If an inclined plane intersects a right circular cone so that the cross section of the intersection is a bounded closed curve, that curve is known as an *ellipse*.



An ellipse can also be defined as the locus of a point such that the sum of the distances from that point to two given points (the *foci*) is a constant.

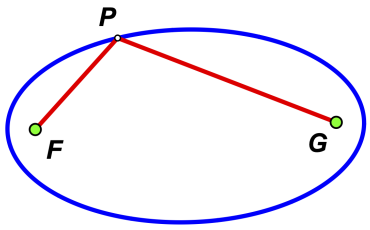


Figure 1. $FP + GP =$ constant

An ellipse may also be defined in terms of one focal point and a line outside the ellipse called the *directrix*. For all points on the ellipse, the ratio between the distance to the focus and the distance to the directrix is a constant, called the *eccentricity*.

Classification Scheme

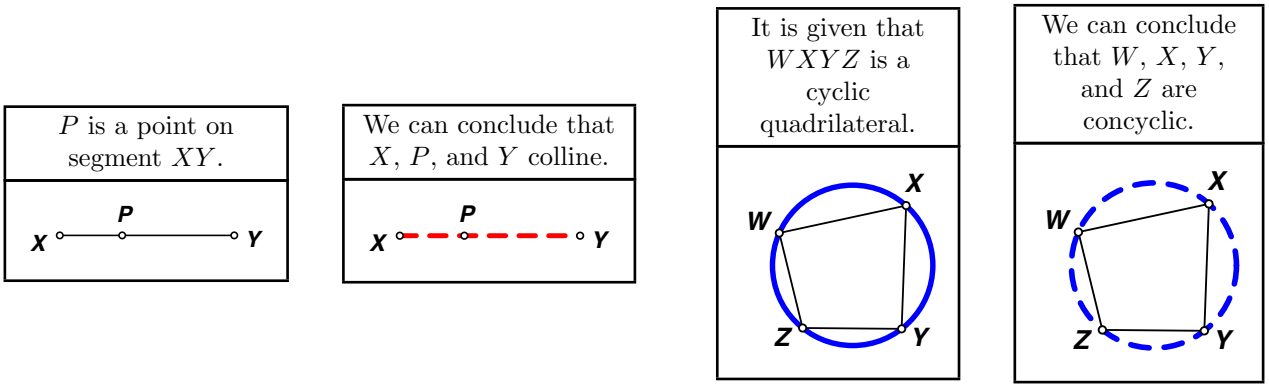
CONTENTS

Scope of this catalog. The mathematical literature is vast. We do not attempt to catalog every property involving an ellipse that appears somewhere in print or on the internet. We do try to catalog any property that is simple or elegant or that can be obtained from the configuration associated with one of our top-level classifications (Ellipse plus point, Ellipse and chord, Ellipse and two tangents, etc.) by applying at most one common geometrical construction (drop a perpendicular, draw an angle bisector, construct a centroid, etc.). Only noncircular ellipses are considered. When analyzing triangle centers, we only look at the common ones, X_1 through X_{20}.

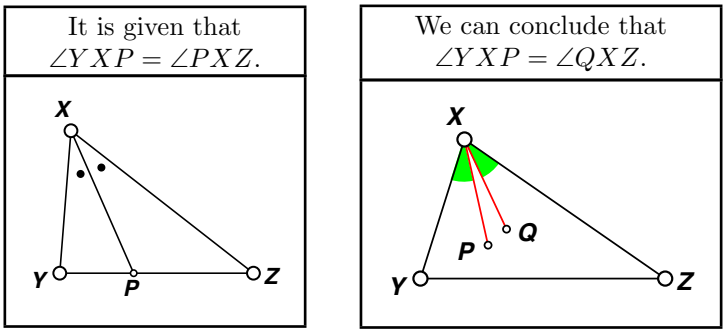
A follow-up paper will cover properties involving circles tangent to ellipses.

Figures. The figures in this paper are *not* decorative. In order to reduce verbiage and declutter the catalog, given information appearing in the figure is not repeated in words. For example, if a figure shows line segments WX and YZ meeting at a point P, we do not state in words that P is the intersection of WX and YZ. This should make it easier for people who don't read English to use this catalog.

A solid line or circle through multiple points means that it is given that these points lie on the same line or circle. A dashed line or circle through multiple points means that the conclusion of the theorem or result is that these points lie on the same line or circle.



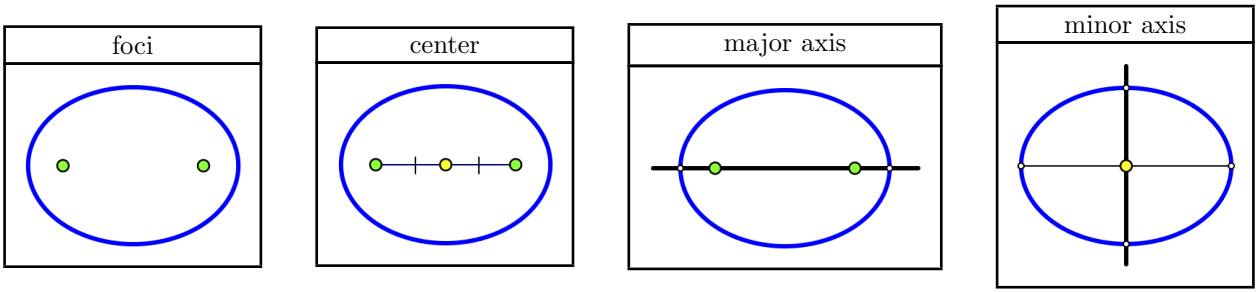
Angles that are given to be equal are marked with the same filled circle. Angles that are concluded to be equal are shaded with the same color.



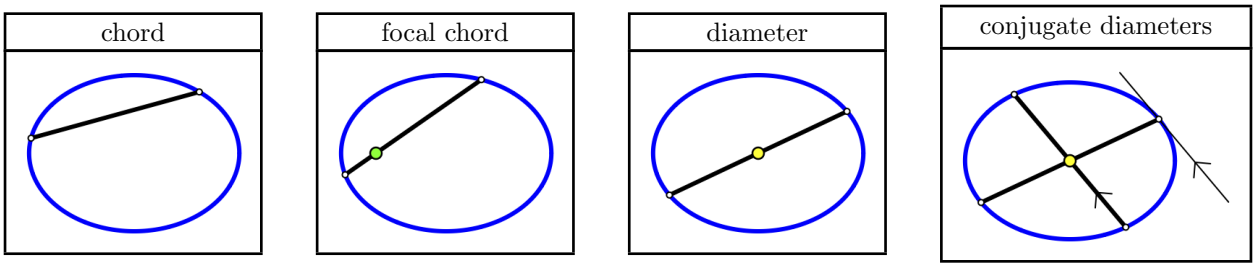
If the title of a section or subsection describes a feature of a figure, then we do not repeat this description if it is obvious from the figure. For example, in a subsection entitled "equilateral triangles", a triangle highlighted in yellow that looks equilateral can be assumed to be an equilateral triangle.

A right-angle marker is used to indicate two lines that are given to be perpendicular. All angles are directed angles. Given information that is not obvious from the associated figure is shown in brown text directly beneath the figure.

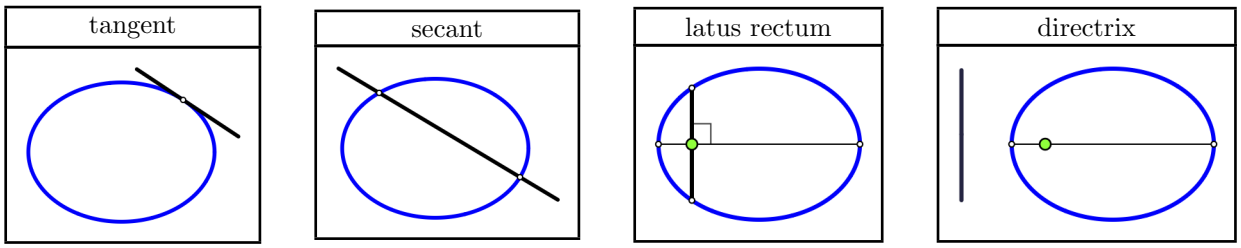
Parts of an ellipse. The *foci* (singular *focus*) of an ellipse were defined above. A noncircular ellipse always has two foci. They will always be shown as green dots. The midpoint of the line joining the foci is called the *center* of the ellipse and will always be shown as a yellow dot. The line joining the two foci is called the *major axis* of the ellipse. The line perpendicular to the major axis at the center of the ellipse is called the *minor axis* of the ellipse.



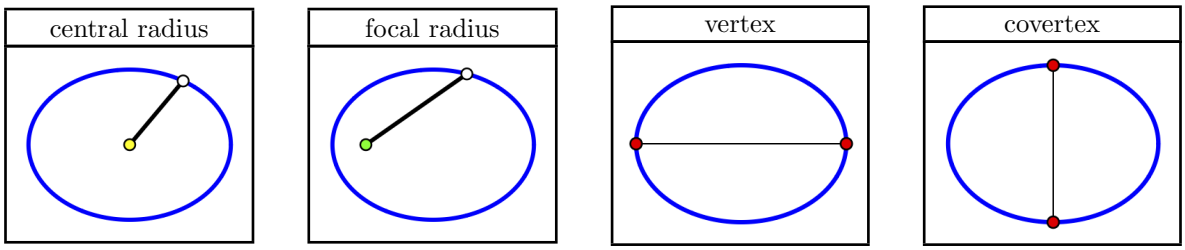
The line segment joining two points on an ellipse is called a *chord*. A chord through a focus is called a *focal chord*. A chord through the center is called a *diameter*. A diameter parallel to the tangent at one endpoint of a given diameter is called the *conjugate diameter* to that diameter.



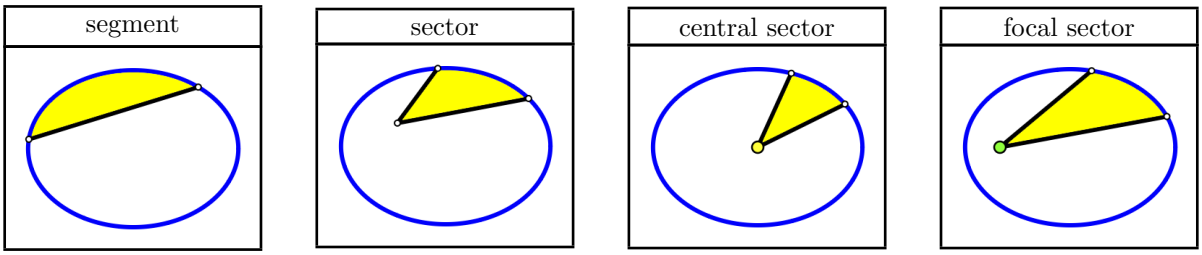
A line that meets an ellipse at just one point is called a *tangent* to the ellipse. The point is called a *touch point*. A line that meets an ellipse at two points is called a *secant*. The chord that is perpendicular to the major axis at a focus is called a *latus rectum* of the ellipse. An ellipse has two *latera recta*. The *directrix* of an ellipse was defined above. An ellipse has two *directrices*, one associated with each focus.



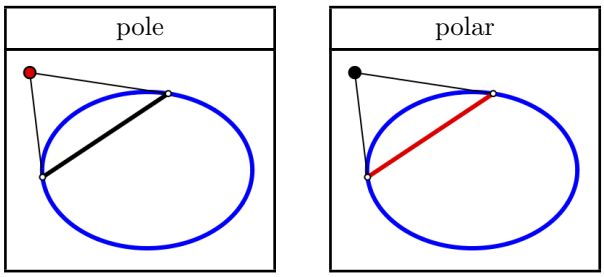
A line segment from the center of an ellipse to a point on the boundary is called a *central radius*. A line segment from a focus to a point on the ellipse is called a *focal radius*. The points where the major axis of an ellipse meets the ellipse are called *vertices* (singular *vertex*). The points where the minor axis of an ellipse meets the ellipse are called *covertices* (singular *covertex*) or *conjugate vertices*.



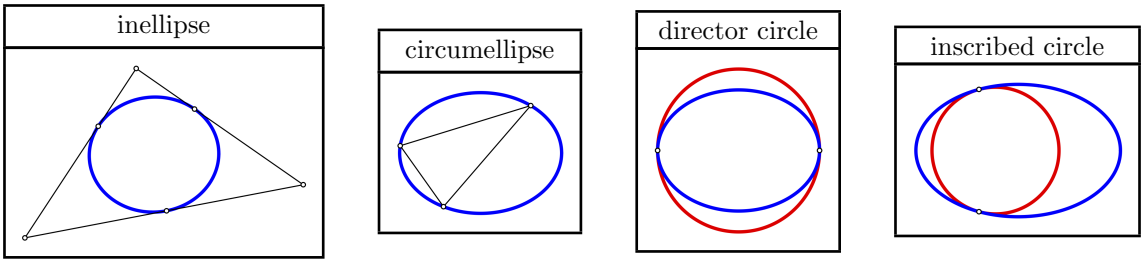
The region bounded by a chord of an ellipse and the ellipse is called a *segment*. If the chord is parallel to the major axis, the segment is called a *horizontal segment*. If the chord is parallel to the minor axis, the segment is called a *vertical segment*. If two line segments are drawn from a given point inside an ellipse to two points on the boundary, the region bounded by those two line segments and the ellipse is called a *sector*. The given point is the *vertex* of the sector. If the vertex of a sector is at the center of the ellipse, the sector is called a *central sector*. If the vertex of a sector is at a focus of the ellipse, the sector is called a *focal sector*.



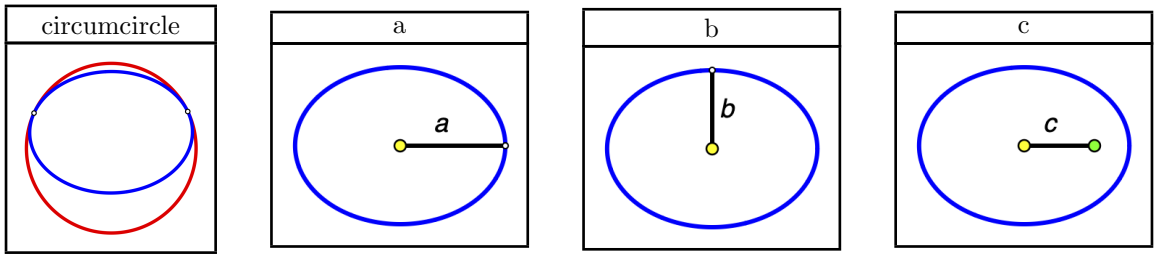
Suppose a given line meets an ellipse at two points. The point of intersection of the tangents to the ellipse at these two points is called the *pole* of the given line. If a given point is outside an ellipse, there are two tangents from that point to the ellipse. The line joining the two touch points of these tangents is called the *polar* of the given point. The definitions of pole and polar can be made more general, but these definitions suffice for this catalog.



An ellipse that is tangent to each side of a polygon is called an *inellipse*. An ellipse that passes through each vertex of a polygon is called a *circumellipse*. A circle that is inside an ellipse and tangent to it at two points is known as an *inscribed circle* or an *incircle*. A circle that is outside an ellipse and tangent to it at two points is known as a *circumscribed circle* or a *circumcircle*. If the points of tangency are the vertices of the ellipse, the circle is known as the *director circle*.



There are several parameters associated with an ellipse. The distance from the center of an ellipse to a vertex is called the *semi-major axis* and is denoted by the letter a. The distance from the center of an ellipse to a covertex is called the *semi-minor axis* and is denoted by the letter b. The distance from the center of an ellipse to a focus is called the *focal distance* and is denoted by the letter c. The ratio c/a is called the *eccentricity* of the ellipse and is denoted by the letter e.



Discoveries. An asterisk after a property number indicates that the property was discovered by computer, either by using GeometricExplorer, Mathematica, or Geometer's Sketchpad. If a reference is given, this means that the result was posted to an online forum in the hope that some forum member might find a geometrical proof of the property.

Arrangement of the catalog.

The catalog is sorted by *starting figures*, such as "Ellipse with a focal chord", "Two tangents drawn to an ellipse from an external point", "A circle tangent to an ellipse", "An ellipse inscribed in a quadrilateral", etc. First we list the intrinsic properties of that starting. An *intrinsic property* of a figure is a property that that figure has without adding any additional constructions (such as finding the midpoint of a line segment, dropping a perpendicular from a point to a line, drawing a tangent to an ellipse, etc.). However, for each ellipse in the starting figure, we may include features of that ellipse when listing a property of that starting figure. The features that can be added are:

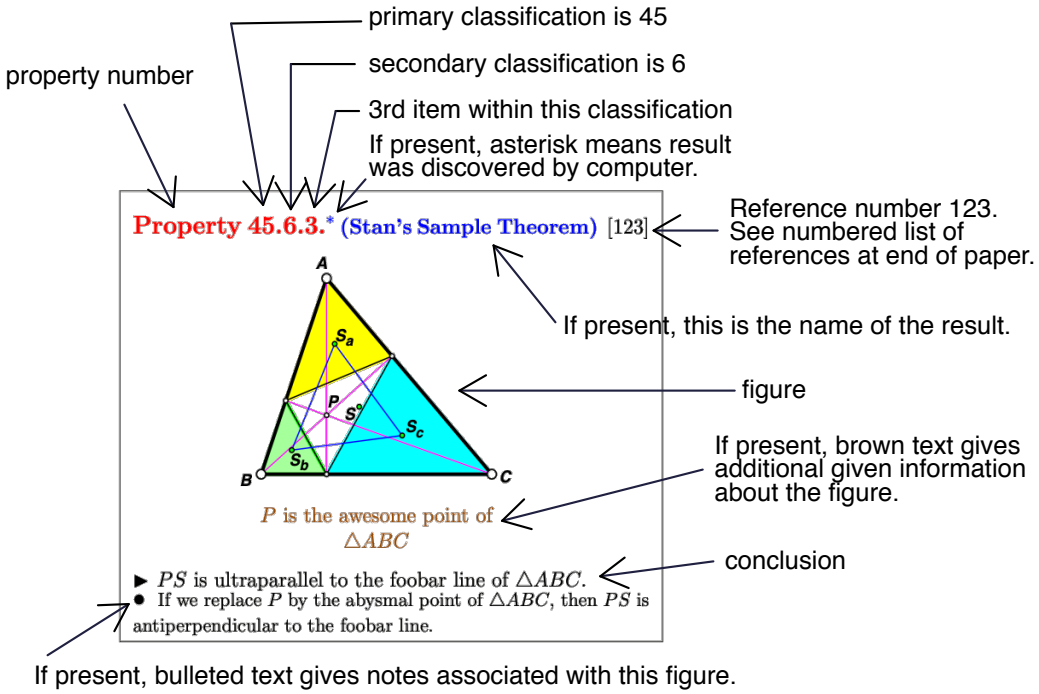
(1) the center
(2) the foci
(3) the major axis
(4) the minor axis
(5) the vertices
(6) the covertices
(7) A line, ray, or line segment joining two points in the starting figure
(8) The point of intersection of two line segments that are in the starting figure.

Features that cannot be added are new constructions, including drawing a line between two points that are not present in the starting figure.

Notation.

Notation used when describing properties	
Notation	**Description**
a	Distance from the center of the ellipse to a vertex
b	Distance from the center of the ellipse to a covertex
c	Distance from the center of the ellipse to a focus
e	Eccentricity of the ellipse (equal to c/a)
$\triangle XYZ$	Triangle XYZ
$\odot XYZ$	The circle through points X, Y, and Z
$[F]$	The area of figure F
ϕ	The golden ratio, $(1+\sqrt{5})/2$
$\sum$red	The sum of the lengths of the red segments
$\prod$red	The product of the lengths of the red segments
$X-Y-Z$	The points X, Y, and Z are collinear.
$\angle XYZ$	Directed angle XYZ. This is the angle through which ray $\overrightarrow{YX}$ must be rotated counterclockwise in order to coincide with ray $\overrightarrow{YZ}$.
X_n	The nth Kimberling center of $\triangle ABC$ (see [17])
$X_n(XYZ)$	The nth Kimberling center of $\triangle XYZ$

Key to the property listings.



Special References.

A reference of the form [Ako m.n] refers to Akopyan's result m.n in [2].

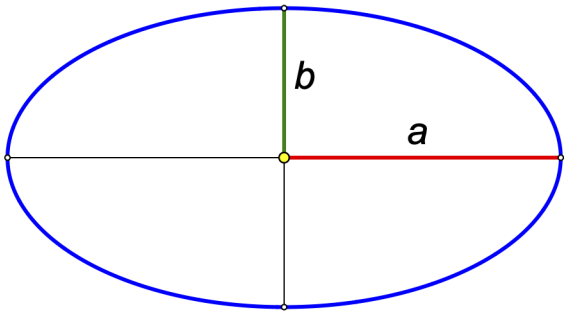
A reference of the form [Apo n] refers to Apollonius' result n in [14].

Basic Properties of an Ellipse

1. Ellipse

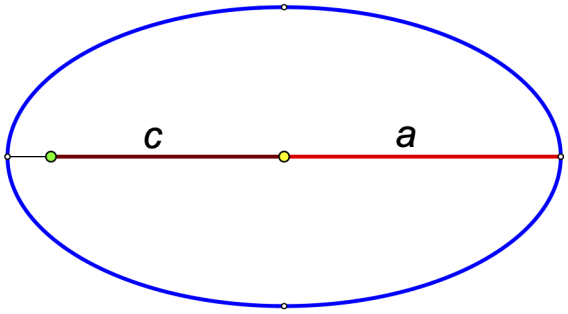
1.1 lengths

Property 1.1.1.



▶ $a > b$

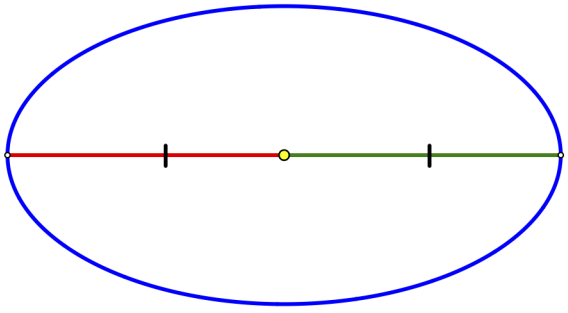
Property 1.1.2. (Eccentricity of an Ellipse)



▶ $e < 1$

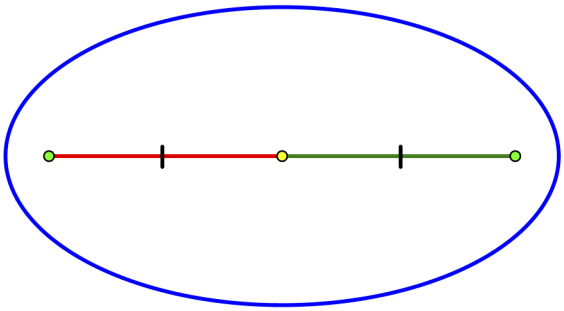
- In other words, $c < a$, since $e = c/a$

Property 1.1.3. (Central Symmetry)



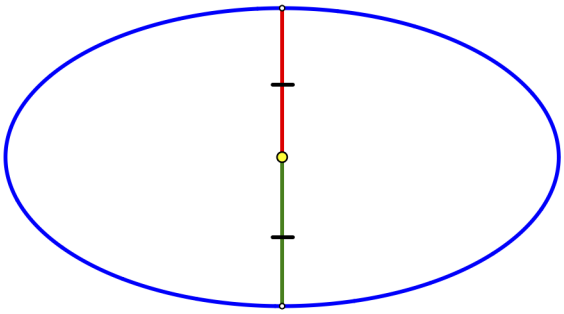
▶ red length = green length

Property 1.1.4. (Focal Symmetry)



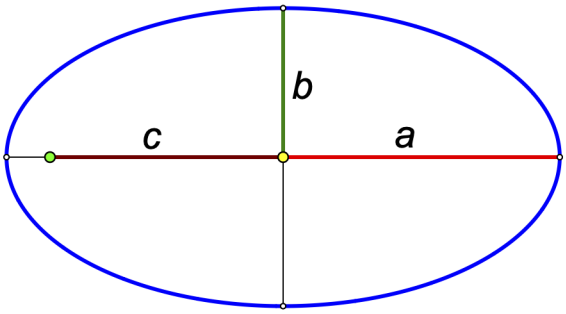
▶ red length = green length

Property 1.1.5. (Central Symmetry)



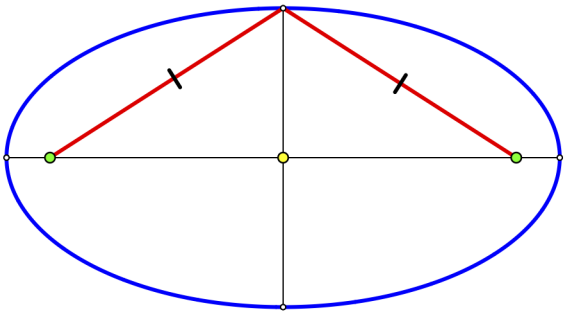
▶ red length = green length

Property 1.1.6. (Relationship Between a, b, and c)



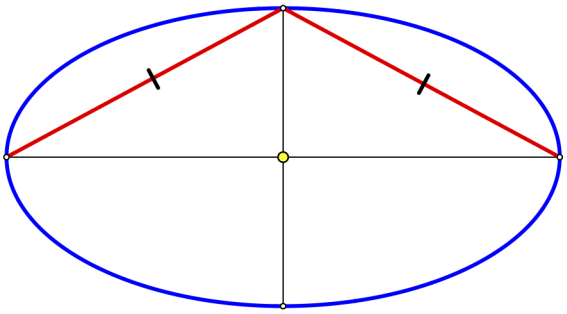
▶ $a^2 = b^2 + c^2$

Property 1.1.7. (Vertical Symmetry)



▶ red lengths are equal

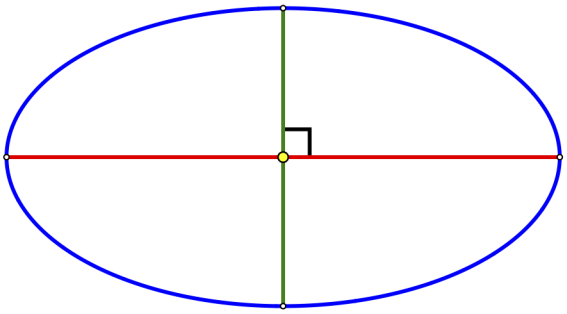
Property 1.1.8. (Vertical Symmetry)



▶ red lengths are equal

1.2 lines

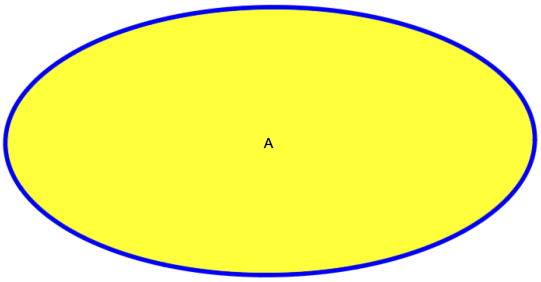
Property 1.2.1.



