Ellipse Constructions and Delights

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Abstract. We show how to perform various geometric constructions involving an ellipse using a dynamic geometry environment such as Geometer's Sketchpad. Many of these can be effected using only straightedge and compasses. This allows us to make drawings of many of the classic results about ellipses. For example, we show how to construct the common tangents to two intersecting ellipses. We also show how to construct various circles thangent to a given ellipse satisfying special conditions. We use these constructions to illustrate interesting properties of the ellipse.

Keywords. ellipse, straightedge and compasses, Geometer's Sketchpad, conics.

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1. Introduction

Japanese geometers of the Edo period were fond of finding results about ellipses. For example, the following theorem was inscribed on a wooden tablet and hung in a temple in the Tochigi prefecture in 1901 [24, problem 6.1]. The notation $O(a, b)$ denotes an ellipse with center O and whose semi-major and semi-minor axes have lengths a and b respectively.

Theorem 1.1. *Let L_1 and L_2 be two parallel tangents to the ellipse $O(a, b)$. A circle with center C is tangent to both L_1 and L_2 and is also externally tangent to the ellipse. Then $OC = a + b$.*

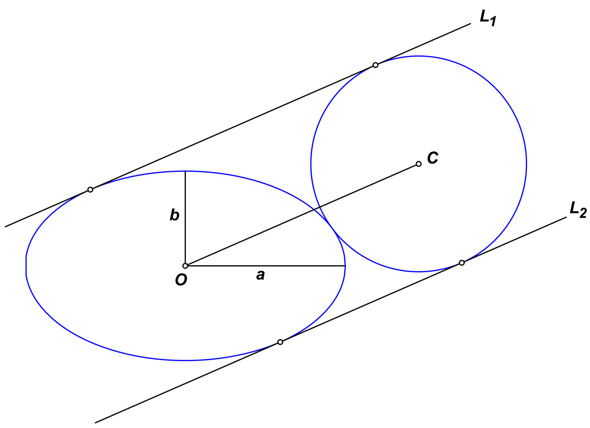


Figure 1. $OC = a + b$

Modern geometers also study ellipses, especially ellipses associated with triangles and triangle centers. Many of these are named after famous mathematicians, such as the Steiner circumellipse, the Brocard inellipse, the Hofstadter ellipse, the Lemoine ellipse, and the Mandart inellipse.

When writing scholarly papers that reference some of these results, it is useful to be able to accompany the result with an accurate drawing (such as Figure 1). Modern geometers often use dynamic geometry environments (DGEs) such as Geometer's Sketchpad (GSP), Cabri, or GeoGebra to create these drawings. Such programs are also useful to geometers by helping them explore configurations and make conjectures about results that appear to be true based on dynamic variation of the points within the configuration. Unfortunately, most of these programs do not include built-in options for drawing and manipulating ellipses, such as constructing an ellipse with a given center inscribed in a triangle or constructing a circle tangent to a given ellipse and two given circles.

These programs all have different capabilities, but they all support the core straightedge and compass constructions promulgated by the ancient Greek geometers and promoted by Euclid in his treatise, The Elements. These include operations like drawing a line between two points and drawing a circle centered at a given point and passing through another given point. Many of these programs allow the user to write scripts or macros that add new capabilities to the language by combining previous constructions.

It is the purpose of this paper to show how many constructions involving ellipses can be performed using DGEs (and GSP in particular). Many of these can be performed using only the basic straightedge and compass tools. Most of these constructions are known and scattered throughout the literature. We collect these here in a single place. We also intersperse constructions with delightful examples that incorporate these constructions.

An ellipse is determined by any five distinct points on its circumference. Since not all DGEs allow displaying an ellipse, we will say that the ellipse has been constructed if we can construct five points on the ellipse. When performing constructions involving an ellipse using straightedge and compasses, we will not assume that the ellipse has been drawn in its entirety, but merely that five points on the ellipse are known. So, for example, it is not trivial to find the points of intersections of a given ellipse and a given straight line, because the complete perimeter of the ellipse is not present in the drawing.

2. Notation and Conventions

Notation for Basic Constructions	
notation	meaning
foot(P, L)	foot of perpendicular from point P to line L
midpt(AB)	midpoint of line segment AB
parallel(P, L)	line through point P parallel to line L
perp(P, L)	line through point P perpendicular to line L
reflect(P, L)	reflect point P about line L
reflect(P, Q)	reflect point P about point Q
dilate(P, L, r)	dilate point P toward line L with ratio r
$O(A)$	circle with center O passing through point A
$O(AB)$	circle with center O with radius the length of AB
$O(r)$	circle with center O with radius r
$X(A, B)$	harmonic conjugate of X with respect to (A, B)
$\odot O$	circle with center O
$\odot(AB)$	circle with diameter AB
$\odot(ABC)$	circle through points A, B, and C
CLP(O, A, L, P)	center of circle tangent to $O(A)$ and L and passing through P
$L_1 \cap L_2$	intersection of lines L_1 and L_2
$\mathcal{C} \cap AB$	if $A \in \mathcal{C}$, this is the 2nd intersection of AB with $\mathcal{C}$
perpBisector(AB)	the perpendicular bisector of segment AB
angleBisector(L_1, L_2)	the bisector of the angle formed by lines L_1 and L_2
$V \in \mathcal{C}$	V is a point on the curve $\mathcal{C}$
$V \notin \mathcal{C}$	V is a point not on the curve $\mathcal{C}$

When specifying algorithms for constructions, the steps for the construction will be accompanied by a figure. In this figure, items (such as points and lines) colored green are inputs to the algorithm, and items colored red denote items that are constructed by the algorithm.

If an ellipse is shown as a dashed curve, this means that the entire ellipse is not given. For example, if an ellipse is implied as one passing through five given points, then the five points will be colored green and the ellipse will be dashed.

If a construction draws a locus, the locus will be shown with brown dots.

Some constructions determine two points. For example, the construction that finds the points of intersection of two circles will return two points. In most DGEs the two points are returned in no specific order. If a construction, f, returns two points P and Q, where the order of the two points is not specified, we will write this as $\{P, Q\} = \mathrm{f}(x, y, z, \ldots)$. If the order of the two points is specified, we will write this as $(P, Q) = \mathrm{f}(x, y, z, \ldots)$. If we write $P = \mathrm{f}(x, y, z, \ldots)$, this means that P is one of the two points and it doesn't matter which one. The same convention holds when a construction returns more than two points.

If `map` is a function that maps points into points and point A gets mapped to point B, then we will write $B = \texttt{map}(A)$ or we will write $\texttt{map} : A \to B$.

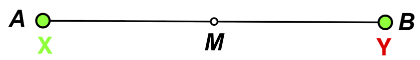
3. HELPER CONSTRUCTIONS

We start by presenting some general constructions that are not specific to ellipses. We will need these constructions later.

3.1. **Points and Lines.**

`Construction Other Endpoint.`

Given: points A and B and a point X known to be one of the endpoints of line segment AB, but you don't know which.
Constructs: the other endpoint Y.

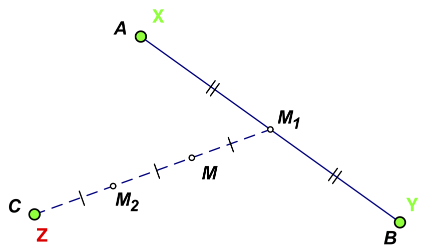


1. $M = \text{midpt}(AB)$.
2. $Y = \text{reflect}(X, M)$.

Note. The figure shows the case when $X = A$.

`Construction Third Vertex.`

Given: $\triangle ABC$ and two points X and Y known to be in $\{A, B, C\}$. That is, X and Y are known to be vertices of $\triangle ABC$, but you don't know which ones.
Constructs: the third vertex Z.
Referenced as: $Z = \text{thirdVertex}(\triangle ABC, X, Y)$



1. $M = \text{centroid}(\triangle ABC)$.
2. $M_1 = \text{midpt}(XY)$.
3. $M_2 = \text{reflect}(M_1, M)$.
4. $Z = \text{reflect}(M, M_2)$.

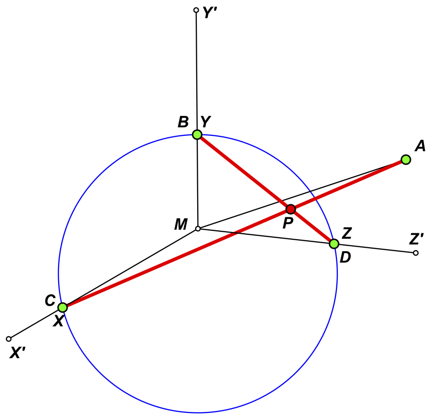
Note. The figure shows the case when $X = A$ and $Y = B$.

The intersection of the diagonals of a convex quadrilateral is known as the *diagonalpoint* of the qudrilateral.

Let $\measuredangle XYZ$ denote the angle between 0° and 360° that ray $\overrightarrow{YX}$ must be rotated counterclockwise about point Y to get it to coincide with ray $\overrightarrow{XZ}$.

`Construction Diagonal Point.`

Given: four points A, B, C, and D. It is known that these points are the vertices of a convex quadrilateral, but you don't know in which order.
Constructs: the diagonals AX and YZ and the diagonal point P.
Referenced as: $P = \text{diagonalPoint}(A, B, C, D)$



1. $M = \text{centroid}(\triangle BCD)$.
2. $\theta_1 = \measuredangle AMB$. $\theta_2 = \measuredangle AMC$. $\theta_3 = \measuredangle AMD$.
3. $(\phi_1, \phi_2, \phi_3) = \text{sort}(\theta_1, \theta_2, \theta_3)$.
4. $Y' = \text{rotate}(A, M, \phi_1)$.
5. $X' = \text{rotate}(A, M, \phi_2)$.
6. $Z' = \text{rotate}(A, M, \phi_3)$.
7. $Y = \overrightarrow{MY'} \cap \odot BCD$.
8. $X = \overrightarrow{MX'} \cap \odot BCD$.
9. $Z = \overrightarrow{MZ'} \cap \odot BCD$.
10. $P = AX \cap YZ$.

Note 1. The figure shows the case when $X = C$, $Y = B$, and $Z = D$.

Note 2. When implementing this construction as a tool in GSP, the units preference for angles must be set to "directed degrees". This allows angles to be measured in the range $(-180°, 180°]$ rather than the range $[0, 180°)$. However, for this construction, we need angles in the range $[0°, 360°)$. To convert an angle measurement θ from GSP's default to the value we need, use the formula

$$\theta + 180°(1 - \text{sgn}(\theta))$$

where $sgn(x)$ is the built-in GSP function signum defined as

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$$

Note 3. Many DGEs do not have a sort function. This function can be emulated in GSP as follows.

$$\begin{aligned}
\min(x, y) &= \frac{x + y - |x - y|}{2} \\
\phi_1 = \min(\theta_1, \theta_2, \theta_3) &= \min\bigl(\min(\theta_1, \theta_2), \theta_3\bigr) \\
\max(x, y) &= \frac{x + y + |x - y|}{2} \\
\phi_2 = \max(\theta_1, \theta_2, \theta_3) &= \max\bigl(\max(\theta_1, \theta_2), \theta_3\bigr) \\
\phi_3 = \text{middle}(\theta_1, \theta_2, \theta_3) &= \theta_1 + \theta_2 + \theta_3 - \min(\theta_1, \theta_2, \theta_3) - \max(\theta_1, \theta_2, \theta_3) \\
\text{sort}(\theta_1, \theta_2, \theta_3) &= (\phi_1, \phi_2, \phi_3)
\end{aligned}$$

3.2. **Circle Constructions.**

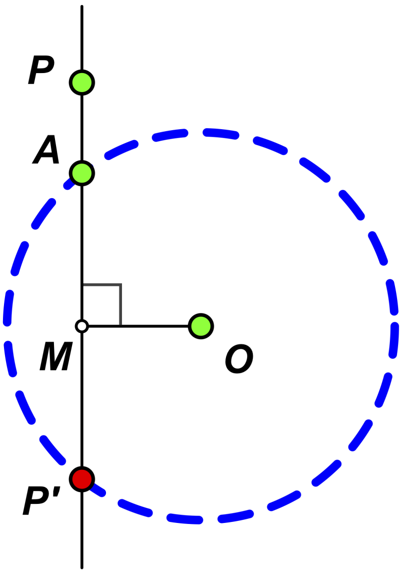
When finding the two intersections of a line with a circle, most DGEs construct the two points in a random order. Sometimes, within scripts, the order is important. The following construction is useful.

`Construction Second Point of Intersection of a Line with a circle.`

Given: points O and A and a point P.

Constructs: the second point of intersection, P', of the line AP with the circle $O(A)$.

Referenced as: $P' = AP \cap O(A)$



1. $M = \text{foot}(O, AP)$.
2. $P' = \text{reflect}(A, M)$.

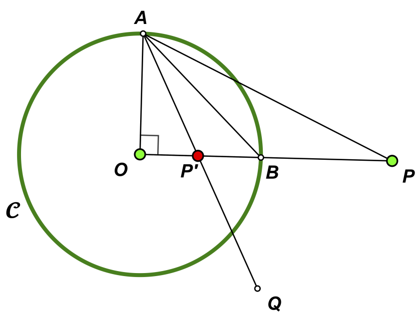
The following construction comes from [22].

`Construction Dutta's Construction for the Inverse of a Point.`

Given: circle $\mathcal{C}$ with center O and any point P other than O.

Constructs: P', the inverse of P with respect to $\mathcal{C}$.

Referenced as: $P' = \text{inverse}(P, \mathcal{C})$



1. $B = \overrightarrow{OP} \cap \mathcal{C}$.
2. $A = \text{perp}(O, OP)$.
3. $Q = \text{reflect}(P, AB)$.
4. $P' = AQ \cap OP$.

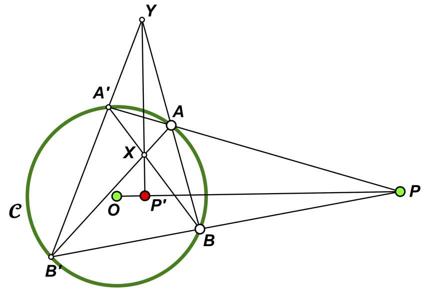
Note 1. A can be any point on $\mathcal{C}$ not on OP.

Note 2. The figure shows the case where P is outside the circle, but the construction works whether P is inside, on, or outside the circle.

The following construction is of interest because it only uses a straightedge, but it is not as general as the previous construction.

`Construction Inverse of a Point Using a Straightedge.`

Given: circle $\mathcal{C}$ with center O and any point P other than O.
Constructs: P', the inverse of P with respect to $\mathcal{C}$.
Referenced as: $P' = \text{inverse}(P, \mathcal{C})$



1. $A \in \mathcal{C}$. $B \in \mathcal{C}$.
2. $X = A'B \cap AB'$.
3. $Y = AB \cap A'B'$.
4. $P' = XY \cap OP$.

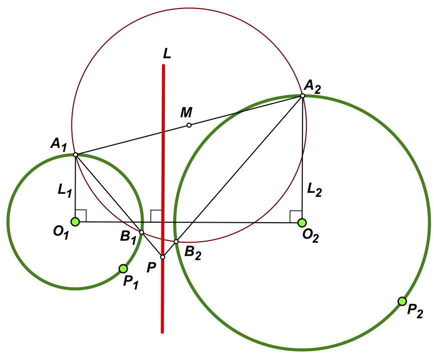
Note 1. This construction fails if A or B lies on OP or if $AB \perp OP$ or if PA or PB is tangent to $\mathcal{C}$ or if $P \in \mathcal{C}$.

The following construction can be used to find the radical axis of two circles. It is derived from [74] where we first construct a circle intersecting the two given circles. The construction works for any two circles. They can be disjoint, intersecting, or even one inside the other.

Roughly speaking, the *radical axis* of two circles is the locus of points from which tangents to the two circles have the same length. When the two circles intersect, this is the common chord.

`Construction Radical Axis.`

Given: two nonconcentric circles $O_1(P_1)$ and $O_2(P_2)$.
Constructs: L, the radical axis of the two circles.

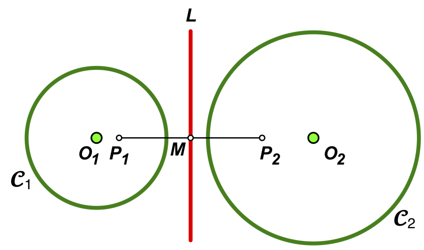


1. $L_1 = \text{perp}(O_1, O_1O_2)$.
2. $A_1 = L_1 \cap O_1(P_1)$.
3. $L_2 = \text{perp}(O_2, O_1O_2)$.
4. $A_2 = L_2 \cap O_2(P_2)$.
5. $M = \text{midpt}(A_1A_2)$.
6. $\{A_1, B_1\} = M(A_1) \cap O_1(P_1)$.
7. $\{A_2, B_2\} = M(A_2) \cap O_2(P_2)$.
8. $P = A_1B_1 \cap A_2B_2$.
9. $L = \text{perp}(P, O_1O_2)$.

A simpler construction comes from [17].

`Construction Radical Axis.`

Given: two nonconcentric circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers O_1 and O_2, respectively.
Constructs: L, the radical axis of the two circles.

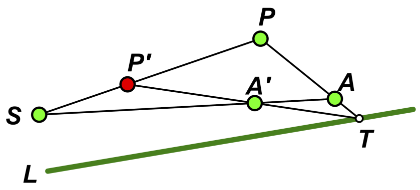


1. $P_1 = \text{inverse}(O_2, \mathcal{C}_1)$.
2. $P_2 = \text{inverse}(O_1, \mathcal{C}_2)$.
3. $M = \text{midpt}(P_1P_2)$.
4. $L = \text{perp}(M, P_1P_2)$.

3.3. Perspectivites.

`Construction Project Point.`

Given: the center of perspectivity S and the axis of perspectivity L of the perspectivity that projects A into A'.
Also given: a point P.
Constructs: the image P' of P under this perspectivity.
Referenced as: $P = \text{project}(P, S, L, A \to A')$



1. $T = PA \cap L$.
2. $P' = TA' \cap SP$.

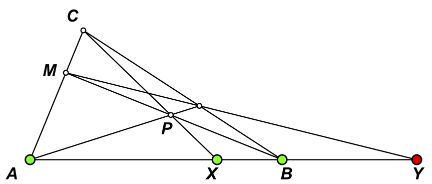
3.4. Involutions.

For some constructions, we will need to know some facts about harmonic conjugates and involutions.

If A, B, C, and D are four points on a straight line, then the pairs (A, B) and (C, D) are said to be *harmonic conjugates* if $AC/BC = AD/BD$.

`Construction Harmonic Conjugate.`

Given: three collinear points A, B, and X.
Constructs: the harmonic conjugate Y of X with respect to (A, B). This means that $AX/XB = AY/YB$.
Referenced as: $Y = X(A, B)$



1. $C \notin AB$.
2. $M \in AC$.
3. $P = BM \cap CX$.
4. $Q = AP \cap BC$.
5. $Y = MQ \cap AB$.

Note 1. This construction can be performed using only a straightedge.

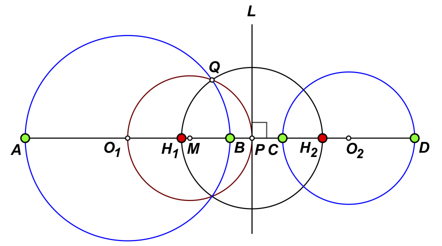
Note 2. GSP does not let you create a random point outside a line when writing a custom tool. If your DGE does not allow you to do the $C \notin AB$ construction inside a script, then change step 1 to $C = A(X) \cap X(A)$.

Note 3. If your DGE does not allow you to do the $M \in AC$ construction inside a script, then change step 2 to $M = \text{midpt}(AC)$.

If A, B, C, and D are four points on a straight line, then there are two points X_1 and X_2 such that (X_1, X_2) is a harmonic conjugate of both (A, B) and (C, D). Using the nomenclature from [40], these points are called common harmonics. They are also called *double points*. In the projective geometry literature, the points A, B, C, and D determine an involution and the points X_1 and Y_1 are called the *foci of the involution*. The midpoint of X_1Y_1 is called the *center of the involution*.

`Construction Common Harmonics.`

Given: four points A, B, C, and D on a straight line.
Constructs: H_1 and H_2, the common harmonics of (A, B) and (C, D).
Referenced as: $\{H_1, H_2\} = \text{double}(A, B, C, D)$

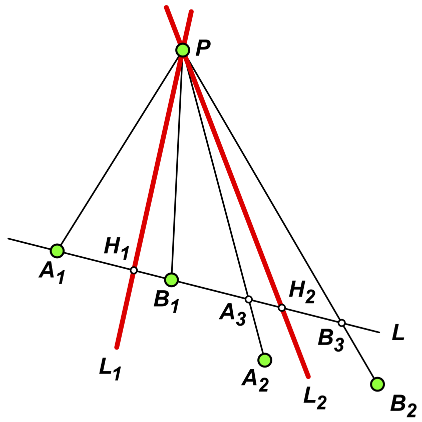


1. $O_1 = \text{midpt}(A, B)$.
2. $O_2 = \text{midpt}(C, D)$.
3. $L = \text{radicalAxis}(O_1(A), O_2(D))$.
4. $P = L \cap AD$.
5. $M = \text{midpt}(PO_1)$.
6. $Q = M(P) \cap O_1(A_1)$.
7. $\{H_1, H_2\} = P(Q) \cap AD$.

Let (L_1, L_2) and (L_3, L_4) be two pair of lines all passing through the same point P. Let L be any line meeting these lines at Q_1, Q_2, Q_3, and Q_4. Let the common harmonics of (Q_1, Q_2) and (Q_3, Q_4) be H_1 and H_2. Then PH_1 and PH_2 are called *double rays*.

`Construction Double Rays.`

Given: four distinct lines PA_1, PB_1, PA_2, and PB_2 through a point P.
Constructs: L_1 and L_2, the double rays of this pencil.
Referenced as: $\{L_1, L_2\} = \text{double}(PA_1, PB_2, PA_2, PB_2)$



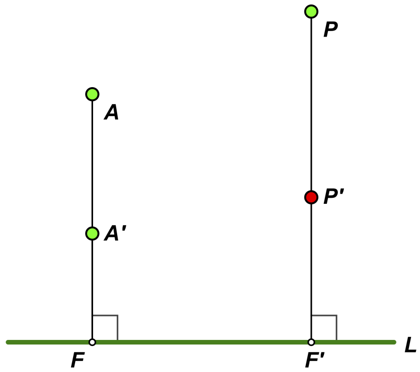
1. $L = A_1B_1$.
2. $A_3 = PA_2 \cap L$.
3. $B_3 = PB_2 \cap L$.
4. $\{H_1, H_2\} = \text{double}(A_1, B_1, A_3, B_3)$.
5. $L_i = PH_i$, $i = 1, 2$.

4. Transformations

A *shear* is a linear map that displaces each point in a fixed direction, by an amount proportional to its signed distance from a line perpendicular to the given direction. More precisely, let L be a fixed line in the plane and let k be a nonzero real number. The line L bounds two regions in the plane. Let one half-plane be considered the positive side of L and the other half-plane the negative side of L. If A is a point not on L, let $D(A)$ denote the perpendicular distance from A to L with a positive sign if A lies on the positive side of L and a negative sign if A lies on the negative side of L. The line L is called the *axis of the shear* and k is the *shear factor*. The direction perpendicular to L is called the *direction of the shear*. Let n be the line through A perpendicular to L. The image of A under the shear is the point B on n such that $|D(A) - D(B)|/D(A) = k$.

Construction Shear

Given: line L and points A, A', and P, with $AA' \perp L$.
Constructs: P', the image of P under the shear with axis L that maps A into A'.
Referenced as: $P' = \text{shear}(L, A \to A', P)$
Also referenced as: $S = \text{shear}(L, A \to A')$; $S : P \to P'$



1. $F = \text{foot}(A, L)$.
2. $F' = \text{foot}(P, L)$.
3. Construct P' on PF' such that $PP'/PF' = AA'/AF$.

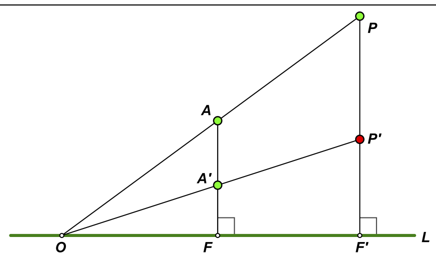
Note 1. A shear is a transformation that maps points into points and lines into lines. A shear maps ellipses into ellipses.

Note 2. The inverse of the shear $\text{shear}(L, A \to A')$ is the shear $\text{shear}(L, A' \to A)$.

If your DGE does not have a shear tool, you can construct a shear using straightedge and compass as follows.

Construction Naive Shear

Given: line L and points A, A', and P, with $AA' \perp L$.
Constructs: P', the image of P under the shear with axis L that maps A into A'.
Referenced as: $P' = \text{shear}(L, A \to A', P)$
Also referenced as: $S = \text{shear}(L, A \to A')$; $S : P \to P'$



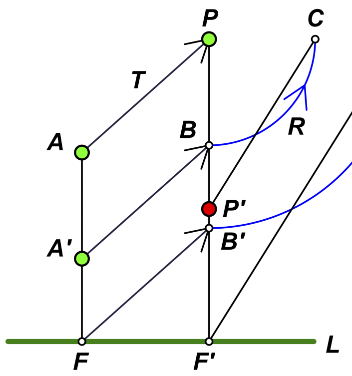
1. $F' = \text{foot}(P, L)$.
2. $O = PA \cap L$.
3. $P' = OA' \cap PF'$.

Note. This construction fails if $PA \parallel L$ or if $P = A$ or if $P \in AA'$.

The following construction using translate and rotate tools never fails.

`Construction Shear via Translate and Rotate`

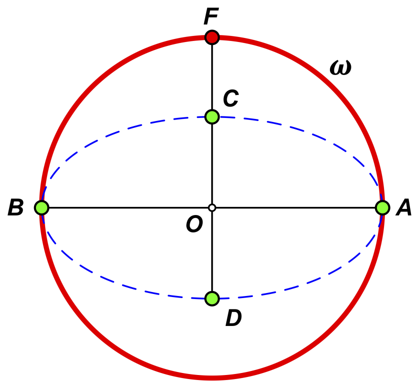
Given: line L and points A, A', and P, with $AA' \perp L$.
Constructs: P', the image of P under the shear with axis L that maps A into A'.
Referenced as: $P' = \text{shear}(L, A \to A', P)$
Also referenced as: $S = \text{shear}(L, A \to A')$; $S : P \to P'$



1. $T = \text{translation}(A \to P)$.
2. $T : A' \to B$. $T : F \to B'$.
3. $R = \text{rotation}(P, 90°)$.
4. $R : B \to C$. $R : B' \to C'$.
5. $P' = \text{parallel}(C, C'F') \cap PF'$.

`Construction Shear Ellipse into a Circle`

Given: ellipse with major axis AB and minor axis C
Constructs: a shear S that maps the ellipse into a circle.



1. $O = AB \cap CD$.
2. $\omega = O(A)$.
3. $F = \overrightarrow{OC} \cap \omega$.
4. $S = \text{shear}(AB, C \to F)$

5. Intersections of Lines and 5-point Conics

Many constructions involving ellipses work for other conics as well (parabolas and hyperbolas). We start by surveying some of the basic constructions involving conics.

The following construction comes from [77, Alg. 12.5.2].

`Construction Second Intersection with 5-point Conic.`

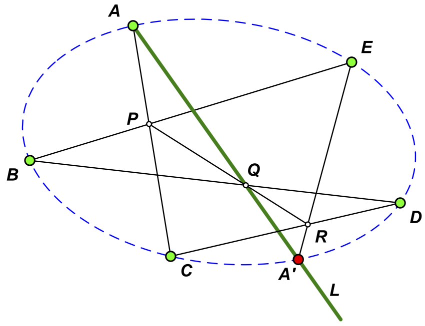
Given: five points, A, B, C, D, and E with no three collinear and given a line L through A.

Constructs: the second intersection, A', of L and the conic, $\mathcal{C}$, through the five points.

Referenced as: $A' = \text{second}(A, B, C, D, E, L)$

Also referenced as: $A' = AP \cap \mathcal{C}$ when $L = AP$

Also referenced as: $A' = L \cap \mathcal{C}$ when it is clear that there is a point A that is on both $\mathcal{C}$ and L



1. $P = AC \cap BE$.
2. $Q = L \cap BD$.
3. $R = PQ \cap CD$.
4. $A' = L \cap ER$.

Note. This construction fails if $AC \parallel BE$. In that case, point P does not exist. (We assume that your DGE does not support points at infinity.) This construction would never fail if we were working in the projective plane, since then any two distinct lines would intersect in a point. But most DGEs work in the Euclidean plane rather than the projective plane. All constructions that we give are for the Euclidean plane.

While it is easy for a human to select the order of the five points in such a way that AC will not be parallel to BE, a computer must be told how to do that. We need an algorithm that can be used within other construction tools and will work for any given five points (no three collinear).

The following construction comes from [41].

`Construction Second Intersection with 5-point Conic.`

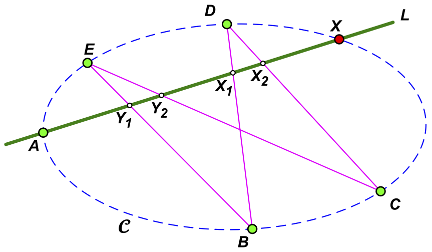
Given: five points, A, B, C, D, and E with no three collinear and given a line L through A that does not pass through B, C, D, or E.

Constructs: the second intersection, X, of L and the conic, $\mathcal{C}$, through the five points.

Referenced as: $A' = \text{second}(A, B, C, D, E, L)$

Also referenced as: $A' = AP \cap \mathcal{C}$ when $L = AP$

Also referenced as: $A' = L \cap \mathcal{C}$ when it is clear that there is a point A that is on both $\mathcal{C}$ and L



Set up a linear coordinate system on L with A as the origin.

1. $X_1 = BD \cap L$. $X_2 = CD \cap L$.
2. $Y_1 = BE \cap L$. $Y_2 = CE \cap L$.
3. $x_1 = |AX_1|$. $x_2 = |AX_2|$.
4. $y_1 = |AY_1|$. $y_2 = |AY_2|$.
5. $x = \dfrac{(x_1 - x_2)y_1y_2 - (y_1 - y_2)x_1x_2}{x_1y_2 - x_2y_1}$
6. X is at coordinate x.

