International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 11, 2026 pp. 1–19
web: http://www.journal-1.eu/
Received 22 Jan. 2026. Published on-line 19 Mar. 2026

Some Geometric Properties of the Yff Points of a Triangle

Stanley Rabinowitz[a] and Francisco Javier García Capitán[b]
[a] 545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com
[b] I. E. S. Álvarez Cubero, 14800 Priego de Córdoba, Spain
e-mail: garciacapitan@gmail.com
web: http://garciacapitan.epizy.com

Abstract. The Yff points of a triangle were introduced by Peter Yff in 1963. Since then, very few new facts have been discovered about these points. We present some geometrical properties of the Yff points of various shaped triangles which were discovered and proved by computer.

Keywords. bicentric pairs, computer-discovered mathematics,Yff points.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

The following result was found by Peter Yff in 1963 [4].

Theorem 1.1. *Let D_1, D_2, E_1, E_2, F_1, F_2 be points on the sides of triangle ABC (as shown in Figure 1) such that $AF_1 = BD_1 = CE_1 = AE_2 = BF_2 = CD_2 = u$ where u is the real root of the equation $x^3 = (a-x)(b-x)(c-x)$. Then AD_1, BE_1, CF_1 meet in a point Y_1 and AD_2, BE_2, CF_2 meet in a point Y_2.*

The points Y_1 and Y_2 have become known as the *Yff Points* of a triangle. If $\triangle ABC$ is named counterclockwise, then Y_1 is the *1st Yff point* and Y_2 is the *2nd Yff point.*

Very little is known about these points that was not found in Yff's original paper in 1963. It is clear from the definition that the Yff points are isotomic conjugates.

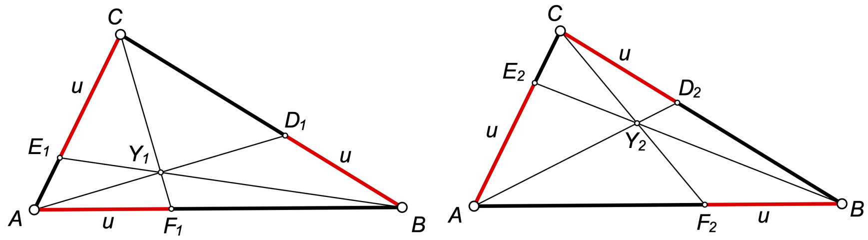


FIGURE 1. Yff Points

A summary of what is known about Yff points can be found in [4], [3], [5, §3.3.2], and [2].

The value u is connected to a, b, and c by the equation

$$u^3 = (a-u)(b-u)(c-u). \tag{1}$$

The value of u can be expressed in terms of radicals as follows.

$$u = \frac{\sqrt[3]{k_1}}{6\sqrt[3]{2}} - \frac{k_4}{3\ 2^{2/3}\sqrt[3]{k_1}} + \frac{1}{6}(a+b+c)$$

where

$$\begin{aligned} k_1 &= k_3 + \sqrt{k_2}, \\ k_2 &= k_3^2 + 4k_4^3, \\ k_3 &= 2(a+b+c)^3 - 18(ab+bc+ca)(a+b+c) + 108abc, \end{aligned}$$

and

$$k_4 = 6(ab+ac+bc) - (a+b+c)^2.$$

This representation is useful when constructing the Yff points in Dynamic Geometry Enviroments such as Geometer's Sketchpad or GeoGebra.

Yff also found the following interesting result, where X_n denotes the n-th named triangle center in the Encyclopedia of Triangle Centers [1].

Theorem 1.2. *Let ABC be an arbitrary triangle (Figure 2). Then $Y_1Y_2 \perp X_1X_3$.*

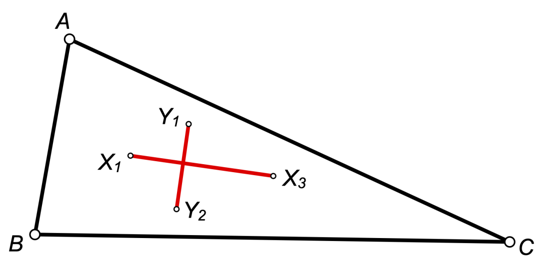


FIGURE 2. red lines are perpendicular

Open Question 1. *Is there a purely synthetic geometric proof of Theorem 1.2 (without using coordinates)?*

In this paper, we will give some new geometric properties of Yff points in various shape triangles.