▶ axes are perpendicular

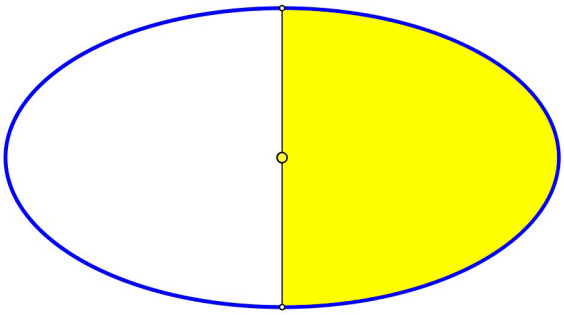
1.3 area

Property 1.3.1. (Area Formula)



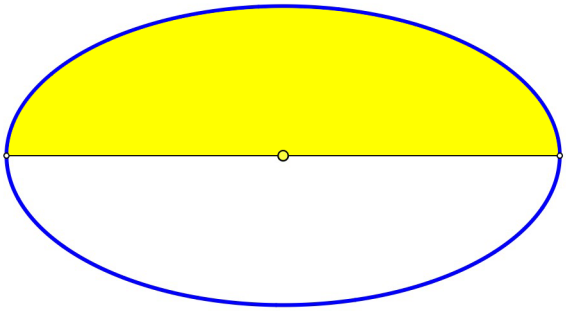
▶ $A = \pi ab$

Property 1.3.2. (Area Formula)



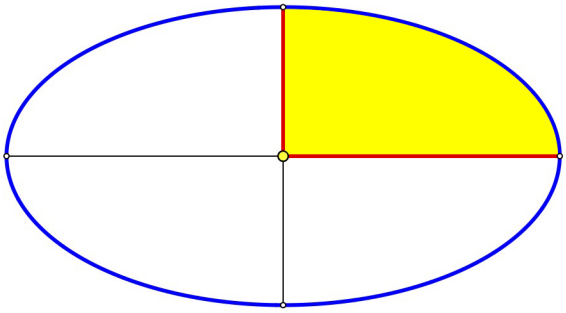
▶ $A = \pi ab/2$

Property 1.3.3. (Area Formula)



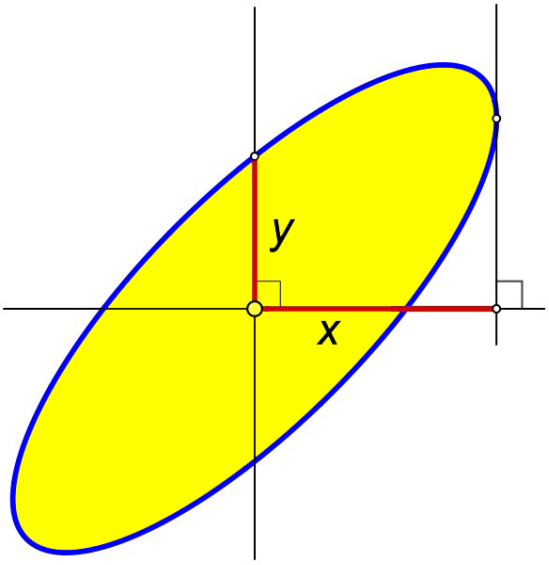
▶ $A = \pi ab/2$

Property 1.3.4. (Area Formula)



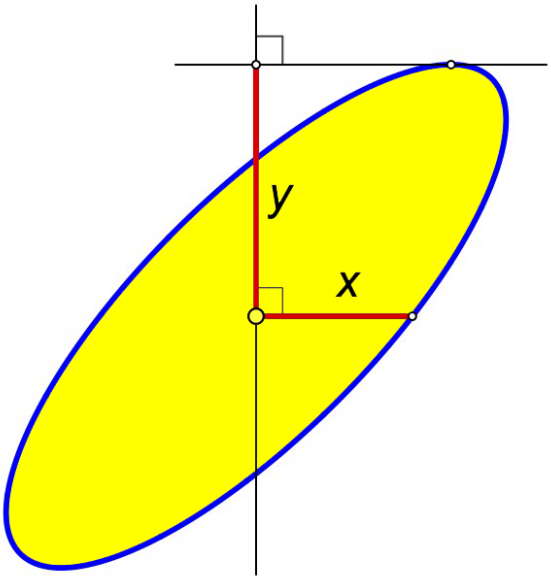
▶ $A = \pi ab/4$

Property 1.3.5. (Tilted Ellipse Area) [65]



▶ $A = \pi xy$

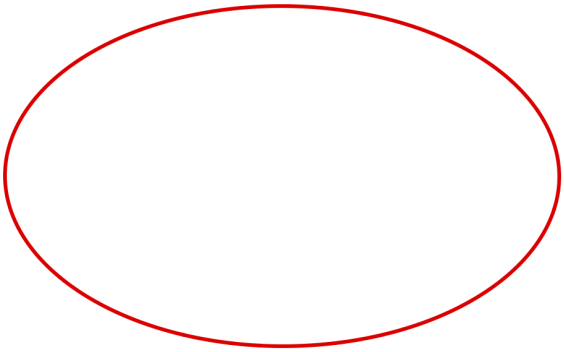
Property 1.3.6. (Tilted Ellipse Area) [65]



▶ $A = \pi xy$

1.4 perimeter

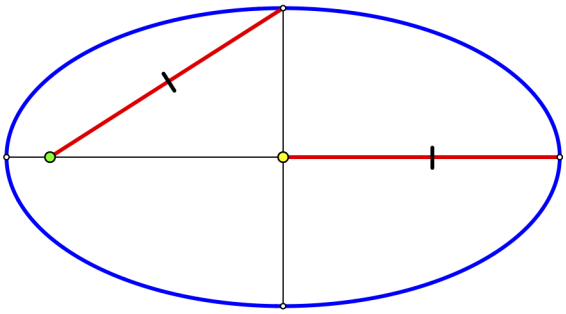
Property 1.4.1.* (Perimeter Formula)



▶ red length $= \int_0^{2\pi} \sqrt{1 - e^2 \sin^2\theta}\, d\theta$

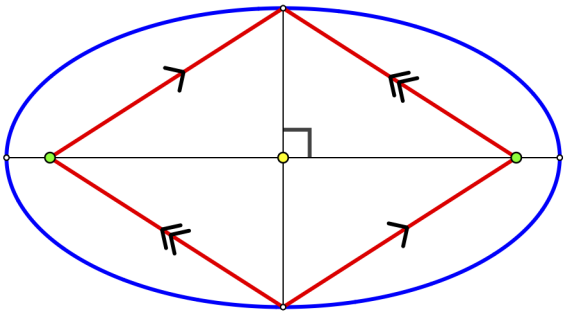
2. Ellipse and Axes

Property 2.0.1.



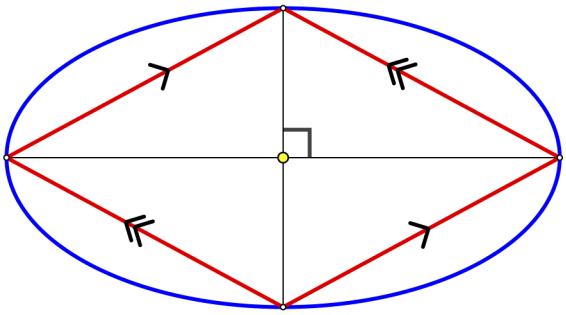
▶ red lengths are equal

Property 2.0.2.



▶ red figure is a rhombus

Property 2.0.3.

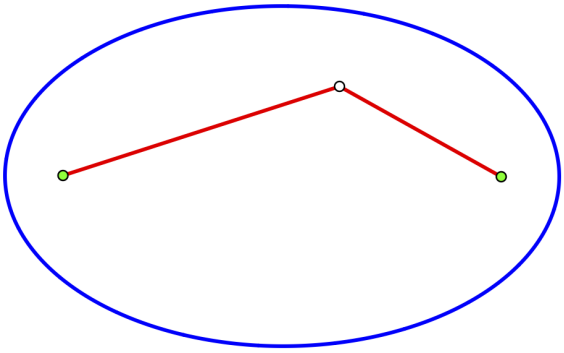


▶ red figure is a rhombus

3. Ellipse and Point

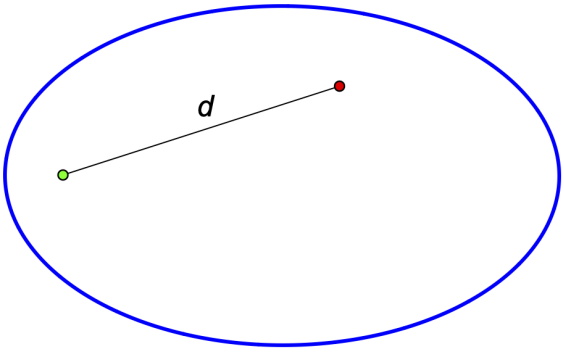
3.1 point inside ellipse

Property 3.1.1. [9, p. 35]



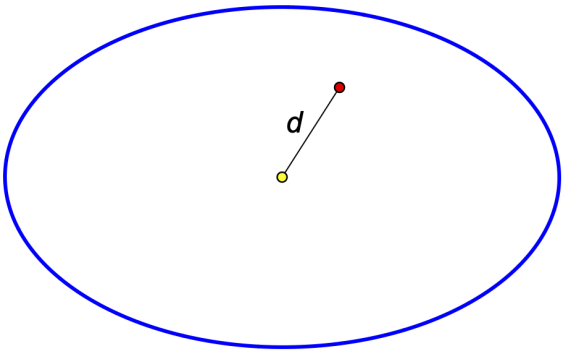
▶ $\sum \text{red} < 2a$

Property 3.1.2.



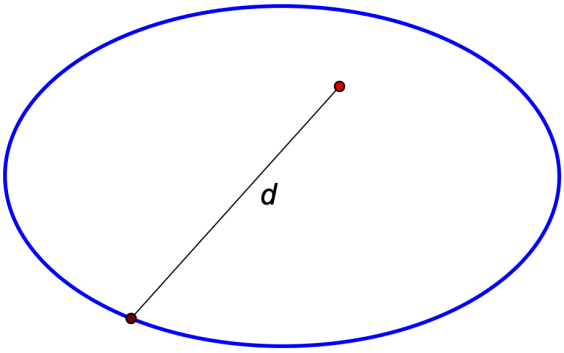
▶ $d < a + c$

Property 3.1.3.



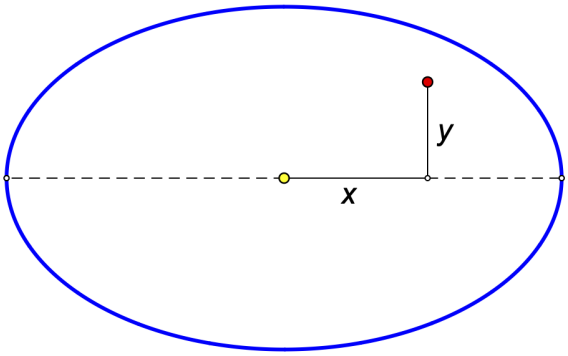
▶ $d < a$

Property 3.1.4.



▶ $d < 2a$

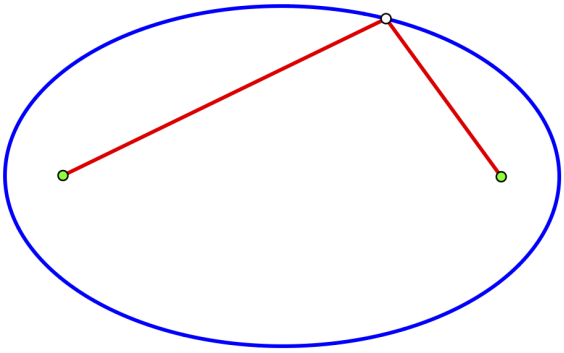
Property 3.1.5.



▶ $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$

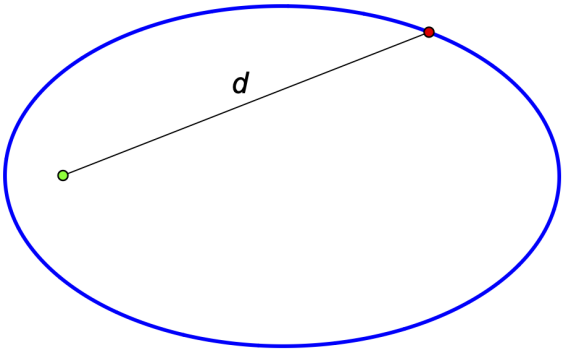
3.2 point on ellipse

Property 3.2.1. (Focal Sum) [Apo 73]



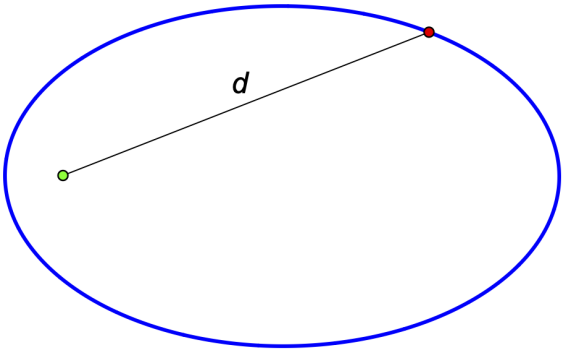
▶ $\sum \text{red} = 2a$

Property 3.2.2. (Apoapsis Inequality)



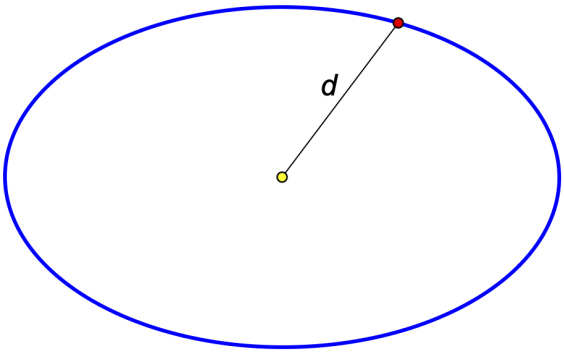
▶ $d \leq a + c$

Property 3.2.3. (Periapsis Inequality)



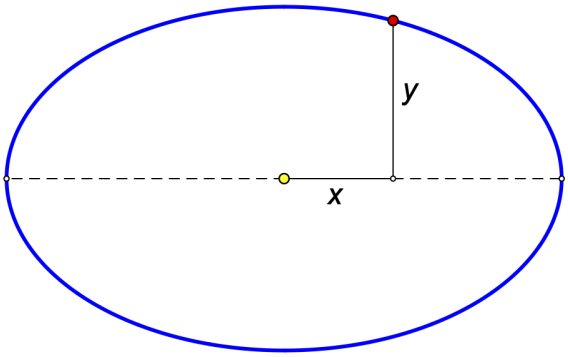
▶ $d \geq a - c$

Property 3.2.4. (Central Radius Bounds)



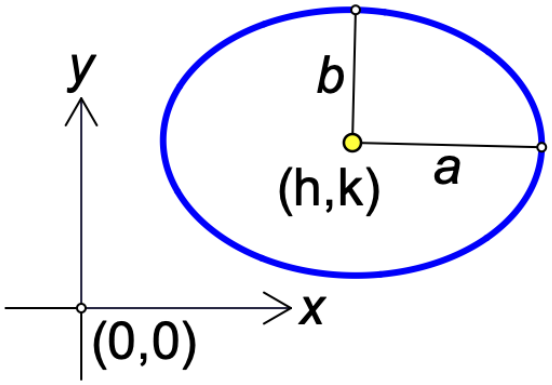
▶ $b \leq d \leq a$

Property 3.2.5. (Standard Form)



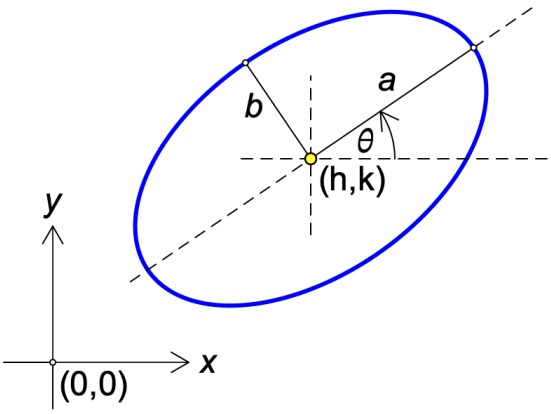
▶ $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Property 3.2.6. (Standard Offset Form)



▶ $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

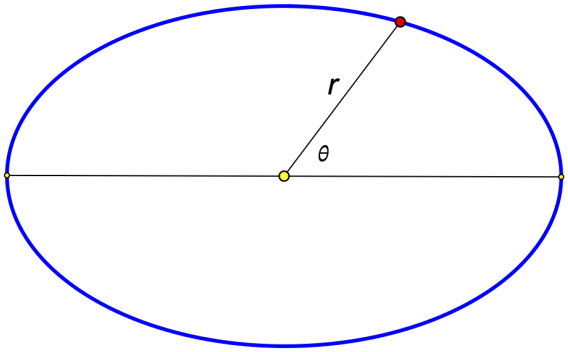
Property 3.2.7. (General Form) [65]



▶ $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

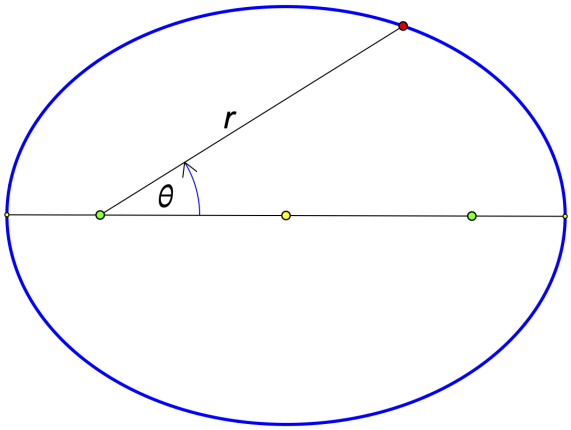
- $A = a^2 \sin^2\theta + b^2 \cos^2\theta$
- $B = (b^2 - a^2) \sin 2\theta$
- $C = a^2 \cos^2\theta + b^2 \sin^2\theta$
- $D = -2Ah - Bk$
- $E = -Bh - 2Ck$
- $F = Ah^2 + Bhk + Ck^2 - a^2b^2$
- $h = (2CD - BE)/\Delta$
- $k = (2AE - BD)/\Delta$
- $\theta = \frac{1}{2}\text{atan2}(-B, C - a)$
- where discriminant $\Delta = B^2 - 4AC < 0$
- and $C \begin{vmatrix} A & B/2 & D/2 \\ B/2 & C/2 & E/2 \\ D/2 & E/2 & F \end{vmatrix} < 0.$

Property 3.2.8. (Central Polar Form)



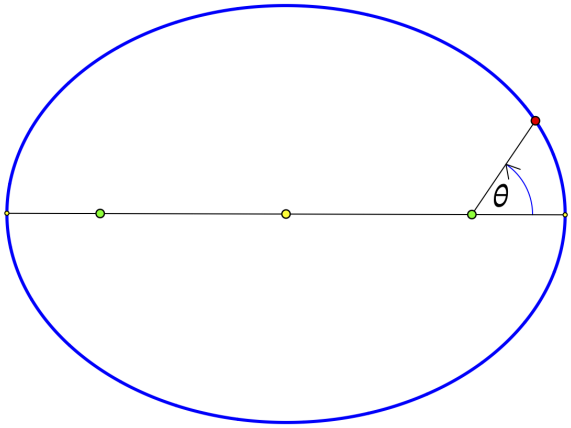
▶ $r = \dfrac{ab}{\sqrt{b^2 \cos^2\theta + a^2 \sin^2\theta}}$

Property 3.2.9. (Focal Polar Form)



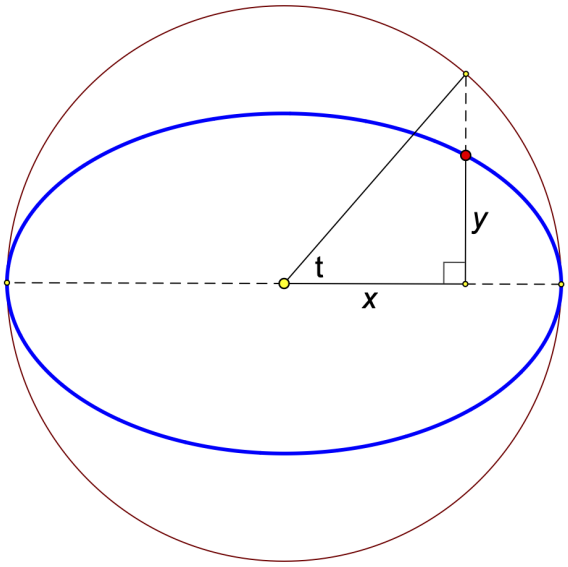
▶ $r = \dfrac{a(1 - e^2)}{1 - e\cos\theta}$

Property 3.2.10. (Focal Polar Form)



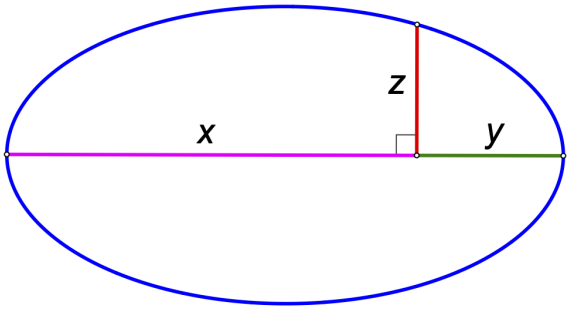
▶ $r = \dfrac{a(1 - e^2)}{1 + e\cos\theta}$

Property 3.2.11. (Parametric Form)



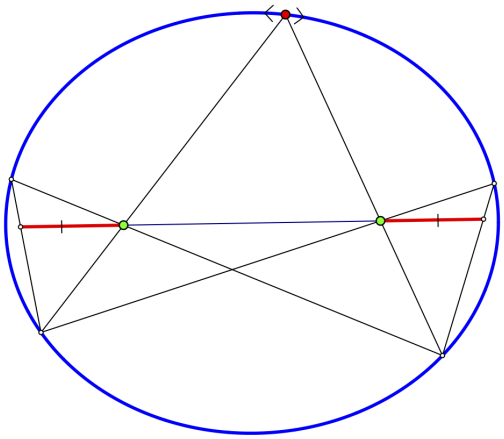
▶
$x = a\cos t$
$y = b\sin t$

Property 3.2.12. [Apo 8]



▶ invariant: $\dfrac{z^2}{xy} = \dfrac{b^2}{a^2}$

Property 3.2.13. (Haruki's Lemma) [50]

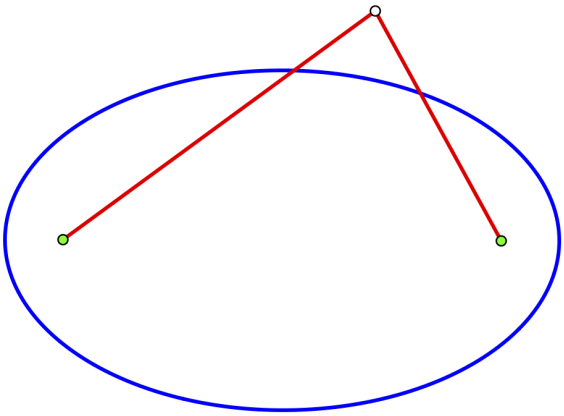


▶ red lengths are equal

● invariant: red length $= \dfrac{2b^2c}{4a^2 - 3b^2}$ [6]

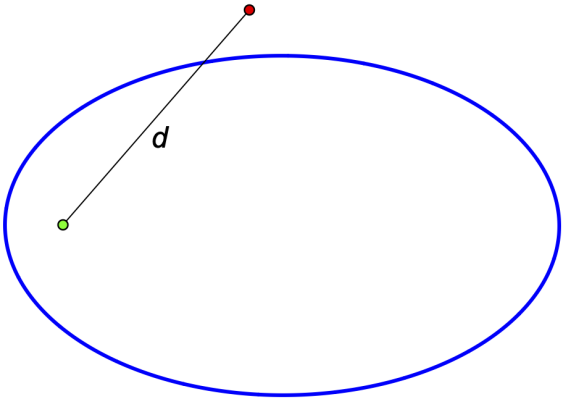
3.3 point outside ellipse

Property 3.3.1. [9, p. 35]



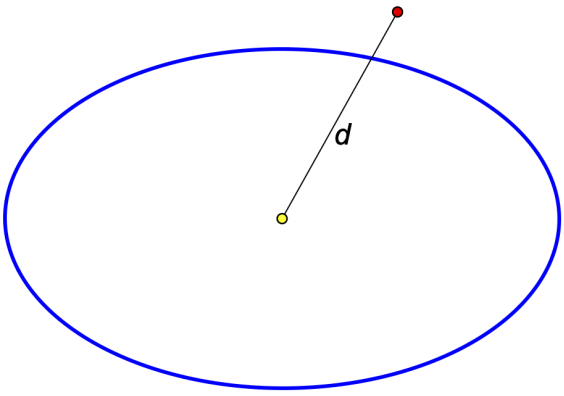
▶ $\sum \text{red} > 2a$

Property 3.3.2.



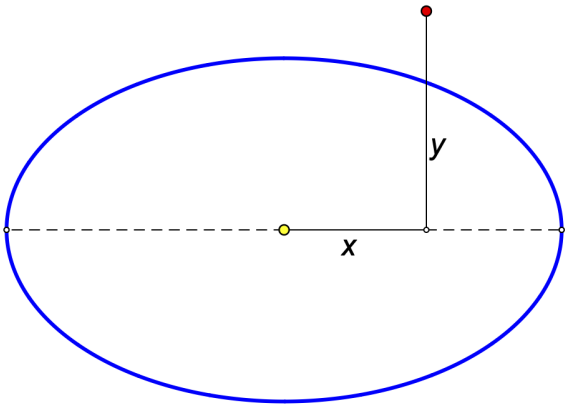
▶ $d > a - c$

Property 3.3.3.



▶ $d > b$

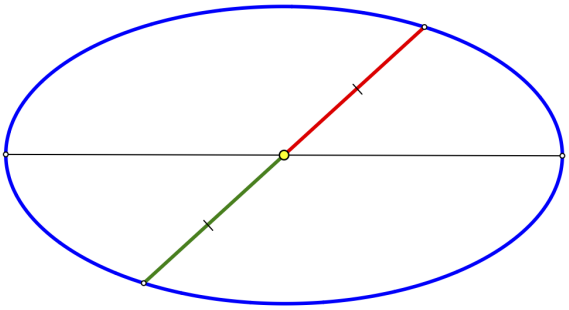
Property 3.3.4.