Note 1. Note that the distances are signed.

Note 2. [41] gives a similar coordinate-based solution to the problem of intersecting any line with a conic.

Note 3. This construction fails if $BD \parallel L$.

Here is another way to find the second intersection of a line with a conic. It is based on the construction given in [20, Art. 212].

`Construction Second Intersection with 5-point Conic.`

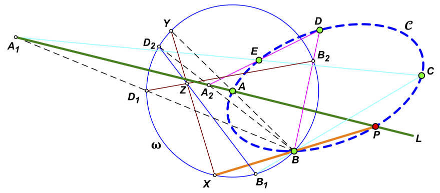
Given: five points, A, B, C, D, and E with no three collinear and given a line L through A that does not pass through B, C, D, or E.

Constructs: the second intersection, P, of L and the conic, $\mathcal{C}$, through the five points.

Referenced as: $A' = \text{second}(A, B, C, D, E, L)$

Also referenced as: $A' = AP \cap \mathcal{C}$ when $L = AP$

Also referenced as: $A' = L \cap \mathcal{C}$ when it is clear that there is a point A that is on both $\mathcal{C}$ and L



1. $A_1 = CE \cap L$. $A_2 = DE \cap L$.
2. $\omega = A(B)$.
3. $A_1 = CE \cap L$. $A_2 = DE \cap L$.
4. $B_1 = BC \cap \omega$. $B_2 = BD \cap \omega$.
5. $D_1 = BA_1 \cap \omega$. $D_2 = BA_2 \cap \omega$.
6. $Y = BA \cap \omega$. $Z = B_1D_2 \cap B_2D_1$.
7. $X = YZ \cap \omega$. $P = XB \cap L$.

Note This construction fails if $CE \parallel L$.

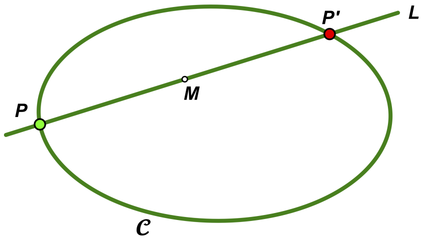
We have now seen several constructions appearing in the literature that allows you to construct (with straightedge and compass) the second point of intersection of a given line through one point of a 5-point conic with that conic. These constructions work most of the time, but each one fails in rare configurations of the five points and the line.

Open Question 1. *Is there a ruler and compass construction in the Euclidean plane that never fails that finds the second point of intersection of a given line through one point of a 5-point conic with that conic?*

If your DGE is like GSP and allows you to find the two intersections of a line with a conic, but does not guarantee the order in which the two intersection points will be returned, then the following construction will be helpful.

`Construction Second Intersection with Conic.`

Given: a point P on a conic $\mathcal{C}$ and a line L that passes through P.
Constructs: the second intersection, P', of L and the conic.
Referenced as: $A' = \text{second}(A, B, C, D, E, L)$
Also referenced as: $A' = AP \cap \mathcal{C}$ when $L = AP$
Also referenced as: $A' = L \cap \mathcal{C}$ when it is clear that there is a point A that is on both $\mathcal{C}$ and L

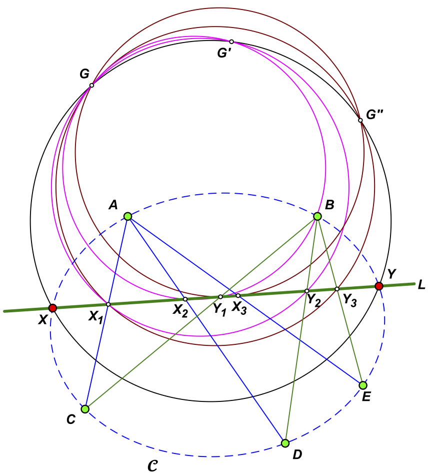


1. $\{P_1, P_2\} = \mathcal{C} \cap L$.
2. $M = \text{midpt}(P_1, P_2)$.
3. $P' = \text{reflect}(P, M)$.

Russell [71, p. 165] explains how to find the intersection of a line with a 5-point conic by constructing two homographic ranges and finding their common points. Milne [29, p. 72] explains how the common points can be constructed geometrically. Combining these two ideas gives us the following construction.

`Construction Line Intersect Conic.`

Given: a conic $\mathcal{C}$ and a line L.
Constructs: points of interesction X and Y, of the line and the conic.
Referenced as: $\{X, Y\} = L \cap \mathcal{C}$



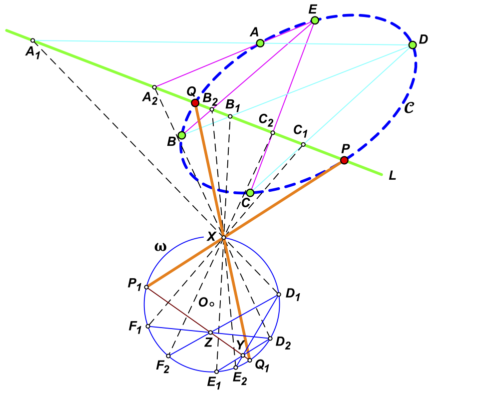
1. $X_1 = AC \cap L$. $X_2 = AD \cap L$. $X_3 = AE \cap L$.
2. $Y_1 = BC \cap L$. $Y_2 = BD \cap L$. $Y_3 = BE \cap L$.
3. $G \notin L$.
4. $G' = \odot GX_2Y_1 \cap \odot GX_1Y_2$.
5. $G'' = \odot GX_3Y_1 \cap \odot GX_1Y_3$.
6. $\{X, Y\} = \odot GG'G'' \cap L$

Note. This construction fails if $AC \parallel L$.

The following algorithm comes from [20, Art. 212].

`Construction Intersection of Line and 5-point Conic.`

Given: five points, A, B, C, D, and E that lie on a conic $\mathcal{C}$ and a line L.
Constructs: the points P and Q where the line L intersects the conic.
Referenced as: $\{P, Q\} = L \cap \mathcal{C}$



1. Let O and X be any two points.
2. $\omega = O(X)$.
3. $A_1 = DA \cap L$. $A_2 = EA \cap L$.
4. $B_1 = DB \cap L$. $B_2 = EB \cap L$.
5. $C_1 = DC \cap L$. $C_2 = EC \cap L$.
6. $D_i = A_iX \cap \omega, i = 1, 2$.
7. $E_i = B_iX \cap \omega, i = 1, 2$.
8. $F_i = C_iX \cap \omega, i = 1, 2$.
9. $Y = D_1E_2 \cap D_2E_1$. $Z = D_1F_2 \cap D_2F_1$.
10. $P_1, Q_1 = YZ \cap \omega$.
11. $P = P_1X \cap L$. $Q = Q_1X \cap L$.

Note 1. This construction can be performed with only a straightedge, assuming that the circle $O(X)$ has already been drawn.

Note 2. If your DGE does not allow you to select random points in the plane when writing a script, you can replace step 1 by $O = \text{midpt}(D, E)$, $X = D$.

Note 3. This construction fails if $DA \parallel L$.

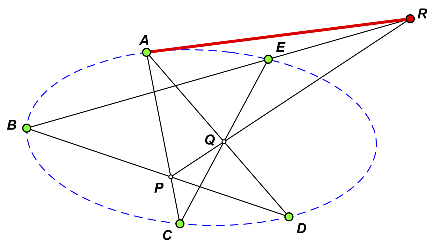
Open Question 2. *Is there a ruler and compass construction in the Euclidean plane that never fails that finds the points of intersection of a given line and a 5-point conic?*

6. Constructions with 5-point Conics

The following construction comes from [77, Alg. 12.5.1].

`Construction Tangent at Point on 5-point Conic.`

Given: five points, A, B, C, D, and E with no three collinear.
Constructs: a line AR that is tangent at point A to the conic through these five points.
Referenced as: $L = \text{tangentAt}(A, B, C, D, E)$

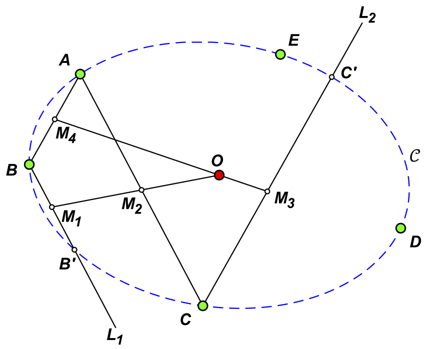


1. $P = AC \cap BD$.
2. $Q = AD \cap CE$.
3. $R = PQ \cap BE$.

Note. This construction fails if $AC \parallel BE$.

We can use the "second" construction to perform many other useful constructions, such as finding the center of a 5-point conic. The following construction comes from [77, Alg. 12.5.3].

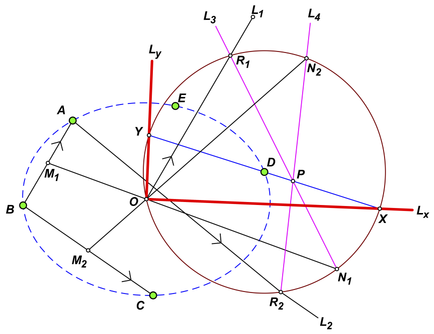
`Construction Center of 5-point Conic.`
Given: five points, A, B, C, D, and E with no three collinear.
Constructs: O, the center of the conic $\mathcal{C}$ through the five points.
Referenced as: center(A, B, C, D, E) or center$(\mathcal{C})$



1. $L_1 = \text{parallel}(B, AC)$.
2. $B' = L_1 \cap \mathcal{C}$.
3. $L_2 = \text{parallel}(C, AB)$.
4. $C' = L_2 \cap \mathcal{C}$.
5. $M_1 = \text{midpt}(BB')$.
6. $M_2 = \text{midpt}(AC')$.
7. $M_3 = \text{midpt}(CC')$.
8. $M_4 = \text{midpt}(AB)$.
9. $O = M_1M_2 \cap M_3M_4$.

The following construction comes from [77, Alg. 12.5.4].

`Construction Axes of 5-point Conic.`
Given: five points, A, B, C, D, and E with no three collinear.
Constructs: the axes L_1 and L_2 of the conic through the five points.
Referenced as: $\{L_1, L_2\} = \text{axes}(A, B, C, D, E)$
Also referenced as: $\{L_1, L_2\} = \text{axes}(\mathcal{C})$ where $\mathcal{C} = \text{conic}(A, B, C, D, E)$

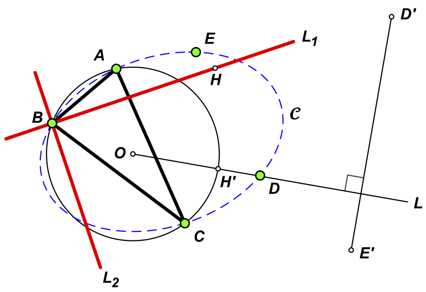


1. $O = \text{center}(A, B, C, D, E)$.
2. $M_1 = \text{midpt}(AB)$.
3. $L_1 = \text{parallel}(O, AB)$.
4. $M_2 = \text{midpt}(BC)$.
5. $L_2 = \text{parallel}(O, BC)$.
6. $\{O, N_1\} = M_1O \cap D(O)$.
7. $\{O, R_1\} = L_1 \cap D(O)$.
8. $\{O, N_2\} = M_2O \cap D(O)$.
9. $\{O, R_2\} = L_2 \cap D(O)$.
10. $P = N_1R_1 \cap N_2R_2$.
11. $\{X_1, X_2\} = DP \cap D(O)$.
12. $L_1 = OX_1$, $L_2 = OX_2$.

Note. This construction does not determine which axis is the major axis.

Here is another way to find the axes. This construction comes from [15].

`Construction Directions of Axes of 5-point Conic.`
Given: five points, A, B, C, D, and E with no three collinear.
Constructs: two lines L_1 and L_2 that are parallel to the axes of the conic $\mathcal{C}$ through the five points.
Referenced as: $\{L_1, L_2\} = \text{axisDirections}(A, B, C, D, E)$



1. $D' = \text{isogonalConj}(D, \triangle ABC)$.
2. $E' = \text{isogonalConj}(E, \triangle ABC)$.
3. $O = \text{center}(\odot ABC)$.
4. $H' = \text{perp}(O, D'E') \cap \odot ABC$.
5. $H = \text{isogonalConj}(H', \triangle ABC)$.
6. $L_1 = BH$. $L_2 = \text{perp}(B, L_1)$.

Note 1. This construction does not determine which axis is parallel to the major axis.

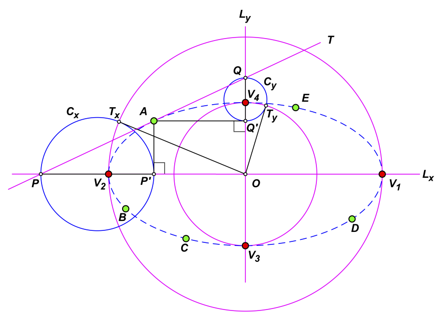
Note 2. The axes themselves can be constructed by first constructing the center of the conic and then drawing lines parallel to L_1 and L_2 through the center.

Note 3. This construction fails if A, B, C, D are concyclic or if A, B, C, E are concyclic.

The following construction comes from [77, Alg. 12.5.5].

`Construction Vertices of 5-point Conic.`

Given: five points, A, B, C, D, and E with no three collinear.
Constructs: the vertices V_1, V_2, V_3, and V_4 of the conic through the five points.
Referenced as: $(V_1, V_2, V_3, V_4) = \text{vertices}(A, B, C, D, E)$
or as $(V_1, V_3) = \text{vertices}(A, B, C, D, E)$ if only two vertices are needed. Points V_1 and V_3 lie at the ends of the major axis of the conic.

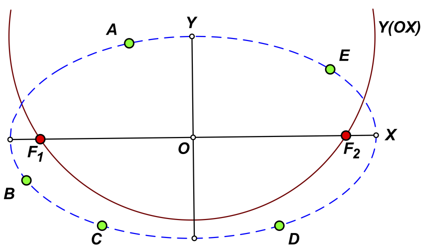


1. $O = \text{center}(A, B, C, D, E)$.
2. $\{L_x, L_y\} = \text{axes}(A, B, C, D, E)$.
3. $T = \text{tangentAt}(A, B, C, D, E)$.
4. $P = T \cap L_x$, $Q = T \cap L_y$
5. $P' = \text{foot}(A, L_x)$. $Q' = \text{foot}(A, L_Y)$.
6. $C_x = \odot(PP')$. $C_y = \odot(QQ')$.
7. OT_x = tangent to C_x from O.
8. OT_y = tangent to C_y from O.
9. $\{V_1, V_2\} = O(T_x) \cap L_x$.
10. $\{V_3, V_4\} = O(T_y) \cap L_y$.

7. Ellipse Specific Constructions

`Construction Foci of 5-point Ellipse.`

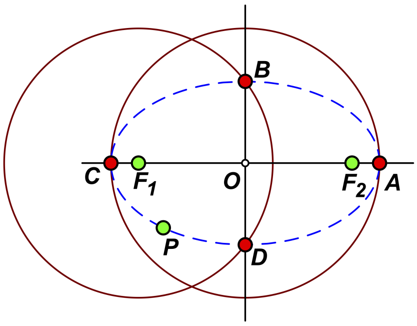
Given: five points, A, B, C, D, and E that lie on an ellipse.
Constructs: the foci F_1 and F_2 of that ellipse.



1. $O = \text{center}(A, B, C, D, E)$.
2. $\{X, Y\} = \text{vertices}(A, B, C, D, E)$.
3. $\{F_1, F_2\} = OX \cap Y(OX)$.

`Construction Vertices.`

Given: the foci F_1 and F_2 of an ellipse and a point P on the ellipse.
Constructs: the vertices A, B, C, and D of that ellipse.

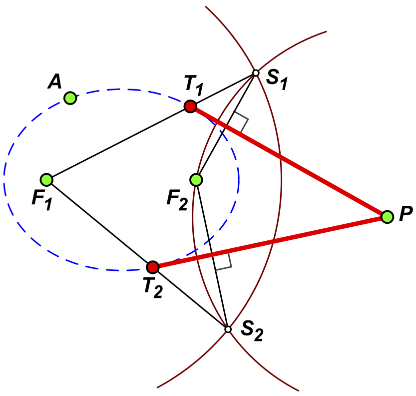


1. $O = \mathrm{midpt}(F_1, F_2)$.
2. $a = (PF_1 + PF_2)/2$.
3. $\{A, C\} = O(a) \cap \overleftrightarrow{F_1F_2}$.
4. $\{B, D\} = F_1(a) \cap \mathrm{perp}(O, F_1F_2)$.

The following construction comes from [30].

`Construction TangentFrom.`

Given: the foci F_1 and F_2 of an ellipse and a point A on the ellipse.
Also given: a point P outside the ellipse.
Constructs: the tangents to the ellipse from P.
The points of contact of the tangents and the ellipse are T_1 and T_2.



1. $k = AF_1 + AF_2$.
2. $\{S_1, S_2\} = P(F_2) \cap F_1(k)$.
3. $T_1 = F_1S_1 \cap \mathrm{perpBisector}(F_2S_1)$.
4. $T_2 = F_1S_2 \cap \mathrm{perpBisector}(F_2S_2)$.

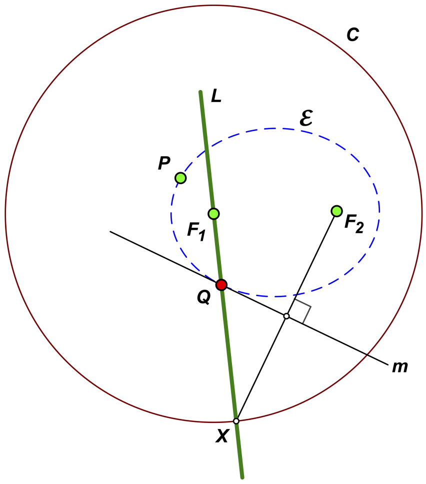
A *chord* of an ellipse is a line segment joining two points on the boundary of the ellipse. A *focal chord* of an ellipse is a chord that passes through a focus. A *focal radius* of an ellipse is a line segment from a focus to a point on the ellipse.

`Construction Endpoint of Focal Radius.`

Given: points F_1, F_2, and P that determine an ellipse $\mathcal{E}$ with foci F_1 and F_2 that passes through P.

Also given: a line L that passes through F_1.

Constructs: the endpoint, Q of a focal radius starting at F_1.



1. $k = PF_1 + PF_2$.
2. $C = F_1(k)$.
3. $X = L \cap C$.
4. $m = \text{perpBisector}(F_2X)$.
5. $Q = m \cap L$.

Note. There are two solutions because there are two possibilities for X in step 3.

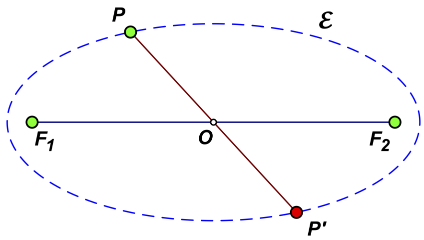
A *diameter* of an ellipse is a chord that passes through the center of the ellipse.

The folowing construction follows from the fact that an ellipse is symmetric about its center.

`Construction 2nd Endpoint of Diameter.`

Given: points F_1, F_2, and P.

Constructs: the point P' so that PP' is a diameter of the ellipse with foci F_1 and F_2 that passes through P.



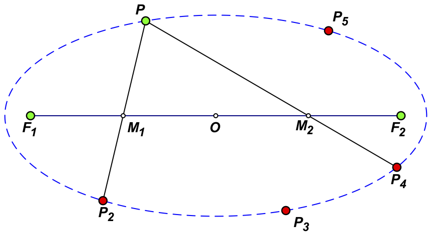
1. $O = \text{midpt}(F_1F_2)$.
2. $P' = \text{reflect}(P, O)$.

Many DGEs (including GSP) will allow you to construct a random point on an ellipse. However, if you need five points on an ellipse and don't want any random constructions (i.e. you want the construction to be repeatable), then you can use the following construction.

Construction Five Points on Ellipse.

Given: three noncollinear points F_1, F_2, and P.

Constructs: four points P_i, $i = 1, 2, 3, 4$, distinct from P that lie on the ellipse with foci F_1 and F_2 that passes through P.



1. $O = \text{midpt}(F_1, F_2)$.
2. $M_1 = \text{midpt}(OF_1)$. $M_2 = \text{midpt}(OF_2)$.
3. $P_2 = \text{reflect}(P, M_1)$. $P_4 = \text{reflect}(P, M_2)$.
4. $P_3 = \text{reflect}(P, O)$. $P_5 = \text{reflect}(P_2, O)$.

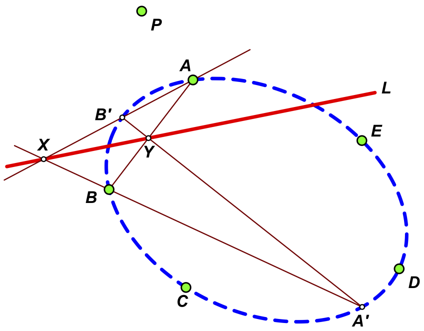
8. POLES AND POLARS

Construction Polar of Point.

Given: five points A, B, C, D, and E determining a conic $\mathcal{C}$, and a point P.

Constructs: L, the polar of the point P with respect to the conic determined by the five points.

Referenced as: $L = \text{polar}(P, \mathcal{C})$ where $\mathcal{C} = \text{conic}(A, B, C, D, E)$



1. $A' = PA \cap \mathcal{C}$.
2. $B' = PB \cap \mathcal{C}$.
3. $X = AB' \cap A'B$.
4. $Y = AB \cap A'B'$.
4. $L = XY$.

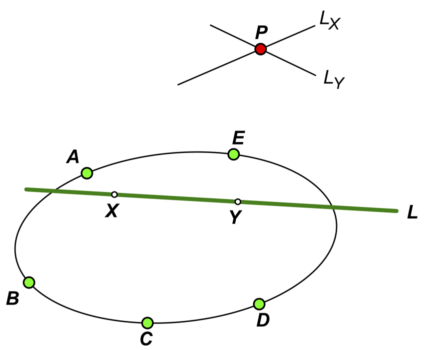
Note. The polar line L has the property that if it intersects the conic, then the points of intersection of L with the conic are the touch points of the two tangents to the conic from point P.

Construction Pole of Line.

Given: five points A, B, C, D, and E determining a conic $\mathcal{C}$, and a line L.

Constructs: P, the pole of the line L with respect to the conic determiend by the five points.

Referenced as: $P = \text{pole}(L, \mathcal{C})$ where $\mathcal{C} = \text{conic}(A, B, C, D, E)$



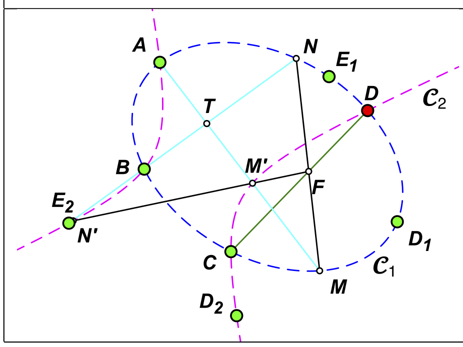
1. $X \in L$. $Y \in L$.
2. $L_X = \text{polar}(X, \mathcal{C})$.
3. $L_Y = \text{polar}(Y, \mathcal{C})$.
4. $P = L_X \cap L_Y$.

9. Intersection of Conics

The points of intersection of two 5-point conics cannot be found with straightedge and compass. See [27] for a simple proof.

However, given three of the points of intersection, we can find the fourth point using only a straightedge using the following construction which comes from [14].

`Construction 4th Point of Intersection of Two Conics.` Suppose conic $\mathcal{C}_1$ (determined by the five points A, B, C, D_1, E_1) and conic $\mathcal{C}_2$ (determined by the five points A, B, C, D_2, E_2) meet at the three points A, B, and C. **Constructs:** D, the 4th point where the two conics intersect.

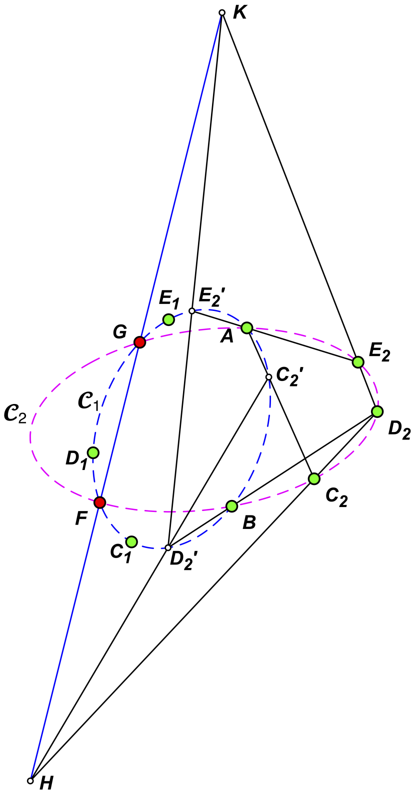


1. Let T be any point.
2. $M = \text{second}(A, B, C, D_1, E_1, AT)$.
3. $M' = \text{second}(A, B, C, D_2, E_2, AT)$.
4. $N = \text{second}(B, A, C, D_1, E_1, AT)$.
5. $N' = \text{second}(B, A, C, D_2, E_2, AT)$.
6. $F = MN \cap M'N'$.
7. $D = \text{second}(C, A, B, D_1, E_1, CF)$.

Note. If your DGE does not allow you to create a random point within a script, you can replace step 1 by $T = \text{midpt}(BD_1)$.

We can also find the 3rd and 4th points of intersection of two conics when we know two points of intersection. This construction comes from [20, Art. 237].

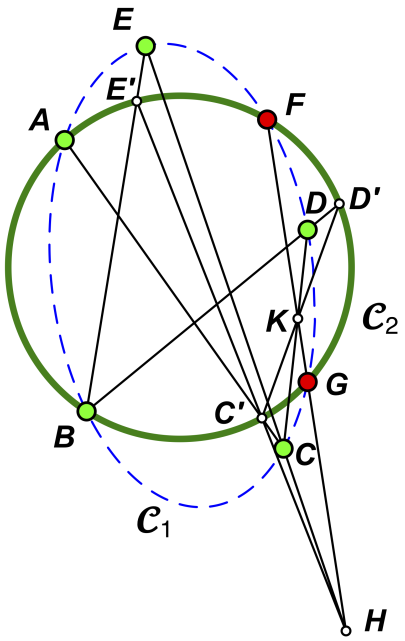
Construction 3rd and 4th Points of Intersection of Two Conics. Suppose conic $\mathcal{C}_1$ (determined by the five points A, B, C_1, D_1, E_1) and conic $\mathcal{C}_2$ (determined by the five points A, B, C_2, D_2, E_2) meet at the two points A and B.
Constructs: F and G, the 3rd and 4th points where the two conics intersect.



1. $C_2' = \text{second}(A, B, C_1, D_1, E_1, AC_2)$.
2. $D_2' = \text{second}(B, A, C_1, D_1, E_1, BD_2)$.
3. $E_2' = \text{second}(A, B, C_1, D_1, E_1, AE_2)$.
4. $H = C_2D_2 \cap C_2'D_2'$.
5. $K = D_2E_2 \cap D_2'E_2'$.
6. $\{F, G\} = HK \cap \mathcal{C}_1$.

GSP allows you to find the intersection of a circle and a conic by selecting the two objects and applying the "intersection" command. If your DGE does not let you find the intersection of a circle and a conic, the following construction from [25, p. 40] may be useful.

`Construction 3rd and 4th Points of Intersection of Conic and Circle.` Suppose conic $\mathcal{C}_1$ (determined by the five points A, B, C, D, E) and a circle $\mathcal{C}_2$ meet at the two points A and B.
Constructs: F and G, the 3rd and 4th points where the circle meets the conic.



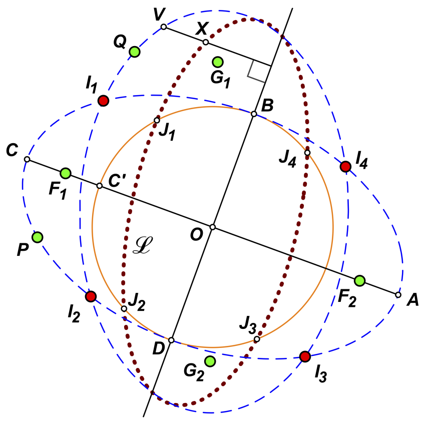
1. $C' = AC \cap \mathcal{C}_2$.
2. $D' = BD \cap \mathcal{C}_2$.
3. $E' = BE \cap \mathcal{C}2$.
4. $H = CE \cap C'E'$.
5. $K = CD \cap C'D'$.
6. $\{F, G\} = HK \cap \mathcal{C}_2$.

Although the points of intersection of two conics cannot be constructed using straightedge and compass, some DGEs permit other operations besides straightedge and compass. Since GSP allows you to find the points of intersection of a circle and an ellipse, we can find the intersection of two ellipses by applying an affine transoformation turning one ellipse into a circle. Then we find the intersection of this circle with the image of the other ellipse and then apply the inverse affine transformation to get the points of intersection of the original two ellipses.

The following construction may only be useful for Geometer's Sketchpad (GSP). GSP lets you find the points of intersection of a circle with a locus by selecting the circle and the locus and applying the "Intersections" command. Unfortunately, GSP does not allow a user to find the intersection of two loci. This is remedied by the following construction which lets you find where two ellipses intersect.

Construction Ellipse Intersect Ellipse.

Given: the foci F_1 and F_2 of an ellipse and a point P on the ellipse.
Also given: the foci G_1 and G_2 of a second ellipse and a point Q on that ellipse.
Constructs: the points I_1, I_2, I_3, and I_4 where the two ellipses intersect.