▶ $\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1$

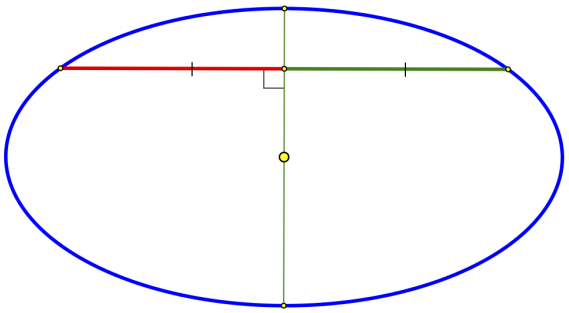
4. Ellipse and Chord

Property 4.0.1. (Central Symmetry) [Apo 10]



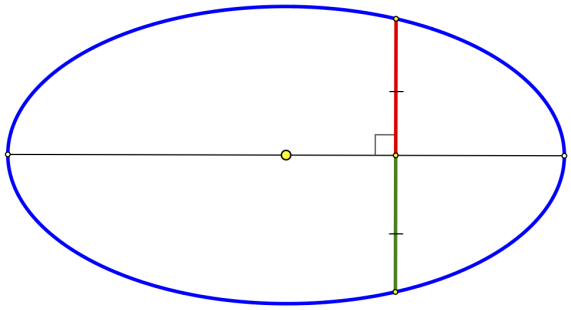
▶ red length = green length

Property 4.0.2. (Horizontal Symmetry)



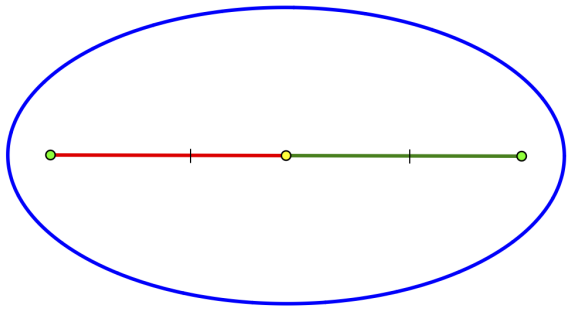
▶ red length = green length

Property 4.0.3. (Vertical Symmetry)



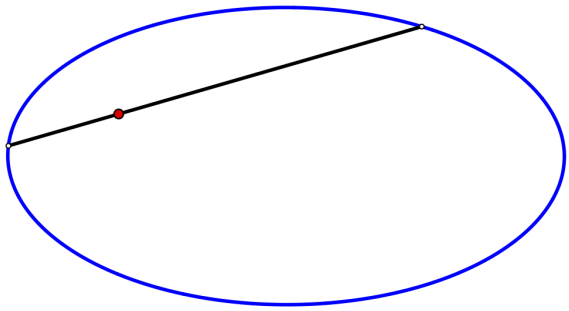
▶ red length = green length

Property 4.0.4. (Focal Symmetry)



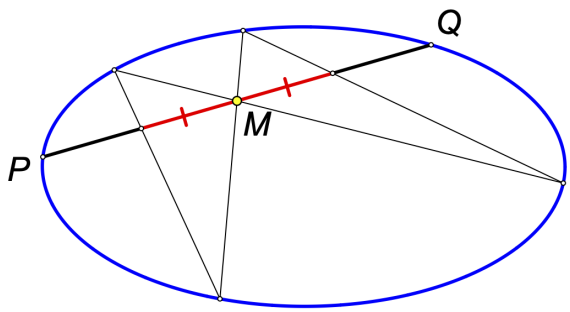
▶ red length = green length

Property 4.0.5. (Convexity Property)



▶ red point lies inside the ellipse

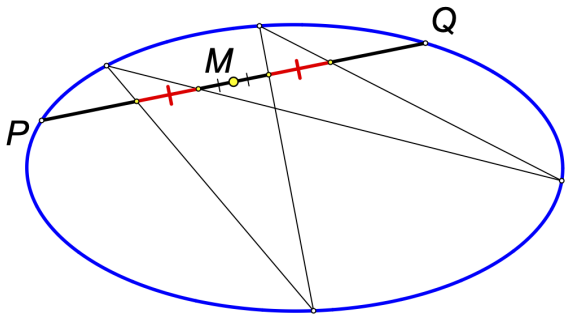
Property 4.0.6. (ButterflyTheorem)



M is the midpoint of PQ

▶ red lengths are equal

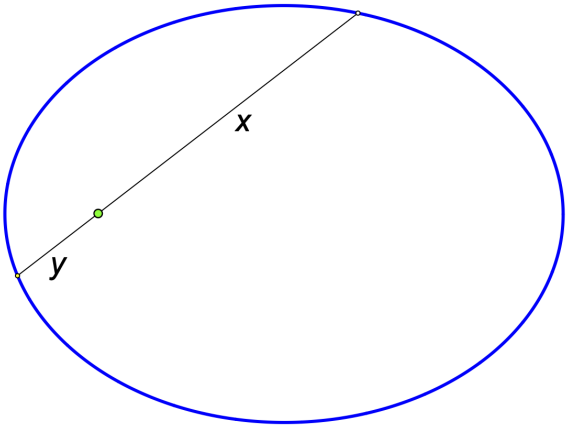
Property 4.0.7. (Generalized ButterflyTheorem)



M is the midpoint of PQ

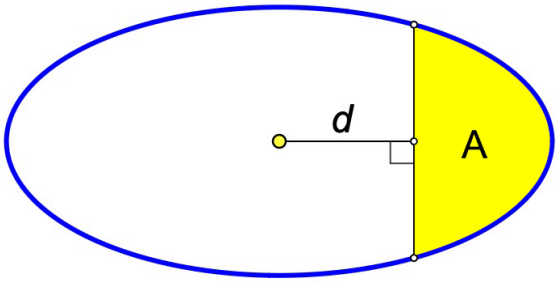
▶ red lengths are equal

Property 4.0.8.



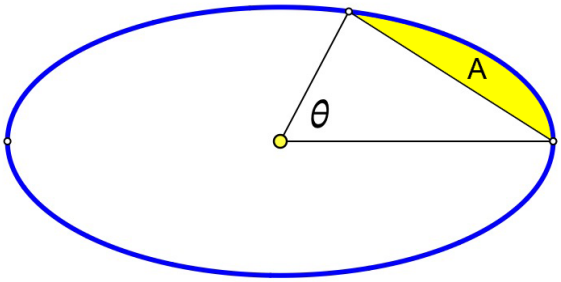
▶ invariant: $\frac{1}{x}+\frac{1}{y}=\frac{2a}{b^2}$

Property 4.0.9. (Area of Vertical Segment)



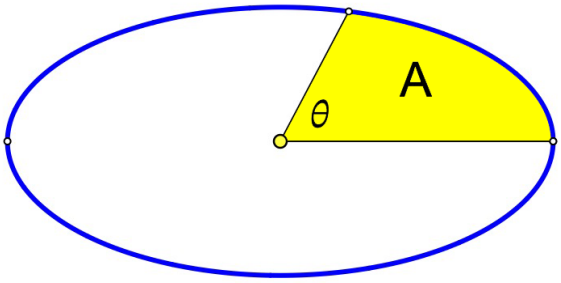
▶ $A = ab\cos^{-1}\left(\frac{d}{a}\right) - bd\sqrt{1-\frac{d^2}{a^2}}$

Property 4.0.10. (Area of Edge Segment)



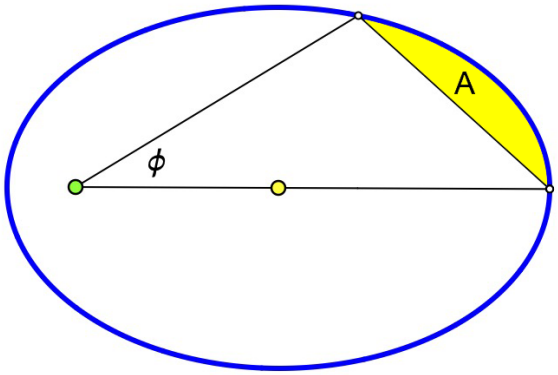
▶ $A = \frac{1}{2}ab\left(\cos^{-1}\left(\frac{b}{x}\right) - \frac{a\tan\theta}{x}\right)$
where $x = \sqrt{b^2+a^2\tan^2\theta}$

Property 4.0.11. (Area of Central Sector)



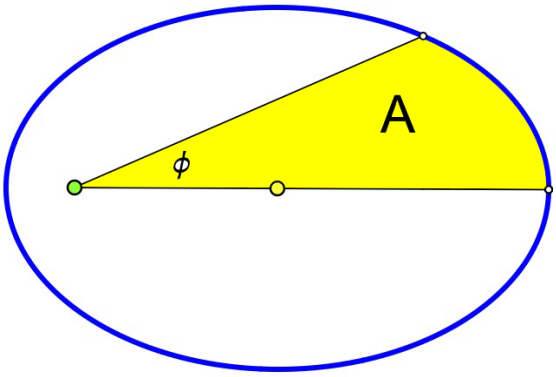
▶ $A = \frac{1}{2}ab\sqrt{1-\frac{b^2}{x^2}} + \frac{1}{2}ab\left(\cos^{-1}\left(\frac{b}{x}\right) - \frac{a\tan\theta}{x}\right)$
where $x = \sqrt{b^2+a^2\tan^2\theta}$

Property 4.0.12. (Area of Focal Segment)



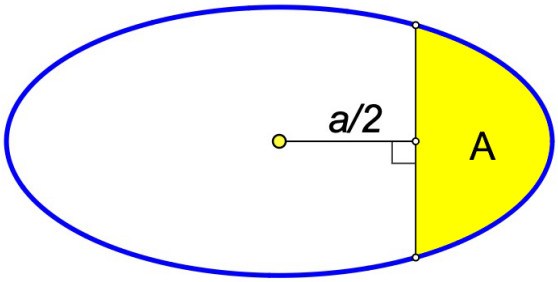
▶ $A = \frac{1}{2}ab\left(\cos^{-1}\psi - \frac{b\tan(\phi)(a\sec(\phi)+c)}{a^2\tan^2(\phi)+b^2}\right)$
where $\psi = \frac{b^2\sec(\phi) - ea^2\tan^2(\phi)}{a^2\tan^2(\phi)+b^2}$ and $e = \frac{c}{a}$

Property 4.0.13. (Area of Focal Sector)



▶ $A = \frac{1}{2}ab\left(\cos^{-1}\psi + e\frac{b\tan(\phi)(a\sec(\phi)+c)}{a^2\tan^2(\phi)+b^2}\right)$
where $\psi = \frac{b^2\sec(\phi) - ea^2\tan^2(\phi)}{a^2\tan^2(\phi)+b^2}$ and $e = \frac{c}{a}$

Property 4.0.14.

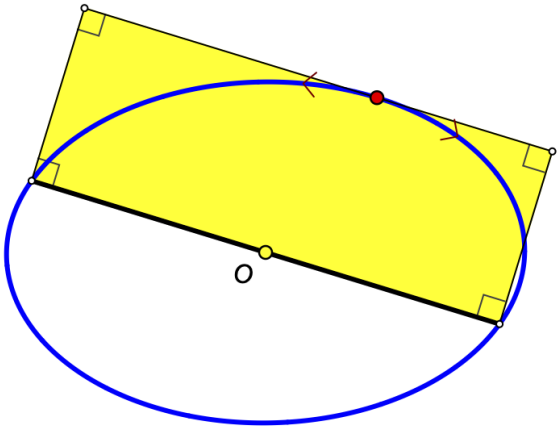


▶ $A = ab\left(3\sqrt{3} - 4\pi\right)/12$

- Special case of Property 4.0.9

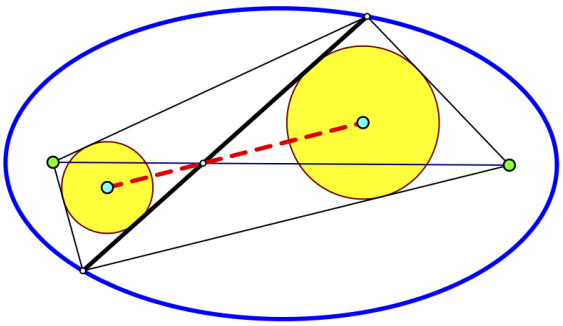
4.1 diameter

Property 4.1.1.



▶ Invariant: yellow area= $2ab$

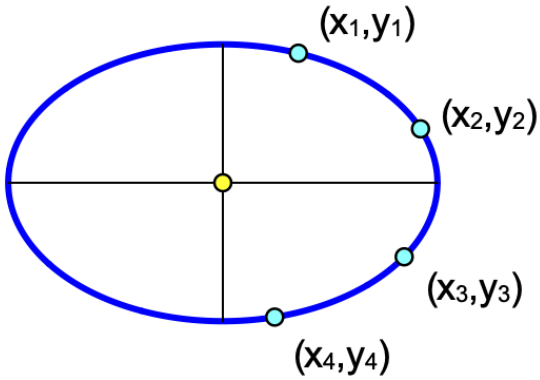
Property 4.1.2. [Ako 11.32]



▶ points are collinear

5. Ellipse and Four Boundary Points

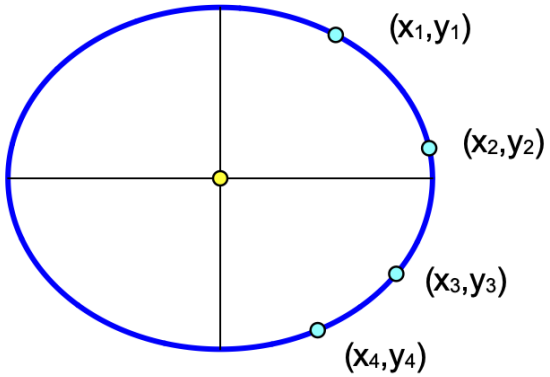
Property 5.0.1. (Four-Point Form) [62]



axes are parallel to coordinate axes

▶ $\begin{vmatrix} x^2 & y^2 & x & y & 1 \\ x_1^2 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0$

Property 5.0.2. (Inscribed Angle Theorem) [65]

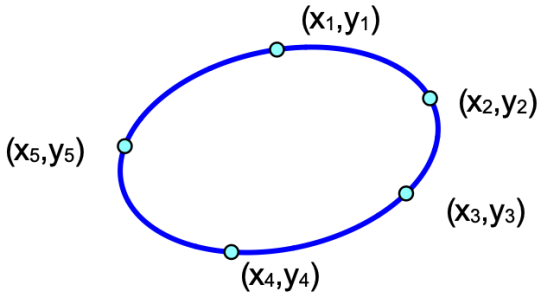


axes are parallel to coordinate axes

▶ $\dfrac{(x_4-x_1)(x_4-x_2)+q(y_4-y_1)(y_4-y_2)}{(y_4-y_1)(x_4-x_2)-(y_4-y_2)(x_4-x_1)} = \dfrac{(x_3-x_1)(x_3-x_2)+q(y_3-y_1)(y_3-y_2)}{(y_3-y_1)(x_3-x_2)-(y_3-y_2)(x_3-x_1)}$
where $q = a^2/b^2$

6. Ellipse and Five Boundary Points

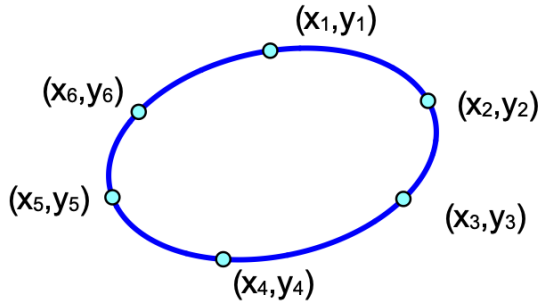
Property 6.0.1. (Five-Point Form)



▶ $\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$

7. Ellipse and Six Boundary Points

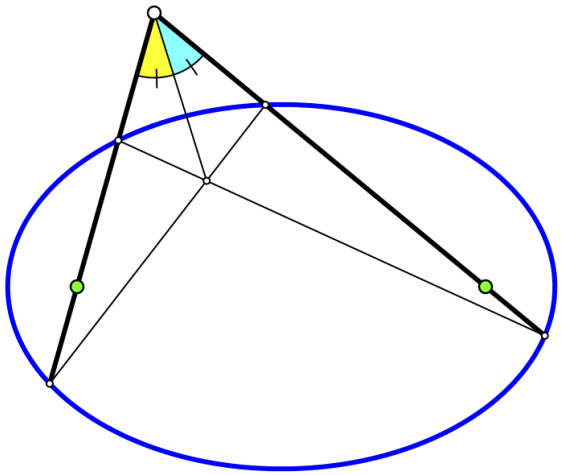
Property 7.0.1.



▶ $\begin{vmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \\ x_6^2 & x_6y_6 & y_6^2 & x_6 & y_6 & 1 \end{vmatrix} = 0$

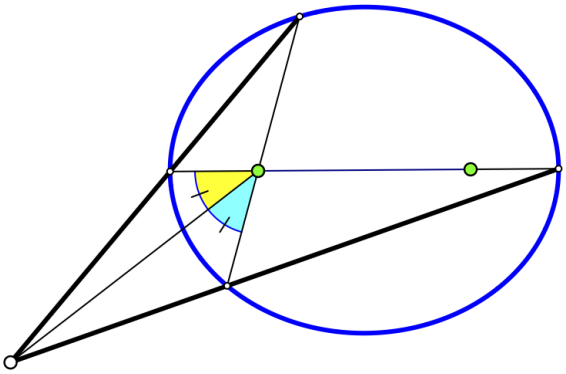
8. Ellipse and Two Secants

Property 8.0.1. [Ako 11.15]



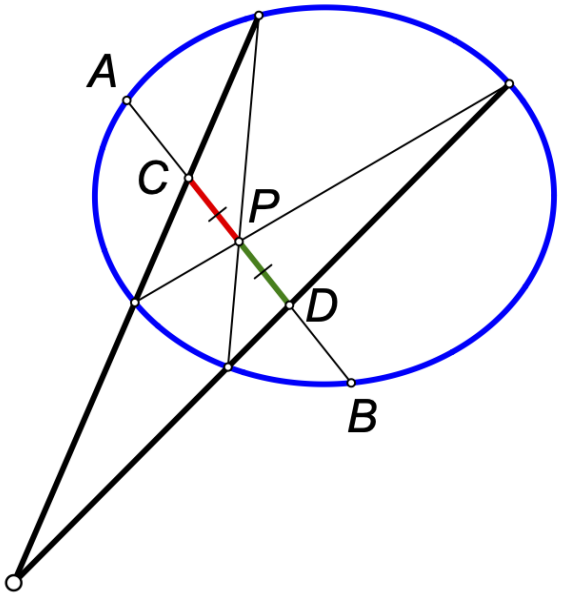
▶ marked angles are equal

Property 8.0.2. [39]



▶ marked angles are equal

Property 8.0.3. (Butterfly Theorem)



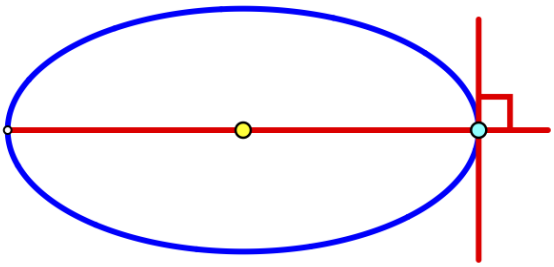
P is midpoint of AB

▶ P is midpoint of CD

9. Ellipse and One Tangent

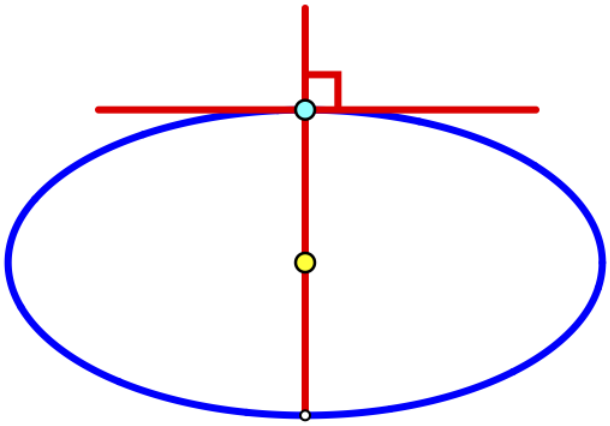
9.1 tangent at a special point

Property 9.1.1.



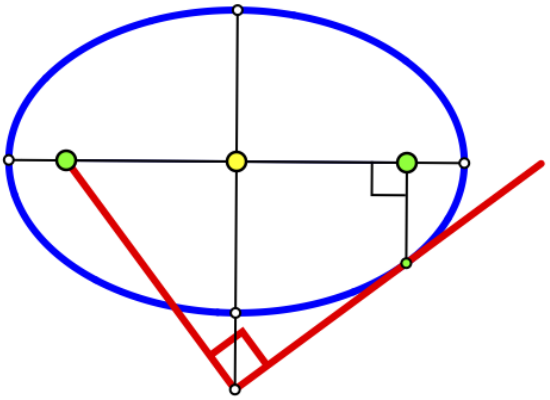
▶ red lines are perpendicular

Property 9.1.2.



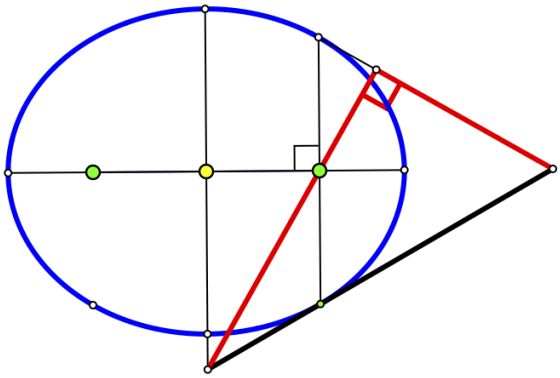
▶ red lines are perpendicular

Property 9.1.3. [59]



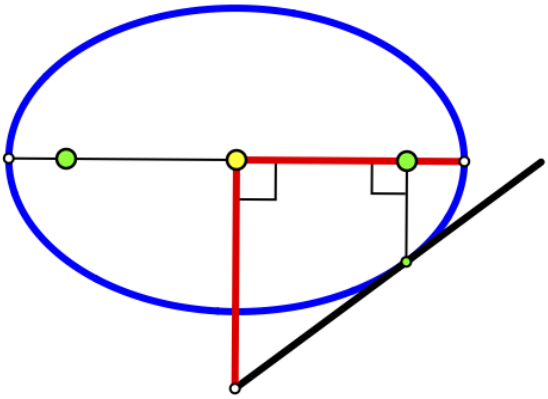
▶ red lines are perpendicular

Property 9.1.4. [46]



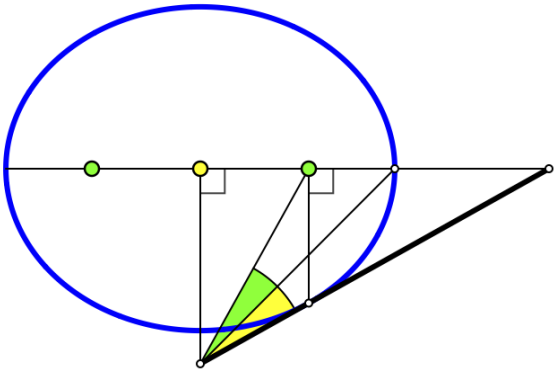
▶ red lines are perpendicular

Property 9.1.5. [47]



▶ red lengths are equal

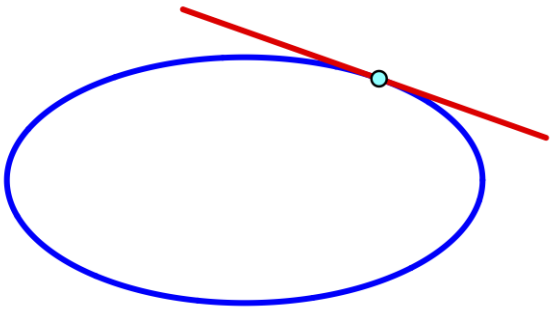
Property 9.1.6. [48]



▶ colored angles are equal

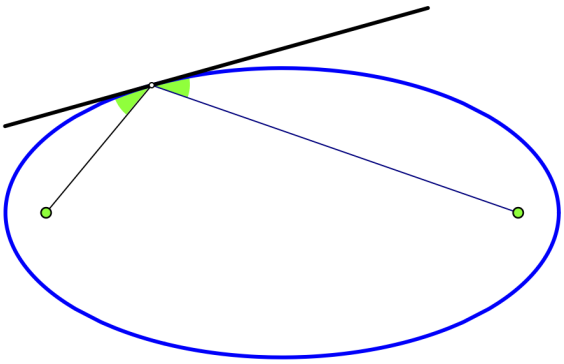
9.2 tangent at an arbitrary point

Property 9.2.1. (Smoothness Criterion)



▶ Ellipse has a tangent at any point on its boundary

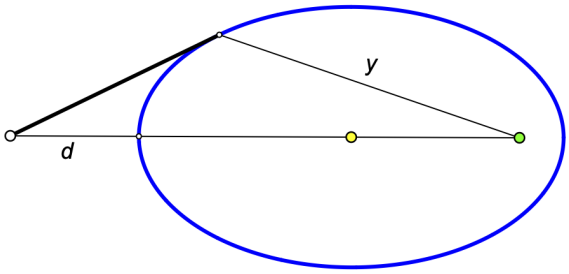
Property 9.2.2. (Reflection Property) [Apo 71]



▶ green angles are equal

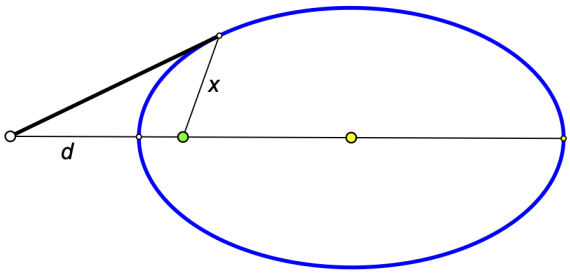
9.3 tangent from a point on the major axis

Property 9.3.1.



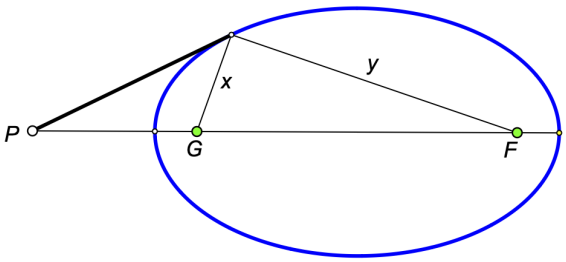
▶ $y = a + \frac{ac}{a+d}$

Property 9.3.2.



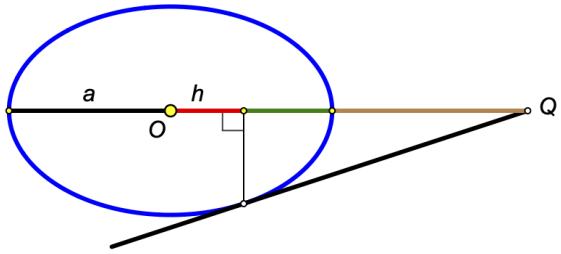
▶ $x = a - \frac{ac}{a+d}$

Property 9.3.3.



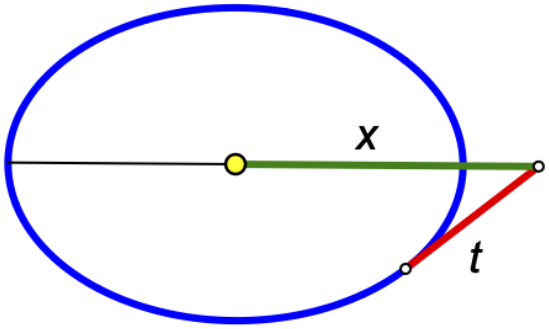
▶ $\frac{x}{y} = \frac{PG}{PF}$

Property 9.3.4.



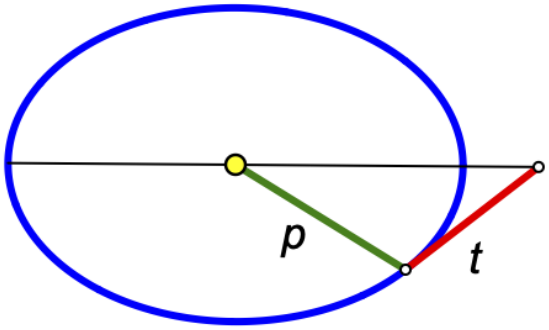
▶ $OQ \cdot h = a^2$

Property 9.3.5. (Length of Tangent from point on major axis) [43]



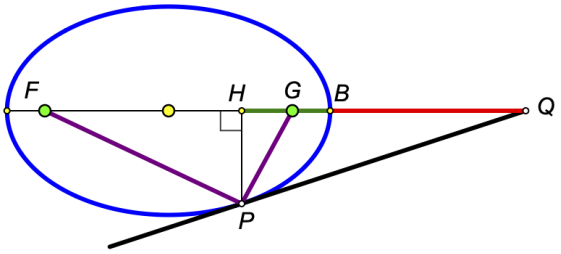
▶ $t = \dfrac{\sqrt{(x^2-a^2)(x^2-c^2)}}{x}$

Property 9.3.6. [56]



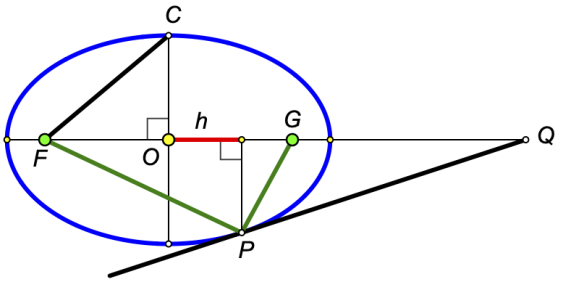
▶ $t = \dfrac{\sqrt{(a^2+b^2-p^2)(a^2-p^2)}}{p^2-b^2}$

Property 9.3.7.