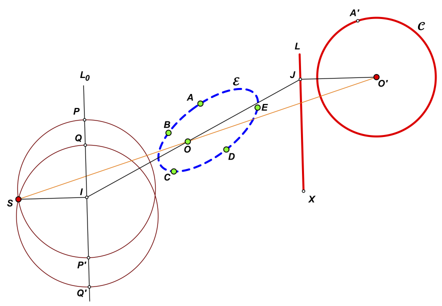


1. $O = \text{midpt}(F_1, F_2)$.
2. $(A, B, C, D) = \text{vertices}(F1, F2, P)$.
3. $C' = O(B) \cap \overrightarrow{OC}$.
4. $\mathcal{E}_2 = \text{conic}(G_1, G_2, Q)$.
5. $V \in \mathcal{E}_2$.
6. $r = OC'/OC$.
7. $X = \text{dilate}(V, BD, r)$.
8. $\mathcal{L} = \text{locus}(X, V, \mathcal{E}_2)$.
8. $\{J_1, J_2, J_3, J_4\} = O(B) \cap \mathcal{L}$
10. $r' = 1/r$.
11. $I_i = \text{dilate}(J_i, BD, r'), i = 1, 2, 3, 4$.

The folowing construction comes from [26, p. 180].

Construction Project Conic into a Circle.

Given: five points A, B, C, D, and E determining a conic $\mathcal{E}$, and a point O inside $\mathcal{E}$.
Constructs: a circle $\mathcal{C}$ with center O' that is the image of $\mathcal{E}$ under some perspective transformation with O' being the image of O.
Also constructs: the center of perspectivity S and the axis of perspectivity L.
Referenced as: $\mathcal{E} \to \mathcal{C}$



1. $L_0 = \text{polar}(O, \mathcal{E})$.
2. $P \in L_0$. $Q \in L_0$.
3. $P' = \text{polar}(P, \mathcal{E}) \cap L_0$.
4. $Q' = \text{polar}(Q, \mathcal{E}) \cap L_0$.
5. $S = \odot(PP') \cap \odot(QQ')$.
6. $I = \text{foot}(S, L_0)$.
7. $X \notin L_0$. $L = \text{parallel}(X, L_0)$.
8. $J = IO \cap L$.
9. $O' = \text{parallel}(J, SI) \cap SO$.
10. $A' = \text{project}(A, S, L, O \to O')$.
11. $\mathcal{C} = O'(A')$.

Note 1. If your DGE does not support the $X \notin L_0$ construction, it can be replaced by $X \in SI$.

Note 2. There are many solutions since X can be any point not on L_0.

Note 3. The *inside of a conic* is the union of the convex hull of all branches.

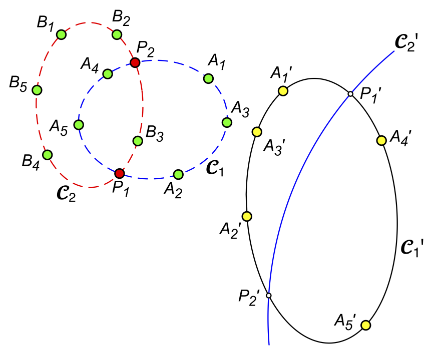
We can now find the intersection of two conics in GSP (or in any DGE that can find the intersection of a circle with a locus). The basic idea is to project one conic into a circle, then find the intersection points of that circle with the image of the other circle. Then project the intersection points back to get the intersection points of the original two conics.

`Construction Conic Intersect Conic.`

Given: five points A_i determining a conic $\mathcal{C}_1$, and five points B_i determining a conic $\mathcal{C}_2$.

Constructs: P_1, P_2, P_3, and P_4, the points where the two conics intersect.

Referenced as: $\{P_1, P_2, P_3, P_4\} = \mathcal{C}_1 \cap C_2$



1. $C_2 \to \mathcal{C}_2'$ (using center S and axis L).
2. $A_i' = \text{project}(A_i, S, L, O \to O')$
   $i = 1, 2, ..., 5$.
3. $\mathcal{C}_1' = \text{conic}(A_1', A_2', A_3', A_4', A_5')$.
4. $\{P_1', P_2', P_3', P_4'\} = \mathcal{C}_1' \cap \mathcal{C}_2'$.
5. $P_i = \text{project}(P_i', S, L, O' \to O)$.
   $i = 1, 2, 3, 4$.

Now that we can construct the point of intersection of two ellipses, this lets us illustrate the following theorems.

The following result comes from [8, Result 11.1.19].

Theorem 9.1. *The green and red ellipses meet at points X, C, Y, and F as shown in Figure 2. A red ellipse passes through X and Y and meets the other two ellipses at B, D, E, and A as shown. Then BE, AD, and CF are concurrent.*

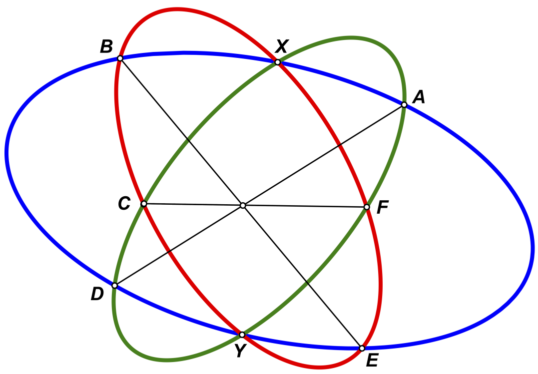


FIGURE 2. black lines concur

The following result comes from [69].

Theorem 9.2. *Let D be the point of contact of the A-excircle of $\triangle ABC$ with side BC. Since $AB + BD = AC + CD$, this means that the ellipse with foci A and D passing through B also passes through C. Call this ellipse E_a. Define E_b and E_c similarly. Let E_b meet E_c at a point A' on the opposite side of BC from A. Define B' and C' similarly. See Figure 3. Then AA', BB', and CC' are concurrent.*

10. CONSTRUCTING AN ELLIPSE FROM POINTS AND LINES

In this section, we survey some constructions for constructing an ellipse with special conditions. We will consider the ellipse to be constructed if either we can find five points on the ellipse or if we can find both foci and one point on the ellipse.

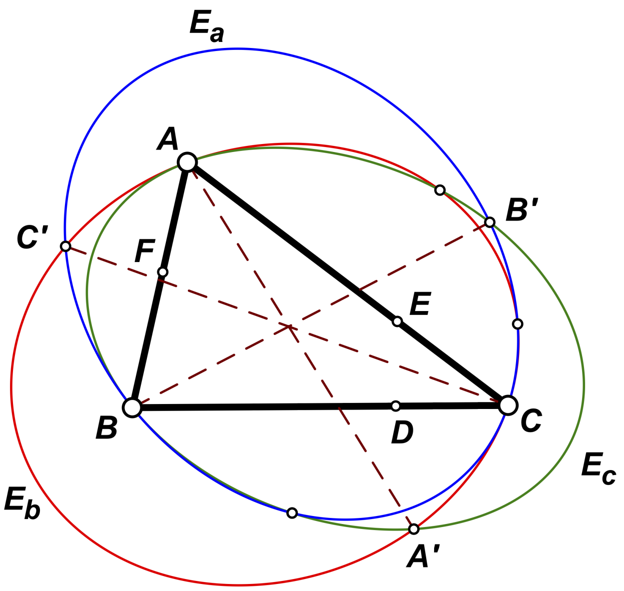


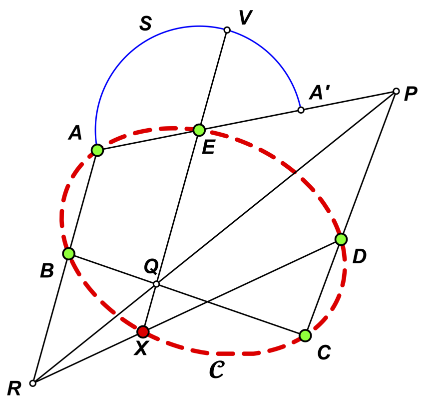
FIGURE 3. three lines concur

We start by considering conditions involving the ellipse passing through specified points or tangent to specified lines.

10.1. **Five Points.**

`Construction conic`

Given: five points, A, B, C, D, and E with no three collinear.
Constructs: the conic $\mathcal{C}$ through these points as a locus.



1. $P = AE \cap CD$.
2. $A' = \text{reflect}(A, E)$.
3. $S = \text{semicircle}(A, A')$.
4. $V \in S$.
5. $Q = VE \cap BC$.
6. $R = PQ \cap AB$.
7. $X = DR \cap EQ$.
8. $\mathcal{C} = \text{locus}(X, V, CS)$.

10.2. **Five Lines.**

The following construction comes from [40].

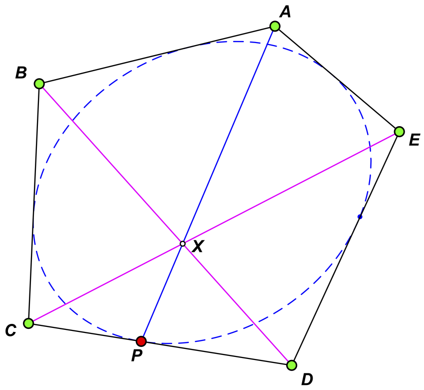
Construction LLLLL.

Given: five points P_1, P_2, P_3, P_4, and P_5.

Constructs: a conic $\mathcal{C}$ that is tangent to the five lines P_1P_2, P_2P_3, P_3P_4, P_4P_5, and P_5P_1.

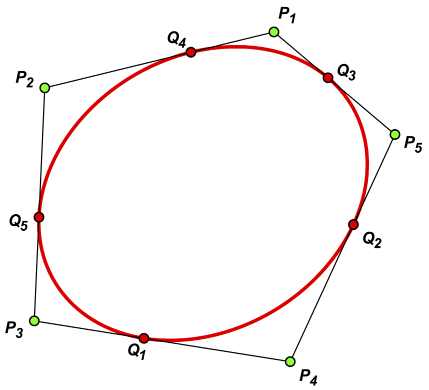
Also constructs: the touch points Q_i of the conic with the lines.

Also constructs: the point F where $\mathcal{C}$ touches AB.



First construct the touch point P of line CD with the conic touching the fiving lines AB, BC, CD, DE, EA. **Referenced as:** $P = \text{opp}(A, B, C, D, E)$

1. $X = BD \cap CE$.
2. $P = AX \cap CD$.



1. $Q_1 = \text{opp}(P_1, P_2, P_3, P_4, P_5)$.
2. $Q_2 = \text{opp}(P_2, P_3, P_4, P_5, P_1)$.
3. $Q_3 = \text{opp}(P_3, P_4, P_5, P_1, P_2)$.
4. $Q_4 = \text{opp}(P_4, P_5, P_1, P_2, P_3)$.
5. $Q_5 = \text{opp}(P_5, P_1, P_2, P_3, P_4)$.
6. $\mathcal{C} = \text{conic}(Q_1, Q_2, Q_3, Q_4, Q_5)$

Note. This construction can be referenced either as $\mathcal{C} = \text{LLLLL}(P_1, P_2, P_3, P_4, P_5)$ or as $\mathcal{C} = \text{LLLLL}(L_1, L_2, L_3, L_4, L_5)$ where $L_1 = P_1P_2$, $L_2 = P_2P_3$, $L_3 = P_3P_4$, $L_4 = P_4P_5$, and $L_5 = P_5P_1$.

Note. If your DGE does not support the $P \in L$ construction, you can replace step 1 with $R_1 = \text{midpt}(DE)$, $R_2 = \text{midpt}(R_1E)$.

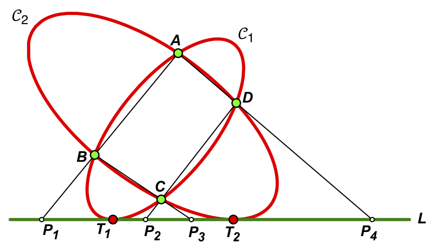
10.3. **Four Points and One Line.**

Construction PPPPL.

Given: four points A, B, C, and D and a line L.

Constructs: the conics $\mathcal{C}_1$ and $\mathcal{C}_2$ that pass through the four points and are tangent to the line.

Also constructs: the touch points of the conics and the line, T_1 and T_2.



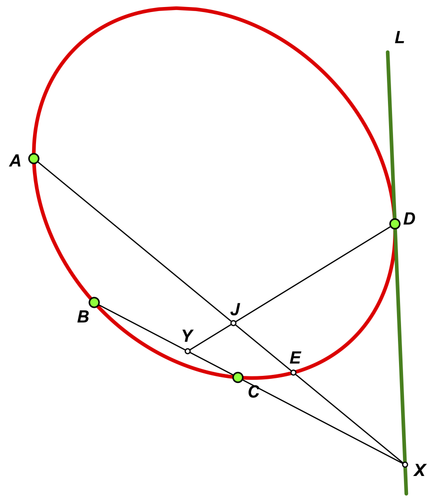
1. $P_1 = AB \cap L$. $P_2 = CD \cap L$.
2. $P_3 = BC \cap L$. $P_4 = AD \cap L$.
3. $\{T_1, T_2\} = \text{double}(P_1, P_2, P_3, P_4)$.
4. $\mathcal{C}_i = \text{conic}(A, B, C, D, T_i)$, $i = 1, 2$.

The following construction comes from [40].

Construction PPPPonL.

Given: three points A, B, and C, and a point D on a line L.

Constructs: the conic $\mathcal{C}$, that passes through the four points and is tangent to L at D.



1. $X = BC \cap L$.
2. $Y = X(B, C)$.
3. $J = AX \cap DY$.
4. $E = A(J, X)$.
5. $\mathcal{C} = \text{conic}(A, B, C, D, E)$.

10.4. Four Lines and One Point.

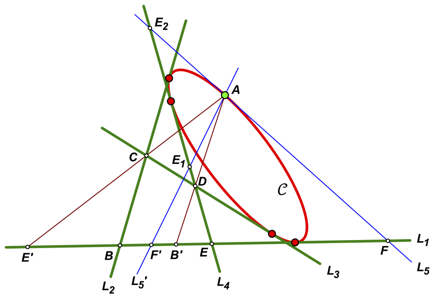
The following construction comes from [40].

`Construction LLLLP.`

Given: four lines L_1, L_2, L_3, L_4 and a point P.

Constructs: a conic $\mathcal{C}$ that is tangent to the four lines and passes through the point P.

Also constructs: the touch points of the conic with the lines.



1. $B = L_1 \cap L_2$. $C = L_2 \cap L_3$.
2. $D = L_3 \cap L_4$. $E = L_4 \cap L_1$.
3. $E' = AC \cap L_1$. $B' = AD \cap L_1$.
4. $\{F, F'\} = \text{double}(B, B', E, E')$.
5. $L_5 = AF$. $L_5' = AF'$.
6. $\mathcal{C} = \text{LLLLL}(L_1, L2, L_3, L_4, L_5)$.
7. $\mathcal{C}' = \text{LLLLL}(L_1, L2, L_3, L_4, L_5')$.

The conics are not necessarily ellipses. It is possible to have two ellipses tangent to four given lines and passing through a given point P as can be seen in Figure 4.

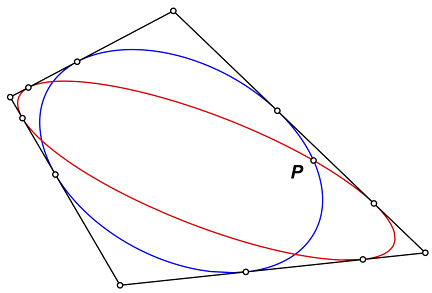


Figure 4. two ellipses passing through P

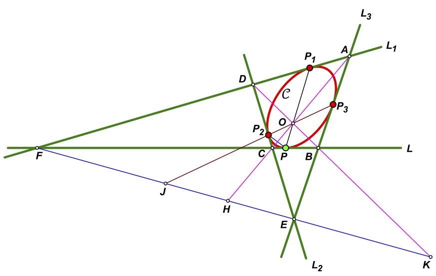
The following construction comes from [40].

`Construction LLLPonL.`

Given: four lines, L_1, L_2, L_3, and L and a point P on L.

Constructs: the conic $\mathcal{C}$, that is tangent to the four lines and touches L at P.

Also constructs: the points of contact (P_1, P_2, and P_3) with the conic.



1. $A = L_1 \cap L_3$. $B = L_3 \cap L$.
2. $C = L_2 \cap L$. $D = L_1 \cap L_2$.
3. $E = L_2 \cap L_3$. $F = L_1 \cap L$.
4. $O = AC \cap BD$.
5. $P_1 = PO \cap L_1$.
6. $K = BD \cap EF$. $H = AC \cap EF$.
7. $J = E(KH)$.
8. $P_2 = JO \cap L_2$. $P_3 = JO \cap L_3$.
5. $\mathcal{C} = \text{PPPPonL}(P_1, P_2, P_3, P, L)$.

10.5. **Two Lines and Three Points.** The following construction comes from [20, Article 221].