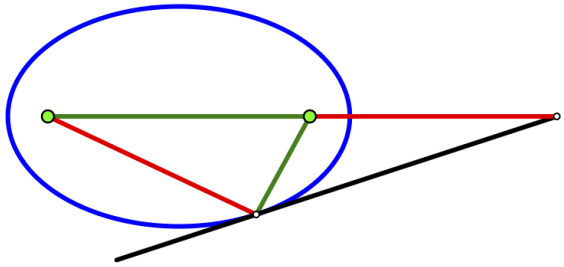
▶ $\dfrac{PG}{GQ} = \dfrac{PF}{FQ} = \dfrac{BH}{BQ}$

Property 9.3.8.



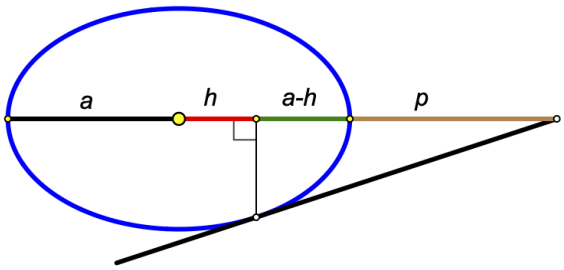
▶ $\dfrac{CF}{h} = \dfrac{GQ}{GP} = \dfrac{FQ}{FP}$

Property 9.3.9. [52]



▶ $\prod \text{red} = \prod \text{green}$

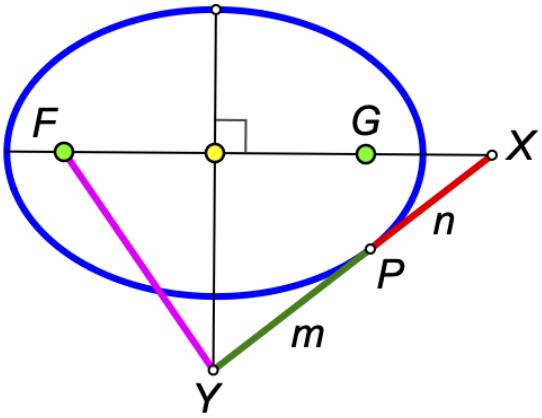
Property 9.3.10.



▶ $p = \dfrac{a(a-h)}{h}$

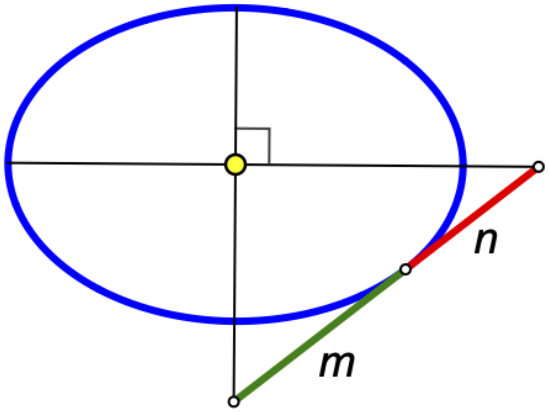
9.4 tangent between two axes

Property 9.4.1. [45]



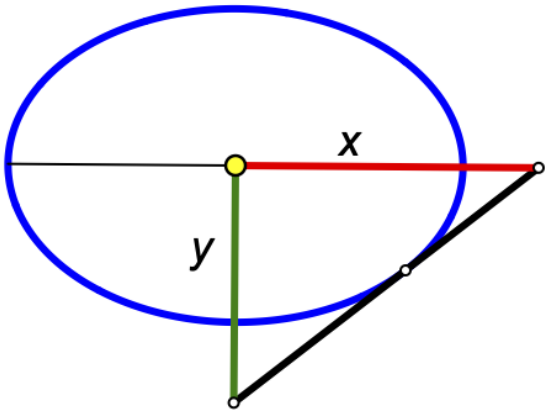
▶ $m(m+n) = (FY)^2$

Property 9.4.2. [55]



▶ $\frac{a^2}{m} + \frac{b^2}{n} = m + n$

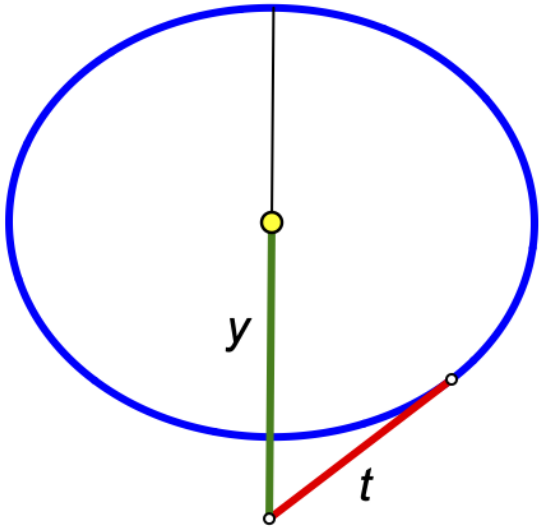
Property 9.4.3. [55]



▶ $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

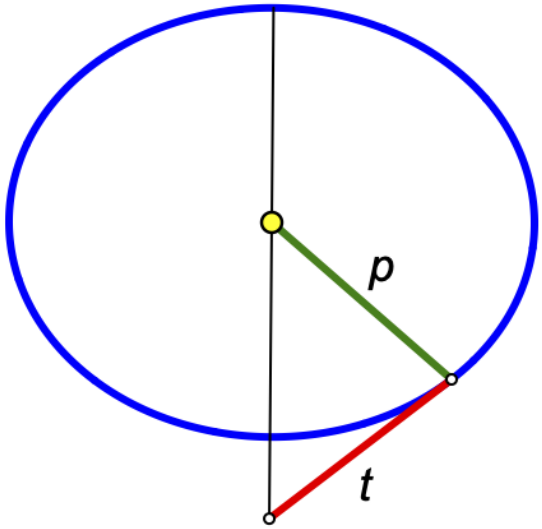
9.5 tangent from a point on the minor axis

Property 9.5.1. (Length of Tangent from point on minor axis) [44]



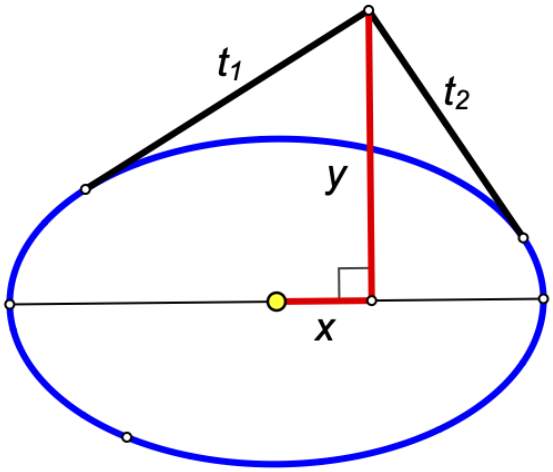
▶ $t = \frac{\sqrt{(y^2 - b^2)(y^2 + c^2)}}{y}$

Property 9.5.2. [57]



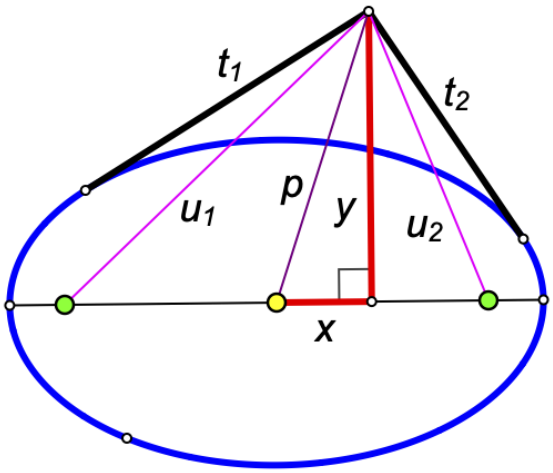
▶ $t = \frac{\sqrt{(a^2 + b^2 - p^2)(p^2 - b^2)}}{a^2 - p^2}$

Property 9.5.3. **(Lengths of Tangents)** [16, p. 70]



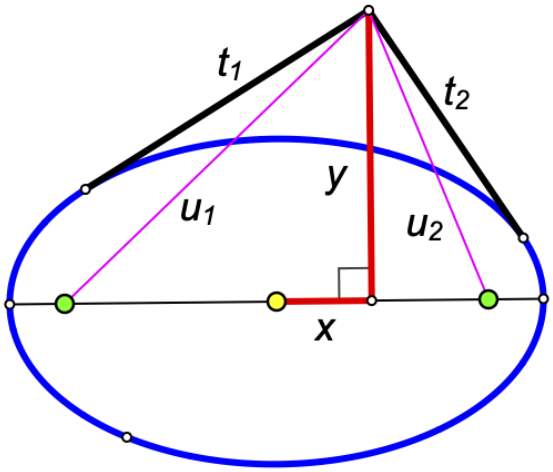
▶ t_1 and t_2 are the roots of the equation
$(f+1)^2t^4 - 2t^2 f\left((x^2+y^2)f + a^2b^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)\right) + f^2\left((x-c)^2+y^2\right)\left((x+c)^2+y^2\right) = 0$
where $f = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

Property 9.5.4. [16, p. 70]



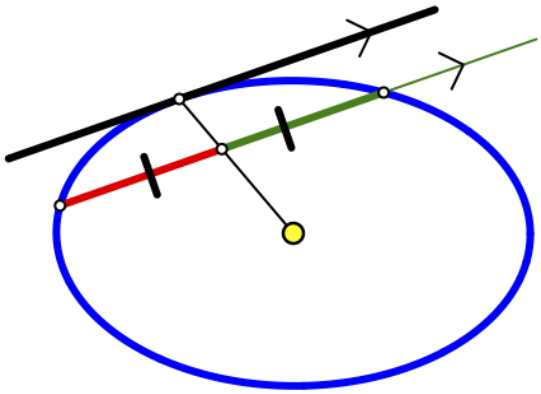
▶ t_1 and t_2 are the roots of the equation
$(f+1)^2t^4 - 2t^2 f\left(p^2 f + a^2b^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)\right) + f^2u_1^2u_2^2 = 0$
where $f = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

Property 9.5.5. [16, p. 70]



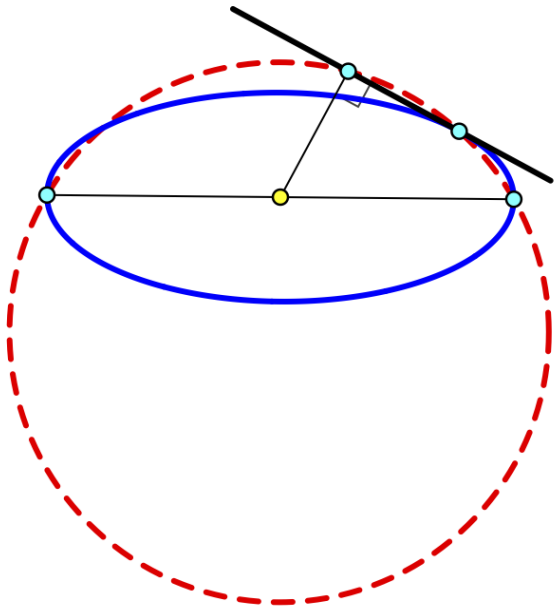
▶ $t_1t_2 = u_1u_2 \cdot \frac{f}{f+1}$ where $f = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

Property 9.5.6. [Apo 21]



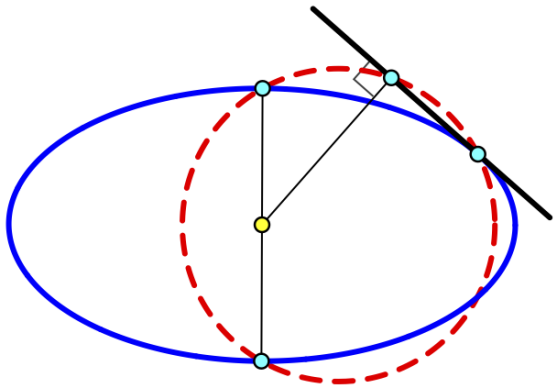
▶ red length = green length

Property 9.5.7. [29]



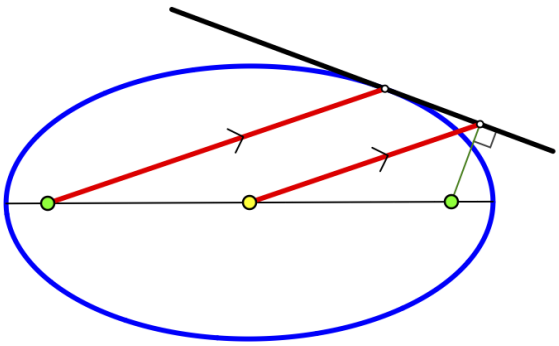
▶ cyan points are concyclic

Property 9.5.8. [30]



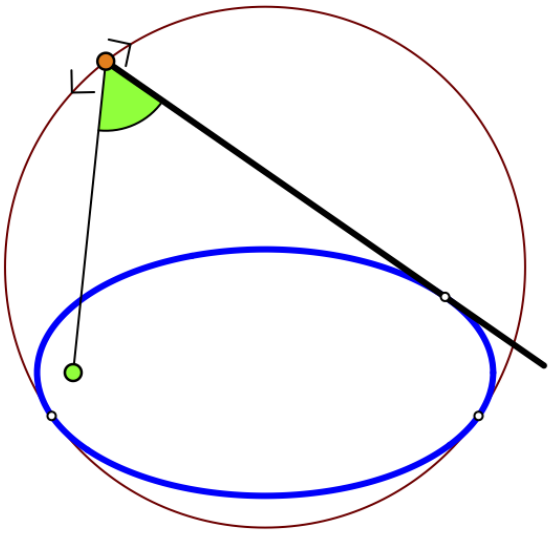
▶ cyan points are concyclic

Property 9.5.9. [49]



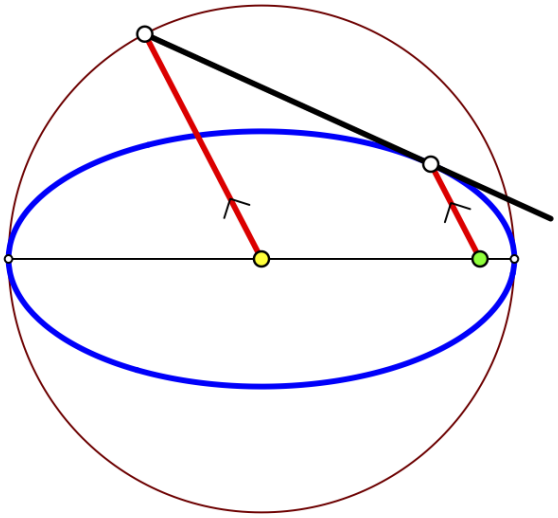
▶ red lines are parallel

Property 9.5.10. [Ako 11.14]



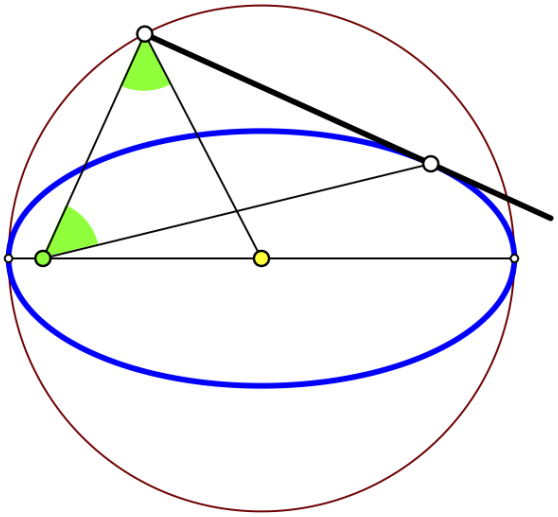
▶ green angle is invariant

Property 9.5.11.



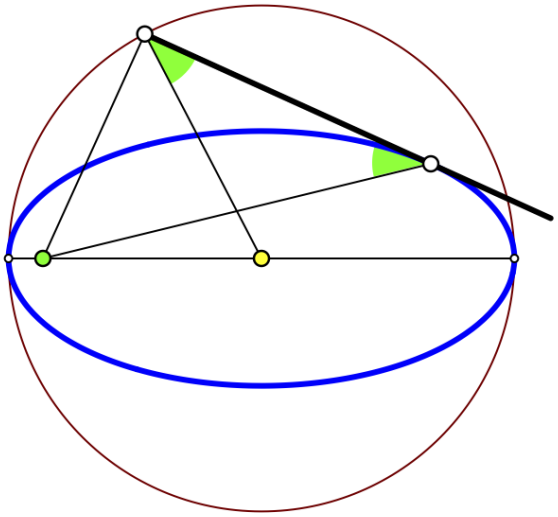
▶ red lines are parallel

Property 9.5.12.



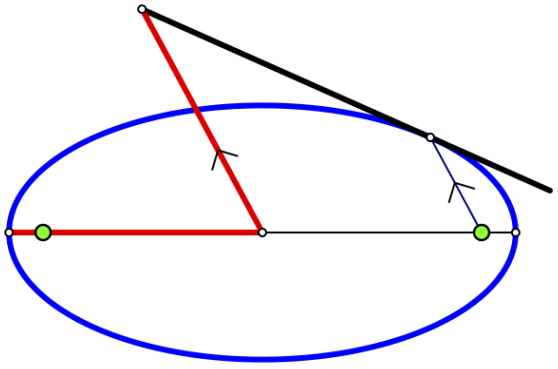
▶ green angles are equal

Property 9.5.13.



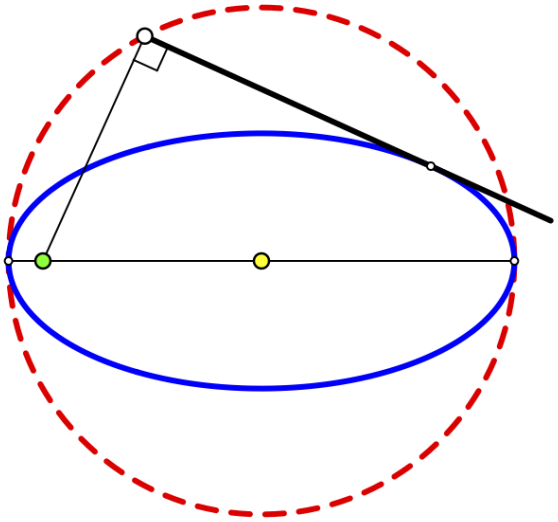
▶ green angles are equal

Property 9.5.14. [Apo 72]



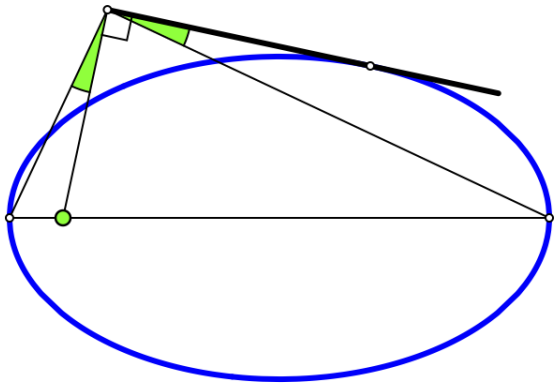
▶ red lengths are equal

Property 9.5.15. [Apo 72]



▶ foot of perpendicular to tangent lies on the director circle

Property 9.5.16.

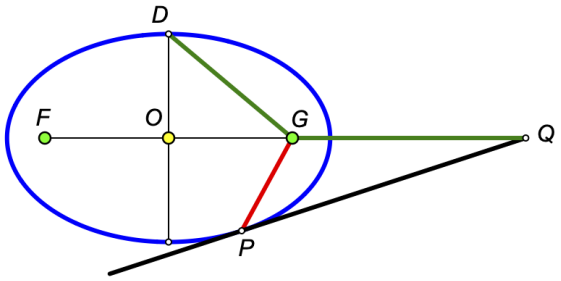


▶ green angles are equal

- Equivalent to Property 9.5.15

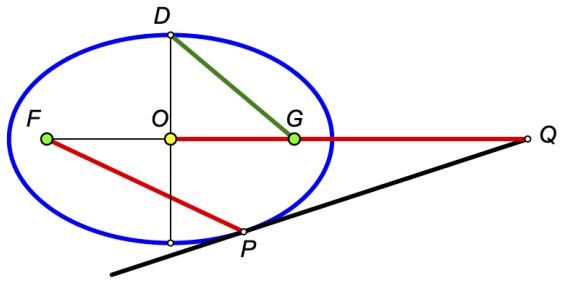
+ minor axis

Property 9.5.17.*



▶ $PG \cdot OQ = DG \cdot GQ$

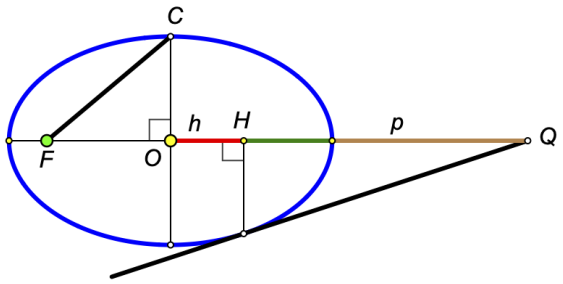
Property 9.5.18.*



▶ $PF \cdot OQ = DG \cdot FQ$

+ perpendicular to major axis

Property 9.5.19.

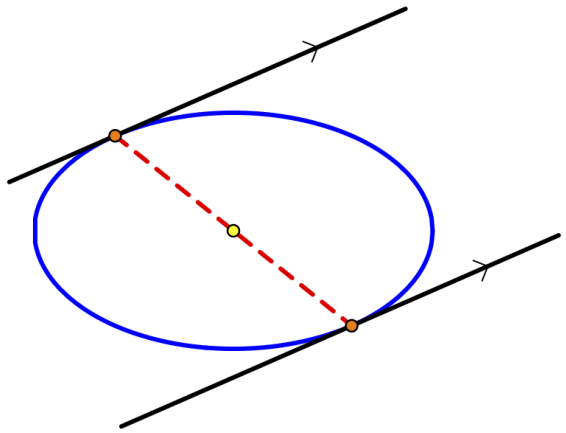


▶ $CF \cdot p = OQ \cdot HQ$

10. Ellipse and Two Tangents

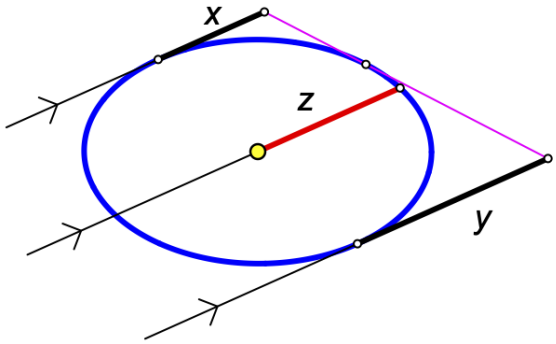
10.1 parallel tangents

Property 10.1.1.



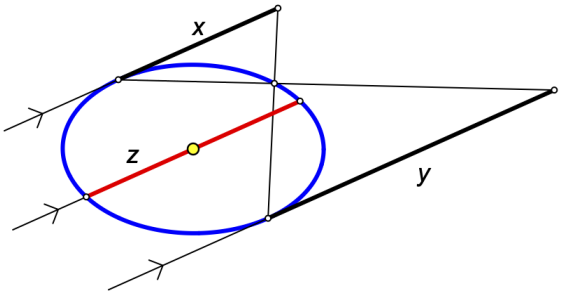
▶ colored points are collinear

Property 10.1.2. [Apo 66]



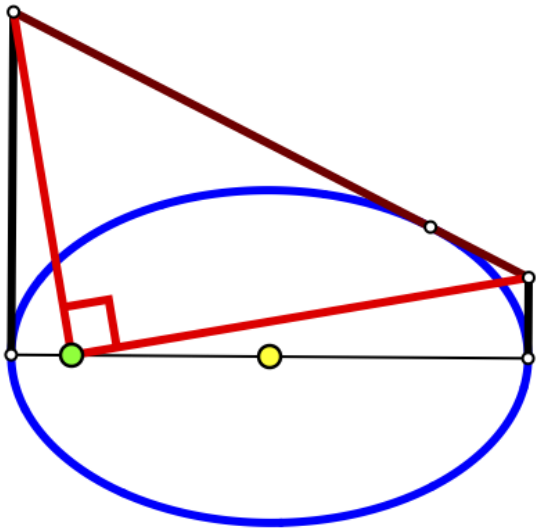
▶ $xy = z^2$

Property 10.1.3. [Apo 74]



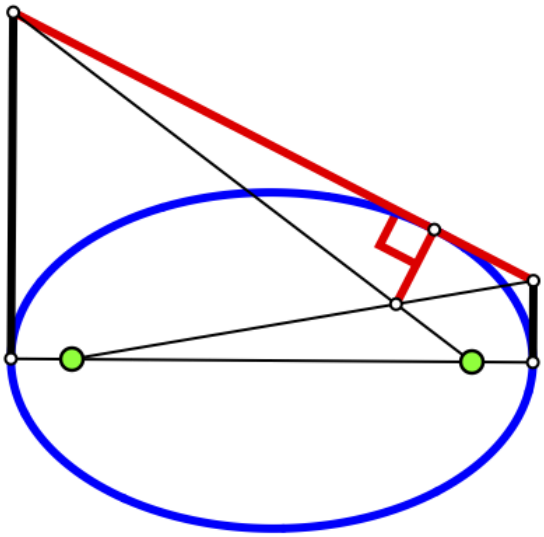
▶ $xy = z^2$

Property 10.1.4. [Apo 69]



▶ red lines are perpendicular

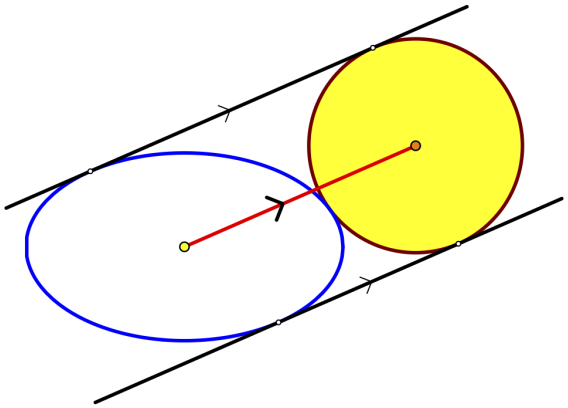
Property 10.1.5. [Apo 70]



▶ red lines are perpendicular

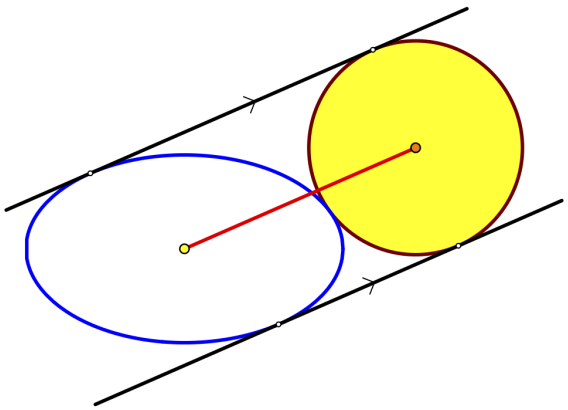
10.2 parallel tangents + tangent circle

Property 10.2.1.