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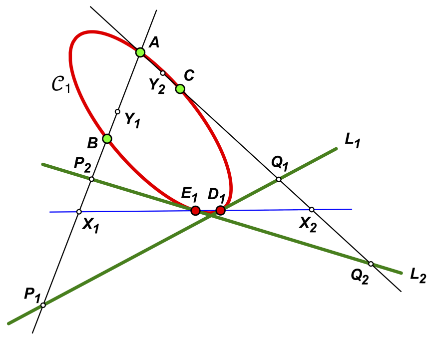
Construction PPPLL.
```

Given: three points A, B, and C, and two lines L_1 and L_2.
Constructs: the four conics $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ that pass through the three points and are tangent to the two lines.
Also constructs: the touch points of the conics and the lines.



1. $P_1 = AB \cap L_1$. $P_2 = AB \cap L_2$.
2. $Q_1 = AC \cap L_1$. $Q_2 = AC \cap L_2$.
3. $\{X_1, Y_1\} = \text{double}(A, B, P_1, P_2)$.
4. $\{X_2, Y_2\} = \text{double}(A, C, Q_1, Q_2)$.
5. $D_1 = X_1Y_1 \cap L_1$ $E_1 = X_1Y_1 \cap L_2$.
6. $D_2 = X_1Y_2 \cap L_1$ $E_2 = X_1Y_2 \cap L_2$.
7. $D_3 = X_2Y_1 \cap L_1$ $E_3 = X_2Y_1 \cap L_2$.
8. $D_4 = X_2Y_2 \cap L_1$ $E_4 = X_2Y_2 \cap L_2$.
9. $\mathcal{C}_i = \text{conic}(A, B, C, D_i, E_i)$, $i = 1, 4$.

The conics are not necessarily ellipses. It is possible to have four ellipses passing through three given points and tangent to two given lines as can be seen in Figure 5.

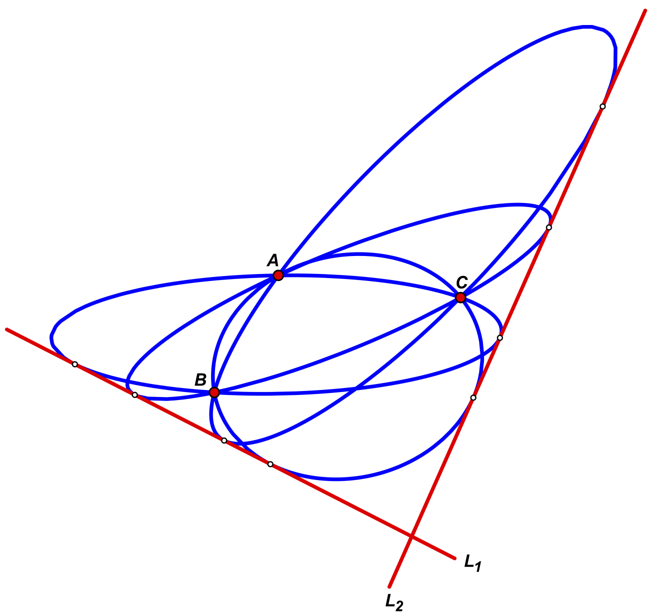


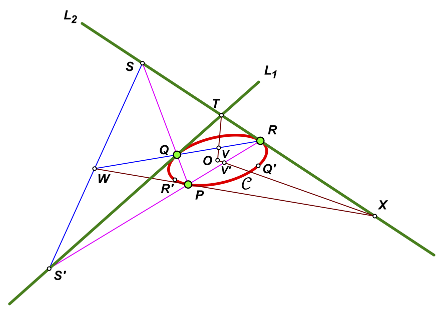
FIGURE 5. 4 ellipses passing through 3 points and tangent to 2 lines

The following construction is based on the one from [23, Problem 81].

`Construction PonLPonLP.`

Given: point Q on line L_1, point R on line L_2, and another point P.

Constructs: the conic $\mathcal{C}$, that is tangent to the two lines at Q and R, respectively, and also passes through P.



1. $T = L_1 \cap L_2$.
2. $S = PQ \cap L_2$. $S' = PR \cap L_1$.
3. $W = SS' \cap QR$.
4. $X = WP \cap L_2$.
5. $V = \text{midpt}(QR)$. $V' = \text{midpt}(PR)$.
6. $O = TV \cap XV'$.
7. $Q' = \text{reflect}(Q, O)$.
8. $R' = \text{reflect}(R, O)$.
9. $\mathcal{C} = \text{conic}(P, Q, R, Q', R')$.

Note. The point O is the center of the conic and WX is the tangent to the conic at P.

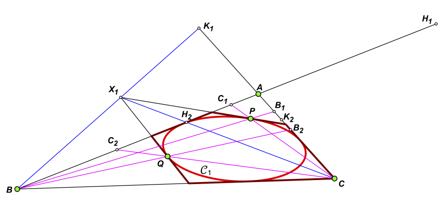
10.6. **Three Lines and Two Points.** The following construction comes from [38].

`Construction LLLPP`

Given: three lines determined by the points A, B, and C.

Also given: two points P and Q.

Constructs: the conics that are tangent to the sides of $\triangle ABC$ and pass through the two points P and Q.



1. $B_1 = BP \cap AC$. $B_2 = BQ \cap AC$.
2. $C_1 = CP \cap AB$. $C_2 = CQ \cap AB$.
3. $\{H_1, H_2\} = \text{double}(A, B, C_1, C_2)$.
4. $\{K_1, K_2\} = \text{double}(A, C, B_1, B_2)$.
5. $X_1 = BK_1 \cap CH_2$. $X_2 = BK_1 \cap CH_1$.
6. $X_3 = BK_2 \cap CH_2$. $X_4 = BK_2 \cap CH_2$.
7. $\mathcal{C}_i = \text{LLLLL}(AB, BC, CA, PX_i, QX_i)$.

Note 1. Only $\mathcal{C}_1$ is shown in the figure.

Note 2. The five lines used for the LLLLL construction are shown in brown.

Note 3. The LLLLL construction also returns the touch points of the conic with the sides of $\triangle ABC$. These are not shown in the figure.

There are typically four conics that meet the given conditions as show in Figure 6.

The following construction comes from from [25, p. 44].

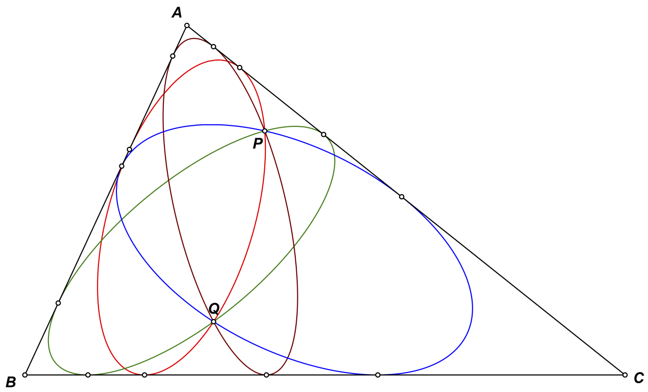
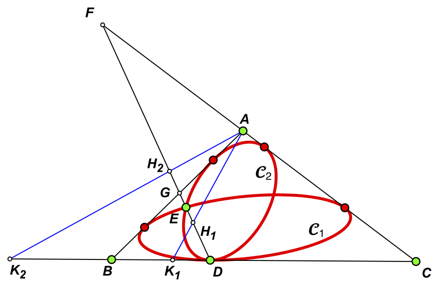


FIGURE 6. 4 ellipses inscribed $\triangle ABC$ passing through P and Q

`Construction PPonLLL.`

Given: triangle ABC and points D and E with D on BC.
Constructs: the conics $\mathcal{C}_1$ and $\mathcal{C}_2$, that are tangent to each side of $\triangle ABC$, pass through E, and touches BC at D.
Also constructs: the touch points with the sides of the triangle.

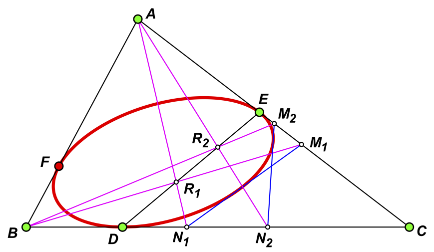


1. $F = DE \cap AC$.
2. $G = DE \cap AB$.
3. $\{H_1, H_2\} = \text{double}(D, E, F, G)$.
4. $K_i = AH_i \cap BC$, $i = 1, 2$.
5. $\mathcal{C}_i = \text{LLLLP}(AB, BC, CA, AK_i, E)$, $i = 1, 2$.

The following construction comes from from [47, Problem 139].

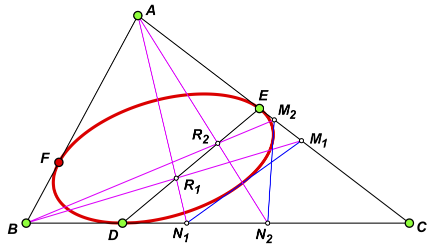
`Construction PonLPonLL.`

Given: triangle ABC and points D and E with D on BC and E on CA.
Constructs: the conic $\mathcal{C}$, that is tangent to each side of $\triangle ABC$, and touches BC at D and CA at E.



1. $R_1 \in DE$. $R_2 \in DE$.
2. $N_1 = AR_1 \cap BC$. $M_1 = BR_1 \cap AC$.
3. $N_2 = AR_2 \cap BC$. $M_2 = BR_2 \cap AC$.
4. $\mathcal{C} = \text{LLLLL}(AB, BC, CA, M_1N_1, M_2N_2)$

This construction allows us to construct inconics given their perspector.

`Construction Perspector.` **Given:** triangle ABC and a point P. Let AP, BP, and CP meet the sides of the triangle at D, E, and F respectively. **Constructs:** the conic $\mathcal{C}$, that is tangent to each side of $\triangle ABC$, and touches BC at D, CA at E, and AB at F.
 1. $\mathcal{C} = \text{PonLPonLL}(ABC, D, E)$.

11. Ellipses Associated with Triangles

Now that we know how to draw a conic through five points, we can use this ability to illustrate various theorems. In general, a conic cannot be drawn through six points, so when we find six points that lie on a conic, that is interesting. Here is a small collection of results that conclude by finding six points that lie on an ellipse (or a conic).

The following result is well known (see, for example, [73, p. 281]).

Theorem 11.1 (Hexagon with Opposite Sides Parallel). *A convex hexagon has its opposite sides parallel (Figure 7). Then the vertices of the hexagon lie on a conic.*

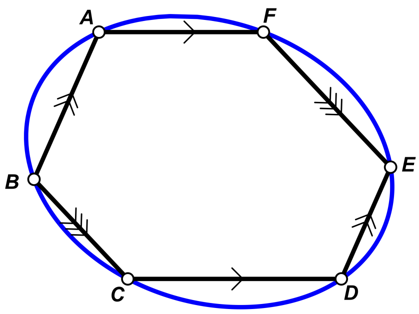


Figure 7. vertices lie on an ellipse

Theorem 11.2 (Two Cevians Conic). *Let P_1 and P_2 be any two points inside $\triangle ABC$ (Figure 8). Let AD_1, BE_1, CF_1 be the cevians through P_1 and let AD_2, BE_2, CF_2 be the cevians through P_2. Then D_1, D_2, E_1, E_2, F_1, and F_2 lie on an ellipse.*

Theorem 11.3 (Three Parallels Conic). *Let X, Y, and Z be any three points on the interior of sides BC, CA, and AB of $\triangle ABC$ (Figure 9). A line through X parallel to AB meets AC at X'. A line through Y parallel to BC meets AB at Y'. A line through Z parallel to AC meets BC at Z'. Then X, Y, Z, X', Y', and Z' lie on an ellipse.*

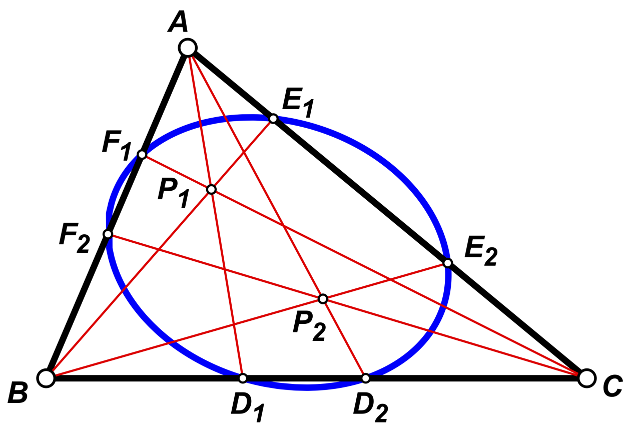


FIGURE 8. six points lie on an ellipse

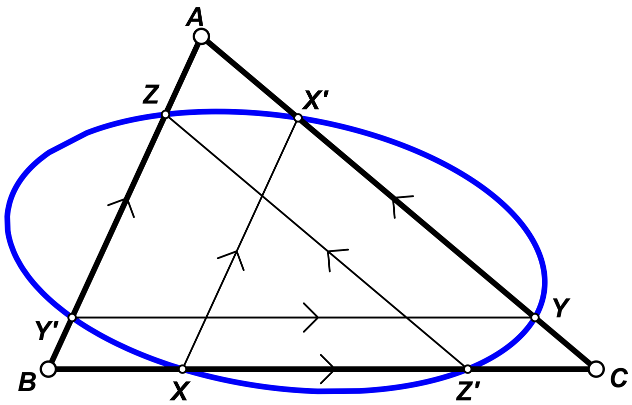


FIGURE 9. six points lie on an ellipse

The following result comes from [43].

Theorem 11.4. *Let I be the incenter of $\triangle ABC$. The cevians through I are AD, BE, and CF. Circles are inscribed in each of the six small triangles formed by these cevians (Figure 10). Then the centers of these circles lie on an ellipse.*

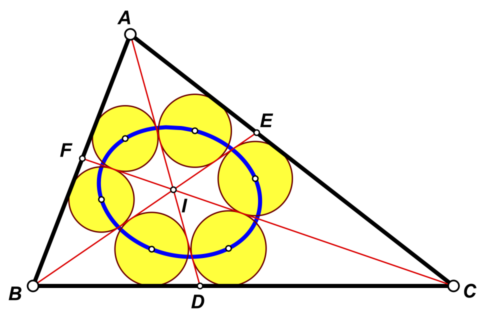


FIGURE 10. six points lie on an ellipse

Note. The point I can be replaced by certain other triangle centers and the result is still true. See [43] for details.

The following result comes from [52].

Theorem 11.5. *Let P be any point inside $\triangle ABC$. The cevians through P are AD, BE, and CF. The angle bisectors of the six angles formed at P meet the sides of the triangle at U, V, W, X, Y, and Z as shown in Figure 11. Then U, V, W, X, Y, and Z lie on an ellipse.*

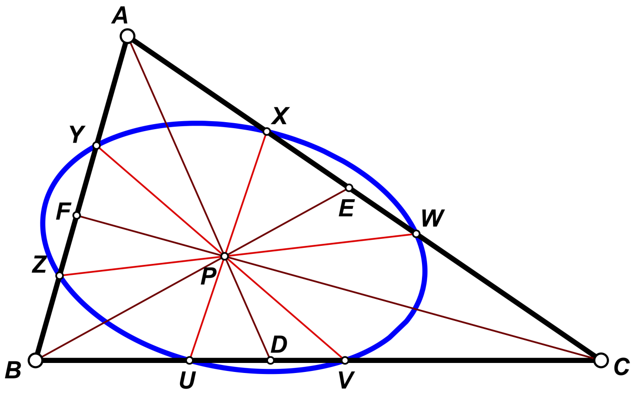


Figure 11. six points lie on an ellipse

The following result comes from [46].

Theorem 11.6. *Let M be the centroid of $\triangle ABC$. The medians through M are AD, BE, and CF (Figure 12). Then the centroids of the six small triangles formed lie on an ellipse.*

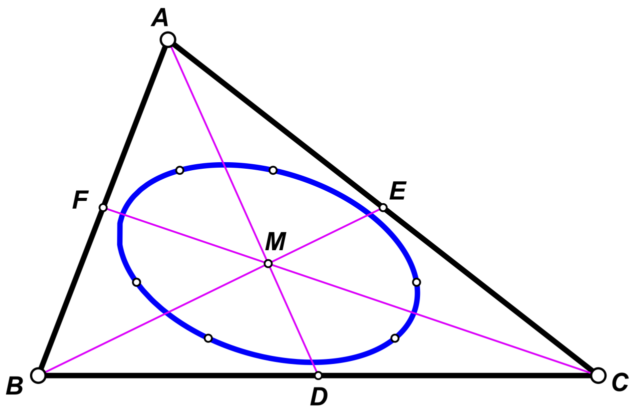


Figure 12. six points lie on an ellipse

The following result comes from [68].

Theorem 11.7. *Let I be the incenter of $\triangle ABC$. The circumcevians through I are AD, BE, and CF. These circumcevians divide the circumcircle of $\triangle ABC$ into six "wedges". A circle is inscribed in each wedge as shown in Figure 13. Then the centers of these circles lie on an ellipse.*

Note. A *circumcevian* of a triangle is a line from a vertex to a point on the circumcircle of that triangle.

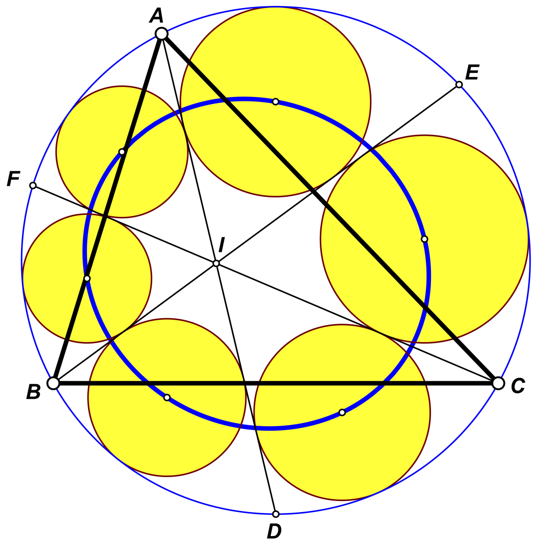


FIGURE 13. six points lie on an ellipse

See [1] for a related result.

The following result comes from [5].

Theorem 11.8. *Let P be any point inside $\triangle ABC$. The circumcevians through P are AD, BE, and CF. Let O_1 through O_6 be the circumcenters of triangles PAF, PBB, PBD, PDC, PCE, and PEA (Figure 14). Then the O_i lie on an ellipse.*

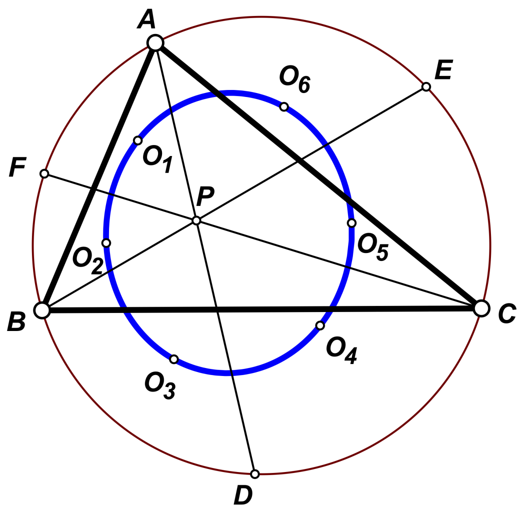


FIGURE 14. six points lie on an ellipse

The following result comes from [6].

Theorem 11.9. *Let H be the orthocenter of $\triangle ABC$. The circumcevians through H are AD, BE, and CF. Circles are constructed so that each one is outside the triangle, tangent to a side of the triangle, tangent to an altitude of the triangle, and tangent internally to the circumcircle, as shown in Figure 15. Then the points of contact of the circles with the sides of the triangle lie on an ellipse.*

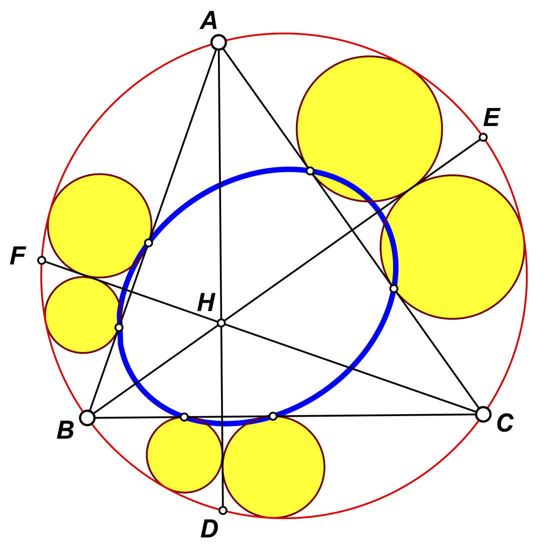


Figure 15. six points lie on an ellipse

The following result comes from [4].

Theorem 11.10. *Let I be the incenter of $\triangle ABC$. The circumcevians through I are AD, BE, and CF. Yellow circles are constructed so that each one is outside the triangle, tangent to a side of the triangle, tangent to a circumcevian, and tangent internally to the circumcircle. Green circles are constructed as shown in Figure 16 so that each is tangent to a side of the triangle, tangent to one of the yellow circles, and tangent internally to the circumcircle. Then the points of contact of the green circles with the sides of the triangle lie on an ellipse.*

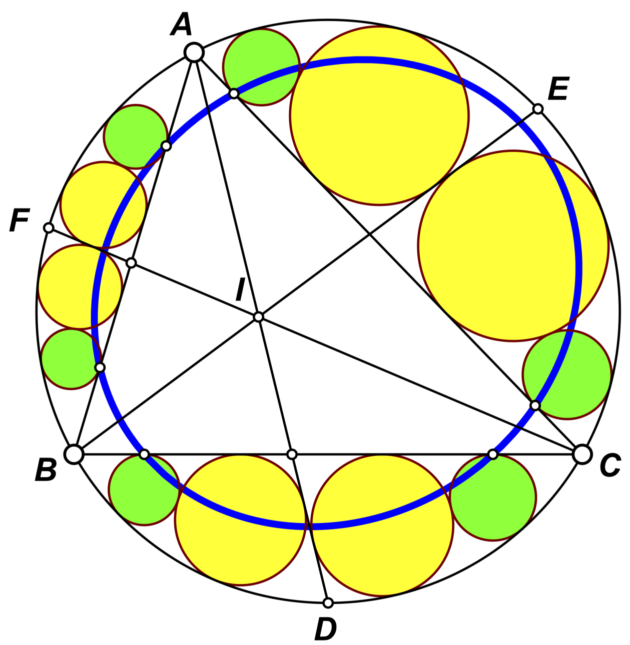


Figure 16. six points lie on an ellipse

The following result comes from [2].

Theorem 11.11. *Let G be the centroid of $\triangle ABC$. The circumcevians through G are AD, BE, and CF. Circles are constructed on GD, GE, and GF as diameters as shown in Figure 17. Then the points of intersection of these circles with the sides of the triangle lie on an ellipse.*

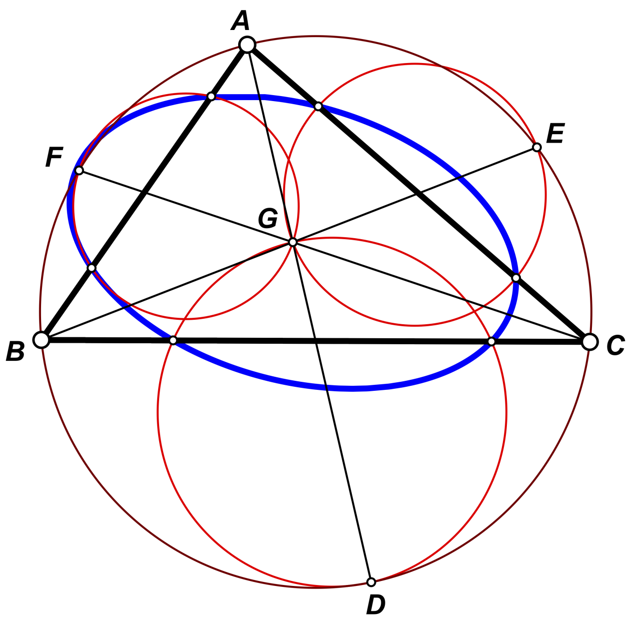


FIGURE 17. six points lie on an ellipse

The following results comes from [12].

Theorem 11.12. *The six points of intersection of the mixtilinear incircles with the sides of a triangle lie on a conic (Figure 18).*

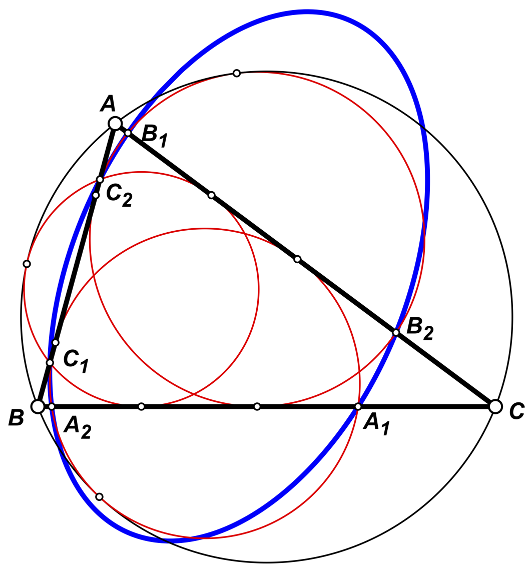


FIGURE 18. six points lie on an ellipse

The following result comes from [3].

Theorem 11.13. *Let I be the incenter of $\triangle ABC$ and let D, E, and F be the touch points of the incircle with the sides of the triangle. Let K_1 be the symmedian point of $\triangle AIE$ and define K_2, K_3, K_4, K_5, and K_6 similarly as shown in (Figure 19). Then the K_i lie on an ellipse.*

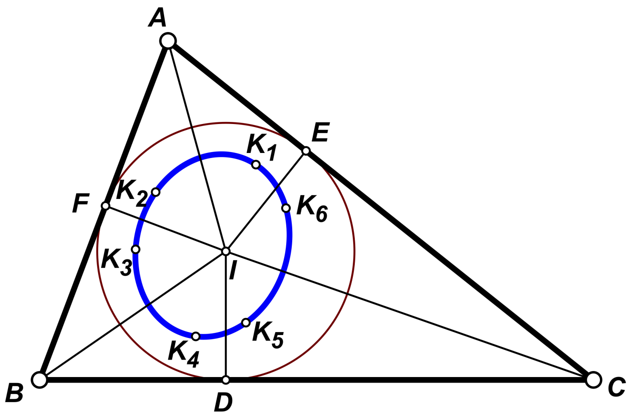


Figure 19. six points lie on an ellipse

The following result comes from [7].

Theorem 11.14. *Let O be the circumcenter of $\triangle ABC$. The diameters of the circumcircle through O are AD, BE, and CF. An ellipse is constructed with foci E and F passing through A as shown in (Figure 20). A second ellipse is constructed with foci D and F passing through B. A third ellipse is constructed with foci D and E passing through C. Then the three ellipses meet in a point.*

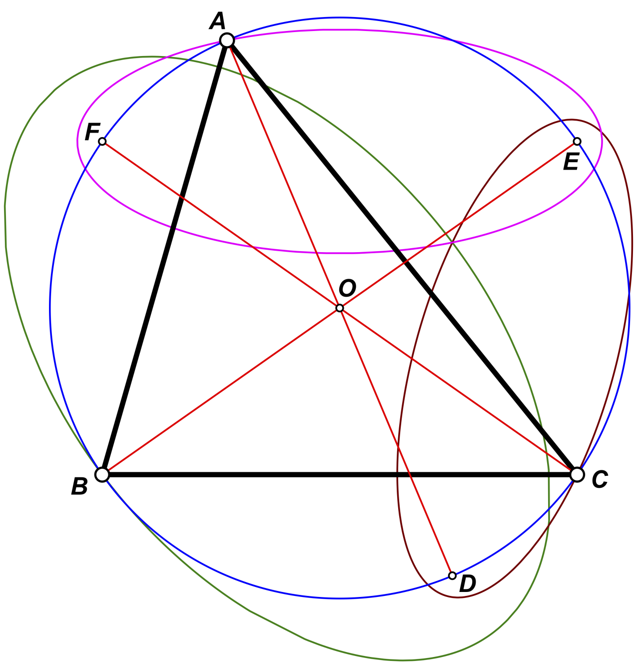


Figure 20. three ellipses meet in a point

Theorem 11.15 (Rabinowitz Conic)**.** *Let P be any point in the plane of $\triangle ABC$. Point U is constructed so that vectors $\overrightarrow{AU}$ and $\overrightarrow{BP}$ have the same direction and $AU = AP$. Points V, W, X, Y, and Z are constructed in a similar manner, as shown in Figure 21. (Line segments colored the same have the same length.) Then there is a conic that passes through the six points U, V, W, X, Y, and Z.*

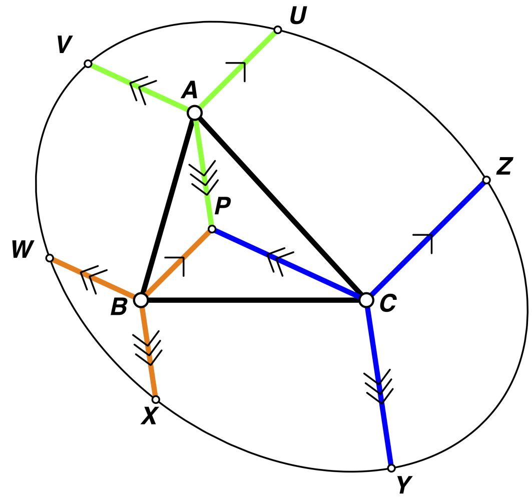


FIGURE 21. Rabinowitz Conic: six points lie on a conic

Theorem 11.16 (Vu's Conic)**.** *Let P be any point in the plane of $\triangle ABC$. Point U is constructed so that vectors $\overrightarrow{AU}$ and $\overrightarrow{BP}$ have the same direction and $AU = BP$. Points V, W, X, Y, and Z are constructed in a similar manner, as shown in Figure 22. (Line segments colored the same have the same length.) Then there is a conic that passes through the six points U, V, W, X, Y, and Z.*

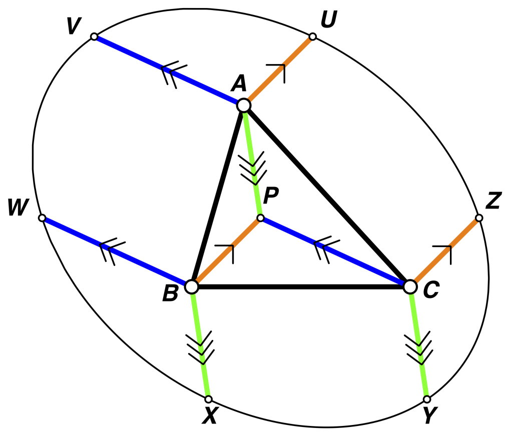


FIGURE 22. Vu's Conic: six points lie on a conic

12. INELLIPSES AND CIRCUMELLIPSES

A conic tangent to each side of a polygon is called an *inconic* of that polygon. A conic that passes through each vertex of a polygon is called a *circumconic* of that polygon. When the conic is an ellipse, these are called *inellipses* and *circumellipses*.

12.1. **Hexagons.**

Using the constructions we have previously enumerated, we can now illustrate some well-known results about inconics and circumconics of hexagons.

The following result concerning the inconic of a hexagon was discovered by Charles Julien Brianchon in 1810.

Theorem 12.1 (Brianchon's Theorem)**.** *Let $ABCDEF$ be a hexagon circumscribed about a conic. Then AD, BE, and CF are concurrent (Figure 23).*

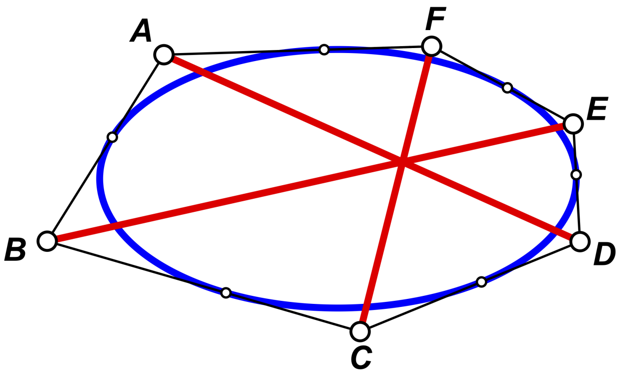


Figure 23. Red lines concur

The following result concerning a circumconic of a hexagon was discovered by Blaise Pascal in 1640.

Theorem 12.2 (Pascal's Theorem)**.** *Let $ABCDEF$ be a hexagon inscribed in a conic. Suppose AB meets DE at X, CD meets FA at Y, and BC meets EF at Z (Figure 24). Then X, Y, and Z are collinear.*

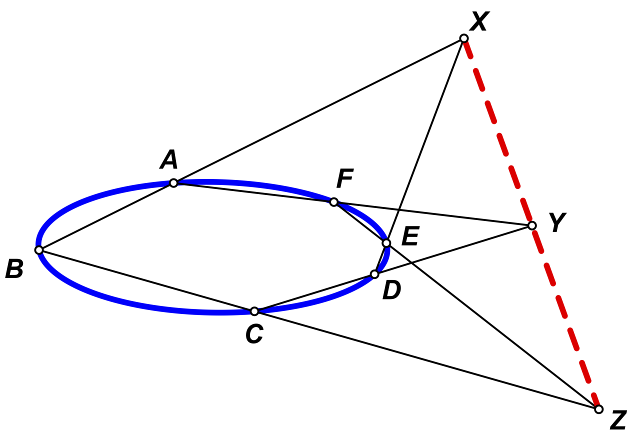


Figure 24. X, Y, and Z colline

12.2. **Pentagons.**

We can construct the circumconic of a pentagon using the `conic` construction. We can construct the inconic of a pentagon using the `LLLLL` construction.

The following result is a limiting case of Brianchon's Theorem.

Theorem 12.3. *Let $ABCDE$ be a pentagon circumscribed about a conic. The conic touches side AE at X. Then AD, BE, and CX are concurrent (Figure 25).*

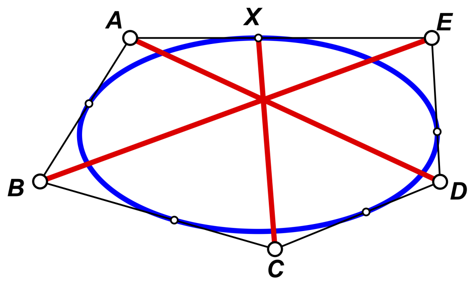


FIGURE 25. Red lines concur

12.3. **Quadrilaterals.**

12.4. **Triangles.**

A triangle has many inellipses. They can be constructed by various techniques.

(1) To construct an inellipse touching two sides of the triangle at specified points, use the `PonLPonLL` construction.
(2) To construct an inellipse with a given perspector, use the `perspector` construction.
(3) To construct an inellipse with a given center, use the `LLLO` construction.
(4) To construct an inellipse with a given focus, use the `FLLL` construction.

The following result comes from [19, p. 149].

Theorem 12.4. *If the diagonals of a quadrilateral intersect at the focus of an ellipse inscribed in the quadrilateral, then the diagonals are perpendicular.*

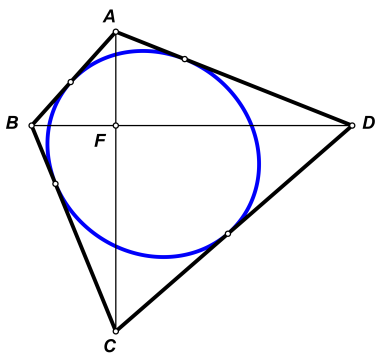


FIGURE 26. F focus $\implies AC \perp BD$

The following result comes from [75].

Theorem 12.5. *Let M be the centroid of $\triangle ABC$ and let K be the symmedian point. The Steiner circumellipse of $\triangle ABC$ meets the circumcircle of the triangle at S. Then the bisector of $\angle KMS$ is the minor axis of the ellipse.*

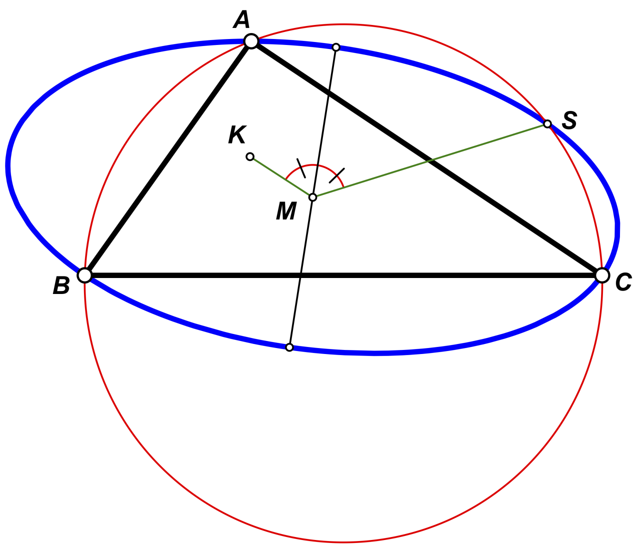


FIGURE 27. Angle bisector of $\angle KMS$ is the minor axis of the Steiner circumellipse

Note. The point S is the Steiner point of the triangle.

The polars of the vertices of $\triangle ABC$ with respect to a conic $\mathcal{C}$ bound a triangle called the *polar triangle* of that triangle with respect to the conic.

Theorem 12.6 (Perspector of Triangle and Conic)**.** *Let $A'B'C'$ be the polar triangle of $\triangle ABC$ with respect to a conic $\mathcal{C}$. Then triangles ABC and $A'B'C'$ are in perspective. That is, AA', BB', and CC' are concurrent at a point P (Figure 28).*

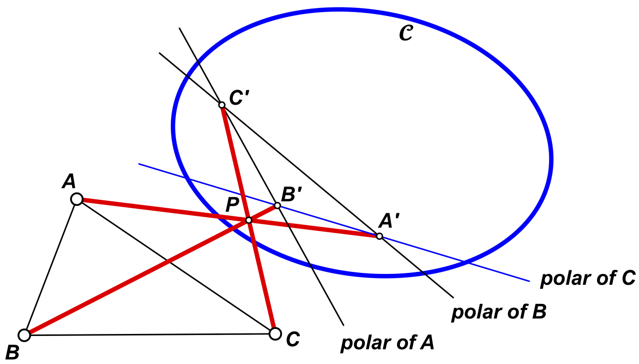
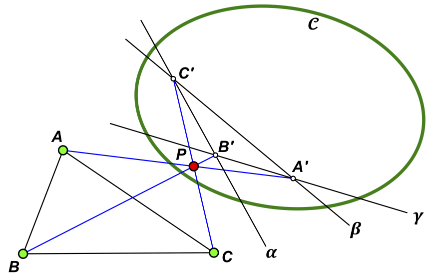


FIGURE 28. red lines concur

The point P is called the *perspector* of the conic with respect to the triangle.

The perspeector can be constructed immediately using the definition.

```
Construction Perpsector
```
Given: $\triangle ABC$ and conic $\mathcal{C}$.
Constructs: the perspector P of the conic with respect to the triangle.

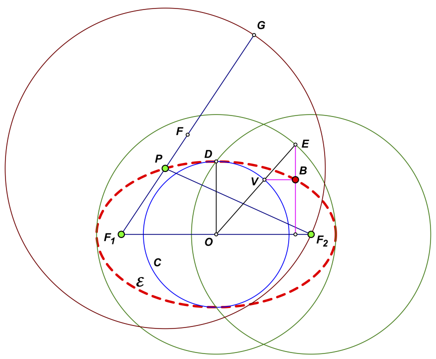


1. $\alpha = \text{polar}(A, \mathcal{C})$.
2. $\beta = \text{polar}(B, \mathcal{C})$.
3. $\gamma = \text{polar}(C, \mathcal{C})$.
4. $A' = \beta \cap \gamma$. $B' = \gamma \cap \alpha$. $C' = \alpha \cap \beta$.
5. $P = AA' \cap BB'$.

13. Constructions Based on Foci

The following construction is the one that GSP actually uses to draw an ellipse as a locus given the two foci and a point on the boundary.

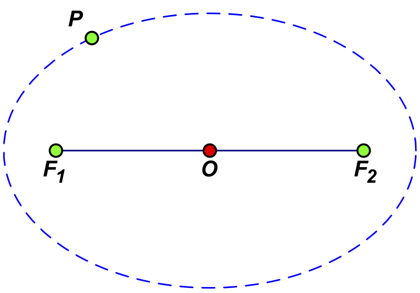
```
Construction ellipse
```
Given: two points F_1 and F_2 and a point P.
Constructs: the ellipse that passes through P and has foci F_1 and F_2 as a locus.



1. $O = \text{midpt}(F_1, F_2)$.
2. $G = \overrightarrow{F_1P} \cap P(F_2)$.
3. $F = \text{midpt}(F_1G)$.
4. $D = \text{perp}(O, F_1F_2) \cap F_2(FF_1)$.
5. $C = O(D)$.
6. $V \in C$.
7. $E = \overrightarrow{OV} \cap O(FF_1)$.
8. $B = \text{parallel}(V, F_1F_2) \cap \text{perp}(E, F_1F_2)$.
9. $\mathcal{E} = \text{locus}(B, V, C)$.

The following two constructions are well known.

```
Construction Center of Ellipse.
```
Given: three points, F_1, F_2, and P that determine an ellipse with foci F_1 and F_2 that passes through P.
Constructs: O the center of the ellipse.

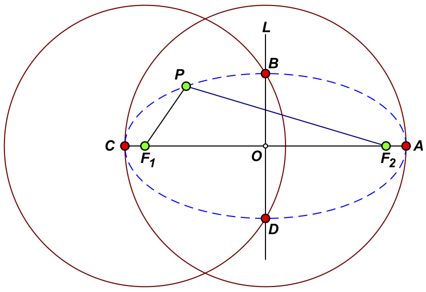


1. $O = \text{midpt}(F_1F_2)$.

`Construction Vertices of Ellipse.`

Given: three points, F_1, F_2, and P that determine an ellipse with foci F_1 and F_2 that passes through P.

Constructs: the vertices A, B, C, and D of the ellipse.



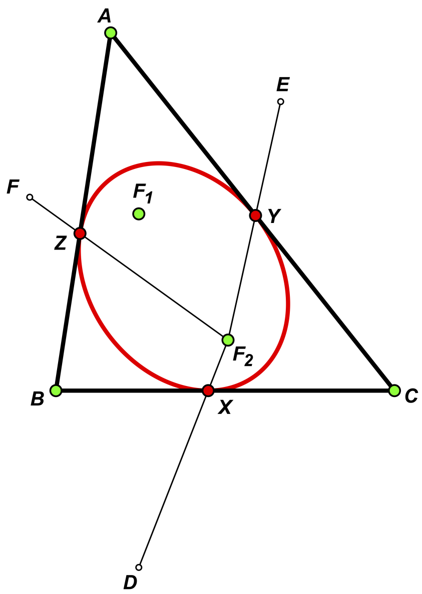
1. $a = (PF_1 + PF_2)/2$.
2. $O = \text{midpt}(F_1F_2)$.
3. $\{A, C\} = O(a) \cap F_1F_2$.
4. $L = \text{perpBisector}(F_1F_2)$.
5. $\{B, D\} = F_1(a) \cap L$.

`Construction FFLLL.`

Given: a triangle ABC and two points F_1 and F_2 known to be the foci of an inellipse of that triangle.

Constructs: the ellipse

Also constructs: the three touch points X, Y, and Z with the sides.

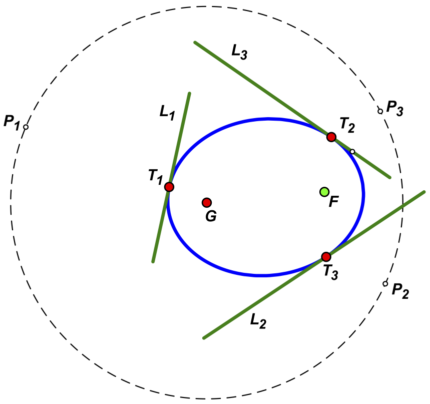


1. $D = \text{reflect}(F_1, BC)$.
2. $E = \text{reflect}(F_1, CA)$.
3. $F = \text{reflect}(F_1, AB)$.
4. $X = F_2D \cap BC$.
5. $Y = F_2E \cap CA$.
6. $Z = F_2F \cap AB$.
7. $\mathcal{E} = \text{ellipse}(F_1, F_2, X)$

The following construction comes from [35].

`Construction FLLL.`

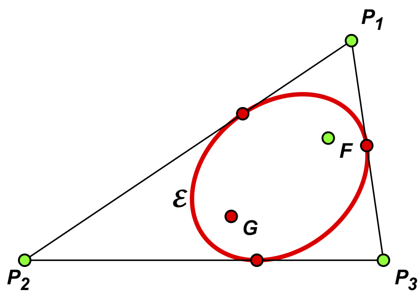
Given: a point F and three lines L_1, L_2, and L_3.
Constructs: the ellipse with one focus at F that is tangent to the three given lines.
Also constructs: the other focus, G.



1. $P_1 = \text{reflect}(F, L_1)$.
2. $P_2 = \text{reflect}(F, L_2)$.
3. $P_3 = \text{reflect}(F, L_3)$.
4. $G = \text{circumcenter}(\triangle P_1P_2P_3)$.
5. $A = L_1 \cap L_2$, $B = L_2 \cap L_3$, $C = L_3 \cap L_1$.
6. $\mathcal{E} = \text{FFLLL}(ABC, F, G)$.

`Construction FPPP.`

Given: a point F and three points P_1, P_2, and P_3.
Constructs: the ellipse $\mathcal{E}$ with one focus at F that passes through the three given points.
Also constructs: the other focus, G.



1. $G = \text{isogonalConj}(P, \triangle P_1P_2P_3)$.
2. $\mathcal{E} = \text{FFLLL}(P_1P_2P_3, F, G)$.

The following result comes from [9, p. 105].

Theorem 13.1 (Isotomic Property of an Inellipse)**.** *An ellipse inscribed in $\triangle ABC$ has center O and touches the sides of the triangle at points D, E, and F as shown in Figure 29. Cevians AD, BE, and CF meet at P. Let M be the centroid of $\triangle ABC$ and let Q be the isotomic conjugate of P. Then O, M, and Q are collinear and $MQ = 2OM$.*

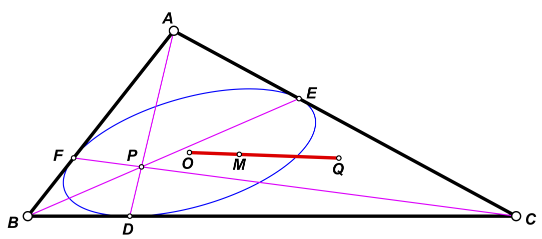
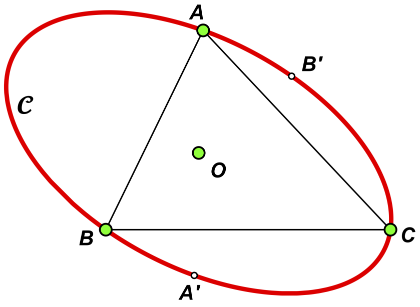


FIGURE 29. Isotomic Property of an inellipse

`Construction PPPO.`

Given: a triangle ABC and a point O not on the boundary of the triangle.
Constructs: the conic $\mathcal{C}$ with center O that passes through A, B, and C.

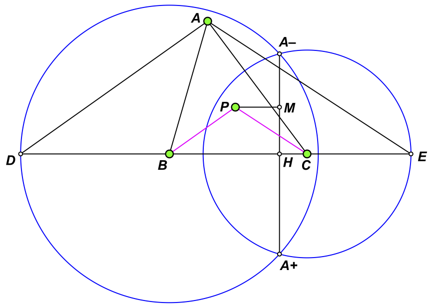


1. $A' = \text{reflect}(A, O)$.
2. $B' = \text{reflect}(B, O)$.
3. $\mathcal{C} = \text{conic}(A, B, C, A', B')$.

The following construction comes from [21].

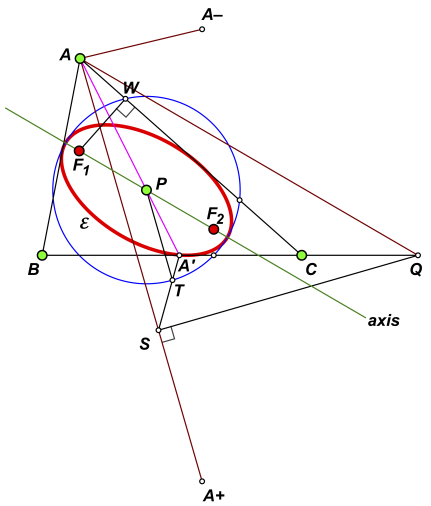
`Construction LLLO.`

Given: a triangle ABC and a point O.
Constructs: the ellipse $\mathcal{E}$ with center O inscribed in the triangle.
Also constructs: the foci, F_1 and F_2.



1. $E = \text{parallel}(A, PC) \cap BC$.
2. $D = \text{parallel}(A, PB) \cap BC$.
3. $\{A^+, A^-\} = C(E) \cap B(D)$.
4. $H = A^+A^- \cap BC$.
5. $M = \text{foot}(P, A^+A^-)$.
6. $A^- = \overrightarrow{HM} \cap C(E)$.
7. $A^+ = \text{reflect}(A^-, H)$

Note. Re



8. $Q = \text{angleBisector}(AA^+, AA^-) \cap BC$.
9. $S = \text{foot}(Q, AA^+)$.
10. $A' = AP \cap BC$.
11. $T = A'S \cap \text{parallel}(P, AA^+)$.
12. $W = P(T) \cap AC$.
13. $F_1 = \text{perp}(W, AC) \cap \text{parallel}(P, AQ)$.
14. $F_2 = \text{reflect}(F_1, P)$
15. $\mathcal{E} = \text{FFLLL}(ABC, F_1, F_2)$.

14. Drawing Lines Tangent to an Ellipse

The following property of an ellipse is well known [8, Thm. 11.3].

Theorem 14.1 (Reflective Property of an Ellipse)**.** *An ellipse has foci F_1 and F_2. A straight line APF is tangent to the ellipse and touches the ellipse at point P (Figure 30). Then $\angle APF_1 = \angle F_2PB$.*

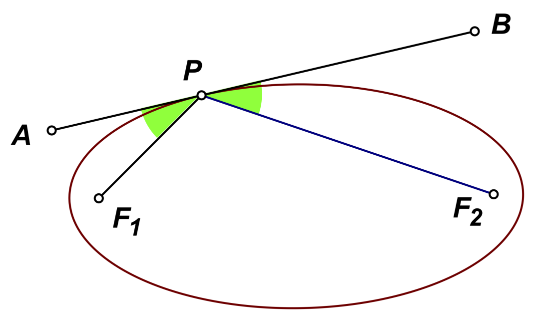


FIGURE 30. Reflective Property of an Ellipse

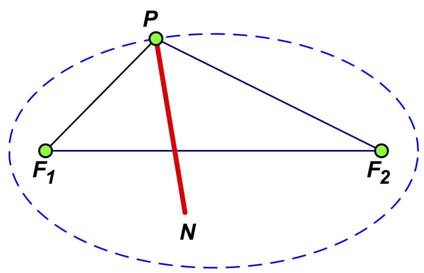
The *normal* to a curve at a point P is the line through P that is perpendicular to the tangent to the curve at P.

Theorem 14.1 gives us an easy way to construct the normal to an ellipse at a point on the circumference.

`Construction NormalAt.`

Given: three points, F_1, F_2, and P that determine an ellipse $\mathcal{E}$ with foci F_1 and F_2 that passes through P.

Constructs: the line N that is the normal to the ellipse at point P.

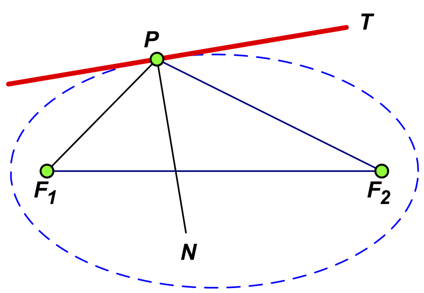


1. $N = \text{angleBisector}(PF_1, PF_2)$.

`Construction TangentAt.`

Given: three points, F_1, F_2, and P that determine an ellipse with foci F_1 and F_2 that passes through P.

Constructs: the line T that is the tangent to the ellipse at point P.



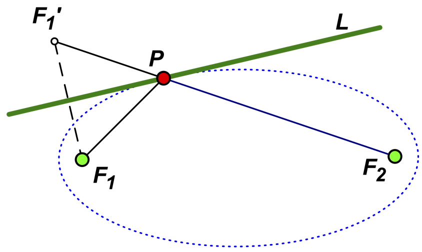
1. $N = \text{normalAt}(\mathcal{E}, P)$
2. $T = \text{perp}(P, N)$.

Theorem 14.1 gives us a way to construct the touch point P given the foci and the tangent.

`Construction FFL.`

Given: the foci of an ellipse, F_1 and F_2 and a line L.

Constructs: a point P on L such that L is tangent to the ellipse $E(F_1, F_2, P)$ at point P.



1. Reflect F_1 about L to get F_1'.
2. $P = F_1'F_2 \cap L$

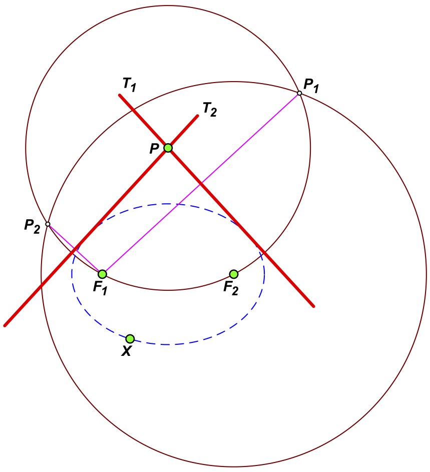
The following construction comes from [78].

`Construction Tangents from Point.`

Given: three points, F_1, F_2, and X that determine an ellipse $\mathcal{E}$ with foci F_1 and F_2 that passes through X.

Also given: a point P outside that ellipse.

Constructs: the tangents T_1 and T_2 from P to the ellipse.



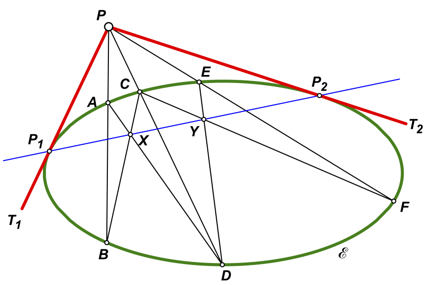
1. $r = XF_1 + XF_2$.
2. $\{P_1, P_2\} = P(PF_1) \cap P(r)$.
3. $T_1 =$ the perp. bisector of P_1F_1.
4. $T_2 =$ the perp. bisector of P_2F_1.

Note. The points of tangency can be found by $P_1F_2 \cap T_1$ and $P_2F_1 \cap T_2$.

`Construction Tangents by Straightedge.`

Given: an ellipse $\mathcal{E}$ and a point P outside the ellipse.

Constructs: the tangents T_1 and T_2 from P to the ellipse using only a straightedge.



1. Draw any three secants PAB, PCD, and PEF to the ellipse
2. $X = AD \cap BC$.
3. $Y = CF \cap DE$.
4. $\{P_1, P_2\} = XY \cap \mathcal{E}$.
5. $T_1 = PP_1$.
6. $T_2 = PP_2$.

Note. This construction presumes that you can find the intersection points of a given line and a given ellipse using a straightedge.

Knowing how to construct a tangent to an ellipse at a given point on the ellipse allows us to illustrate the following interesting results.

The following result comes from [8, Thm. 11.12].

Theorem 14.2. *Let P be any point on an ellipse with focus F and major axis AB. The tangent to the ellipse at P meets the circle with diameter AB at Q as shown in Figure 31. Then $FQ \perp QP$.*

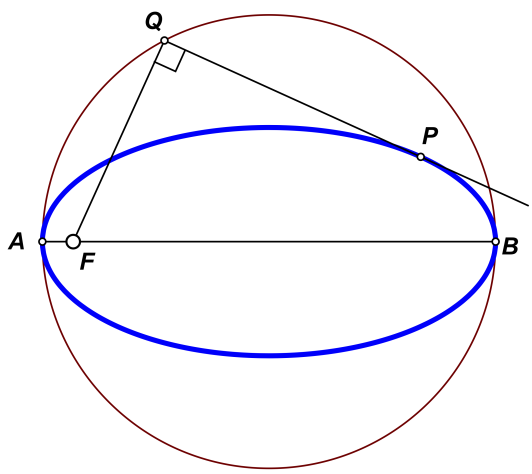


FIGURE 31. $FQ \perp QP$

The following result comes from [8, Thm. 11.10].

Theorem 14.3. *Let XY be any chord of an ellipse that passes through a focus F. Tangents to the ellipse at points X and Y meet at P as shown in Figure 32. Then $PF \perp XY$.*

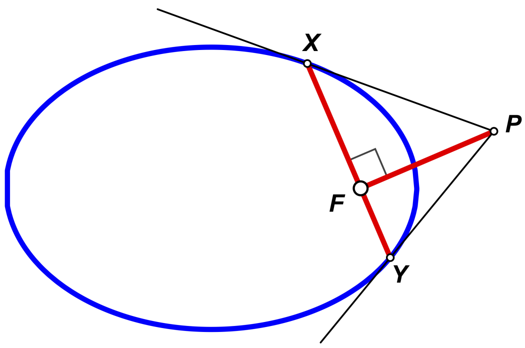


FIGURE 32. $PF \perp XY$

The following result comes from [8, Thm. 11.11].

Theorem 14.4. *Let XY be any chord of an ellipse with foci F_1 and F_2. Tangents to the ellipse at points X and Y meet at P. Let Q be the foot of the perpendicular from P to XY as shown in Figure 33. Then $\angle XQF_1 = \angle YQF_2$.*

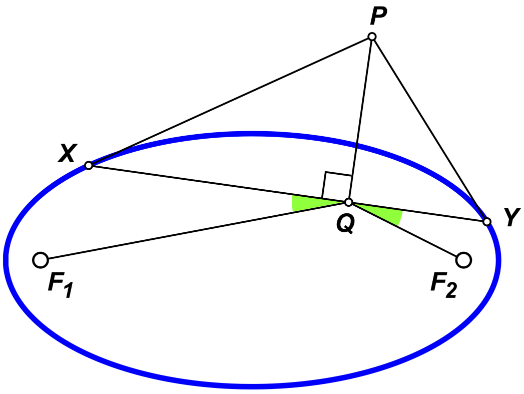


Figure 33. green angles are equal

The following result is a consequence of [56].

Theorem 14.5. *Let P be a point outside an ellipse with center O. The tangents to the ellipse from P touch the ellipse at A and B (Figure 34). Then PO bisects AB.*

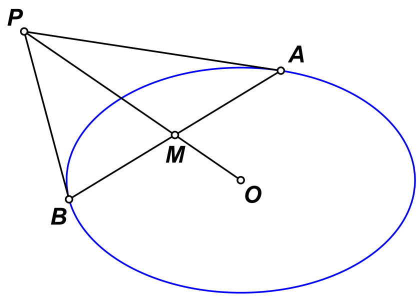


Figure 34. $BM = AM$

The following result comes from [58] and is an immediate consequence of Theorem 14.5.

Theorem 14.6. *Let P be a point outside an ellipse with center O. The tangents to the ellipse from P touch the ellipse at A and B (Figure 35). Then $[APO] = [BPO]$.*

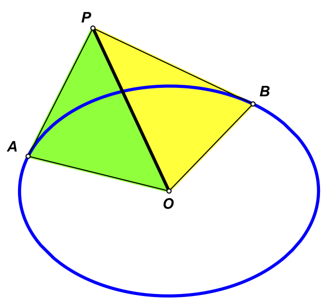


Figure 35. green area = yellow area

The following result comes from [10].

Theorem 14.7 (Bradley's Theorem). *Let E be an ellipse inside $\triangle ABC$. The tangents to E from A, B, and C meet the sides of the triangle at A_1, A_2, B_1, B_2, C_1, and C_2 as shown in Figure 36. Then A_1, A_2, B_1, B_2, C_1, and C_2 lie on ellipse.*

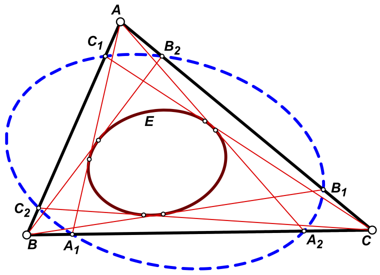


Figure 36. Six points lie on an ellipse

The following result comes from [57].

Theorem 14.8. *Let P be a point outside an ellipse with center O and foci F_1 and F_2. The tangents to the ellipse from P touch the ellipse at A and B (Figure 37). Then $[APX] + [OYF_2] = [BPY] + [OXF_1]$.*

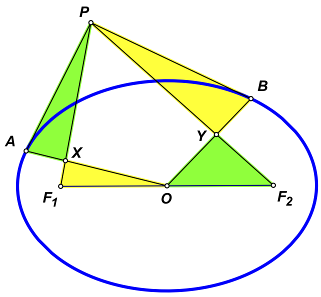


Figure 37. green area = yellow area

The following result comes from [59].

Theorem 14.9. *Let P be a point outside an ellipse with foci F_1 and F_2. The tangents to the ellipse from P touch the ellipse at A and B (Figure 38). Then $(PA)(PF_1)(BF_2) = (PB)(PF_2)(AF_1)$.*

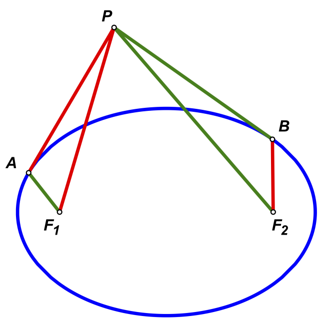


Figure 38. product of red lengths = product of green lengths

The following result comes from [60].

Theorem 14.10. *Let P be a point outside an ellipse with with foci F_1 and F_2. The tangents to the ellipse from P touch the ellipse at A and B (Figure 39). Then $(PA)(PF_2)(BF_1) = (PB)(PF_1)(AF_2)$.*

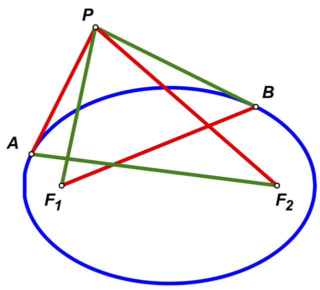


FIGURE 39. product of red lengths = product of green lengths

The following result comes from [8, Thm. 11.5].

Theorem 14.11. *Let P be a point outside an ellipse with with foci F_1 and F_2. The tangents to the ellipse from P touch the ellipse at A and B (Figure 40). Then $\angle APF_1 = \angle F_2PB$.*

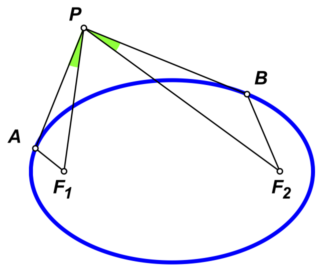


FIGURE 40. green angles are equal

The following result comes from [8, Thm. 11.6].

Theorem 14.12. *Let P be a point outside an ellipse with focus F. The tangents to the ellipse from P touch the ellipse at A and B (Figure 38). Then $\angle PFA = \angle PFB$.*

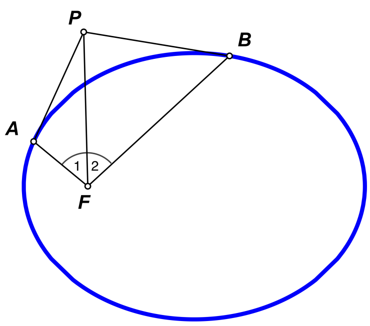


FIGURE 41. $\angle 1 = \angle 2$

The following result comes from [61].

Theorem 14.13. *Let P be a point outside an ellipse with foci F_1 and F_2. The tangents to the ellipse from P touch the ellipse at A and B (Figure 42). Lines BF_1 and AF_2 meet at Q. Then $\angle PQA = \angle PQB$.*

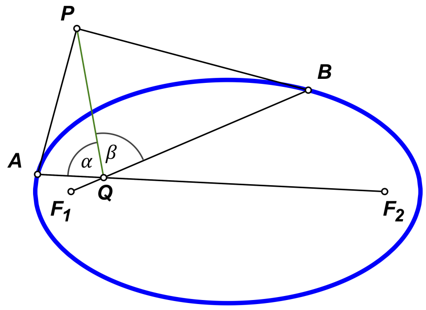


FIGURE 42. $\angle\alpha = \angle\beta$

The following result comes from [62].

Theorem 14.14. *Let P be a point outside an ellipse with foci F_1 and F_2. The tangents to the ellipse from P touch the ellipse at A and B (Figure 43). The perpendicular from P to F_1F_2 meets F_1F_2 at H. Then $\angle AHF_1 = \angle BHF_2$.*

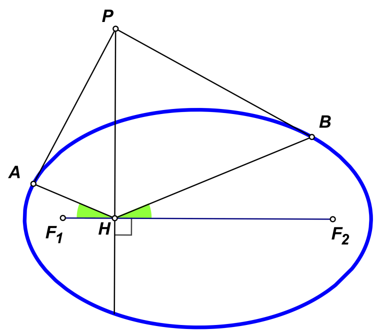


FIGURE 43. green angles are equal

The following result comes from [64].

Theorem 14.15. *Let P be a point outside an ellipse with major axis AB. The tangents to the ellipse from P touch the ellipse at C and D (Figure 44). The perpendicular from P to AB meets AB at H. Then AD, BC, and PH are concurrent.*

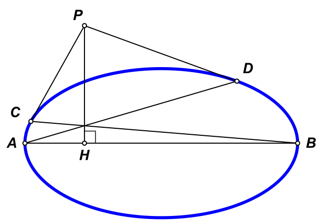


FIGURE 44. AD, BC, PH concur

The following result comes from [65].

Theorem 14.16. *Let P be a point outside an ellipse with major axis AB. The tangents to the ellipse from P touch the ellipse at C and D (Figure 45). Lines AC abd BD meet at E. Lines AD abd BC meet at F. Then points E, P, and F are collinear, $EF \perp AB$, and $EP = PF$.*

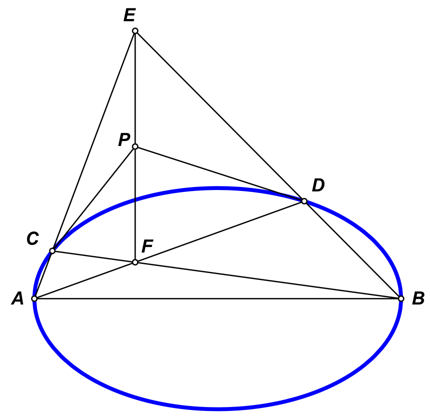


Figure 45. $EP = PF$

The following result comes from [9, p. 67].

Theorem 14.17. *A conic meets the sides of $\triangle ABC$ at points A_1, A_2, B_1, B_2, C_1, C_2 as shown in Figure 46. Tangents to the conic at A_1 and A_2 meet at A'. Points B' and C' are defined similarly. Then AA', BB', and CC' are concurrent.*

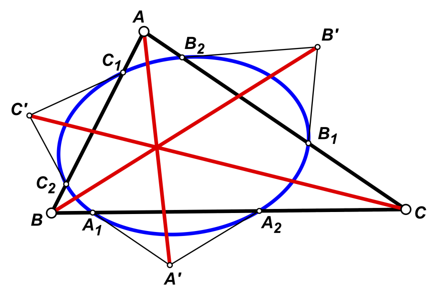


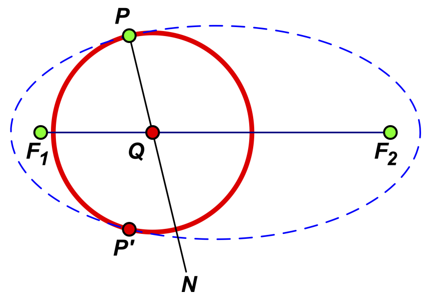
Figure 46. red lines concur

15. Circles Tangent to an Ellipse

A circle is said to be *inscribed* in an ellipse if it is inside the ellipse and is tangent to the ellipse at two points. The center of the circle is necessarily on the major axis of the ellipse.

`Construction Incircle at Point.`

Given: three points, F_1, F_2, and P that determine an ellipse with foci F_1 and F_2 that passes through P.
Constructs: a circle $\mathcal{C}$ inscribed in the ellipse tangent to the ellipse at P.
Also constructs: the center of the circle Q and P' the second point of tangency of the circle with the ellipse.



1. N = normal to ellipse at P using construction `Normal at point`.
2. $Q = F_1F_2 \cap N$.
3. $\mathcal{C} = Q(P)$.
4. $P' = \text{reflect}(P, F_1F_2)$.

Knowing how to construct an incircle of an ellipse allows us to illustrate the following interesting result that comes from [37].

Theorem 15.1. *Two circles, centers O_1 and O_2, are inscribed in an ellipse, touching the ellipse at points T_1 and T_2 as shown in Figure 47. Tangents to the ellipse at T_1 and T_2 meet at P. Then $\angle T_1PO_1 = \angle T_2PO_2$.*

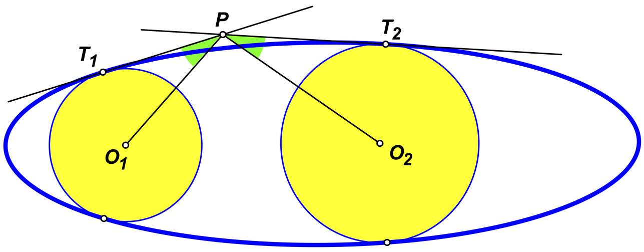
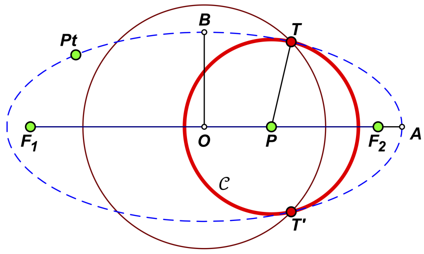


FIGURE 47. green angles are equal

`Construction Incircle Around Point.`

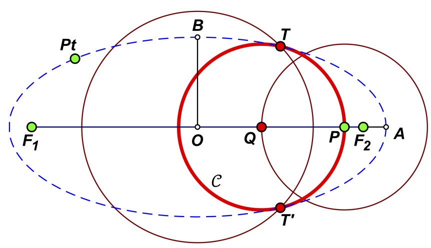
Given: three points, F_1, F_2, and Pt that determine an ellipse with foci F_1 and F_2 that passes through Pt.
Also given: a point P on the major axis of the ellipse.
Constructs: a circle $\mathcal{C}$ inscribed in the ellipse with center at P.
Also constructs: the points of tangency, T and T', of the circle with the ellipse.
Referenced as: $\mathcal{C} = \text{incircleAround}(F_1, F_2, Pt, P)$



1. O = midpoint of F_1F_2.
2. $\{A, B\}$ = vertices of ellipse using construction `Vertices of Ellipse`.
3. $a = OA$, $b = OB$, $c = OF_2$, $d = OP$.
4. $r = b\sqrt{c^2 - d^2}/c$.
5. $s = \sqrt{b^2c^2 + a^2d^2}/c$.
6. $\mathcal{C} = P(r)$.
7. $\{T, T'\} = O(s) \cap \mathcal{C}$.

`Construction Incircle Through Point.`

Given: three points, F_1, F_2, and Pt that determine an ellipse with foci F_1 and F_2 that passes through Pt.
Also given: a point P on the major axis of the ellipse.
Constructs: a circle $\mathcal{C}$ inscribed in the ellipse that passes through P.
Also constructs: the points of tangency, T and T', of the circle with the ellipse.



1. O = midpoint of F_1F_2.
2. $\{A, B\}$ = vertices of ellipse using construction `Vertices of Ellipse`.
3. $a = OA$, $b = OB$, $c = OF_2$, $p = OP$.
4. $r = \left(b^2p + bc\sqrt{a^2 - p^2}\right)/a^2$.
5. $d = p - r$.
6. $s = \sqrt{b^2c^2 + a^2d^2}/c$.
7. $Q = P(r) \cap F_1F_2$.
8. $\mathcal{C} = Q(r)$.
9. $\{T, T'\} = O(s) \cap \mathcal{C}$.

Combining the two constructions, Incircle Around Point and Incircle Through Point allows us to construct two tangent incircles and lets us illustrate the following result which comes from [24, p. 151].

Theorem 15.2 (Relationship Between Two Tangent Incircles)**.** *Two tangent circles with radii r_1 and r_2 are each inscribed in the ellipse $E(a, b)$ as shown in Figure 48. Then*

$$a^4(r_1^2 + r_2^2) - 2a^2(a^2 - 2b^2)r_1r_2 = 4b^4(a^2 - b^2).$$

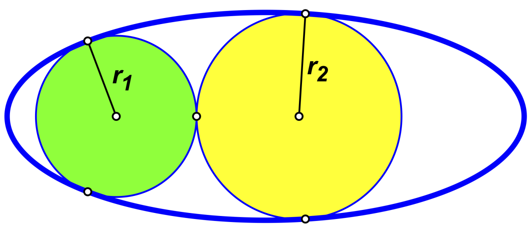


FIGURE 48. $a^4(r_1^2 + r_2^2) - 2a^2(a^2 - 2b^2)r_1r_2 = 4b^4(a^2 - b^2)$

If r_2 is fixed, then there are two values for r_1. Let us call them r_1 and r_3. Using the formula for the sum of the roots of a quadratic gives the following corollary.

Corollary 15.3 (Relationship Between Three Tangent Incircles)**.** *A chain of three circles (radii r_1, r_2, and r_3) are tangent in succession. Each circle is inscribed in the ellipse $E(a, b)$ as shown in Figure 49. Then*

$$r_1 + r_3 = \frac{2(a^2 - 2b^2)}{a^2} r_2.$$

We can take the equations from the previous two results and eliminate either a or b to get the following corollary.

Corollary 15.4. *A chain of three circles (radii r_1, r_2, and r_3) are tangent in succession. Each circle is inscribed in the ellipse $E(a, b)$ as shown in Figure 49.*

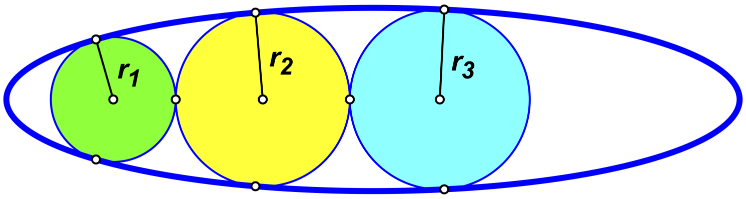


FIGURE 49. $r_1 + r_3 = \dfrac{2(a^2 - 2b^2)}{a^2} r_2$

Then

$$a^2(r_1 - 2r_2 + r_3)^2(r_1 + 2r_2 + r_3) = 16r_2^3(r_2^2 - r_1r_3)$$

and

$$b^2(r_1 - 2r_2 + r_3)(r_1 + 2r_2 + r_3) = 4r_2^2(r_1r_3 - r_2^2).$$

The following theorem comes from the proof of Example 6.5 in [24].

Theorem 15.5 (Relationship Between Four Tangent Incircles)**.** *A chain of four circles (radii r_1, r_2, r_3, and r_4) are tangent in succession. Each circle is inscribed in a given ellipse as shown in Figure 50. Then*

$$\frac{r_1 + r_3}{r_2} = \frac{r_2 + r_4}{r_3}.$$

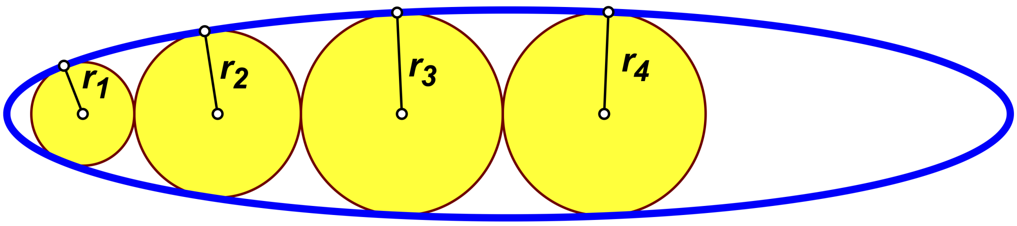


FIGURE 50. $(r_1 + r_3)/r_2 = (r_2 + r_4)/r_3$

Picking a random point on the ellipse, we can use construction `Incircle at Point` to construct the first circle. This circle meets the major axis of the ellipse at a point from which we can use construction `Incircle Through Point` to construct the next circle in the chain.

Note. It is also shown in [24, p. 151] that the radii form a second order linear recurrence

$$r_n = cr_{n-1} - r_{n-2} \qquad \text{where } c = \frac{2(a^2 - 2b^2)}{a^2}$$

if the ellipse is of the form $E(a, b)$. If $w_n = Pw_{n-1} - Qw_{n-2}$ is an arbitrary second order linear recurrence with constant coefficients, then it is known ([49, section 17], The Recurrence for Multiples) that if $W_n = w_{kn}$ for some positive integer k, then

$$W_n = v_k W_{n-1} - Q^k W_{n-2}$$

where v_k depends only on k. In this case, $Q = 1$, so $r_{kn} = v_k r_{k(n-1)} - r_{k(n-2)}$, so

$$\frac{r_{kn} + r_{k(n+2)}}{r_{k(n-1)}}$$