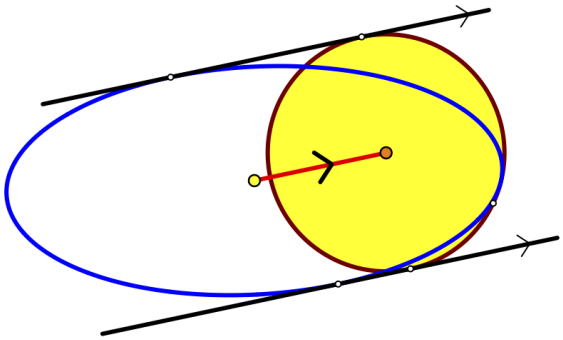
▶ red line is parallel to tangents

Property 10.2.2.



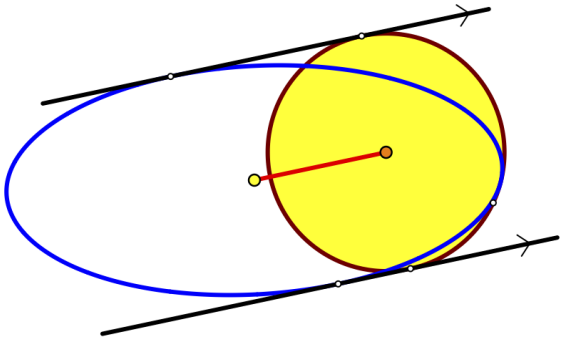
▶ red length $= a + b$

Property 10.2.3.



▶ red line is parallel to tangents

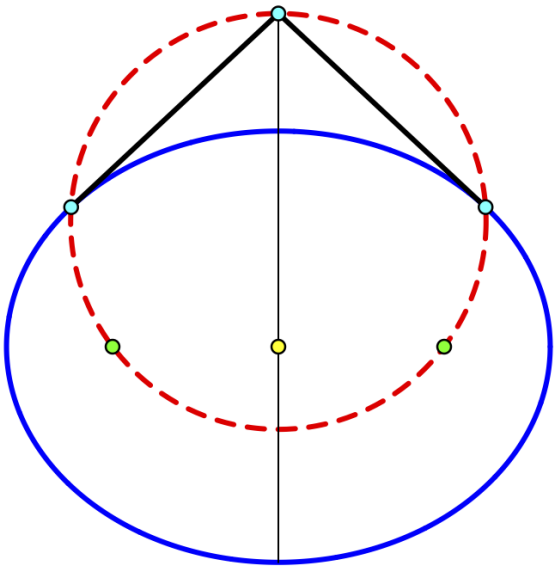
Property 10.2.4.



▶ red length $= a - b$

10.3 two tangents from a point on the minor axis

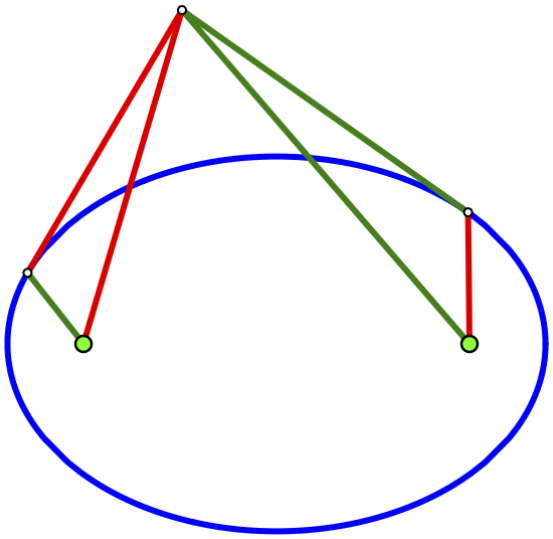
Property 10.3.1.



▶ cyan and green points lie on a circle

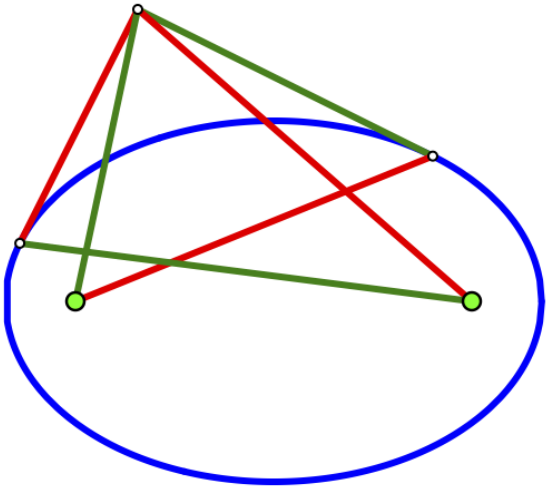
10.4 two tangents from the same point

Property 10.4.1.



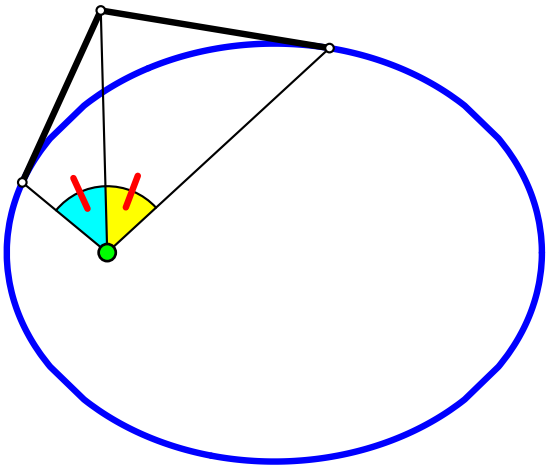
▶ $\prod$ red $=$ $\prod$ green

Property 10.4.2.



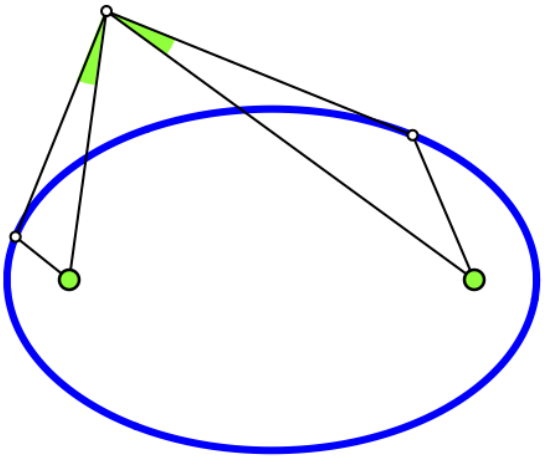
▶ $\prod$ red $=$ $\prod$ green

Property 10.4.3. [Ako 11.6]



▶ colored angles are equal

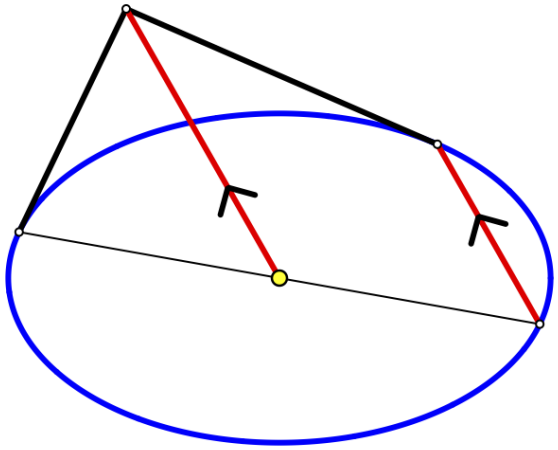
Property 10.4.4.



▶ green angles are equal

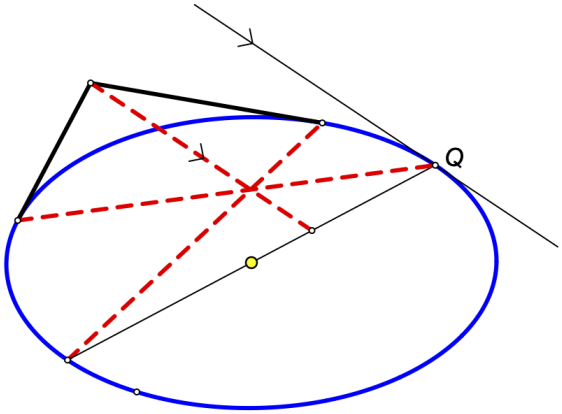
+ diameter

Property 10.4.5. [53]



▶ red lines are parallel

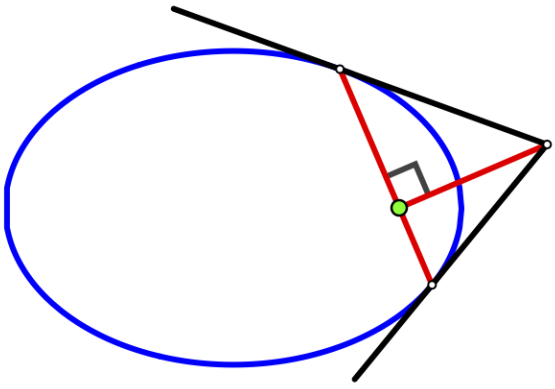
Property 10.4.6.



▶ red lines are concurrent

+ focal chord

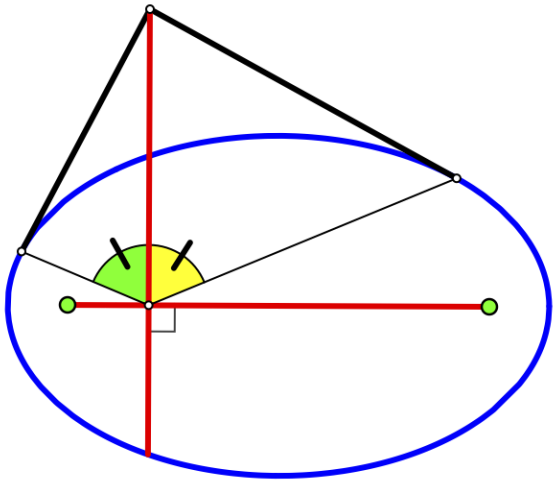
Property 10.4.7. [Ako 11.10]



▶ red lines are perpendicular

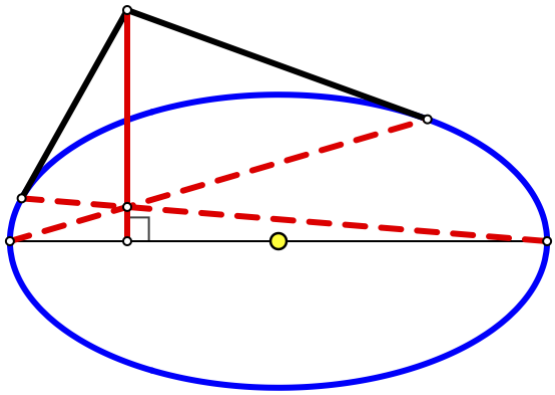
+ perpendicular

Property 10.4.8.



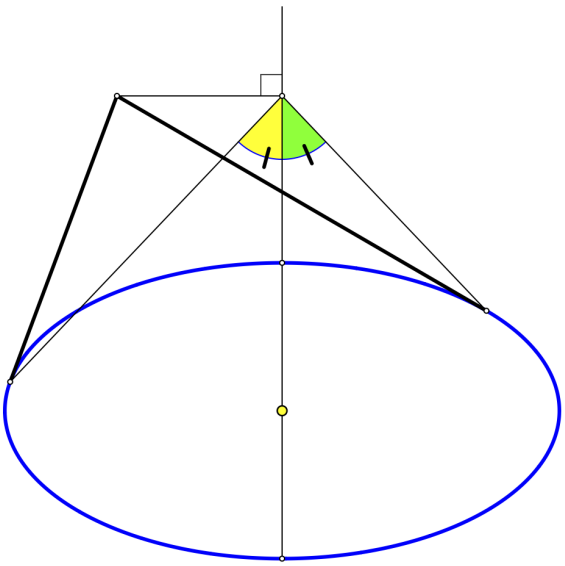
▶ colored angles are equal

Property 10.4.9.



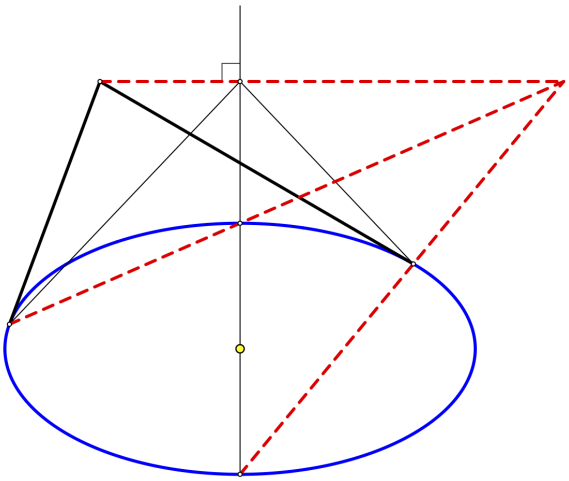
▶ red lines are concurrent

Property 10.4.10.



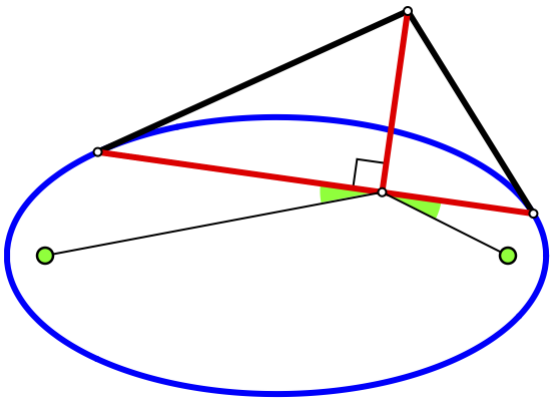
▶ colored angles are equal

Property 10.4.11.



▶ red lines are concurrent

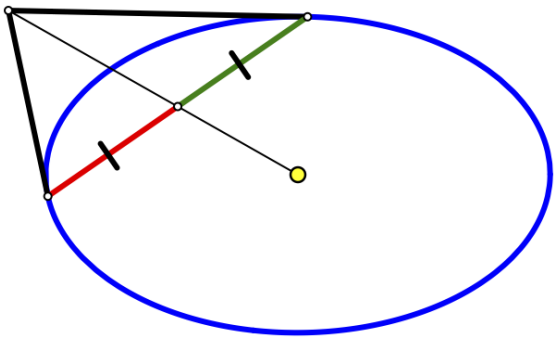
Property 10.4.12. [Ako 11.11]



▶ green angles are equal

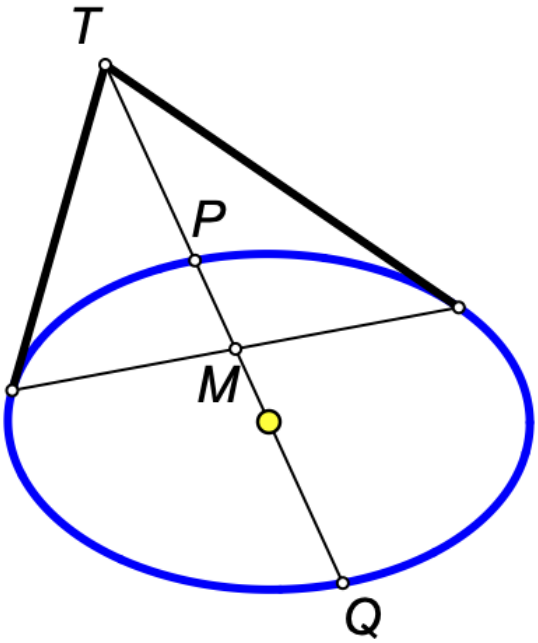
+ line to center

Property 10.4.13.



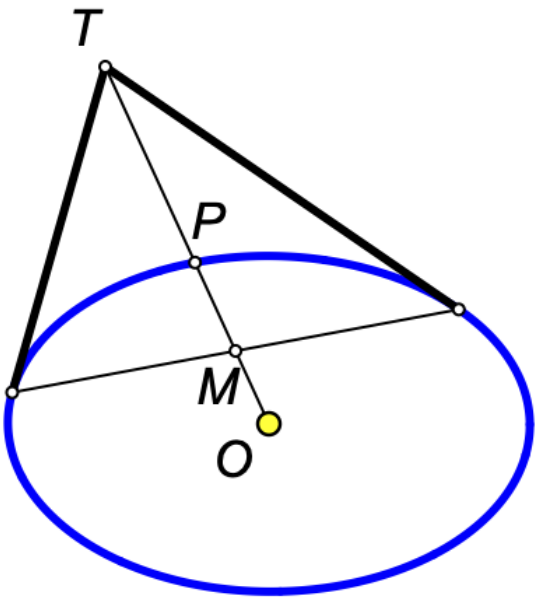
▶ red length = green length

Property 10.4.14. [Apo 13]



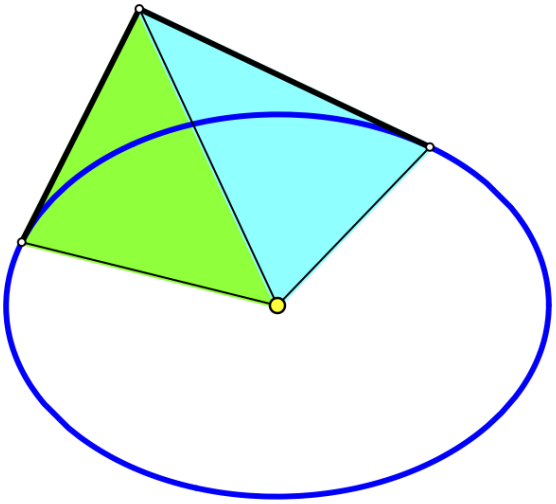
▶ $\frac{TP}{TQ} = \frac{PM}{MQ}$

Property 10.4.15. [Apo 14]



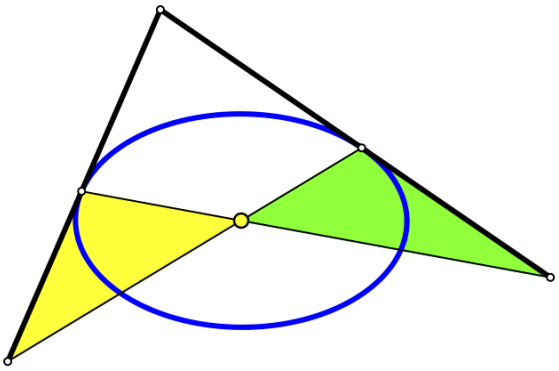
▶ $OM \cdot OT = (OP)^2$

Property 10.4.16.



▶ shaded areas are equal

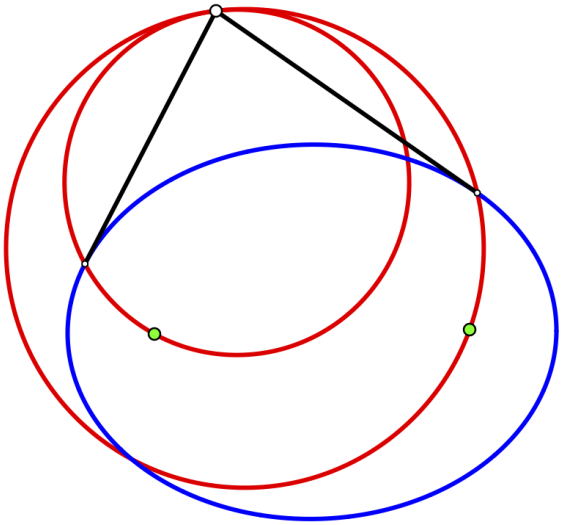
Property 10.4.17. [Apo 53]



▶ shaded areas are equal

+ circles

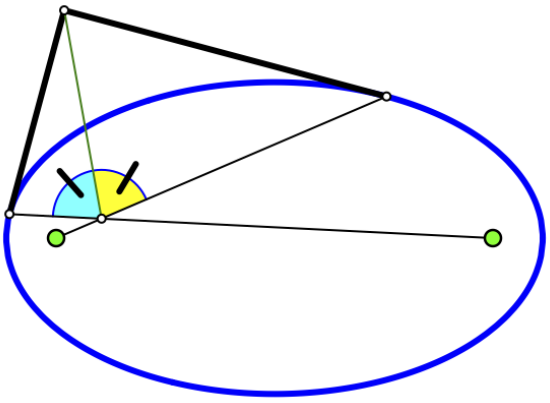
Property 10.4.18.



▶ red circles are tangent

+ intersection of focal radii

Property 10.4.19.

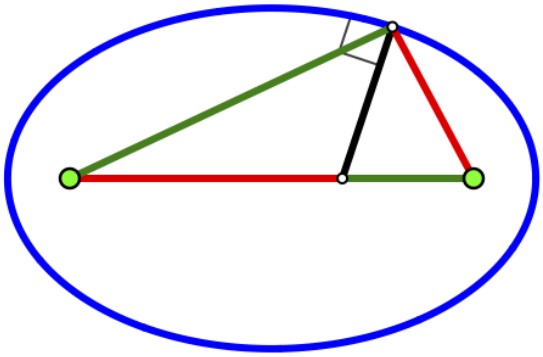


▶ colored angles are equal

11. Ellipse and Normals

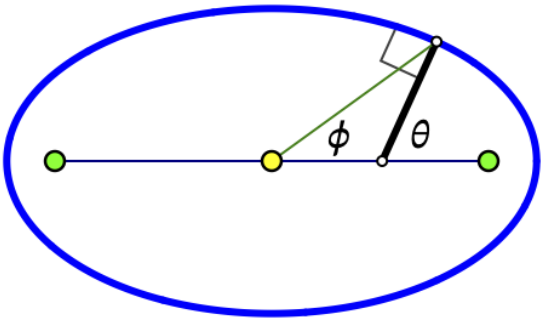
11.1 normal meets major axis

Property 11.1.1.



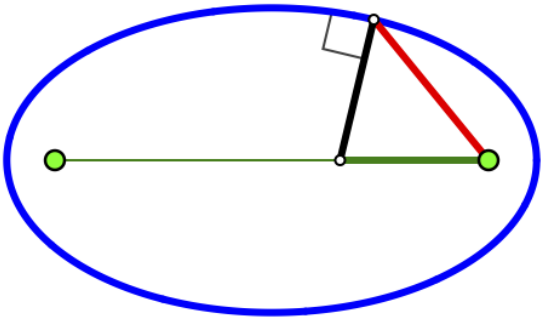
▶ $\prod \text{red} = \prod \text{green}$

Property 11.1.2. [68]



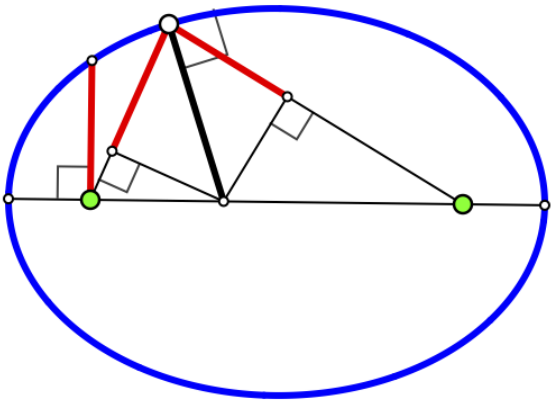
▶ invariant: $\dfrac{\tan\theta}{\tan\phi} = \dfrac{a^2}{b^2}$

Property 11.1.3. [32]



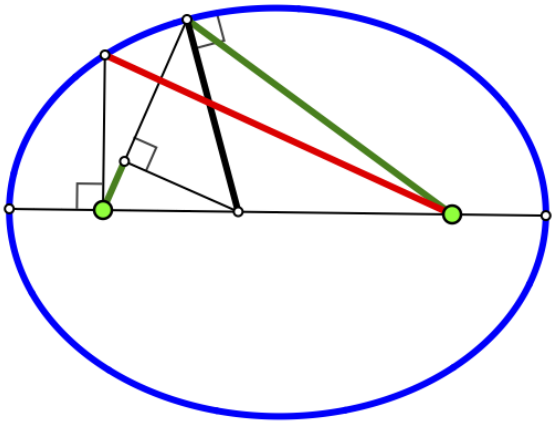
▶ invariant: $\dfrac{\text{green}}{\text{red}} = e$

Property 11.1.4. [35]



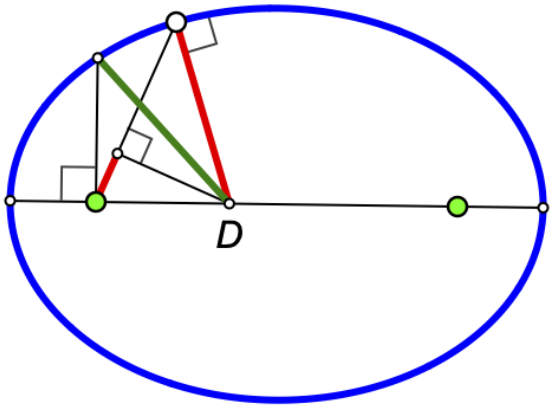
▶ red lengths are equal

Property 11.1.5. [37]



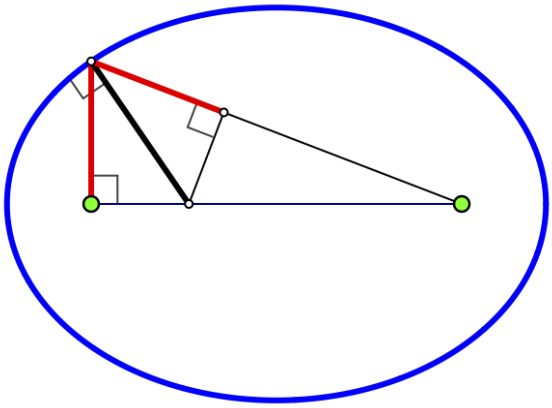
▶ $\sum \text{green} = \text{red}$

Property 11.1.6. [36]



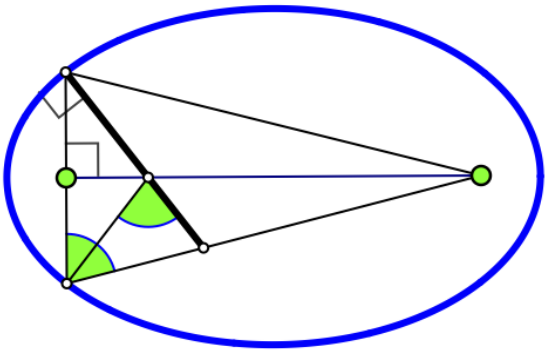
▶ $\sum \text{red}^2 = \text{green}^2$

Property 11.1.7. [36]



► red lengths are equal

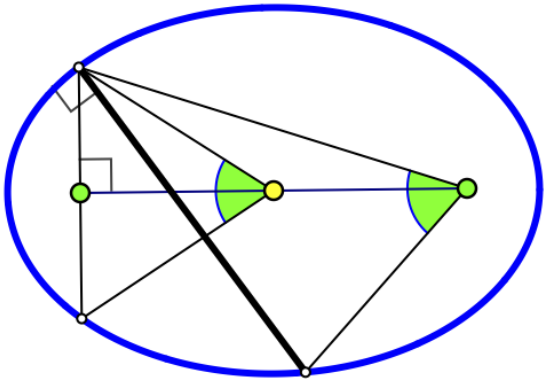
Property 11.1.8. [54]



► green angles are equal

11.2 normal meets ellipse

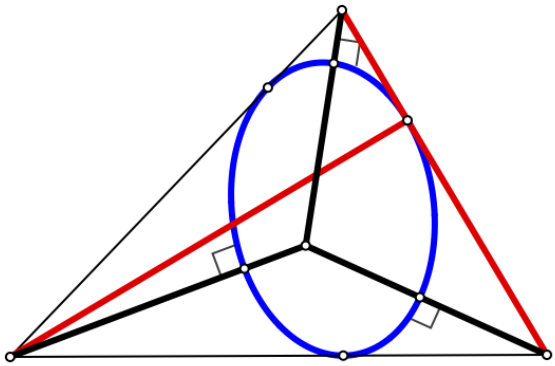
Property 11.2.1. [38]



► green angles are equal

11.3 3 normals

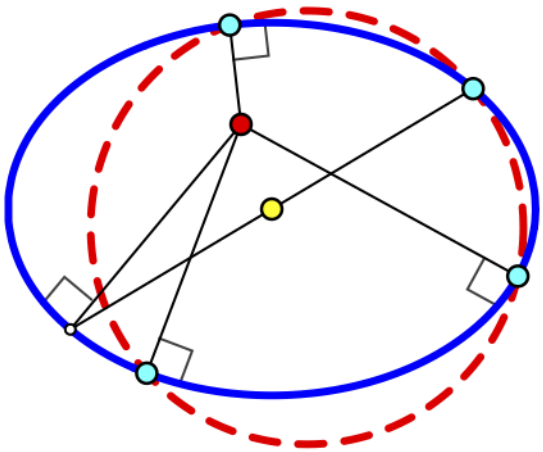
Property 11.3.1. [20]



► red lines are perpendicular

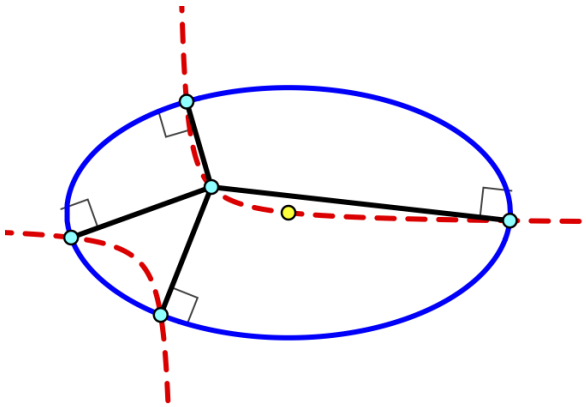
11.4 4 normals

Property 11.4.1.