is independent of n. This implies that

$$\frac{r_{kn} + r_{k(n+2)}}{r_{k(n+1)}} = \frac{r_{k(n+1)} + r_{k(n+3)}}{r_{k(n+2)}}.$$

Letting n and k have specific values such as $n = 1$ and $k = 5$, say, gives us interesting relations like

$$r_{15}(r_5 + r_{15}) = r_{10}(r_{10} + r_{20})$$

connecting the radii of circles in the chain. This generalizes a sangaku written in 1842 in the Aichi prefecture [24, Example 6.4] in which $k = 3$.

The following theorem comes from [31, Problem 6.2.5].

Theorem 15.6. *A chain of five circles are tangent in succession. Each circle is inscribed in a given ellipse as shown in Figure 51. The common tangent between the i-th circle and the $(i+1)$-th circle meets the ellipse at P_i and Q_i. If $t_i = P_iQ_i$, then*

$$\frac{t_1 + t_3}{t_2} = \frac{t_2 + t_4}{t_3}.$$

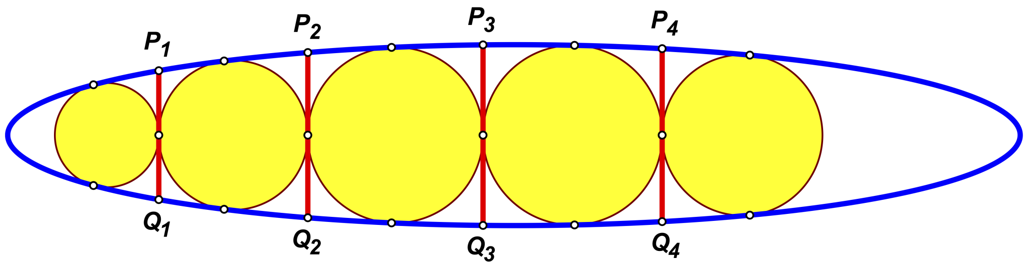


FIGURE 51. $P_iQ_i = t_i \implies (t_1 + t_3)/t_2 = (t_2 + t_4)/t_3$

The following theorem comes from [24, p. 152].

Theorem 15.7. *A chain of five circles are tangent in succession. Each circle is inscribed in a given ellipse as shown in Figure 52. A circle radius r_i is inscribed in the region bounded by the ellipse, the i-th circle and the $(i+1)$-th circle. Then*

$$\frac{r_1 + r_3}{r_2} = \frac{r_2 + r_4}{r_3}.$$

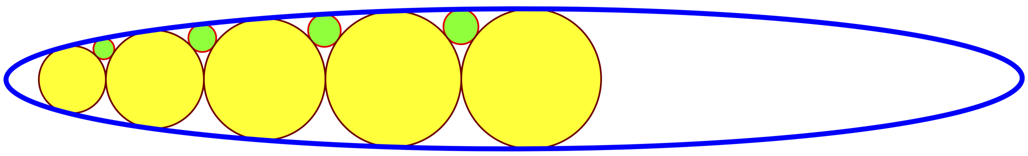


FIGURE 52. r_i = radius of i-th green circle $\implies (r_1 + r_3)/r_2 = (r_2 + r_4)/r_3$

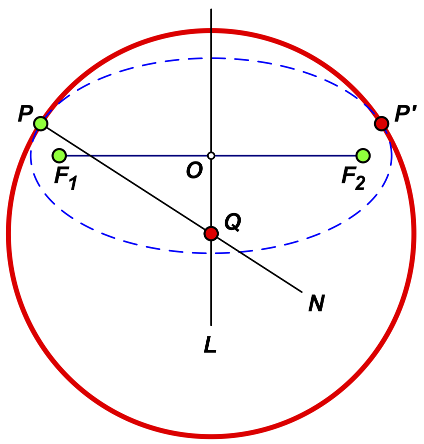
A circle is said to be *circumscribed* about an ellipse if it is outside the ellipse and is tangent to the ellipse at two points. The center of such a circle is necessarily on the minor axis of the ellipse.

`Construction Circumscribed Circle.`

Given: three points, F_1, F_2, and P that determine an ellipse with foci F_1 and F_2 that passes through P.

Constructs: a circle $\mathcal{C}$ circumscribed about the ellipse tangent to the ellipse at P.

Also constructs: the center of the circle Q and P' the second point of tangency of the circle with the ellipse.



1. N = normal to ellipse at P using construction `Normal at point`.
2. L = perpendicular bisector of F_1F_2.
3. $Q = L \cap N$.
4. $\mathcal{C} = Q(P)$.
5. Reflect P about L to get P'.

Knowing how to construct a circle circumscribed to an ellipse, we can now illustrate an interesting property of such a circumcircle.

The following result comes from [8, Thm. 11.14].

Theorem 15.8 (Equal Angle Property of a Circumcircle of an Ellipse)**.** *Let F be a focus of an ellipse. Let P be any point on a fixed circle circumscribed about the ellipse. Let PT be the tangent to the ellipse from point P such that F–P–T proceeds clockwise as shown in Figure 53. Then $\angle FPT$ remains constant as P moves along the circle.*

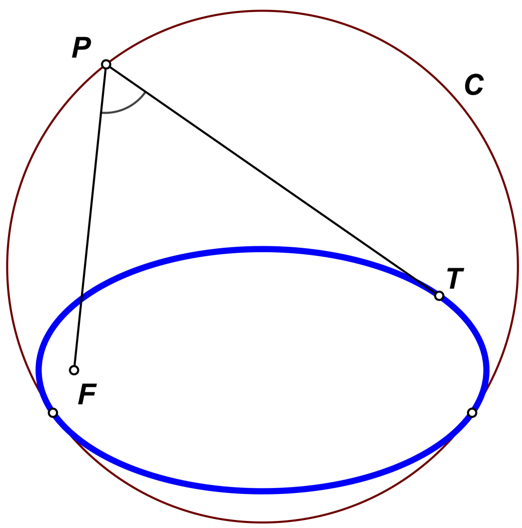


FIGURE 53. $\angle P$ remains constant as P moves along C

The following result that comes from [70].

Theorem 15.9. *Two circles, centers O_1 and O_2, are circumscribed about an ellipse E, touching the ellipse at points T_1 and T_2 as shown in Figure 54. Tangents to the ellipse at T_1 and T_2 meet at P. Then $\angle T_1PO_2 = \angle T_2PO_1$.*

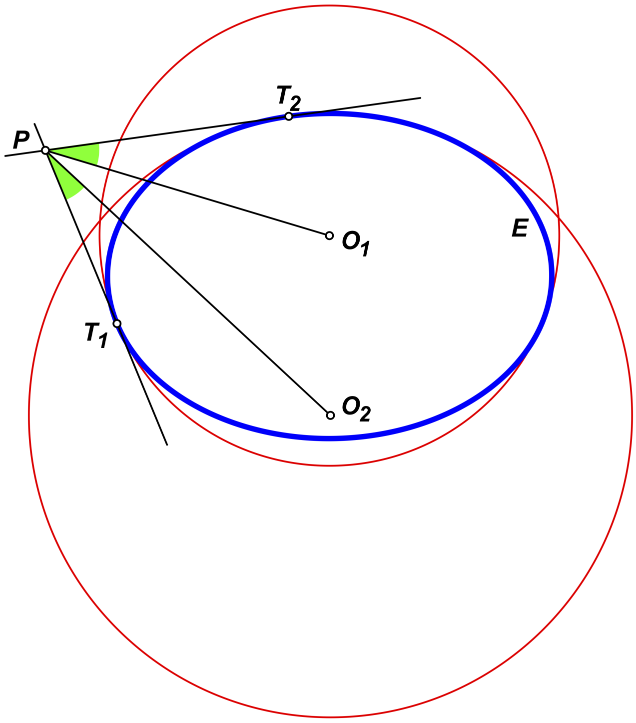


Figure 54. green angles are equal

The following result comes from [77, Thm. 11.3.1].

Theorem 15.10 (Bickart Property of the Steiner Circumellipse)**.** *Let F be a focus of the Steiner circumellipse of $\triangle ABC$. The cevians through F meet the sides at points D, E, and F as shown in Figure 55. Then $AD = BE = CF$.*

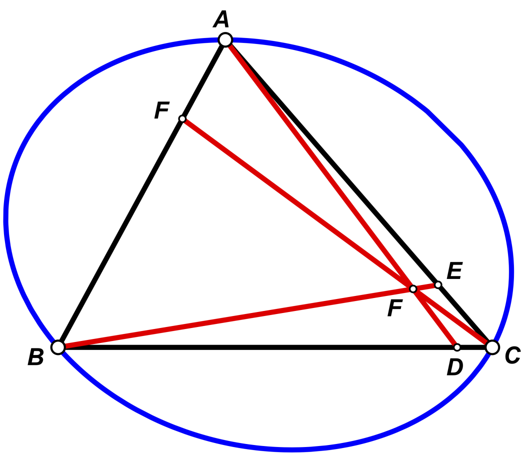


Figure 55. red lengths are equal

The following result comes from [54].

Theorem 15.11. *An ellipse with center O is inscribed in rectangle $ABCD$. The axes of the ellipse are PQ and RS (Figure 56). Then $OB = PR$.*

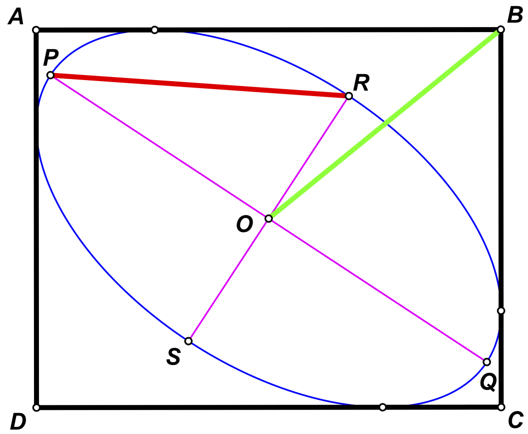


FIGURE 56. red length = green length

The following result comes from [8, Thm. 11.15].

Theorem 15.12. *Let P be any point outside an ellipse with foci F_1 and F_2. Two secants, PAB and PCD pass through the foci as shown in Figure 57. Let BC meet AD at E. Then $\angle BPE = \angle DPE$.*

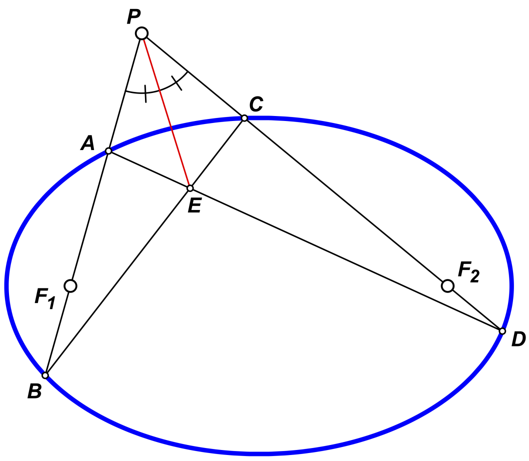


FIGURE 57. $\angle BPE = \angle DPE$

The following result comes from [8, Thm. 11.32].

Theorem 15.13. *Let PQ be a chord of an ellipse with foci F_1 and F_2. PQ meets F_1F_2 at R as shown in Figure 57. Let O_1 be the incenter of $\triangle PQF_1$ and let O_2 be the incenter of $\triangle PQF_2$. Then O_1, R, and O_2 are collinear.*

The following construction is believed to be new.

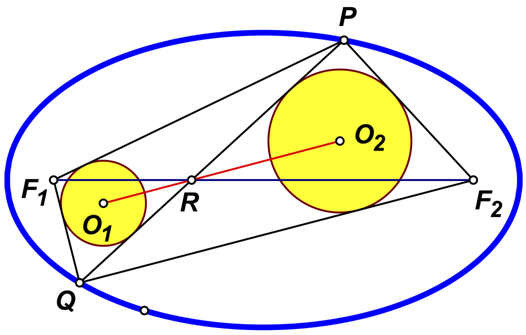


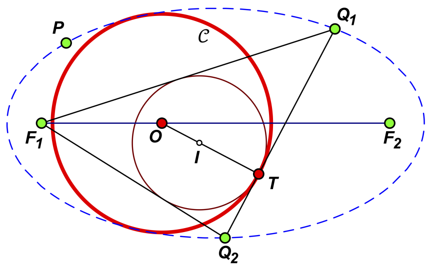
Figure 58. O_1, R, and O_2 are collinear

`Construction Incircle of Segment.`

Given: three points, F_1, F_2, and P that determine an ellipse with foci F_1 and F_2 that passes through P.

Also given: a chord Q_1Q_2 of that ellipse.

Constructs: a circle $\mathcal{C}$ (with center O) inscribed in the ellipse with center on the major axis of the ellipse and tangent to the chord.



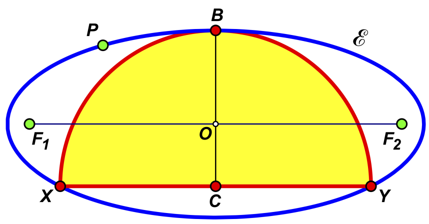
1. $I =$ incenter of $\triangle F_1Q_1Q_2$.
2. $T =$ foot of perp. from I to Q_1Q_2.
3. $O = F_1F_2 \cap TI$.
4. $Q = L \cap N$.
5. $\mathcal{C} = O(T)$.

The following construction comes from [33, Problem 28].

`Construction Inscribed Semicircle.`

Given: three points, F_1, F_2, and P that determine an ellipse $\mathcal{E}$ with foci F_1 and F_2 that passes through P.

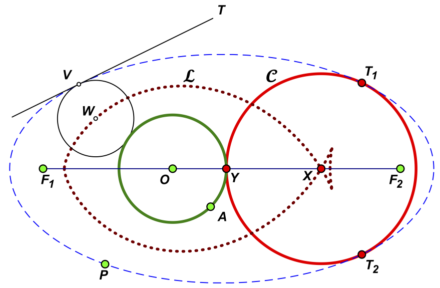
Constructs: a semicircle (with center C) with base XY parallel to the major axis of the ellipse tangent to the ellipse (at point B).



1. $O = \text{midpt}(F_1, F_2)$.
2. $B = \text{perp}(O, F_1F_2) \cap \mathcal{E}$.
3. $a = BF_2$. $b = OB$.
4. $c = 2a^2b^2/(a^2 + b^2)$.
5. $C = B(c) \cap \overrightarrow{BO}$.
6. $\{X, Y\} = C(B) \cap \mathcal{E}$.

If your DGE allows drawing a locus and can find the intersection of a line with a locus, then we have the following construction.

Construction Inscribed Circle Tangent to Circle.
Given: three points, F_1, F_2, and P that determine an ellipse $\mathcal{E}$ with foci F_1 and F_2 that passes through P. **Also given:** a circle with center O that passes through A. **Constructs:** a circle $\mathcal{C}$ inscribed in the ellipse and tangent to the circle. **Also constructs:** points T_1 and T_2.



1. Let V be a variable point on $\mathcal{E}$.
2. $T = \text{tangentAt}(F_1, F_2, V)$.
3. $W = \text{CLP}(O(A), T, V)$.
4. $\mathcal{L} = \text{locus}(W, V, \mathcal{E})$.
5. $X = \mathcal{L} \cap F_1F_2$.
6. $\mathcal{C} = \text{incircleAround}(F_1, F_2, P, X)$.
7. $Y = \mathcal{C} \cap \overrightarrow{XO}$.

Note. Construction `CLP` is the Apollonius construction that constructs a circle tangent to a given circle and tangent to a given line and passing through a given point.

The following result comes from [33, Problem 28]. It describes a sangaku hung in the Yamagata prefecture in 1799.

Theorem 15.14. *A semicircle C_1 is inscribed in an ellipse with its base parallel to the major axis of the ellipse. A circle C_2 is inscribed in the ellipse touching the ellipse in two points and touching the semicircle. A circle C_3 with center on the minor axis of the ellipse is tangent to the ellipse and touches the base of the semicircle as shown in Figure 59. Let r_i be the radius of C_i, $i = 1, 2, 3$. Then $r_1 = 2r_2 + 6r_3$.*

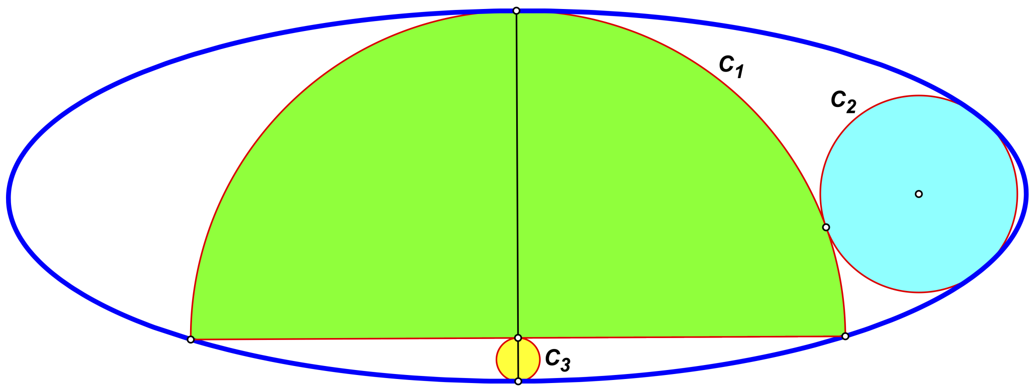


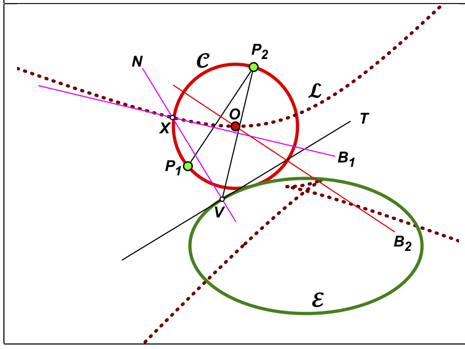
FIGURE 59. $r_1 = 2r_2 + 6r_3$

If your DGE allows drawing a locus and can find the intersection of a line with a locus, then we have the following constructions.

`Construction EPP.`

Given: a conic $\mathcal{E}$ and two points P_1 and P_2.

Constructs: a circle $\mathcal{C}$ with center O tangent to the conic and passing through the two points.



1. Let $V \in \mathcal{E}$.
2. $T = \text{tangentAt}(\mathcal{E}, V)$.
3. $N = \text{perp}(T, V)$.
4. $B_1 = \text{perpBisector}(VP_2)$.
5. $X = N \cap B_1$.
6. $\mathcal{L} = \text{locus}(X, V, \mathcal{E})$.
7. $B_2 = \text{perpBisector}(P_1P_2)$.
8. $O = B_2 \cap \mathcal{L}$.
9. $\mathcal{C} = O(P_1)$.

Note 1. The locus $\mathcal{L}$ represents all points that are equidistant from $\mathcal{E}$ and P_2. The perpendicular bisector B_2 represents all points equidistant from P_1 and P_2.

Note 2. The name "EPP" is a mnemonic for "Ellipse/Point/Point", however, the construction works for all conics, not just ellipses.

Note 3. There are typically two solutions. There are usually two points where $\mathcal{L}$ meets B_2. Figure 60 shows two circles tangent to an ellipse and passing through two fixed points inside the ellipse.

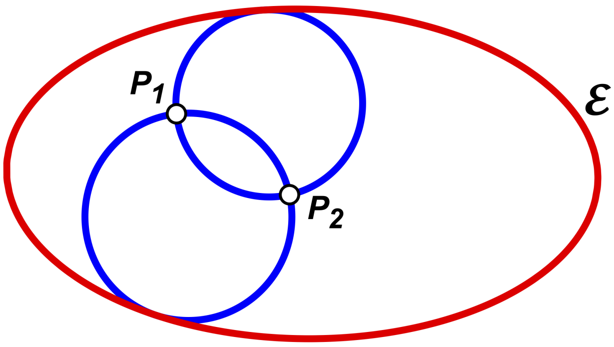
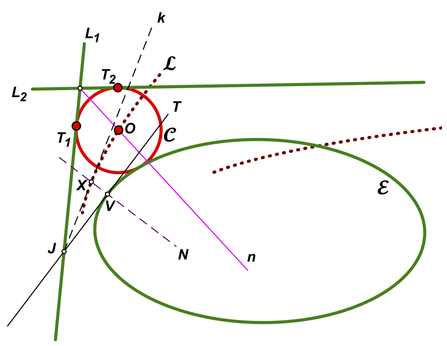


Figure 60. two circles tangent to $\mathcal{E}$ passing through P_1 and P_2

`Construction ELL.`

Given: a conic $\mathcal{E}$ and two lines L_1 and L_2.
Constructs: a circle $\mathcal{C}$ with center O tangent to both lines and the ellipse.



1. Let $V \in \mathcal{E}$.
2. $T = \text{tangentAt}(\mathcal{E}, V)$.
3. $J = T \cap L_1$.
4. $k = \text{angleBisector}(L_1, T)$.
5. $N = \text{perp}(T, V)$.
6. $X = N \cap k$.
7. $\mathcal{L} = \text{locus}(X, V, \mathcal{E})$.
8. $n = \text{angleBisector}(L_1, L_2)$.
9. $O = \mathcal{L} \cap n$.
10. $T_1 = \text{foot}(O, L_1)$. $T_2 = \text{foot}(O, L_2)$.
11. $\mathcal{C} = O(T_1)$.

Note 1. The locus $\mathcal{L}$ represents all points that are equidistant from $\mathcal{E}$ and L_1. The angle bisector n represents all points equidistant from L_1 and L_2.

Note 2. The name "ELL" is a mnemonic for "Ellipse/Line/Line", however, the construction works for all conics, not just ellipses.

Note 3. There can be more than one solution. There are two choices for each angleBisector construction and there may be multiple points where $\mathcal{L}$ meets n. Figure 61 shows 8 circles tangent to two lines and an ellipse.

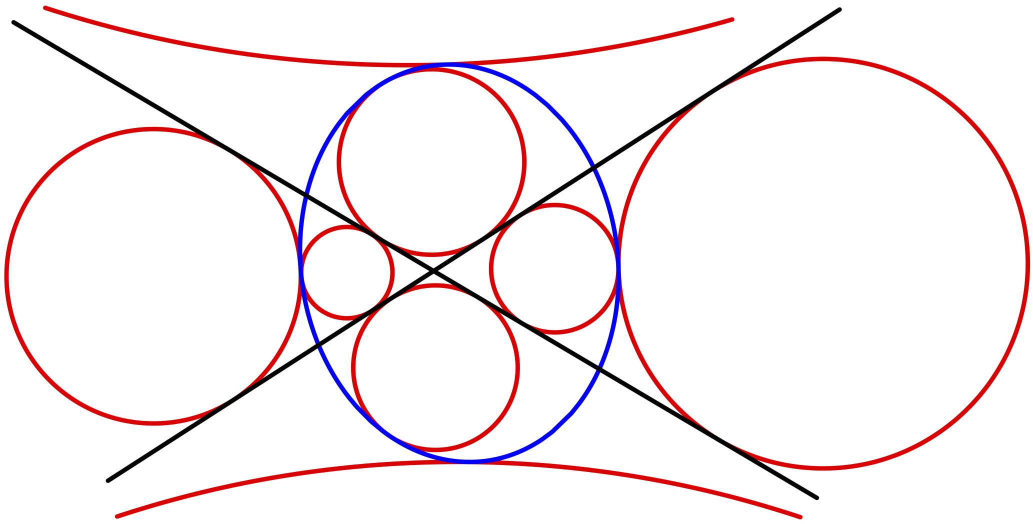


FIGURE 61. 8 circles tangent to two lines and an ellipse

The following result comes from [24, Problem 8.1.4].

Theorem 15.15. *An ellipse is inscribed in rectangle $ABCD$ as shown in Figure 62 Circle c_1 is externally tangent to the ellipse and tangent to AB and BC. Circle c_2 is internally tangent to the ellipse and tangent to AB and AD. Circle c_3 is internally tangent to the ellipse and tangent to AD and CD. Circle c_4 is externally tangent to the ellipse and tangent to BC and CD. Let r_i be the radius of circle c_i. Then*

$$\sqrt{r_1} + \sqrt{r_2} = \sqrt{r_3} + \sqrt{r_4}.$$

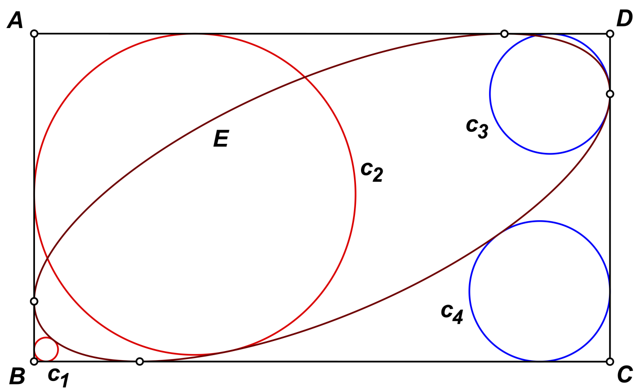
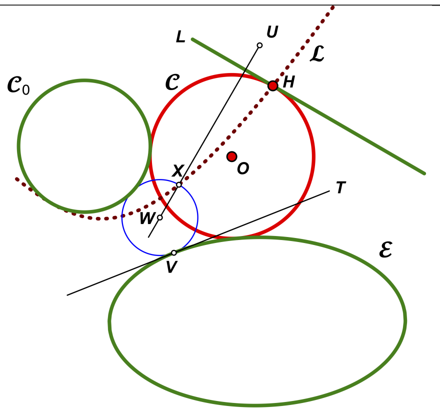


FIGURE 62. $\sqrt{r_1} + \sqrt{r_2} = \sqrt{r_3} + \sqrt{r_4}$

`Construction ECL.`

Given: a conic $\mathcal{E}$, a circle $\mathcal{C}_0$, and a line L.
Constructs: a circle $\mathcal{C}$ with center O tangent to the conic, circle, and line.



1. $V \in \mathcal{E}$.
2. $T = \text{tangentAt}(\mathcal{E}, V)$.
3. $\odot W = \text{CLP}(\mathcal{C}, T, V)$.
4. $U = \text{perp}(L, W)$.
5. $X = U \cap \odot W$.
6. $\mathcal{L} = \text{locus}(X, V, \mathcal{E})$.
7. $H = L \cap \mathcal{L}$.
8. $\mathcal{C} = O(H)$.

Note 1. The locus $\mathcal{L}$ represents all points that are the foot of the perpendicular to the line L from the center of a circle tangent to $\mathcal{C}_0$ and T.

Note 2. The name "ECL" is a mnemonic for "Ellipse/Circle/Line", however, the construction works for all conics, not just ellipses.

Note 3. There are many solutions. In step 3, the CLP construction can produce two circles. In step 5, the line U normally meets the circle W in two points. In step 7, the line L can meet the locus $\mathcal{L}$ in many points (as many as 6 points). Figure 63 shows several circles tangent to an ellipse and passing through a fixed circle and line.

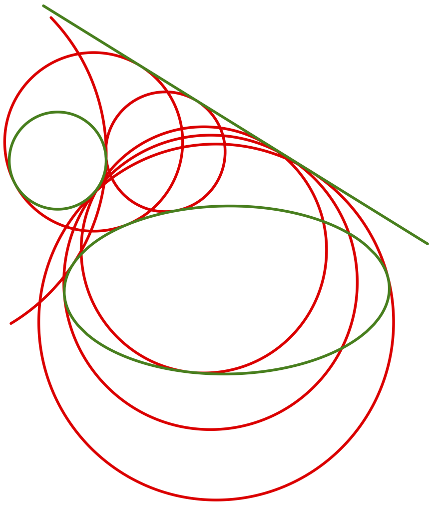
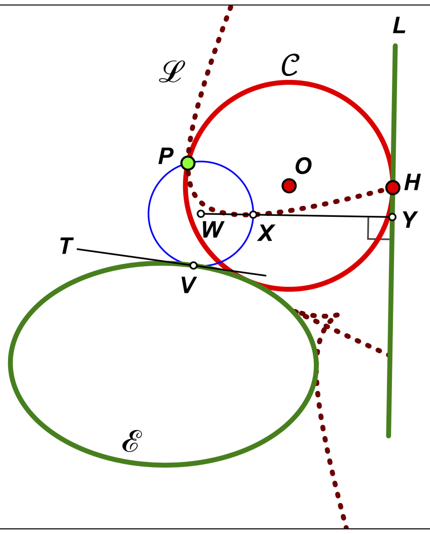


FIGURE 63. several circles tangent to an ellipse, circle, and line

`Construction ELP.`

Given: a conic $\mathcal{E}$, a point P, and a line L.

Constructs: a circle $\mathcal{C}$ with center O tangent to the conic and line and also passing through P.



1. $V \in \mathcal{E}$.
2. $T = \text{tangentAt}(\mathcal{E}, V)$.
3. $\odot W = \text{LPP}(T, V, P)$.
4. $Y = \text{perp}(L, W) \cap L$.
5. $X = \overrightarrow{WY} \cap \odot W$.
6. $\mathcal{L} = \text{locus}(X, V, \mathcal{E})$.
7. $H = L \cap \mathcal{L}$.
8. $\odot O = \text{LPP}(L, H, P)$.
9. $\mathcal{C} = O(H)$.

Note 1. The locus $\mathcal{L}$ represents all points that are the foot of the perpendicular to the line L from the center of a circle tangent to T and passing through P..

Note 2. The name "ELP" is a mnemonic for "Ellipse/Line/Point", however, the construction works for all conics, not just ellipses.

Note 3. There are several solutions. In step 7, the line L can meet the locus $\mathcal{L}$ in as many as four points. Figure 64 shows four circles tangent to an ellipse and a line and passing through a fixed point.

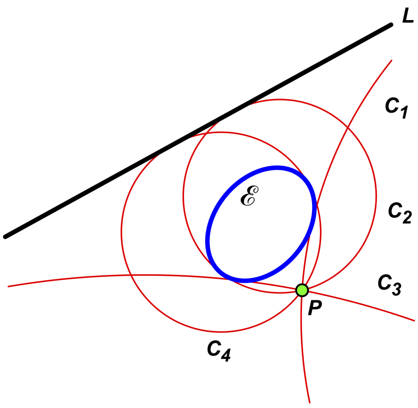
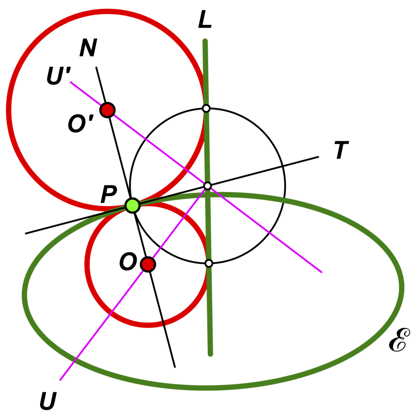


Figure 64. four circles tangent to $\mathcal{E}$ and L passing through P

Construction PonEL.

Given: a point P on a conic $\mathcal{E}$ and a line L.

Constructs: a circle $O(P)$ with center O tangent to the conic at P and tangent to the line.



1. $T = \text{tangentAt}(\mathcal{E}, P)$.
2. $N = \text{perp}(T, P)$.
3. $U = \text{angleBisector}(L, T)$.
4. $O = U \cap N$.

Note. There are two solutions because two angle bisectors can be constructed in step 3.

Open Question 3. *A chain of circles is inscribed in a semiellipse as shown in Figure 65. Is there a simple algebraic relationship between the radii of these circles, similar to the relationship in Theorem xxx?*

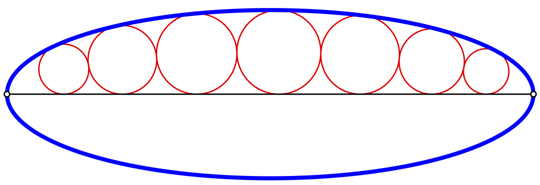


Figure 65. chain of circles inscribed in a semiellipse

The following result comes from [24, Thm. 6.1.1].

Theorem 15.16. *Line segment POQ is a diameter of an ellipse $O(a,b)$ that is parallel to a tangent to the ellipse at point T. A circle with radius r touches the ellipse at T and touches diameter PQ as shown in Figure 66. Then $PQ \cdot r = a \cdot b$.*

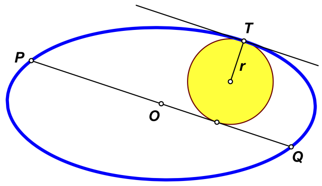


Figure 66. $PQ \cdot r = a \cdot b$

The following result comes from [8, Thm. 11.45].

Theorem 15.17. *A circle is inscribed in an ellipse as shown in Figure 67. A focus of the ellipse, F, lies outside the circle. A tangent from F to the circle meets the circle at A and meets the ellipse at points B and C. Then $AB = CF$.*

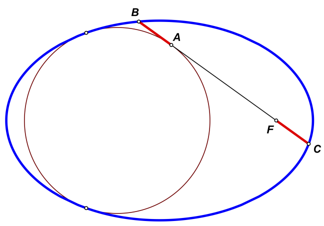


Figure 67. red lengths are equal

The following result comes from [31, Thm. 6.0.3].

Theorem 15.18. *Two non-intersecting circles are inscribed in an ellipse as shown in Figure 68. A common internal tangent to the two circles meets meets the ellipse at points A and B. A common external tangent to the two circles meets meets the ellipse at points C and D. Then $AB = CD$.*

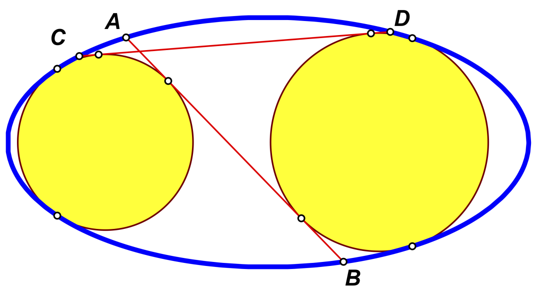


Figure 68. red chords are equal

The following result comes from [8, Thm. 11.46].

Theorem 15.19. *Two non-intersecting circles are inscribed in an ellipse as shown in Figure 69. A common internal tangent to the two circles meets the circles at points C and D and meets the ellipse at points A and B. Then $AC = BD$.*

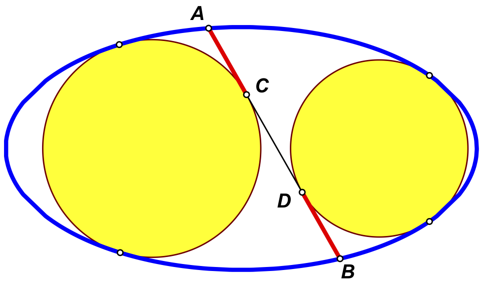


Figure 69. red lengths are equal

The result is also true if the common internal tangent is replaced by a common external tangent.

Theorem 15.20. *Two circles are inscribed in an ellipse as shown in Figure 70. A common external tangent to the two circles meets the circles at points C and D and meets the ellipse at points A and B. Then $AC = BD$.*

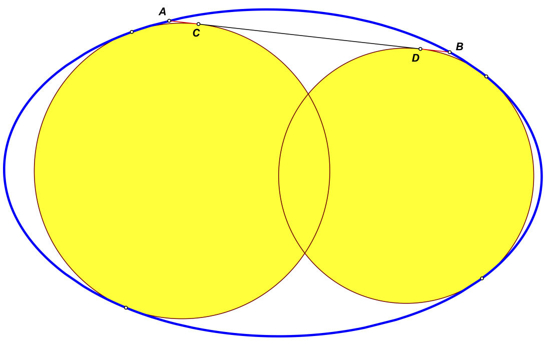


Figure 70. red lengths are equal

The following result comes from [8, Thm. 11.4].

Theorem 15.21. *Two circles are inscribed in an ellipse as shown in Figure 71. A variable point P lies on the ellipse. Tangents from P to the two circles meet the circles at points S and T. Then $PS + PT$ remains constant as P varies on the circumference of the ellipse between the two circles.*

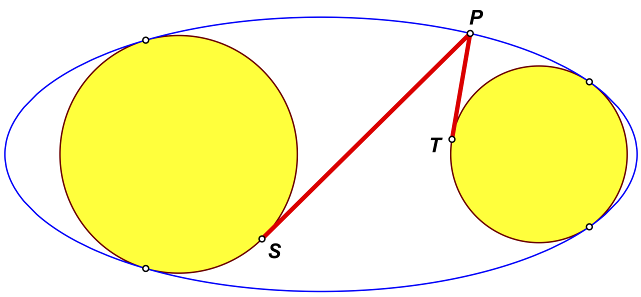


Figure 71. $PS + PT$ =const

An immediate corollary of Theorem 15.21 is the following.

Theorem 15.22. *Two circles are inscribed in an ellipse as shown in Figure 72. Then the length of the tangent from the point where one circle touches the ellipse to the other circle is equal to the length of the tangent from the point where the second circle touches the ellipse to the first circle.*

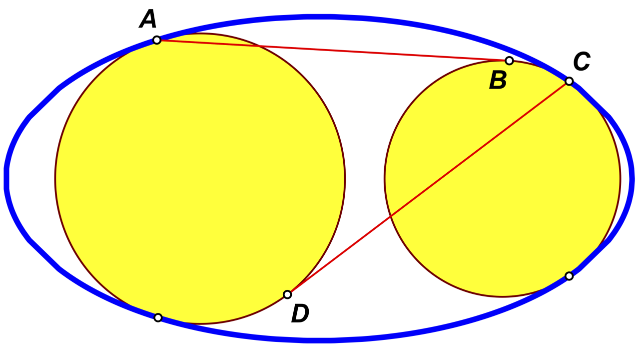


FIGURE 72. red lengths are equal

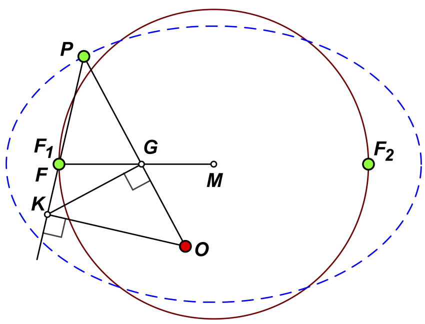
16. Circles of Curvature

The early Japanese geometers created many sangaku involving circles of curvature in an ellipse. For example, see [24, pp. 59–60] and [31, pp.50–54]. To illustrate these results, we need to be able to create circles of curvature.

The following two constructions come from [23, Problem 88].

`Construction CenterOfCurvature`

Given: are three points, F_1, F_2, and P.
Constructs: the center of curvature, O, of the ellipse passing through P with foci F_1 and F_2 at the point P.



1. $G = \text{normalAt}(P, F_1, F_2) \cap F_1F_2$.
2. $M = \text{midpt}(F_1, F_2)$.
3. $F = \overrightarrow{MG} \cap M(F_1)$.
4. $K = \text{perp}(G, PG) \cap PF$.
5. $O = \text{perp}(K, PK) \cap PG$.

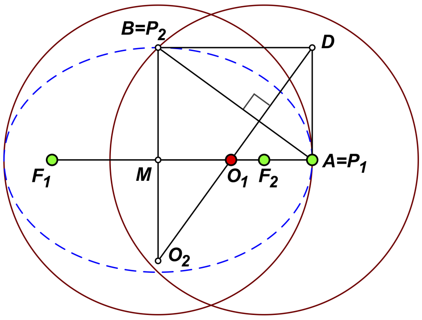
Note 1. The circle through P centered at O is called the *circle of curvature* at the point P.

Note 2. The only reason for constructing the circle $M(F_1)$ is so that we can find F, the focus closest to G.

Note 3. This construction fails if P lies on F_1F_2 or if P lies on the perpendicular bisector of F_1F_2. In those cases, use the following construction.

```
Construction CenterOfCurvAtVertex
```

Given: are three points, F_1, F_2, and A, where A either lies on F_1F_2 of A lies on the perpendicular bisector of F_1F_2.
Constructs: the center of curvature, O, of the ellipse passing through P with foci F_1 and F_2 at the point P.



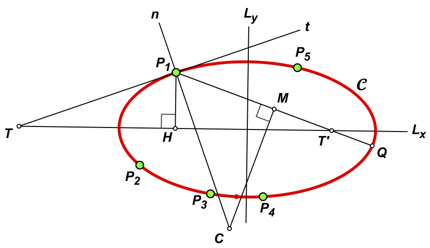
1. $M = \text{midpt}(F_1, F_2)$.
2. $a = (F_1P + F_2AP)/2$.
3. $A = \overrightarrow{MF_2} \cap M(a)$.
4. $B = \text{perp}(M, F_1F_2) \cap F_2(a)$.
5. $D = \text{perp}(B, BM) \cap \text{perp}(A, AM)$.
6. $O = \text{perp}(D, AB) \cap PM$.

Note. P is one of the vertices of the ellipse. The figure shows the case where P is on the line F_1F_2.

The following construction comes from [26, p. 210].

```
Construction Curvature of 5-pt Conic
```

Given: are five points, P_1, P_2, P_3, P_4, and P_5, no three collinear.
Constructs: the center of curvature, C, of the conic $\mathcal{C}$ passing through these five points at the point P_1.



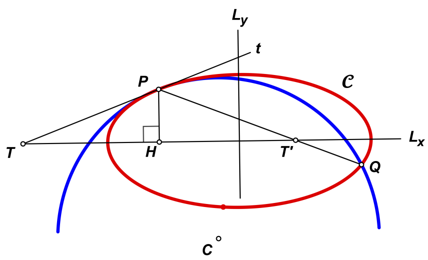
1. $t = \text{tangentAt}(P_1, \mathcal{C})$.
2. $n = \text{perp}(P_1, t)$.
3. $(L_x, L_y) = \text{axes}(\mathcal{C})$.
4. $T = t \cap L_x$.
5. $H = \text{foot}(P_1, L_x)$.
6. $T' = \text{reflect}(T, H)$.
7. $Q = \text{second}(P_1, P_2, P_3, P_4, P_5, P_1L$.
8. $C = \text{perpBisector}(P_1Q) \cap n$.

Note. This construction fails if P_1 is a vertex of the conic.

The following construction is useful if your DGE cannot construct the intersection of a circle and a conic.