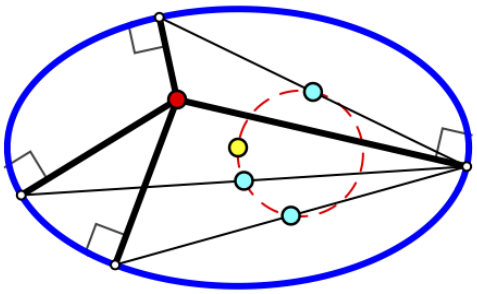
► cyan points are concyclic

Property 11.4.2. [33]



► conic through cyan points passes through center of ellipse

Property 11.4.3. [34]

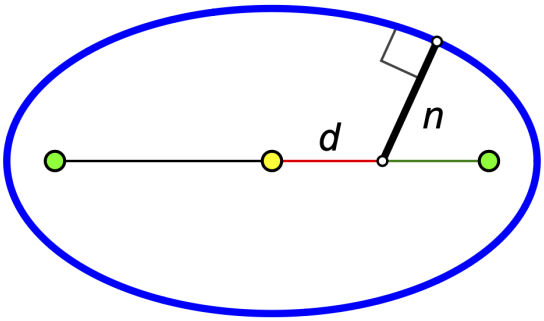


cyan points are midpoints

▶ cyan points and center of ellipse are concyclic

● In other words, the center of the ellipse is the Gergonne Steiner point of the quadrilateral formed by the feet of the four normals.

Property 11.4.4. (Length of Normal)

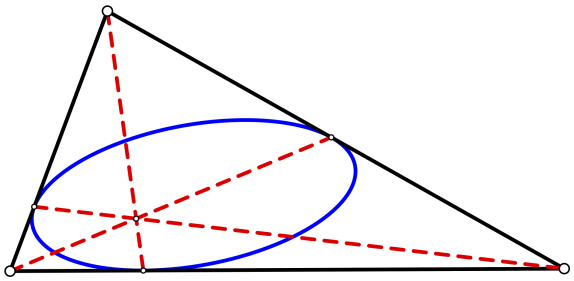


▶ $n = \frac{b}{c}\sqrt{c^2 - d^2}$

12. Inellipses

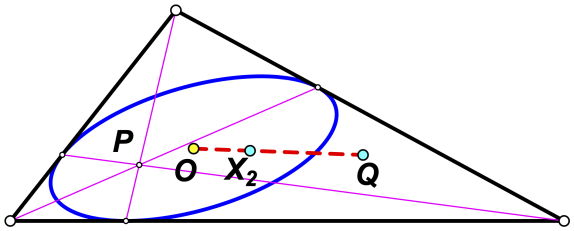
12.1 ellipse inscribed in a triangle

Property 12.1.1. (Perspector of Inconic)



▶ red lines are concurrent

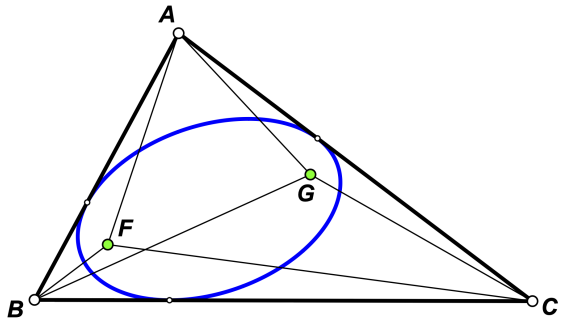
Property 12.1.2. (Isotomic Property) [3, p. 105]



P and Q are isotomic conjugates

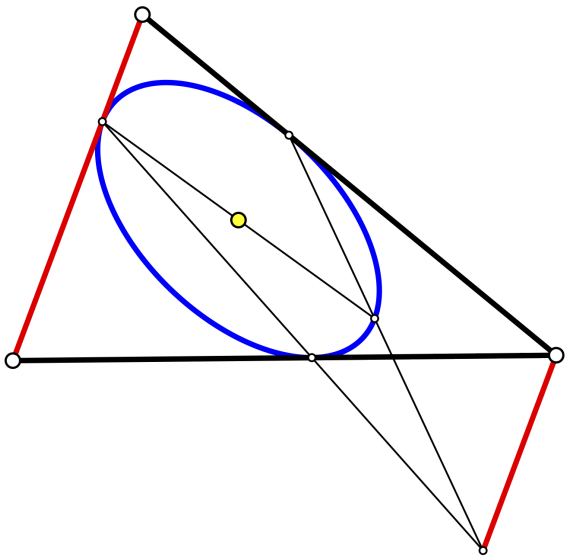
▶ $QX_2 = 2 \cdot X_2O$

Property 12.1.3. [1]



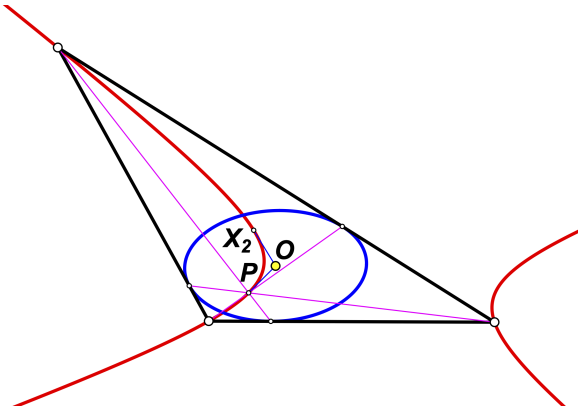
▶ $(BC)(AF)(AG) + (CA)(BF)(BG) + (AB)(CF)(CG) = (AB)(BC)(CA)$

Property 12.1.4. [8]



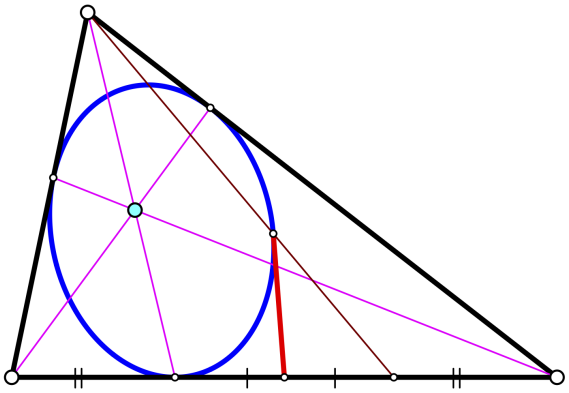
▶ red lines are parallel

Property 12.1.5. [3, p. 106]



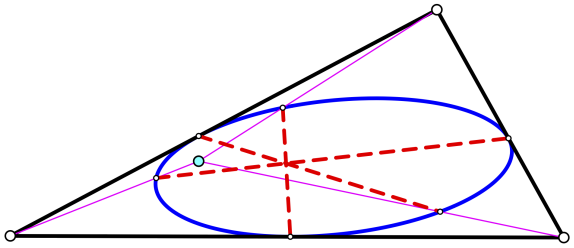
▶ tangents to circumconic at P and X_2 meet at O

Property 12.1.6. [5]



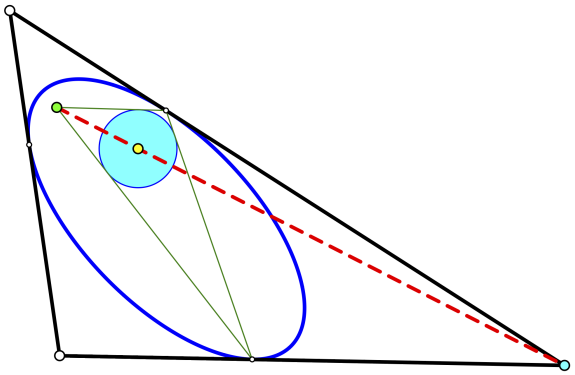
▶ red line is tangent to inconic.

Property 12.1.7. [22]



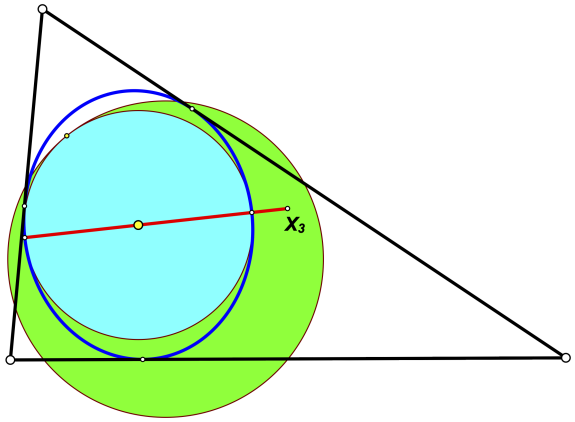
▶ red lines are concurrent.

Property 12.1.8.



▶ colored points are collinear.

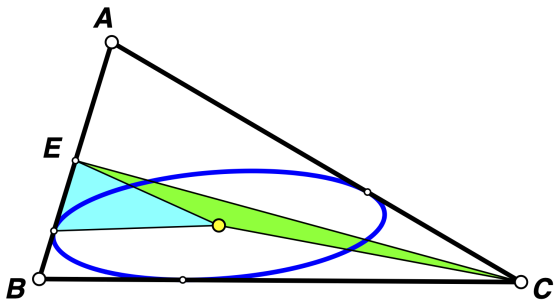
Property 12.1.9. [60]



red line is an axis of inellipse
green circle is nine-point circle

▶ blue and green circles are tangent.

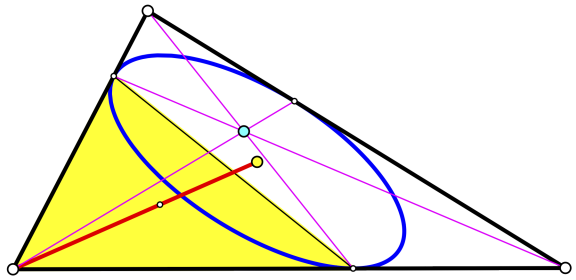
Property 12.1.10. [23]



E is midpoint of AB

▶ colored areas are equal.

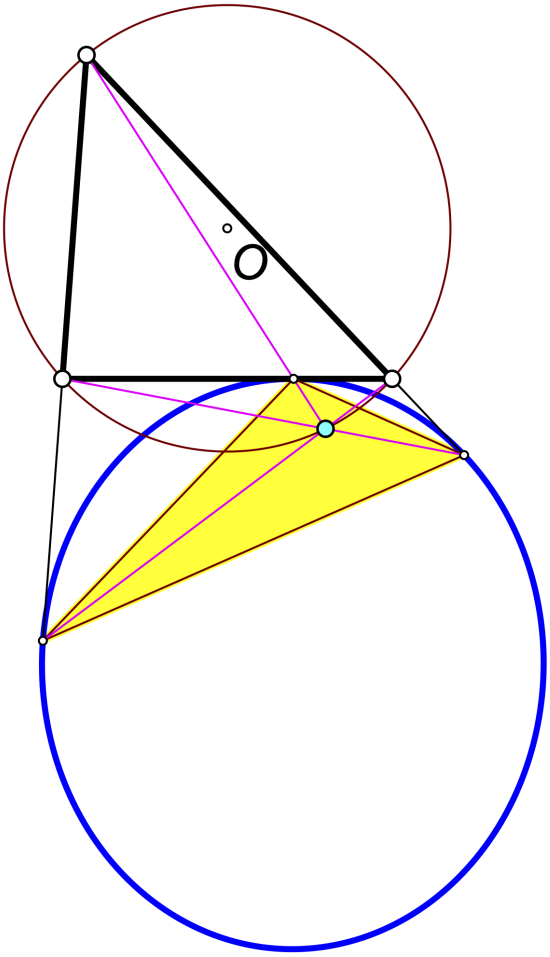
Property 12.1.11. [26]



▶ centroid of yellow triangle lies on red line.

12.2 Perspector on circumcircle

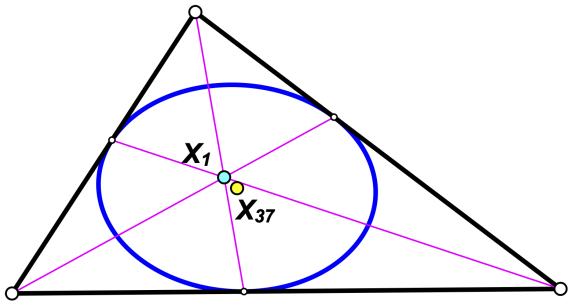
Property 12.2.1. [13]



▶ O is orthocenter of yellow triangle.

12.3 Perspector X_1 (incenter)

Property 12.3.1.

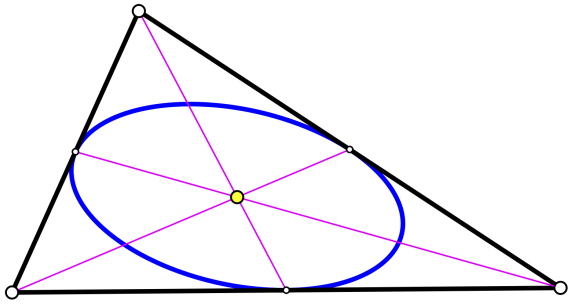


▶ X_{37} is center of inellipse.

12.4 Perspector X_2 (centroid)

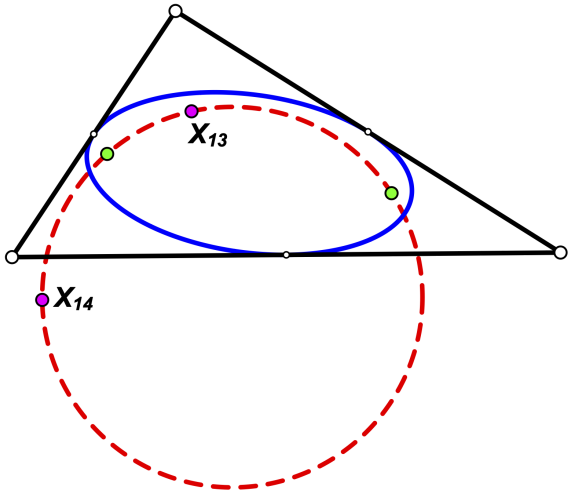
- Steiner inellipse

Property 12.4.1.



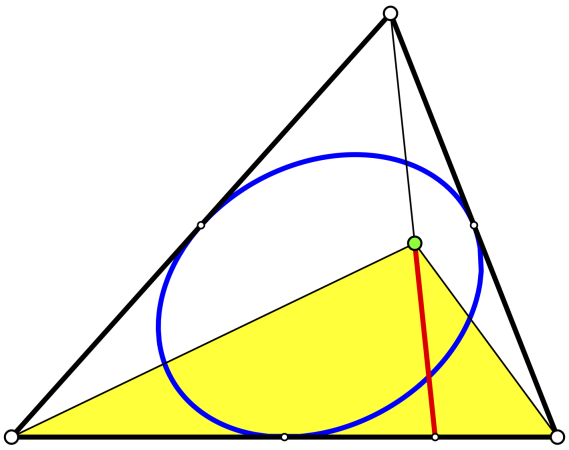
▶ X_2 is the center of the inellipse and coincides with the perspector.

Property 12.4.2. [66]



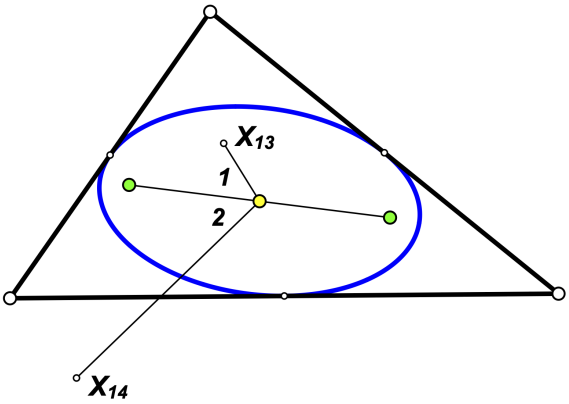
▶ colored points lie on a circle.

Property 12.4.3. [21]



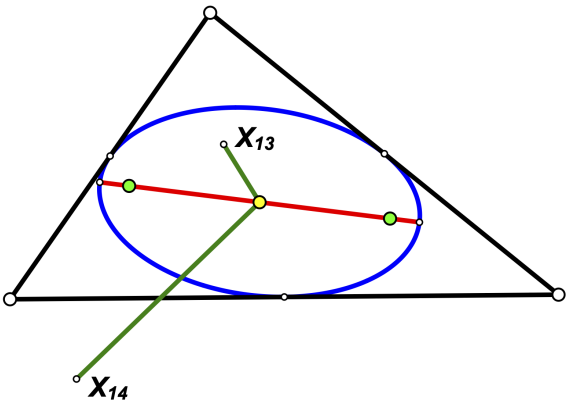
▶ Red line is symmedian of yellow triangle.

Property 12.4.4. [66]



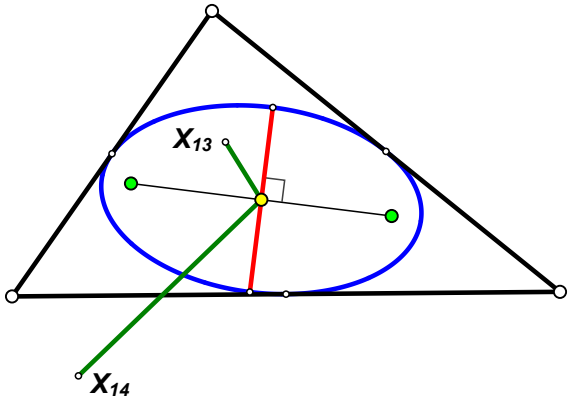
▶ $\angle 1 = \angle 2$

Property 12.4.5. [66]



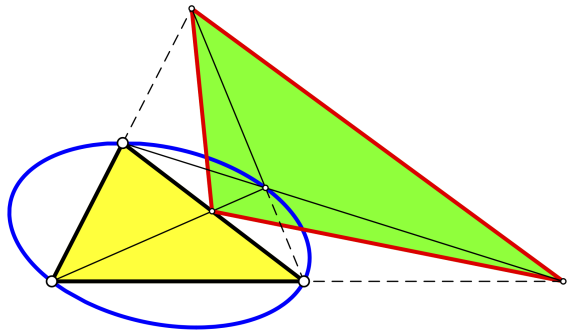
▶ red length = sum of green lengths

Property 12.4.6. [66]



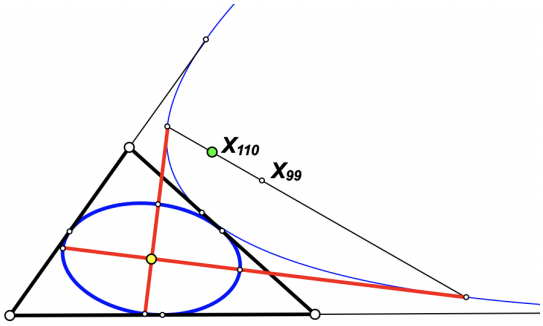
▶ red length = difference of green lengths

Property 12.4.7.



▶ area of inellipse = $\frac{\pi}{3\sqrt{3}}$ times area of triangle.

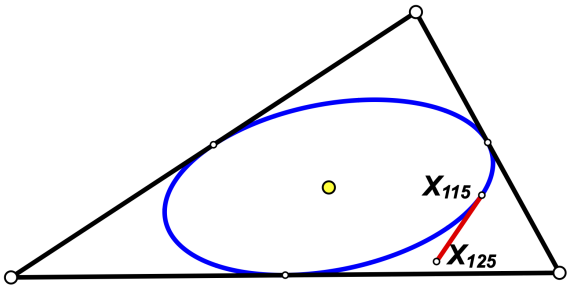
Property 12.4.8. [58]



▶ axes of inellipse are tangent to Kiepert parabola

- chord passes through focus and perceptor of parabola

Property 12.4.9. [17]

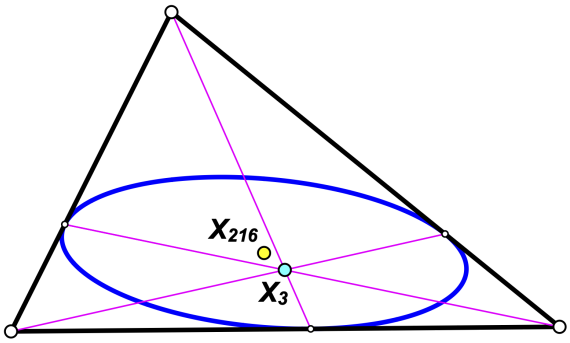


▶ X_{115} lies on the Steiner inellipse

● The tangent to the ellipse at X_{115} passes through X_{125}

12.5 Perspector X_3 (circumcenter)

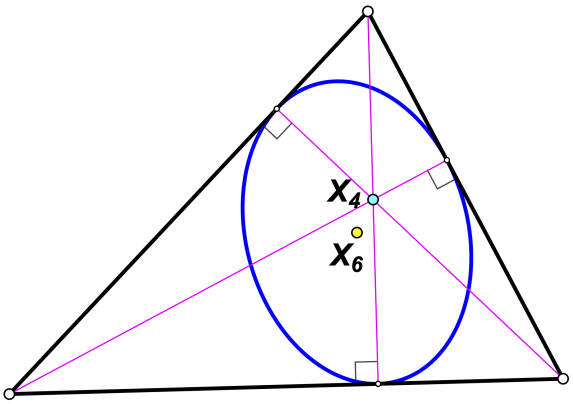
Property 12.5.1.



▶ X_{216} is center of inellipse.

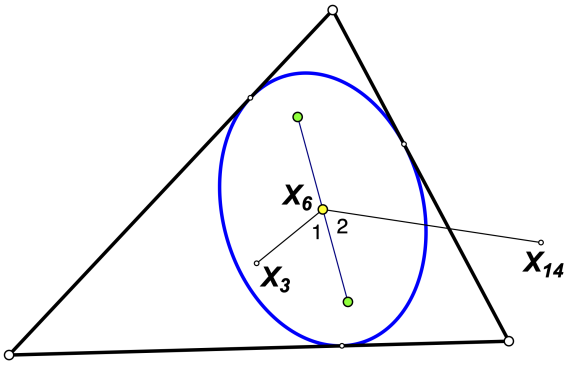
12.6 Perspector X_4 (orthocenter)

Property 12.6.1. [17]



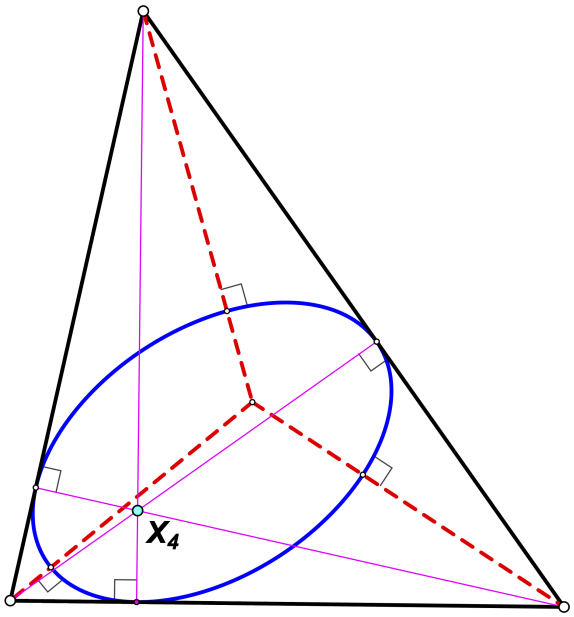
▶ X_6 is center of the inconic.

Property 12.6.2.



▶ $\angle 1 = \angle 2$

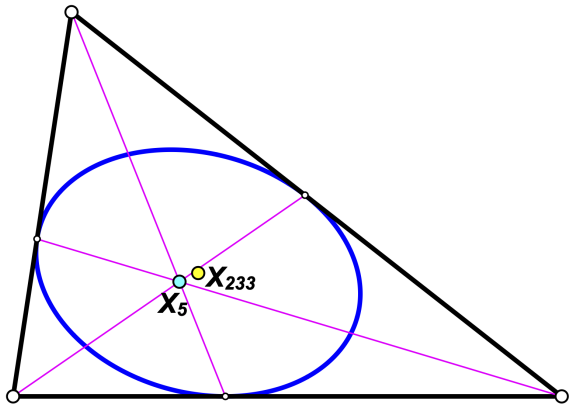
Property 12.6.3. [20]



▶ red normals are concurrent.

12.7 Perspector X_5 (nine-point center)

Property 12.7.1.

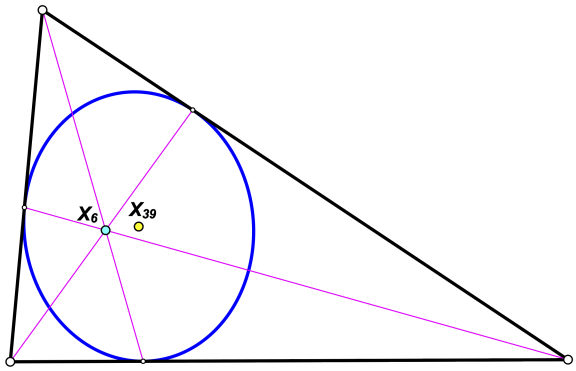


▶ X_{233} is center of inellipse.