```
Construction 2ndPoint on Circle of Curvature
```

Given: a point P on a conic $\mathcal{C}$.
Constructs: Q, the second point of intersection of the conic with the circle of curvature at P.



1. $t = \text{tangentAt}(P, \mathcal{C})$.
2. $(L_x, L_y) = \text{axes}(\mathcal{C})$.
3. $T = t \cap L_x$.
4. $T' = \text{reflect}(T, \text{perp}(P, L_x))$.
5. $C = \text{centerOfCurvature5}(P, \mathcal{C})$.
6. $Q = C(P) \cap PT'$.

The following result is derived from [28, p. 179, Ex. 8].

Theorem 16.1. *Let P be a point on an ellipse with center O such that $OC \perp PC$ where C is the center of curvature of the ellipse at P as shown in Figure 73. Then the yellow area is equal to the green area.*

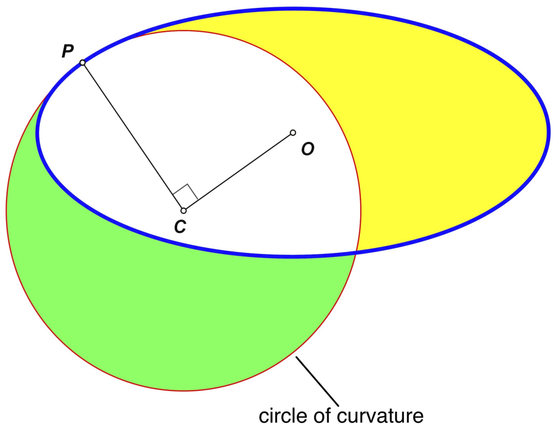
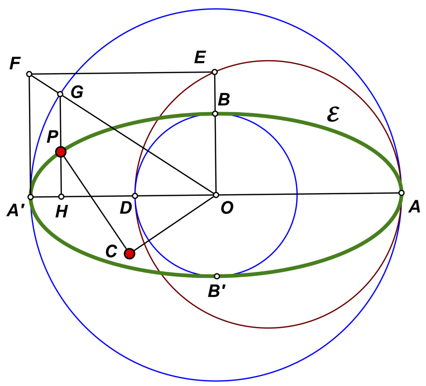


FIGURE 73. yellow area = green are

This figure cannot be drawn using any of the constructions we have listed so far. The following construction, found by Kousik Sett [72], allowed us to illustrate the figure.

`Construction OrthoRadiusOfCurvature`

Given: an ellipse $\mathcal{E}$ with center O and vertices A, B, A', and B'.
Constructs: a point P on the ellipse such that $OC \perp PC$ where C is the center of curvature of the ellipse at P.



1. $D = \overrightarrow{OA'} \cap O(B)$.
2. $E = \odot(DA) \cap \overrightarrow{OB}$.
3. $F = \text{parallel}(E, OA') \cap \text{perp}(A', OA')$.
4. $G = \overrightarrow{OF} \cap O(A')$.
5. $P = \text{perp}(G, OA') \cap \mathcal{E}$.
6. $C = \text{centerOfCurvature}(\mathcal{E}, P)$.

Note. If $OA = a$ and $B = b$, this construction creates E, P, and C so that $OE = A'F = CP = \sqrt{ab}$ and $\tan \angle FOA' = \sqrt{b/a}$.

17. ELLIPSES ASSOCIATED WITH TRIANGLES

The following result is well known.

Theorem 17.1 (Perspector of Inconic)**.** *The touchpoints of an ellipse inscribed in $\triangle ABC$ are D, E, and F as shown in Figure 74. Then AD, BE, and CF are concurrent.*

Note. The point of concurrence is known as the *perspector* of the ellipse with respect to the triangle.

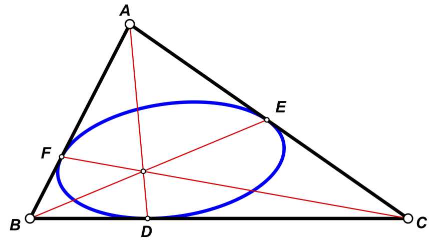


Figure 74. AD, BE, CF concur

The following result comes from [16].

Theorem 17.2. *An ellipse with perspector P is inscribed in $\triangle ABC$. Let M be the midpoint of BC and let D be the point where the ellipse touches BC. Let D' be the reflection of D about M. Let AD' meet the ellipse at T (closer to D') as shown in Figure 75. Then MT is tangent to the ellipse at T.*

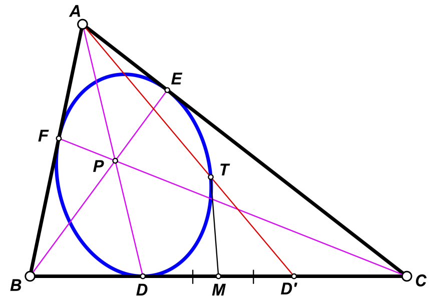


Figure 75. MT is tangent to the ellipse

The following result comes from [8, Result 11.1.22].

Theorem 17.3. *Two ellipses meet at points A, B, C, and D as shown in Figure 76. A third ellipse is tangent to the two ellipses at points W, X, Y, and X as shown. Then AC, BD, XZ, and WY are concurrent.*

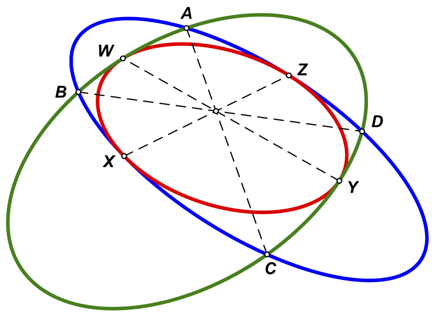


Figure 76. dashed lines concur

The following result comes from [50].

Theorem 17.4. *An ellipse with center D is inscribed in $\triangle ABC$. Let E be the midpoint of AB and let F be the point where the ellipse touches AB (Figure 77). Then $[CDE] = [DEF]$.*

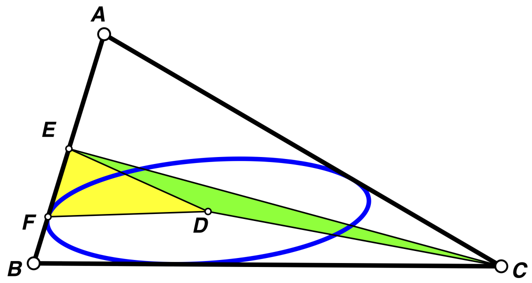


FIGURE 77. green area = yellow area

The following result comes from [42].

Theorem 17.5 (Symmedian Property of the Steiner Inellipse)**.** *Let F be a focus of the Steiner inellipse of $\triangle ABC$ (the ellipse tangent to each side of the triangle at its midpoint). Let AF meet BC at D. Then FD is a symmedian of $\triangle BFC$.*

Note: The *symmedian* of a triangle is the reflection of a median about the angle bisector.

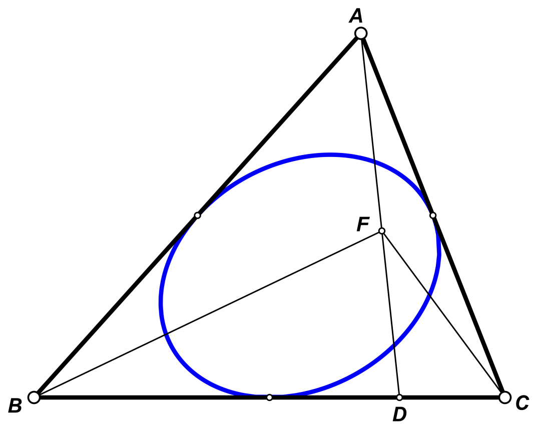


FIGURE 78. FD is a symmedian of $\triangle BFC$

The following three results come from [34].

Theorem 17.6 (Feuerbach Property of the Mandart Inellipse)**.** *Let D, E, and F be the points where the excircles of $\triangle ABC$ touch the sides of the triangle (Figure 79). Then the inellipse that is tangent to the sides of the triangle at these points passes through Fe, the Feuerbach point of the triangle.*

Note: The *Feuerbach point* of a triangle is the point where the incircle touches the nine-point circle. The *nine-point circle* is the circle through the midpoints of the sides.

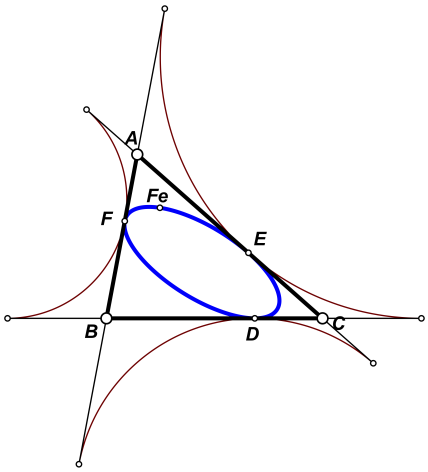


Figure 79. Fe lies on the inellipse

Theorem 17.7 (Feuerbach Property of the Circumellipse of the Medial and Incentral Triangles)**.** *Let D, E, and F be the points where the incircle of $\triangle ABC$ touch the sides of the triangle (Figure 80). Let X, Y, and Z be the midpoints of the sides. Then the ellipse that passes through D, E, F, X, Y, and Z also passes through Fe, the Feuerbach point of the triangle.*

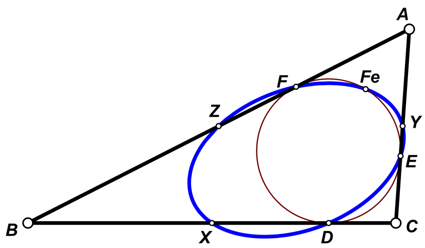


Figure 80. Fe lies on the ellipse

Theorem 17.8 (Feuerbach Property of the Bicevian Conic of X_1 and X_2)**.** *Let I be the incenter of $\triangle ABC$. Let D, E, and F be the points where the cevians through I meet the sides of the triangle (Figure 81). Let X, Y, and Z be the midpoints of the sides. Then the ellipse that passes through D, E, F, X, Y, and Z also passes through Fe, the Feuerbach point of the triangle.*

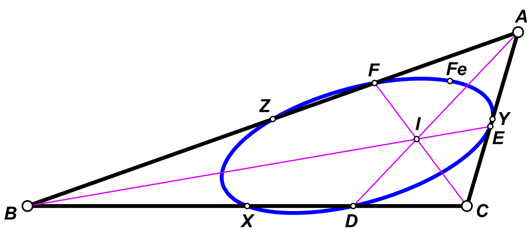


Figure 81. Fe lies on the ellipse

Theorem 17.9. *Let O be the center of of an ellipse inscribed in convex quadrilateral $ABCD$. Let E and F be the midpoints of the diagonals. Then O lies on EF.*

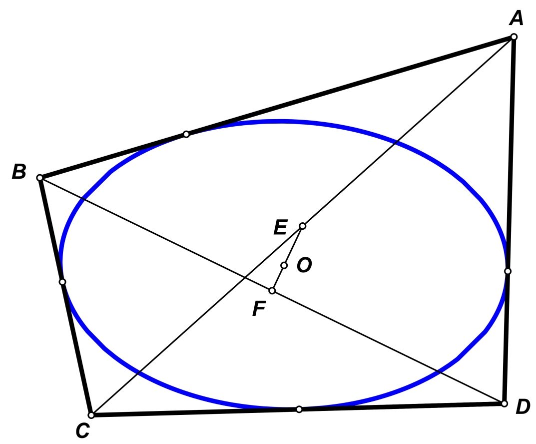


FIGURE 82. O lies on line joining the midpoints of the diagonals

The following result comes from[8, Theorem 11.1.7].

Theorem 17.10. *An ellipse is inscribed in quadrilateral $ABCD$. The touch points with the sides are W, X, Y, and Z as shown in Figure 83. Then AC, BD, WY, and XZ are concurrent.*

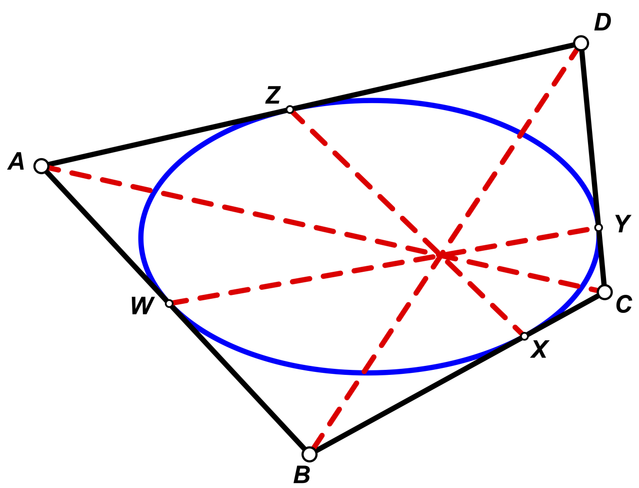


FIGURE 83. red lines concur

The following result is an affine transformation of [48].

Theorem 17.11. *An ellipse is inscribed in $\triangle ABC$. The touch points with BC, CA, and AB are D, E, and F, respectively. Point P is any point inside the ellipse. Line segments from P to the vertices of the triangle meet this ellipse at points X, Y, and Z as shown in Figure 84. Then DX, EY, and FZ are concurrent.*

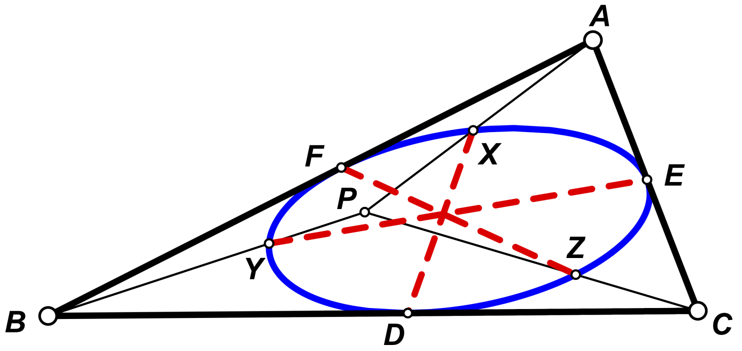
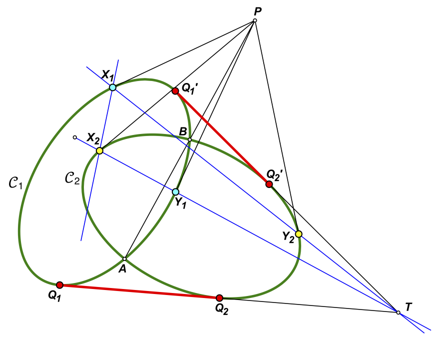


Figure 84. red lines concur

`Construction CommonTangent.`

Given: two conics $\mathcal{C}_1$and $\mathcal{C}_2$ that meet in at least two points A and B.
Constructs: a common tangent Q_1Q_2 to the two conics.



1. $P = \text{reflect}(A, B)$.
2. $\{X_1, Y_1\} = \text{tangentFrom}(P, \mathcal{C}_1)$.
3. $\{X_2, Y_2\} = \text{tangentFrom}(P, \mathcal{C}_2)$.
4. $T = X_1Y_2 \cap X_2Y_1$.
5. $Q_1 = \text{tangentFrom}(T, \mathcal{C}_1)$.
6. $Q_2 = \text{tangentFrom}(T, \mathcal{C}_2)$.

Note 1. Actually, P can be most any point on the common chord.

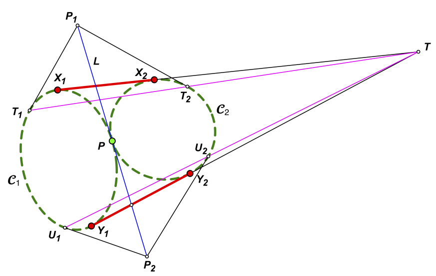
Note 2. If the conics meet in exactly two points (and are not tangent), there will be two solutions as shown in the figure.

Note 3. If the conics meet in four points, the other two tangents will meet at $T' = X_1X_2 \cap T_1Y_2$.

The construction can be modified to handle the case when the two conics are tangent.

`Construction CommonTangentToTouchingConics.`

Given: two 5-point conics $\mathcal{C}_1$and $\mathcal{C}_2$ that are tangent externally at a point P.
Constructs: a common tangent X_1X_2 to the two conics.



1. $L = \text{tangentAt}(P, \mathcal{C}1)$.
2. $P_1 \in L$. $P_2 \in L$.
3. $\{T_1, P\} = \text{tangentFrom}(P_1, \mathcal{C}_1)$.
4. $\{T_2, P\} = \text{tangentFrom}(P_1, \mathcal{C}_2)$.
5. $\{U_1, P\} = \text{tangentFrom}(P_2, \mathcal{C}_1)$.
6. $\{U_2, P\} = \text{tangentFrom}(P_2, \mathcal{C}_2)$.
7. $T = T_1T_2 \cap U_1U_2$.
8. $X_1 = \text{tangentFrom}(T, \mathcal{C}_1)$.
9. $X_2 = \text{tangentFrom}(T, \mathcal{C}_2)$.

Constructing a common tangent to two non-intersecting conics is a bit more difficult.

Points A and B are said to be *conjugate points* or *polar conjugates* with respect to a conic if each lies on the pole of the other. A point can have many polar conjugates.

If P is a point and $\mathcal{C}_1$ and C_2 are two conics, then there is a unique point Q such that P and Q are polar conjugates with respect to both conics. The points P and Q are said to be *common conjugates* with respect to the two conics.

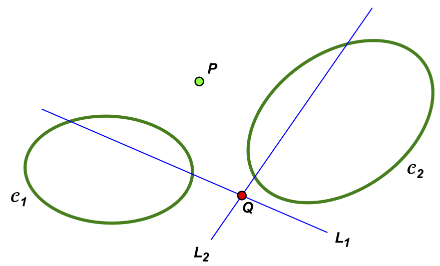
The following construction from [47, Art. 103] explains how to construct the common conjuate of a given point.

`Construction Common Conjugate of Point with respect to Two Conics.`

Given: two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ and a point P.

Constructs: the common conjugate, Q, of point P with respect to the two conics.

Referenced as: $Q = \text{conjugate}(P, \mathcal{C}_1, \mathcal{C}_2)$



1. $L_1 = \text{polar}(\mathcal{C}_1)$.
2. $L_2 = \text{polar}(\mathcal{C}_2)$.
3. $Q = L_1 \cap L_2$.

Given a line L and two conics $\mathcal{C}_1$ and $\mathcal{C}_2$, the locus of the common conjugate of P as P moves along L is a conic known as the *conjugate conic* of the line with respect to the two conics.

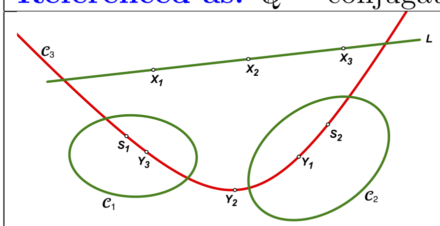
The following construction from [47, Art. 107] explains how to construct the conjugate conic of a given line.

`Construction Conjugate Conic of a Line with respect to Two Conics.`

Given: two conics $\mathcal{C}_1$ and $\mathcal{C}_2$ and a line L.

Constructs: the conjugate conic, $\mathcal{C}_3$, of the line L with respect to the two conics.

Referenced as: $Q = \text{conjugateConic}(L, \mathcal{C}_1, \mathcal{C}_2)$.



1. $X_i \in L$, $i = 1, 2, 3$.
2. $Y_i = \text{conjugate}(X_i, \mathcal{C}_1, \mathcal{C}_2)$, $i = 1, 2, 3$.
3. $S_i = \text{pole}(L, \mathcal{C}_i)$, $i = 1, 2$.
4. $\mathcal{C}_3 = \text{conic}(Y_1, Y_2, Y_3, S_1, S_2)$.

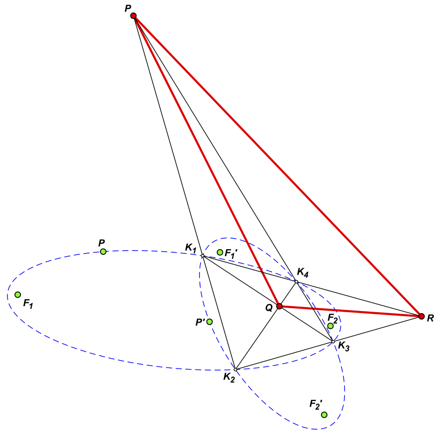
A triangle is said to be *self-polar* with respect to a conic if each side of the triangle is the polar of the opposite vertex with respect to the conic. If a triangle is self-polar with respect to two conics, it is called a *common self-polar triangle*.

`Construction Common Self-Polar Triangle of Intersecting Ellipses.`

Given: two ellipses $\mathcal{E}_1$ (defined by foci F_1, F_2, and point P) and $\mathcal{E}_2$ (defined by foci F'_1, F'_2, and point P') that meet in four points.

Constructs: the triangle PQR that is self-polar with respect to both ellipses.

Referenced as: $\triangle PQR = \text{commonSelfPolar}(\mathcal{E}_1, \mathcal{E}_2)$.



1. $\{K_1, K_2, K_3, K_4\} = \mathcal{E}_1 \cap \mathcal{E}_1$.
2. $Q = K_1K_3 \cap K_2K_4$.
3. $R = K_1K_4 \cap K_2K_3$.
4. $P = K_1K_2 \cap K_3K_4$.

Note. The points K_1, K_2, K_3, and K_4 may be situated in any order, not necessarily as shown in the figure.

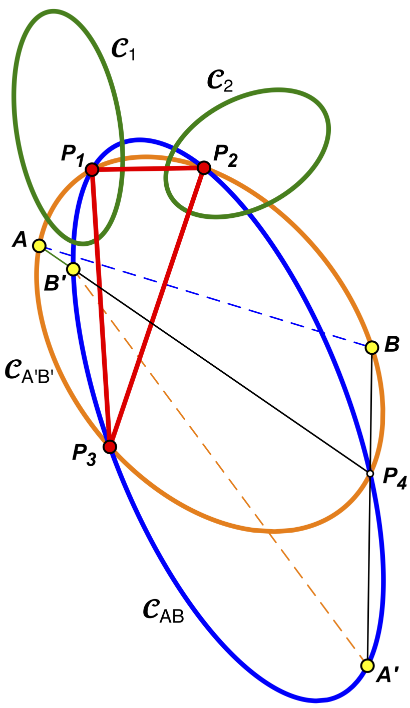
The construction is more complicated if the conics do not intersect. The following constructions comes from [47, Art. 110].

`Construction Common Self-Polar Triangle.`

Given: two non-intersecting conics $\mathcal{C}_1$ and $\mathcal{C}_2$.

Constructs: the triangle XYZ that is self-polar with respect to both conics.

Referenced as: $\{X, Y, Z\} = \text{commonSelfPolar}(\mathcal{C}_1, \mathcal{C}_2)$.



1. Pick any two points A and B.
2. $A' = \text{conjugate}(A, \mathcal{C}_1, \mathcal{C}_2)$.
3. $B' = \text{conjugate}(B, \mathcal{C}_1, \mathcal{C}_2)$.
4. $C = AB' \cap A'B$.
5. $\mathcal{C}_{AB} = \text{conjugateConic}(AB, \mathcal{C}_1, \mathcal{C}_2)$.
6. $\mathcal{C}_{A'B'} = \text{conjugateConic}(A'B', \mathcal{C}_1, \mathcal{C}_2)$.
7. $\{P_1, P_2, P_3, P_4\} = \mathcal{C}_{AB} \cap \mathcal{C}_{A'B'}$.
8. $\{X, Y, Z\} = \{P_1, P_2, P_3, P_4\} \setminus C$.

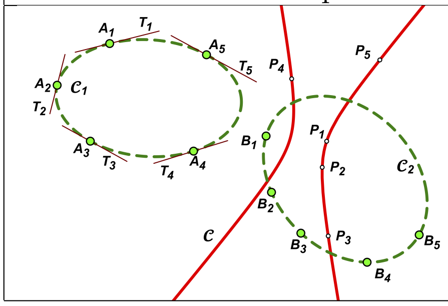
Note. In the figure, AB' meets $A'B$ at C so $C = P_4$, and $\{X, Y, Z\} = \{P_1, P_2, P_3\}$ is the common self-polar triangle to the two green conics.

The fact that four points are constructed (P_1, P_2, P_3, P_4) and we have to remove the one that corresponds to C is awkward for use in scripts.

A better construction comes from [11]. First we need to define a polar conic. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two conics. Let V be a variable point on C_1 and let T be the tangent to $\mathcal{C}_1$ at V. The *polar conic* of $\mathcal{C}_1$ with respect to $\mathcal{C}_2$ is the locus of the pole of T as V varies along $\mathcal{C}_1$. It can be constructed as follows even if your DGE does not have a locus command.

`Construction Polar Conic.`

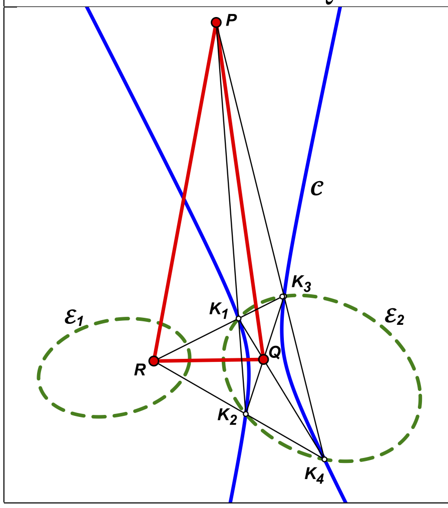
Given: two 5-point conics $\mathcal{C}_1 = \text{conic}(A_1, A_2, A_3, A_4, A_5)$ and $\mathcal{C}_2 = \text{conic}(B_1, B_2, B_3, B_4, B_5)$.
Constructs: the polar conic, $\mathcal{C}$, of $\mathcal{C}_1$ with respect to $\mathcal{C}_2$.
Referenced as: $\mathcal{C} = \text{polarConic}(\mathcal{C}_1, \mathcal{C}_2)$.



1. $T_i = \text{tangentAt}(A_i, \mathcal{C}_1),\ i = 1, 2, 3, 4, 5.$
2. $P_i = \text{pole}(T_i, \mathcal{C}_2),\ i = 1, 2, 3, 4, 5.$
3. $\mathcal{C} = \text{conic}(P_1, P_2, P_3, P_4, P_5).$

`Construction Common Self-Polar Triangle.`

Given: two non-intersecting ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$.
Constructs: the triangle PQR that is self-polar with respect to both ellipses.
Referenced as: $\triangle PQR = \text{commonSelfPolar}(\mathcal{E}_1, \mathcal{E}_2)$.



1. $\mathcal{C} = \text{polarConic}(\mathcal{E}_1, \mathcal{E}_2).$
2. $\{K_1, K_2, K_3, K_4\} = \mathcal{C} \cap \mathcal{E}_2.$
3. $P = K_1K_2 \cap K_3K_4.$
4. $Q = K_1K_4 \cap K_2K_3.$
5. $R = K_1K_3 \cap K_2K_4.$

Note 1. This construction works for any two conics, but we need at least the second one to be an ellipse for step 2.

Note 2. This construction constructs the points P, Q, and R in some random order. It is known that one of the vertices of the common self-polar triangle lies inside ellipse E_1, one lies inside E_2, and one lies outside both ellipses. In the figure, it is Q that lies inside ellipse E_2, but that is only because of the way K_1, K_2, K_3, and K_4 are situated on E_2. Step 2 does not determine an order.

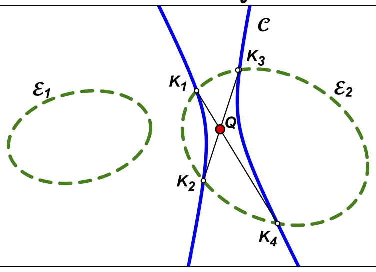
To distinguish the three vertices of the common self-polar triangle, we can use the following two constructions..

`Construction Inner Vertex.`

Given: two non-intersecting ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$.

Constructs: the vertex Q of the common self-polar triangle that is inside $\mathcal{E}_2$.

Referenced as: $Q = \text{innerVertex}(\mathcal{E}_1, \mathcal{E}_2)$.



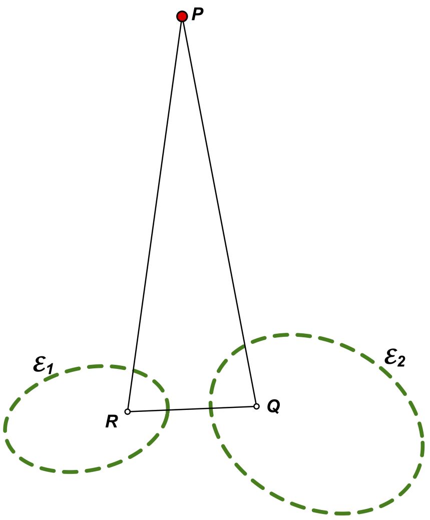
1. $\mathcal{C} = \text{polarConic}(\mathcal{E}_1, \mathcal{E}_2)$.
2. $\{K_1, K_2, K_3, K_4\} = \mathcal{C} \cap \mathcal{E}_2$.
3. $Q = \text{diagonalPoint}(K_1, K_2, K_3, K_4)$.

`Construction Outer Vertex.`

Given: two non-intersecting ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$.

Constructs: the vertex P of the common self-polar triangle that is outside both ellipses.

Referenced as: $P = \text{outerVertex}(\mathcal{E}_1, \mathcal{E}_2)$.



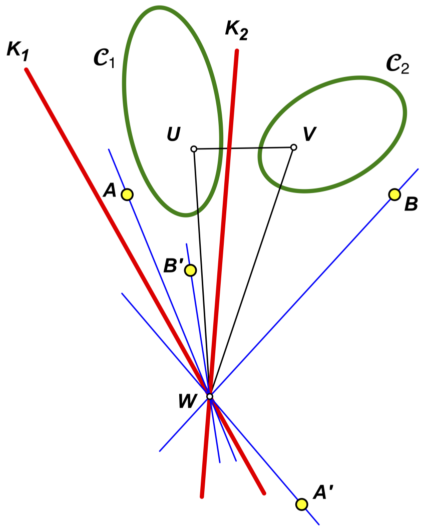
1. $Q = \text{innerVertex}(\mathcal{E}_1, \mathcal{E}_2)$.
2. $R = \text{innerVertex}(\mathcal{E}_2, \mathcal{E}_1)$.
3. $P = \text{pole}(QR, \mathcal{E}_1)$.

`Construction Common Chord.`

Given: two non-intersecting ellipses $\mathcal{C}_1$ and $\mathcal{C}_2$.

Constructs: their common chords K_1 and K_2.

Referenced as: $\{K_1, K_2\} = \text{commonChord}(\mathcal{C}_1, \mathcal{C}_2)$.



1. Pick any two points A and B.
2. $A' = \text{conjugate}(A, \mathcal{C}_1, \mathcal{C}_2)$.
3. $B' = \text{conjugate}(B, \mathcal{C}_1, \mathcal{C}_2)$.
4. $W = \text{outerVertex}(\mathcal{C}_1, \mathcal{C}_2)$.
5. $\{K_1, K_2\} = \text{double}(WA, WA', WB, WB')$.

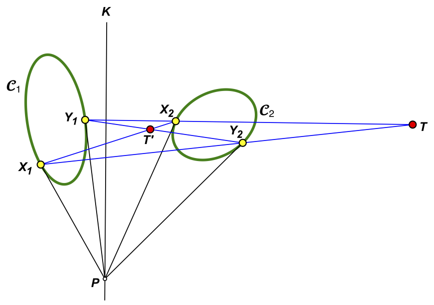
Given two ellipses, one outside the other, there are four common tangents. The two external tangents meet at a point called the *external tangent center*. The two internal tangents meet at a point called the *internal tangent center*.

`Construction Tangent Centers.`

Given: two non-intersecting ellipses $\mathcal{C}_1$ and $\mathcal{C}_2$, one outside the other.

Constructs: their internal and external tangent centers T and T'.

Referenced as: $\{T, T'\} = \text{tangentCenters}(\mathcal{C}_1, \mathcal{C}_2)$.



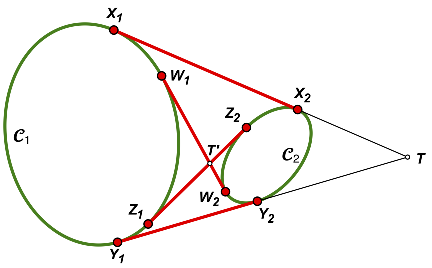
1. $K = \text{commonChord}(\mathcal{C}_1, \mathcal{C}_2)$.
2. $P \in K$.
3. $\{X_1, Y_1\} = \text{tangentFrom}(P, \mathcal{C}_1)$.
4. $\{X_2, Y_2\} = \text{tangentFrom}(P, \mathcal{C}_2)$.
5. $T = X_1Y_2 \cap X_2Y_1$.
6. $T' = X_1X_2 \cap Y_1Y_2$.

`Construction Common Tangents.`

Given: two non-intersecting ellipses $\mathcal{C}_1$ and $\mathcal{C}_2$, one outside the other.

Constructs: their common tangents X_1X_2, Y_1Y_2, Z_1Z_2 and W_1W_2.

Referenced as: $\{X_1X_2, Y_1Y_2, Z_1Z_2, W_1W_2\} = \text{commonTangents}(\mathcal{C}_1, \mathcal{C}_2)$.



1. $\{T, T'\} = \text{tangentCenters}(\mathcal{C}_1, \mathcal{C}_2)$.
2. $\{X_1, Y_1\} = \text{tangentFrom}(T, \mathcal{C}_1)$.
3. $\{X_2, Y_2\} = \text{tangentFrom}(T, \mathcal{C}_2)$.
4. $\{Z_1, Z_2\} = \text{tangentFrom}(T', \mathcal{C}_1)$.
5. $\{W_1, W_2\} = \text{tangentFrom}(T', \mathcal{C}_2)$.

Open Question 4. *Is there a way to project two nonintersecting ellipses into two circles?*

If we could do that, then we could project two ellipses into circles, construyct the common tangents, and then project back.

18. Drawing Normals to an Ellipse

The normal to an ellipse from an external point cannot be constructed with straightedge and compass.

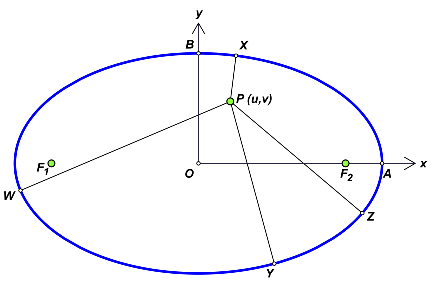
Nevertheless, DGEs can construct such normals since they can perform more operations in addition to straightedge and compass constructions.

`Construction normalFromPoint.`

Given: an ellipse and its foci, F_1 and F_2, and a point P for which there exists four normals to the ellipse from P.

Constructs: the four normals

1. Let O be the midpoint of F_1F_2.
2. Set up a coordinate system with the origin at O and F_2 on the positive x-axis.
3. Let the coordinates of P be (u, v).
4. A =ellipse $\cap$ positive x-axis, B =ellipse $\cap$ positive y-axis, $a = OA$, $b = OB$
5. $A = 2(au + a^2 - b^2)/(bv)$, $B = 2(au - a^2 + b^2)/(bv)$
6. $\Delta = -256 - 27A^4 - 192AB + 6A^2B^2 - 4A^3B^3 - 27B^4$
7. $\alpha = -(AB + 4)\sqrt[3]{2}$, $\beta = -54\alpha^3 + (27C)^2$, $\gamma = -(A^3 + 8B)$, $C = B^2 - A^2$



If $\Delta > 0$, then proceeed as follows:

10. $r = \sqrt{-(3AB + 12)}$, $\theta = 27C/(2r^3)$
12. $T = \cos\left(\frac{1}{3}\cos^{-1}\theta\right)$
13. $p = -3A^2/8$, $q = A^3/B + 8$
14. $X = 2rT/3$, $S = \frac{1}{2}\sqrt{A^2/4 + X}$
15. $r^+ = \sqrt{-4S^2 - 2p - q/S}$
16. $r^- = \sqrt{-4S^2 - 2p + q/S}$
17. $s^{++} = -A/4 + S + r^+/2$
18. $s^{+-} = -A/4 + S - r^+/2$
19. $s^{-+} = -A/4 - S + r^-/2$
20. $s^{--} = -A/4 - S - r^-/2$
21. $\epsilon^{++} = 2\tan^{-1} s^{++}$
22. $\epsilon^{+-} = 2\tan^{-1} s^{+-}$
23. $\epsilon^{-+} = 2\tan^{-1} s^{-+}$
24. $\epsilon^{--} = 2\tan^{-1} s^{--}$

If $\Delta < 0$, then proceeed as follows:

12. $D_0 = \left(27C + \sqrt{\beta}\right)^{1/3}$
13. $D_1 = \alpha/D_0 + D_0/(3\sqrt[3]{2})$
14. $D_2 = \sqrt{A^2/4 + D_1}$
15. $\delta^+ = \sqrt{A^2/2 - D_1 + \gamma/(4D_2)}$
16. $\delta^- = \sqrt{A^2/2 - D_1 - \gamma/(4D_2)}$
17. $t^{++} = -A/4 + D_2/2 + \delta^+/2$
18. $t^{+-} = -A/4 + D_2/2 - \delta^+/2$
19. $t^{-+} = -A/4 - D_2/2 + \delta^-/2$
20. $t^{--} = -A/4 - D_2/2 - \delta^-/2$
21. $\epsilon^{++} = 2\tan^{-1} t^{++}$
22. $\epsilon^{+-} = 2\tan^{-1} t^{+-}$
23. $\epsilon^{-+} = 2\tan^{-1} t^{-+}$
24. $\epsilon^{--} = 2\tan^{-1} t^{--}$

Then $X =$

$Z = (a\cos\epsilon^{+-}, b\sin\epsilon^{+-})$, $W = (a\cos\epsilon^{--}, b\sin\epsilon^{--})$

The following result comes from [9, p. 114].

Theorem 18.1 (Joachimsthal's Circle)**.** *Let P be a point in the plane of an ellipse with center O such that four normals can be drawn from P to the ellipse. The* *four normals meet the ellipse at points W, X, Y, and Z. Let W' be the reflection of W about O (Figure 85). Then W', X, Y, and Z lie on a circle.*

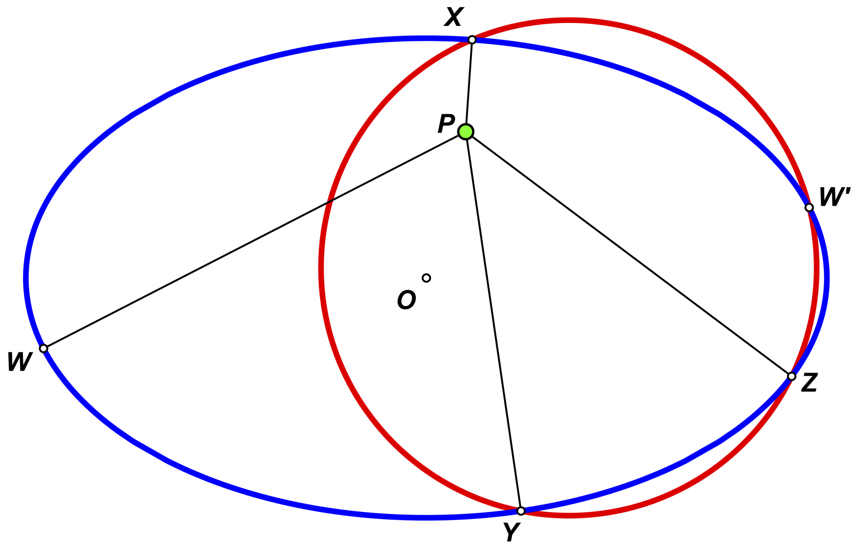


FIGURE 85. Joachimsthal's Circle

The following result comes from [45].

Theorem 18.2 (Pompe's Rectangle)**.** *An ellipse is inscribed in $\angle XPY$ (Figure 86). There are four circles that can also be inscribed in this angle and are tangent to the ellipse. Then the four touch points with the ellipse form a rectangle.*

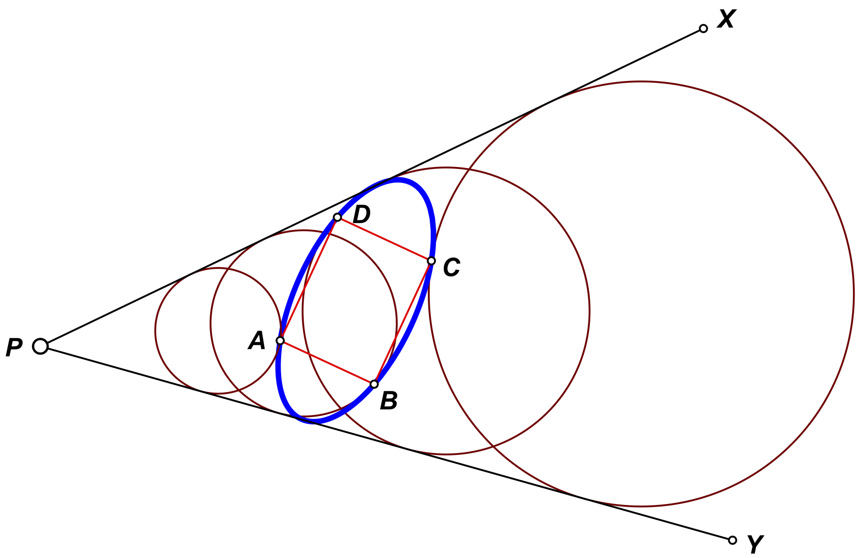


FIGURE 86. Pompe's Rectangle

Theorem 18.3. *Let K be the symmedian point of $\triangle ABC$. An ellipse has center K and is inscribed in $\triangle ABC$ (Figure 87). Let D be the closest point on the ellipse to A. Define E and F similarly. Then AD, BE, and CF are concurrent.*

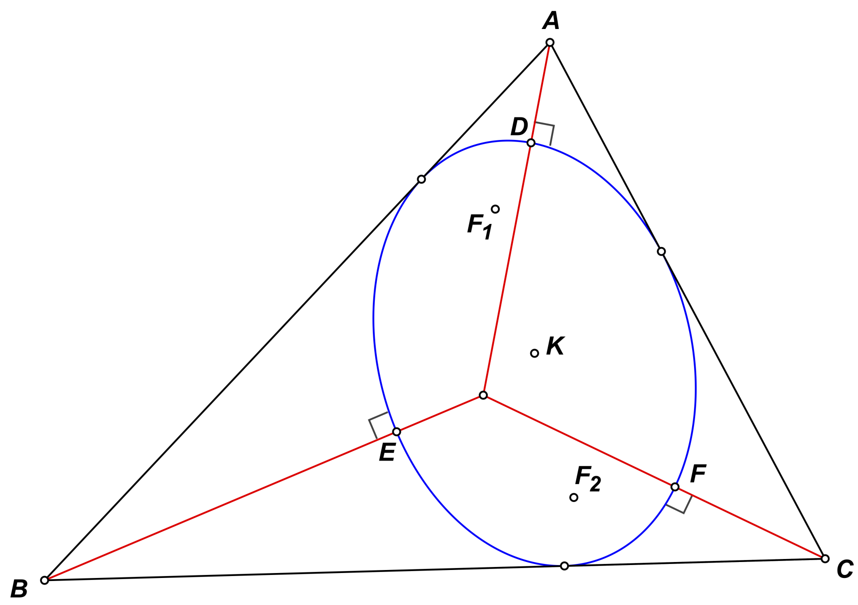


FIGURE 87.

Theorem 18.4. *Let K be the symmedian point of $\triangle ABC$. The inellipse to $\triangle ABC$ with center K touches the sides at D, E, and F (Figure 88). Then AD, BE, and CF are concurrent at the orthocenter of the triangle.*

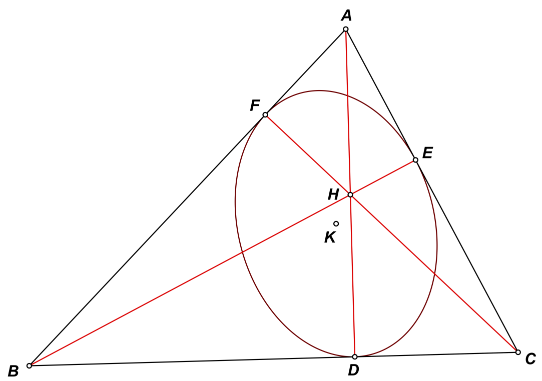


FIGURE 88. red lines meet at orthocenter

19. A Few More Delights

We conclude this paper with a few more delightful results about ellipses that are hard to draw.

The following results comes from [13].

Theorem 19.1. *A circle meets an ellipse at points A, B, C, and D. Another circle meets an ellipse at points A, B, E, and F (Figure 89). Then $CD \parallel EF$.*

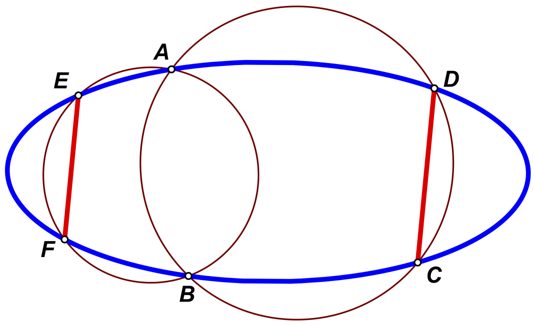


Figure 89. red lines are parallel

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