12.8 Perspector X_6 (symmedian point)

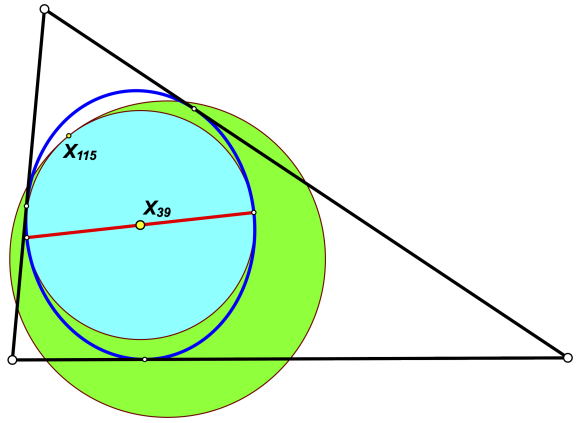
- Brocard inellipse

Property 12.8.1.



▶ X_{39} is the center of the inellipse.

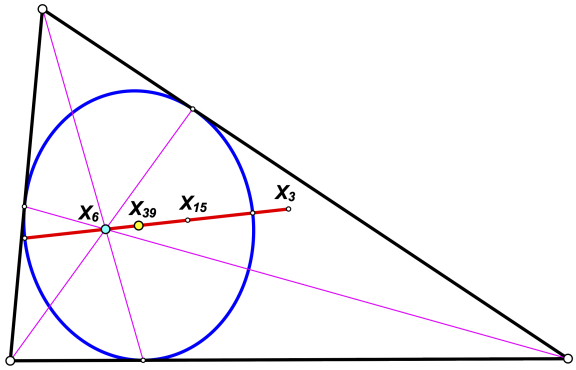
Property 12.8.2.



red line is minor axis of inellipse
green circle is nine-point circle

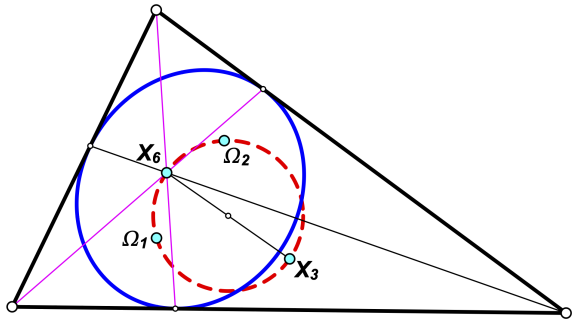
▶ blue and green circles are tangent.

Property 12.8.3.



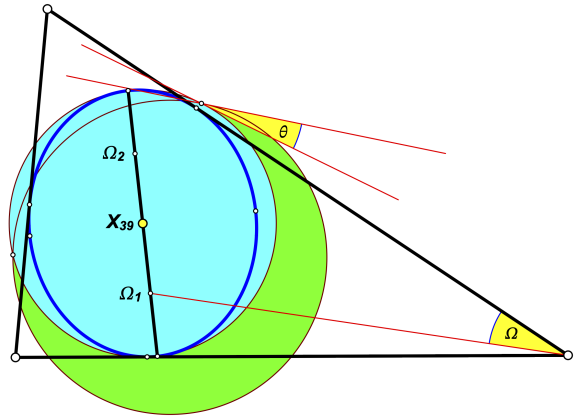
▶ Brocard axis is minor axis of Brocard inellipse.

Property 12.8.4.



▶ colored points lie on circle with diameter X_3X_6.

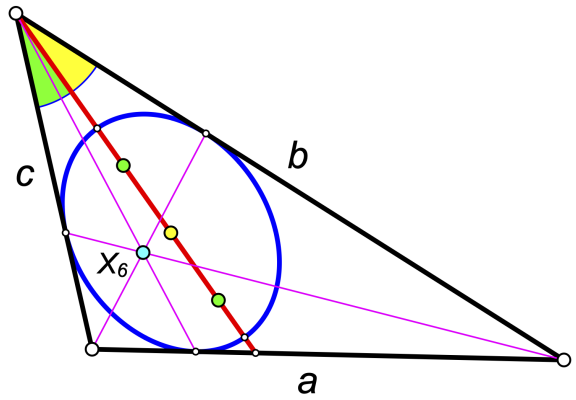
Property 12.8.5.



θ is angle between green circle
and blue circle
green circle is nine-point circle

▶ $3\Omega + \theta = 90°$

Property 12.8.6. [28]

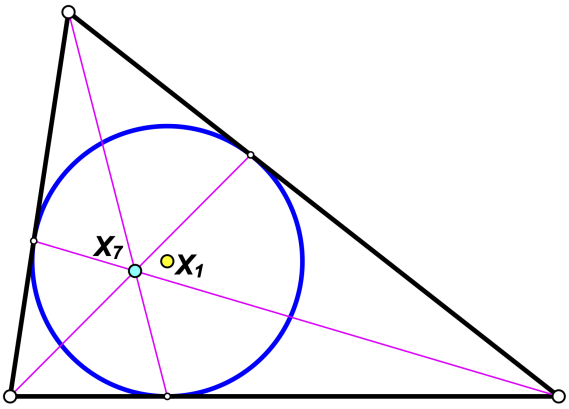


$a^2 = bc$

▶ colored angles are equal.

12.9 Perspector X_7 (Gergonne point)

Property 12.9.1.

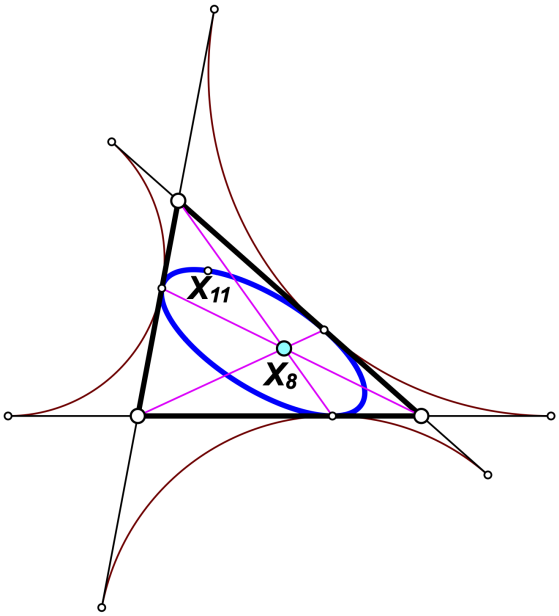


▶ inellipse is a circle with center X_1.

12.10 Perspector X_8 (Nagel point)

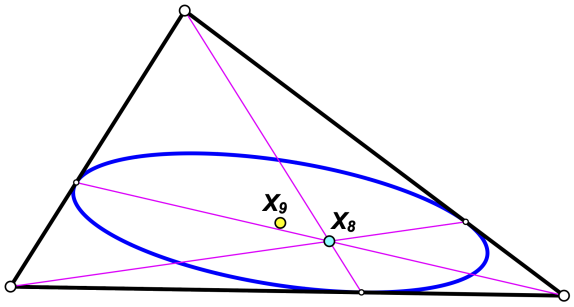
- Mandart inellipse

Property 12.10.1. [63]



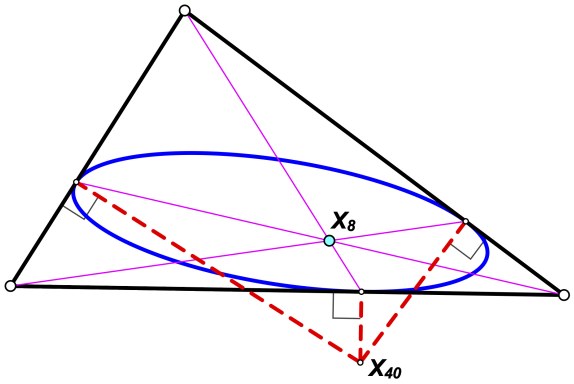
▶ X_{11} lies on inellipse.

Property 12.10.2. [63]



▶ X_9 is center of inellipse.

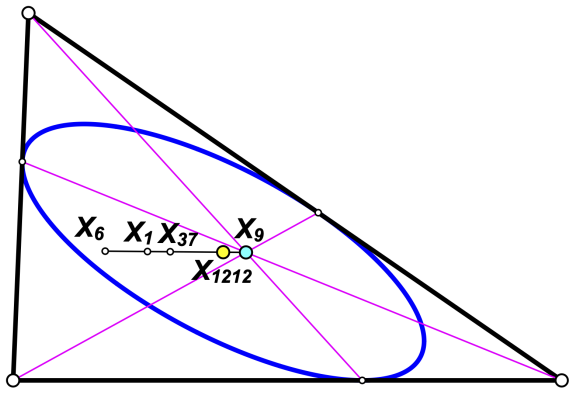
Property 12.10.3.



▶ red normals are concurrent.

12.11 Perspector X_9 (mittenpunkt)

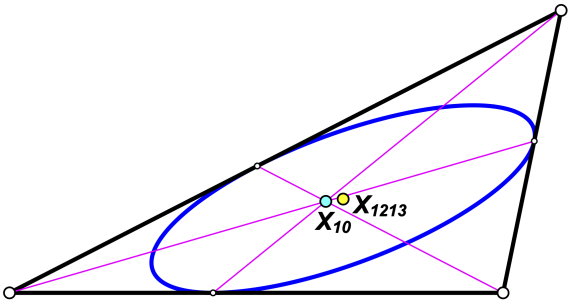
Property 12.11.1.



▶ X_{1212} is center of inellipse and lies on IK-line.

12.12 Perspector X_{10} (Spieker center)

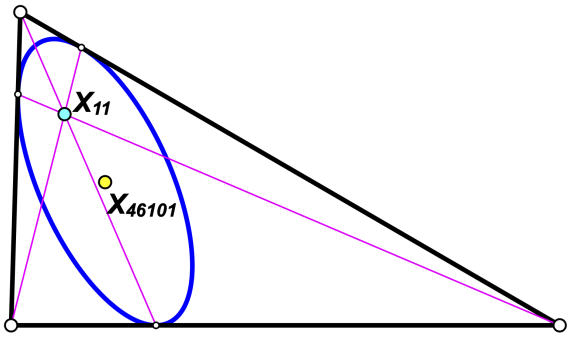
Property 12.12.1.



► X_{1213} is center of inellipse.

12.13 Perspector X_{11} (Feuerbach point)

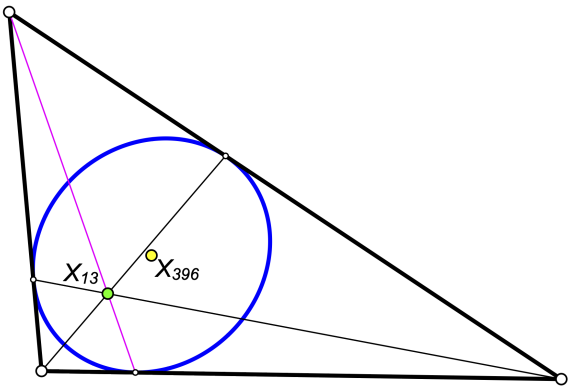
Property 12.13.1.



► X_{46101} is center of inellipse.

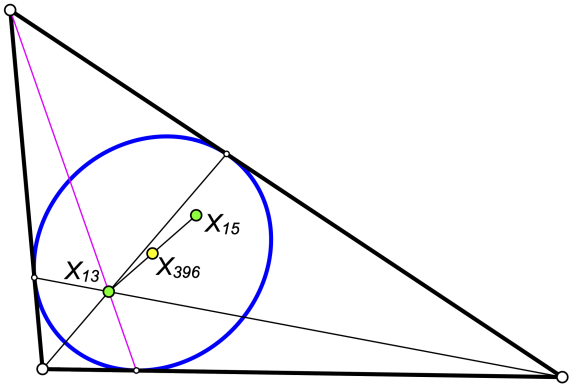
12.14 Perspector X_{13} (1st Fermat point)

Property 12.14.1.



► X_{396} is the center of the inconic.

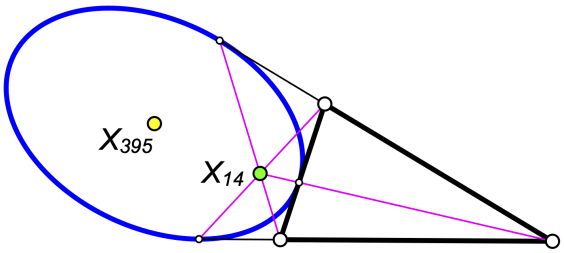
Property 12.14.2. [31]



► X_{13} and X_{15} are the foci of the inconic.

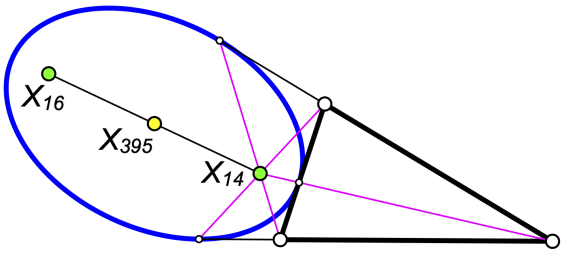
12.15 Perspector X_{14} (2nd Fermat point)

Property 12.15.1.



► X_{395} is the center of the inconic.

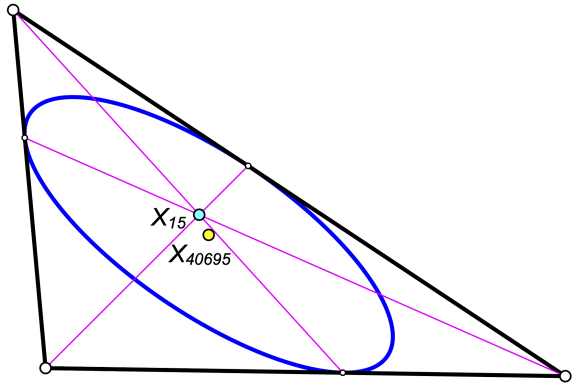
Property 12.15.2.



► X_{14} and X_{16} are the foci of the inconic.

12.16 Perspector X_{15} (1st isodynamic point)

Property 12.16.1.

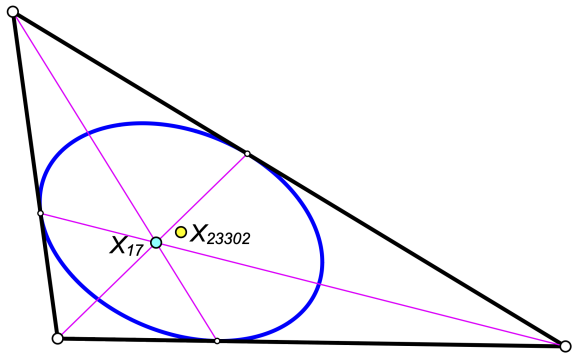


▶ X_{40695} is the center of the inconic.

● X_{40696} is the center of the inconic with perspector X_{16}.

12.17 Perspector X_{17} (1st Napoleon point)

Property 12.17.1.

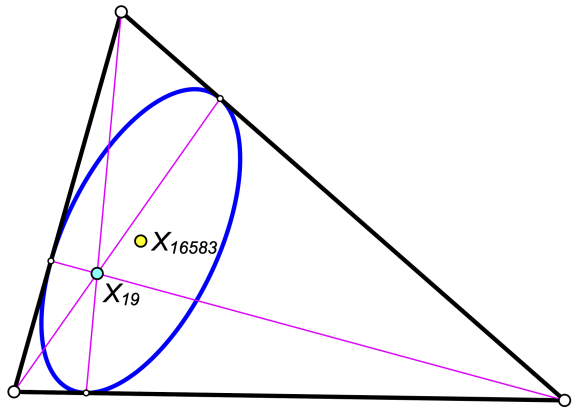


▶ X_{23302} is the center of the inconic.

● X_{23303} is the center of the inconic with perspector X_{18}.

12.18 Perspector X_{19} (Clawson point)

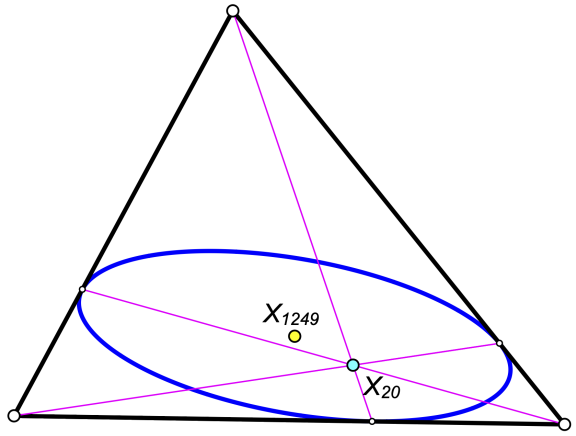
Property 12.18.1.



▶ X_{16583} is the center of the inconic.

12.19 Perspector X_{20} (de Longchamps point)

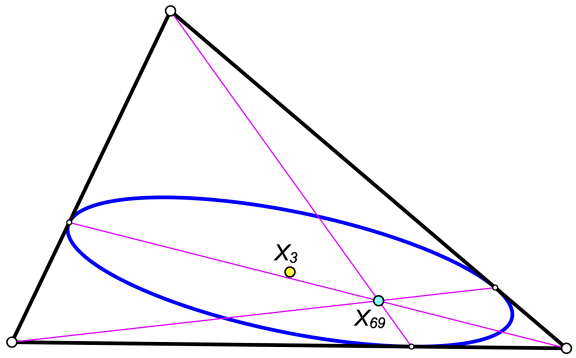
Property 12.19.1.



▶ X_{1249} is the center of the inconic.

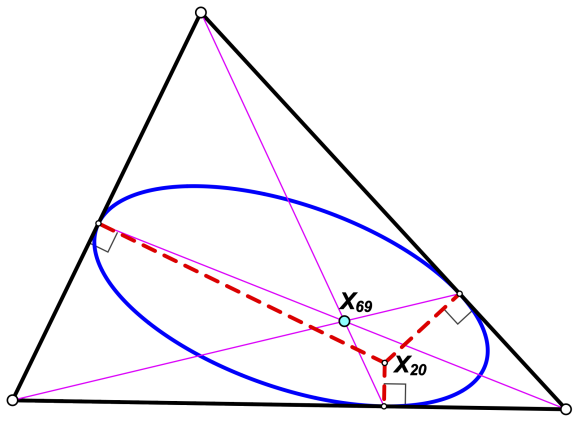
12.20 Perspector X_{69}

Property 12.20.1.



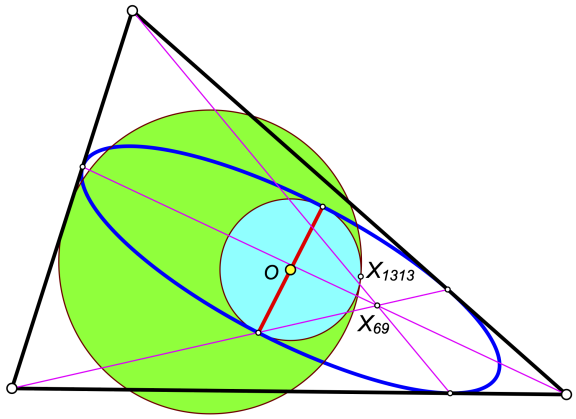
▶ X_3 is center of inellipse.

Property 12.20.2.



▶ red normals are concurrent.

Property 12.20.3. [19]



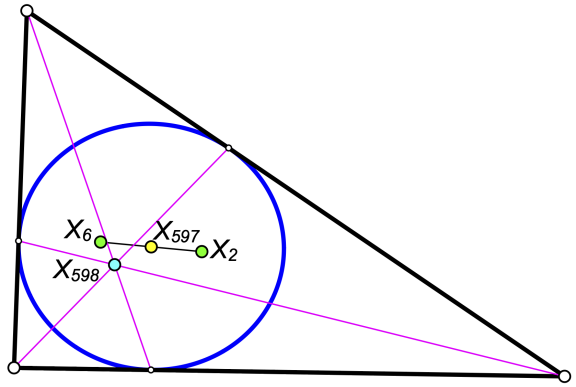
red line is minor axis of inellipse
green circle is nine-point circle

▶ blue and green circles are tangent.

12.21 Perspector X_{598}

- Lemoine inellipse

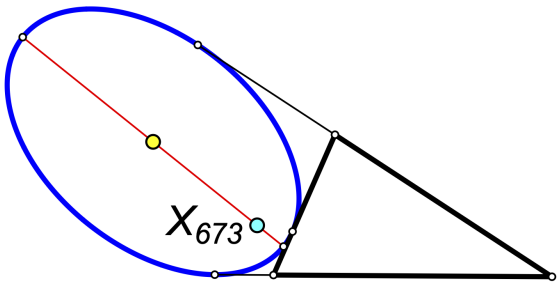
Property 12.21.1.



▶ X_{597} is the center of the inellipse.

12.22 Perspector X_{673}

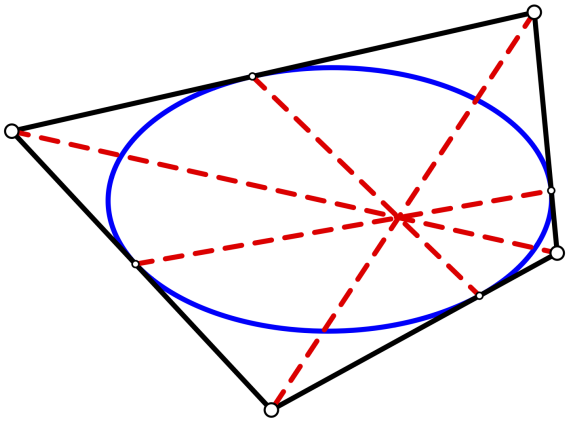
Property 12.22.1. [18]



▶ red line is major axis of inellipse.

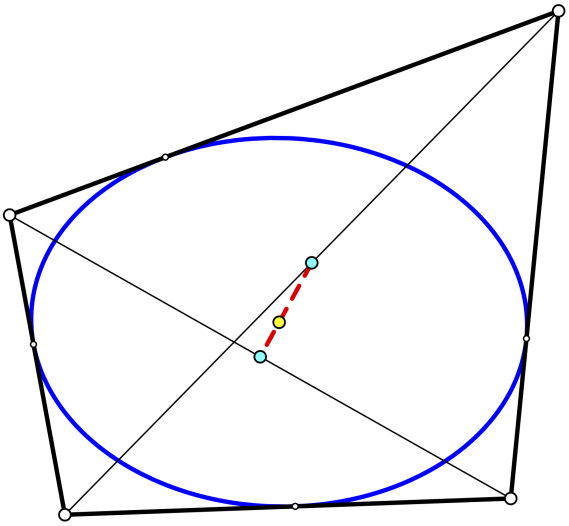
12.23 ellipse inscribed in a quadrilateral

Property 12.23.1. [Ako 11.1.7]



▶ red lines are concurrent.

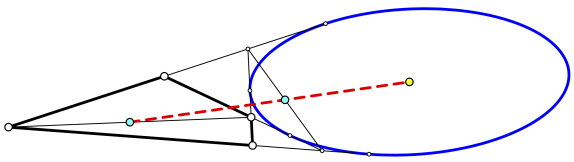
Property 12.23.2.



cyan points are midpoints

▶ colored points are collinear

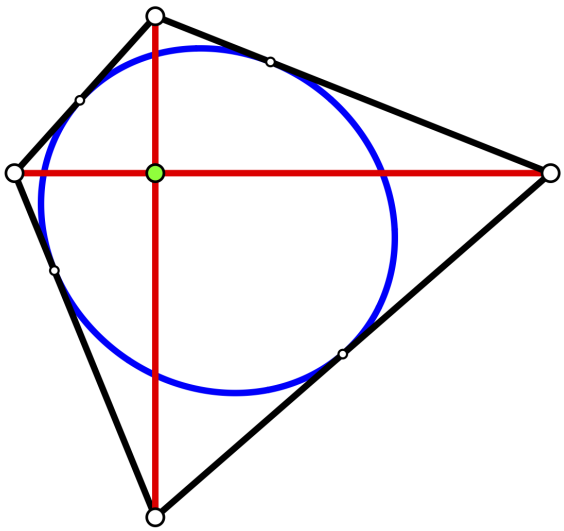
Property 12.23.3.



cyan points are midpoints

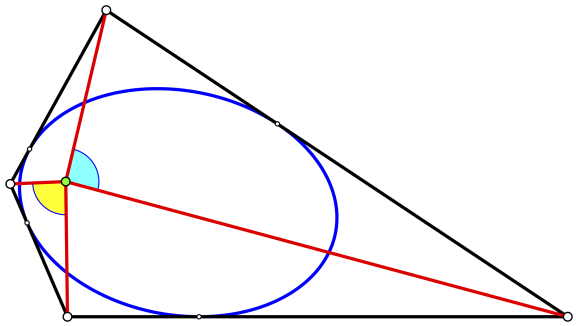
▶ colored points are collinear

Property 12.23.4. [7, p. 149]



▶ red lines are perpendicular.

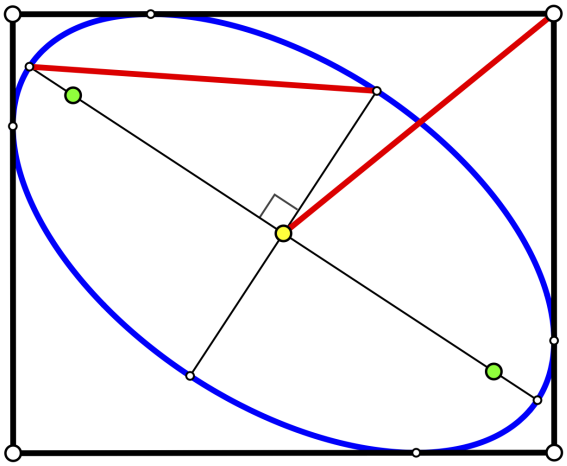
Property 12.23.5. [Ako 11.7]



▶ colored angles are supplementary.

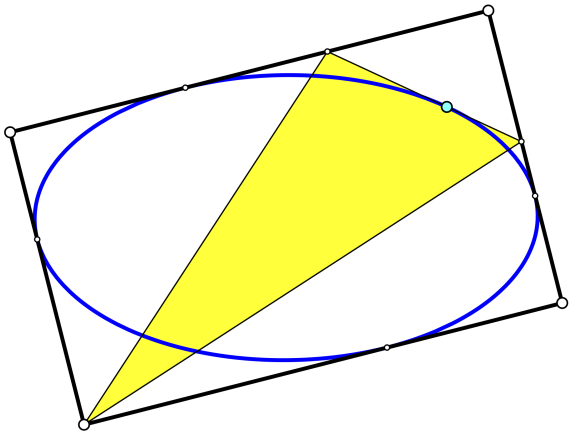
12.24 ellipse inscribed in a rectangle

Property 12.24.1. [24]



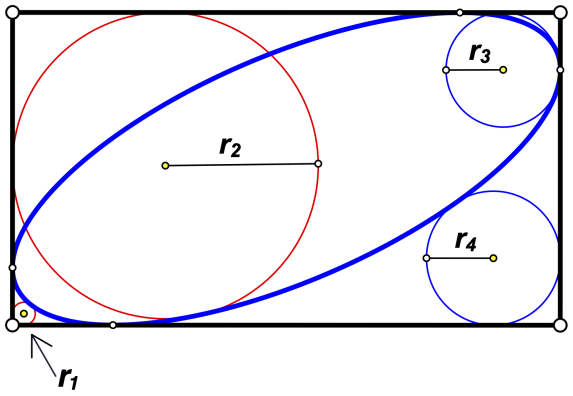
▶ red lengths are equal.

Property 12.24.2. [25]



▶ yellow area is invariant.

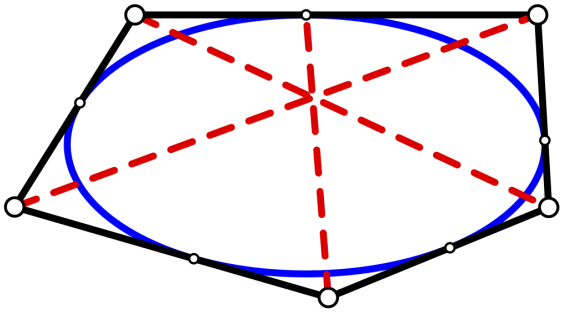
Property 12.24.3. [11, prob. 8.1.4]



▶ $\sqrt{r_1} + \sqrt{r_2} = \sqrt{r_3} + \sqrt{r_4}$

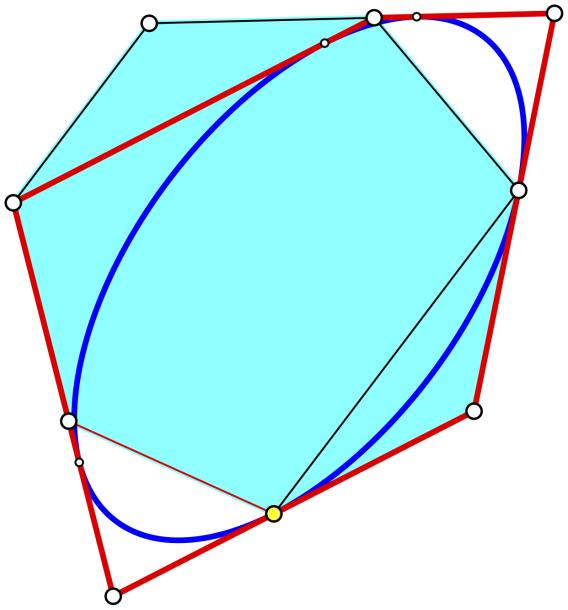
12.25 ellipse inscribed in a pentagon

Property 12.25.1.



▶ red lines are concurrent.

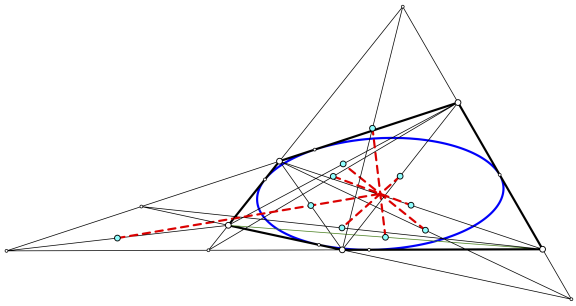
Property 12.25.2.



blue heptagon is regular

▶ inellipse to red pentagon touches it at yellow point.

Property 12.25.3.

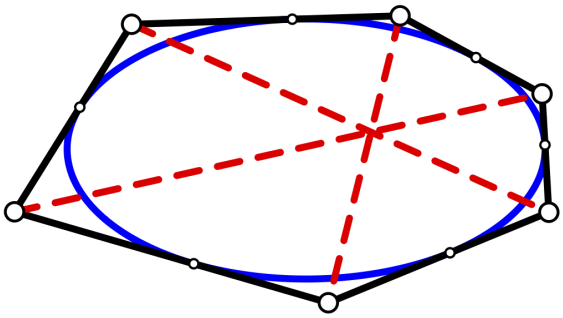


cyan points are midpoints

▶ red lines are concurrent

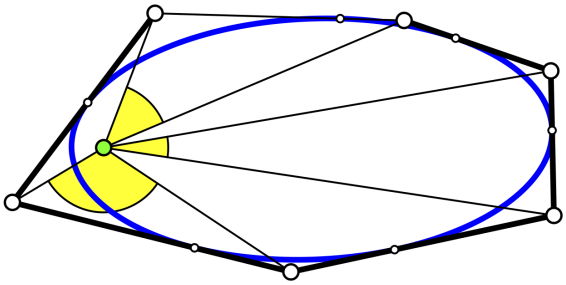
12.26 ellipse inscribed in a hexagon

Property 12.26.1. (Brianchon's Theorem)



▶ red lines are concurrent.

Property 12.26.2. [Ako 11.8]

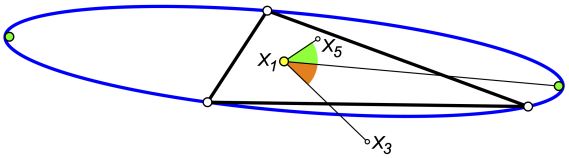


▶ sum of yellow angles = 180°

13. Circumellipses

13.1 Center X_1 (circumellipse)

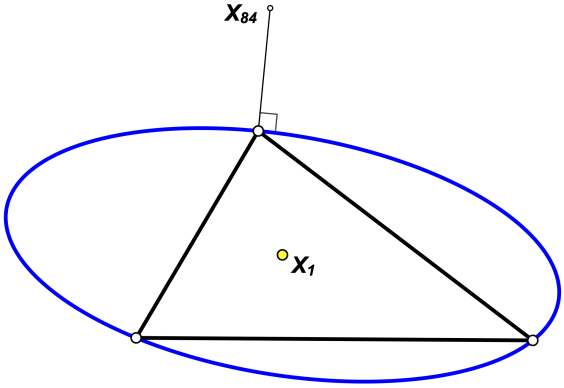
Property 13.1.1.* [42]



▶ colored angles are equal

● Also true for many other centers

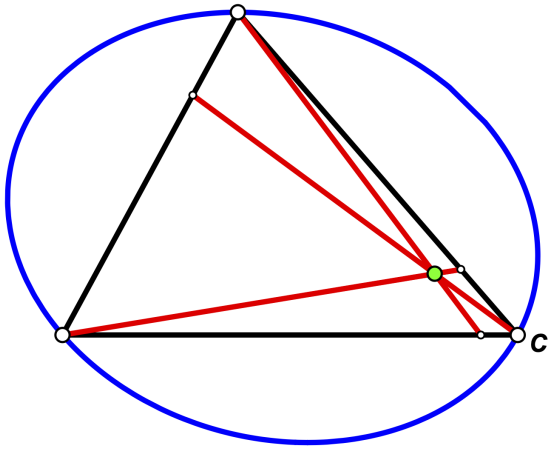
Property 13.1.2.



▶ normals concur

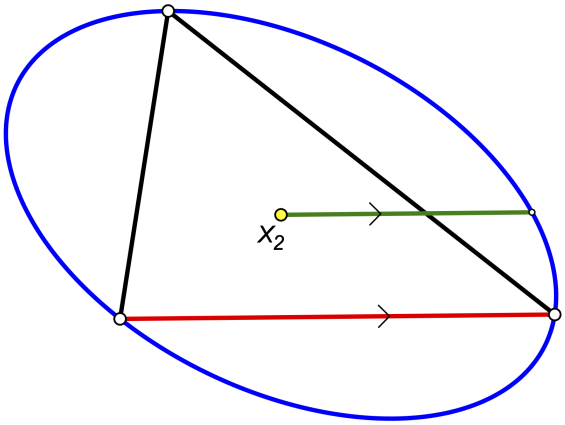
13.2 Center X_2 (Steiner circumellipse)

Property 13.2.1. [67, section 11.3.1]



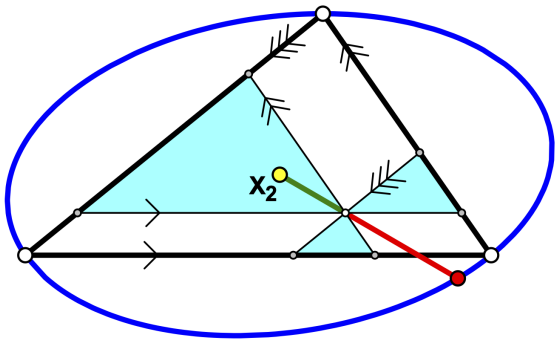
▶ red lengths are equal

Property 13.2.2. [27]



▶ red length = green length $\times\sqrt{3}$

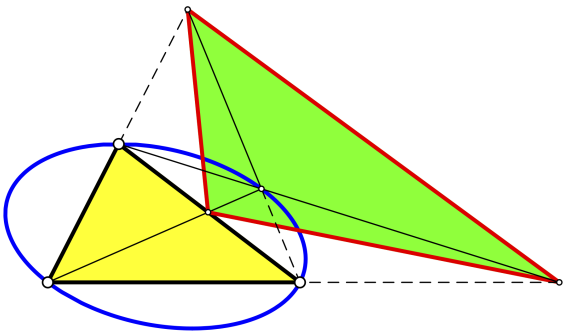
Property 13.2.3.



red length/green length = constant

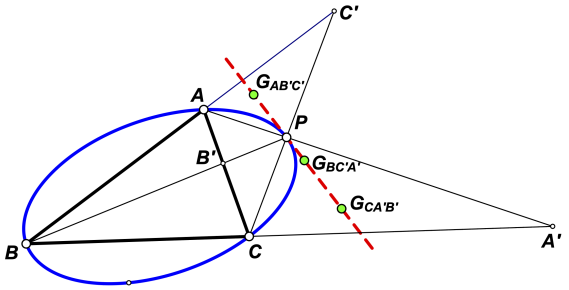
▶ $\sum$cyan areas is constant as red point moves

Property 13.2.4. [64]



▶ green = 2× yellow

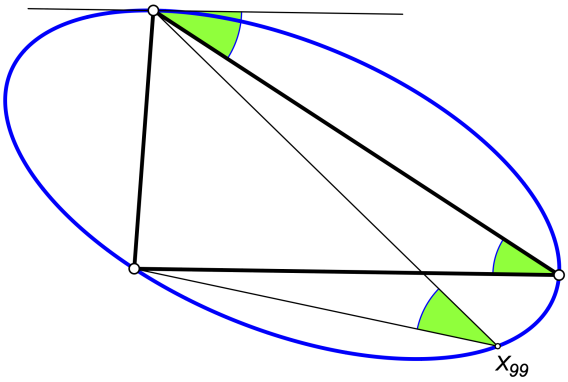
Property 13.2.5. [4]



▶ green centroids are collinear

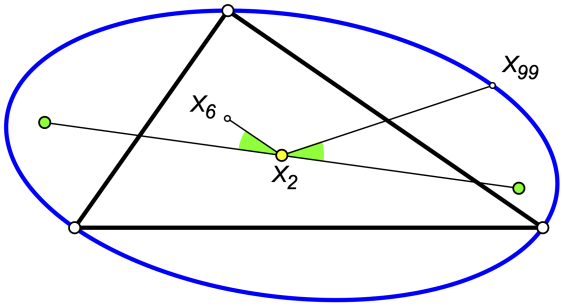
● red line is tangent to ellipse

Property 13.2.6. [51]



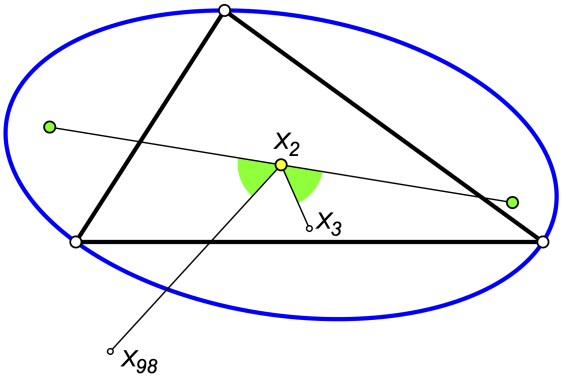
▶ green angles are equal

Property 13.2.7. [64]



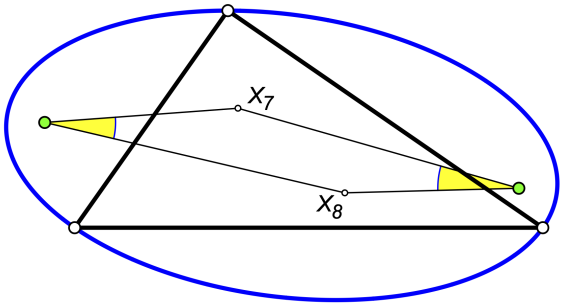
▶ green angles are equal

Property 13.2.8. [64]



▶ green angles are equal

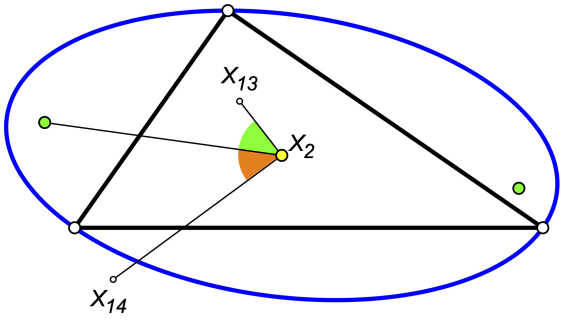
Property 13.2.9.* [40]



▶ yellow angles are equal

● It is interesting to note that the quadrilateral is not a parallelogram.

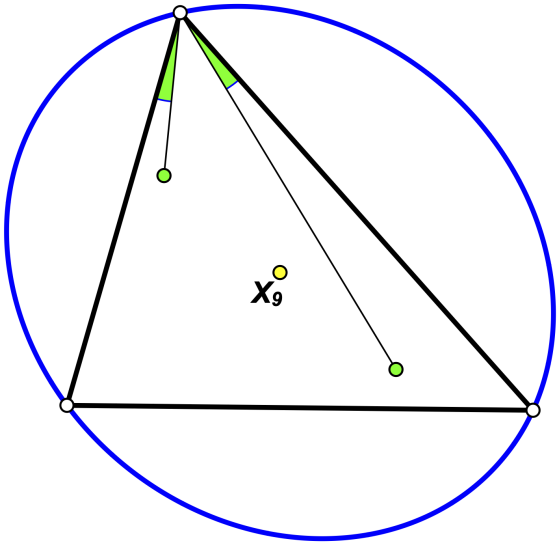
Property 13.2.10.* [41]



► colored angles are equal

13.3 Center X_9 (circumellipse)

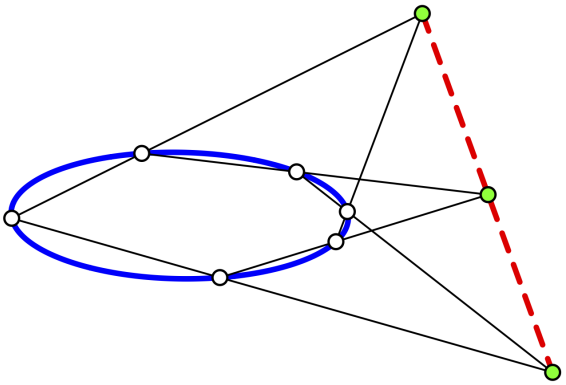
Property 13.3.1.*



► green angles are equal

13.4 Circumconic of hexagon

Property 13.4.1. (Pascal's Theorem)



► green points are collinear

References

[1] Patricia R. Allaire, Junmin Zhou, and Haishen Yao. "Proving a Nineteenth Century Ellipse Identity." The Mathematical Gazette **96**(2012)161–65.
http://www.jstor.org/stable/23249547

[2] Arseniy V. Akopyan, Geometry in Figures, 2nd edition. 2017.

[3] Arseny V. Akopyan and Alexey A. Zaslavsy, Geometry of Conics, American Mathematical Society, 2007.

[4] Francisco Javier García Capitán, A Property of Steiner Circumellipse, Facebook, Feb. 18, 2016.
https://www.facebook.com/fjavier.garcia.capitan/posts/pfbid02VqGvxbPMrVmRAgxCniyXLtoe4Z5znRtrFqMUaqWDo19mFtz9RVQSzpa5rz8VEc9Gl

[5] Francisco Javier García Capitán, Problem 8212, Romantics of Geometry (Facebook Group), May 20, 2021.
https://www.facebook.com/groups/parmenides52/posts/4007812992665710

[6] Ichung Chen, Comment, Romantics of Geometry (Facebook Group), Nov. 8, 2024.
https://www.facebook.com/groups/parmenides52/posts/a-and-b-are-the-semi-major-axis-and-semi-minor-axis-of-the-ellipse/8710723869041242/

[7] Arthur Cockshott and F. B. Walters, A Treatise on Geometrical Conics, Macmillan, 1889.
https://books.google.com/books?id=6_x6tlejDEUC

[8] Jean-Pol Coulon, Proposed Problem, Romantics of Geometry (Facebook Group), Feb. 25, 2025.
https://www.facebook.com/groups/parmenides52/posts/9331781043602185/

[9] William Henry Drew, A Geometrical Treatise on Conic Sections: With Numeous Examples, Macmillan and Company, 1869.
https://books.google.com/books?id=UnL8l8-rbdEC

[10] Lawrence S. Evans, *A Conic Through Six Triangle Centers*. Forum Geometricorum. 2(2002)89–92.
https://forumgeom.fau.edu/FG2002volume2/FG200211.pdf

[11] Hidetosi Fukagawa and Dan Pedoe, Japanese Temple Geometry Problems, Charles Babbage Research Centre, 1989.

[12] Andreas Hatzipolakis, *Euclid message 3072: An Ellipse*. Nov. 10, 2021.
https://groups.io/g/euclid/message/3072

[13] Elias Hagos, Comment on Problem 15114, Romantics of Geometry (Facebook Group), Sept. 6, 2024.
https://www.facebook.com/groups/parmenides52/posts/8212283245551976/

[14] Thomas Little Heath, Apollonius of Perga: Treatise on Conice Sections, Cambridge University Press: 1896.
https://books.google.com/books?id=0cxZUnwuTWIC

[15] Tran Quang Hung, *Post 4750 in ADGEOM: Six points are on Ellipse* June 23, 2018.
http://adgeom.epizy.com/message.php?msg=4750

[16] Alfred Clement Jones, Notes on Analytical Geometry: An Appendix, Clarendon Press, 1903.
https://books.google.com/books?id=yXwLAAAAYAAJ

[17] Clark Kimberling, *Encyclopedia of Triangle Centers*.
https://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[18] Stanley Rabinowitz, Problem SR-72, Plane Geometry Research (Facebook Group), Sept. 8, 2024.
https://www.facebook.com/groups/2008519989391030/posts/3751435425099469/

[19] Stanley Rabinowitz, Problem SR-74, Plane Geometry Research (Facebook Group), Oct. 21, 2024.
https://www.facebook.com/groups/2008519989391030/posts/3797658200477191/

[20] Ivan Pavlov, Problem 10668, Romantics of Geometry (Facebook Group), Aug. 8, 2022.
https://www.facebook.com/groups/parmenides52/posts/5356235731156756/

[21] Ivan Pavlov, Problem 11096, Romantics of Geometry (Facebook Group), Nov. 4, 2022.
https://www.facebook.com/groups/parmenides52/posts/5606266972820296/

[22] Stanley Rabinowitz, Problem 1364, Mathematics Magazine, **64**(1991)60.

[23] Stanley Rabinowitz, Problem 4781, Romantics of Geometry (Facebook Group), Mar. 18, 2020.
https://www.facebook.com/groups/parmenides52/posts/2838126969634324/

[24] Stanley Rabinowitz, Problem 10971, Romantics of Geometry (Facebook Group), Oct. 8, 2022.
https://www.facebook.com/groups/parmenides52/posts/5529390950507899/

[25] Stanley Rabinowitz, Problem 11036, Romantics of Geometry (Facebook Group), Oct. 22, 2022.
https://www.facebook.com/groups/parmenides52/posts/5571445282969132/

[26] Stanley Rabinowitz, Comment on Problem 11505, Romantics of Geometry (Facebook Group), Jan. 12, 2023.
https://www.facebook.com/groups/parmenides52/posts/5811999952246996/

[27] Stanley Rabinowitz, Problem 15113, Romantics of Geometry (Facebook Group), Sept. 8, 2024.
https://www.facebook.com/groups/parmenides52/posts/8224907697622864/

[28] Stanley Rabinowitz, Problem 15115, Romantics of Geometry (Facebook Group), Sept. 8, 2024.
https://www.facebook.com/groups/parmenides52/posts/8224907697622864/

[29] Stanley Rabinowitz, Problem 15062, Romantics of Geometry (Facebook Group), Aug. 21, 2024.
https://www.facebook.com/groups/parmenides52/posts/8109444369169198/

[30] Stanley Rabinowitz, Comment on Problem 15062, Romantics of Geometry (Facebook Group), Aug. 21, 2024.
https://www.facebook.com/groups/parmenides52/posts/8109444369169198/

[31] Stanley Rabinowitz, Problem 15112, Romantics of Geometry (Facebook Group), Sept. 4, 2024.
https://www.facebook.com/groups/parmenides52/posts/8196223420491292/

[32] Stanley Rabinowitz, Problem 15128, Romantics of Geometry (Facebook Group), Sept. 18, 2024.
https://www.facebook.com/groups/parmenides52/posts/8303586823088284/

[33] Stanley Rabinowitz, Problem 15130, Romantics of Geometry (Facebook Group), Sept. 20, 2024.
https://www.facebook.com/groups/parmenides52/posts/8322124264567873/

[34] Stanley Rabinowitz, Problem 15131, Romantics of Geometry (Facebook Group), Sept. 20, 2024.
https://www.facebook.com/groups/parmenides52/posts/8322308851216081/

[35] Stanley Rabinowitz, Problem 15138, Romantics of Geometry (Facebook Group), Sept. 23, 2024.
https://www.facebook.com/groups/parmenides52/posts/8349530928493873/

[36] Stanley Rabinowitz, Problem 15140, Romantics of Geometry (Facebook Group), Sept. 23, 2024.
https://www.facebook.com/groups/parmenides52/posts/8350458758401090/
[37] Stanley Rabinowitz, Problem 15141, Romantics of Geometry (Facebook Group), Sept. 23, 2024.
https://www.facebook.com/groups/parmenides52/posts/8350710278375938/
[38] Stanley Rabinowitz, Problem 15148, Romantics of Geometry (Facebook Group), Sept. 27, 2024.
https://www.facebook.com/groups/parmenides52/posts/8390274834419482/
[39] Stanley Rabinowitz, Problem 15150, Romantics of Geometry (Facebook Group), Sept. 28, 2024.
https://www.facebook.com/groups/parmenides52/posts/8399313230182309/
[40] Stanley Rabinowitz, Problem 15160, Romantics of Geometry (Facebook Group), Oct. 5, 2024.
https://www.facebook.com/groups/parmenides52/posts/8468163799963918/
[41] Stanley Rabinowitz, Problem 15161, Romantics of Geometry (Facebook Group), Oct. 5, 2024.
https://www.facebook.com/groups/parmenides52/posts/8468313959948902/
[42] Stanley Rabinowitz, Problem 15162, Romantics of Geometry (Facebook Group), Oct. 5, 2024.
https://www.facebook.com/groups/parmenides52/posts/8468573783256253/
[43] Stanley Rabinowitz, Problem 15501, Romantics of Geometry (Facebook Group), Nov. 30, 2024.
https://www.facebook.com/groups/parmenides52/posts/8834306033349691/
[44] Stanley Rabinowitz, Problem 15502, Romantics of Geometry (Facebook Group), Nov. 30, 2024.
https://www.facebook.com/groups/parmenides52/posts/8834388286674799/
[45] Stanley Rabinowitz, Problem 15504, Romantics of Geometry (Facebook Group), Dec. 3, 2024.
https://www.facebook.com/groups/parmenides52/posts/8851305184983109/
[46] Stanley Rabinowitz, Problem 15506, Romantics of Geometry (Facebook Group), Dec. 4, 2024.
https://www.facebook.com/groups/parmenides52/posts/8857370654376562/
[47] Stanley Rabinowitz, Problem 15507, Romantics of Geometry (Facebook Group), Dec. 4, 2024.
https://www.facebook.com/groups/parmenides52/posts/8857466197700341/
[48] Stanley Rabinowitz, Problem 15508, Romantics of Geometry (Facebook Group), Dec. 4, 2024.
https://www.facebook.com/groups/parmenides52/posts/8857806274333000/
[49] Stanley Rabinowitz, Problem 15063, Romantics of Geometry (Facebook Group), Aug. 21, 2024.
https://www.facebook.com/groups/parmenides52/posts/8110501565730145/
[50] Stanley Rabinowitz, Problem 15100, Romantics of Geometry (Facebook Group), Nov. 8, 2024.
https://www.facebook.com/groups/parmenides52/posts/8710723869041242/
[51] Stanley Rabinowitz, Problem 15126, Romantics of Geometry (Facebook Group), Sept. 17, 2024.
https://www.facebook.com/groups/parmenides52/posts/8296909363756030/
[52] Stanley Rabinowitz, Problem 15136, Romantics of Geometry (Facebook Group), Sept. 21, 2024.
https://www.facebook.com/groups/parmenides52/posts/8332785040168462/
[53] Stanley Rabinowitz, Problem 15137, Romantics of Geometry (Facebook Group), Sept.22, 2024.
https://www.facebook.com/groups/parmenides52/posts/8341358995977733/
[54] Stanley Rabinowitz, Problem 15149, Romantics of Geometry (Facebook Group), Sept.27, 2024.
https://www.facebook.com/groups/parmenides52/posts/8390423464404619/
[55] Kousik Sett, Problem 15505, Romantics of Geometry (Facebook Group), Dec. 2, 2024.
https://www.facebook.com/groups/parmenides52/posts/8844365425677085/
[56] Vincenzo Sicari, Comment on Problem 15501, Romantics of Geometry (Facebook Group), Nov. 30, 2024.
https://www.facebook.com/groups/parmenides52/posts/8834306033349691/
[57] Vincenzo Sicari, Comment on Problem 15502, Romantics of Geometry (Facebook Group), Nov. 30, 2024.
https://www.facebook.com/groups/parmenides52/posts/8834388286674799/
[58] Benedetto Scimemi, "Simple relations regarding the Steiner inellipse of a triangle." Forum Geometricorum. **10**(2010)55–77.
[59] Vincenzo Sicari, Problem 16001, Romantics of Geometry (Facebook Group), Dec. 4, 2024.
https://www.facebook.com/groups/parmenides52/posts/8855137544599873/
[60] Victor Thébault, Problem 3874, American Mathematical Monthly, **45**(1938)323.
[61] H. F. Sandham, Problem 4043, American Mathematical Monthly, **49**(1942)408.
[62] Eric W. Weisstein, *Ellipse*, From MathWorld–A Wolfram Web Resource.
https://mathworld.wolfram.com/Ellipse.html
[63] Eric W. Weisstein, *Mandart Inellipse*, From MathWorld–A Wolfram Web Resource.
https://mathworld.wolfram.com/MandartInellipse.html
[64] Eric W. Weisstein, *Steiner Circumellipse*, From MathWorld–A Wolfram Web Resource.
https://mathworld.wolfram.com/SteinerCircumellipse.html
[65] Wikipedia contributors, *Ellipse*, Wikipedia, The Free Encyclopedia.
https://en.wikipedia.org/w/index.php?title=Ellipse&oldid=1254592540
[66] Wikipedia contributors, *Steiner inellipse*, Wikipedia, The Free Encyclopedia.
https://en.wikipedia.org/w/index.php?title=Steiner_inellipse&oldid=1258902987
[67] Paul Yiu, *Introduction to the Geometry of the Triangle*, 2004.
http://math.fau.edu/yiu/Oldwebsites/MSTGeometry2004/000geombookmaster.pdf
[68] 2′5 9′2, Answer to "Normal to Ellipse and Angle at Major Axis", Mathematics Stack Exchange, Oct. 25, 2014.
https://math.stackexchange.com/q/990013