We found these properties using Mathematica. We started with a random numerical triangle. As n and m ranged from 1 to 30, we looked at the triangle plus the Yff points plus X_n and X_m and checked (numerically) for various properties in the resulting figure, such as whether two lines were parallel, two segments had equal length, two angles had equal measure, etc. We did not find any results that were not equivalent to Theorem 1.2.

We did the same thing for various shaped triangles, such as right triangles, heptagonal triangles, and triangles whose sides are in arithmetic progression. The results are presented in the next sections. Since these results were found numerically, valid to 10 decimal places, the computations did not constitute a proof that the result was true. At this point, we then used Mathematica again, this time using exact symbolic computations to prove formally that each result was true.

2. Right Triangles

Theorem 2.1. *Let ABC be a right triangle (named counterclockwise) with right angle at B (Figure 3). Then AY_2, CY_1, and X_8X_{20} are concurrent.*

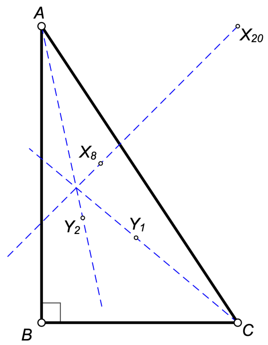


Figure 3. dashed lines are concurrent

Proof. Set up a barycentric coordinate system using $\triangle ABC$ as the reference triangle, so that he coordinates for A, B, and C are $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$, respectively. The coordinates for X_8 and X_{20} can be found in [1].

$$X_8 = a - b - c ::$$
$$X_{20} = 3a^4 - 2a^2b^2 - 2a^2c^2 - b^4 + 2b^2c^2 - c^4 ::$$

where "$f(a, b, c)$::" denotes the coordinates $\big(f(a, b, c) : f(b, c, a) : f(c : a : b)\big)$.

The coordinates for Y_1 and Y_2 can be found in [4].

$$Y_1 = \left(\frac{c-u}{b-u}\right)^{1/3} ::$$
$$Y_2 = \left(\frac{b-u}{c-u}\right)^{1/3} ::$$

where u is given in Theorem 1.1. We will call these the *symmetric coordinates* for Y_1 and Y_2.

Yff also gives simpler coordinates for Y_1 that are not as symmetric: $\left(\frac{u}{b-u} : \frac{a-u}{u} : 1\right)$. Eliminating fractions and finding similar coordinates for Y_2 give

$$\begin{aligned} Y_1 &= \big(u^2 : (a-u)(b-u) : u(b-u)\big) \\ Y_2 &= \big((a-u)(b-u) : u^2 : u(a-u)\big) \end{aligned} \tag{2}$$

We will call these the *simple coordinates* for Y_1 and Y_2.

If we represent the line $px + qy + rz = 0$ by the triple $(p : q : r)$, then, we get the nice symmetric properties that the line through two points is given in Mathematica by the vector cross product `Cross[point1,point2]` and the point of intersection of two lines is given by `Cross[line1,line2]`. Having found the barycentric coordinates for all the points in Figure 3, we can now calculate the equations of the three dashed lines. To avoid cube roots, we use the coordinates for Y_1 and Y_2 as given in display (2) .

From [5, §4.3], we have that three lines $(p_i : q_i : r_i)$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0.$$

In our case, the condition for the three dashed lines to be concurrent is

$$\begin{aligned} &- 2a^7bu + 2a^7u^2 + 2a^6b^2u - 4a^6bcu + 4a^6bu^2 + 4a^6cu^2 - 6a^6u^3 + 4a^5b^3u \\ &- 4a^5b^2u^2 + 6a^5bcu^2 - 4a^5bu^3 - 6a^5cu^3 + 4a^5u^4 - 4a^4b^4u - 4a^4b^3u^2 \\ &+ 6a^4b^2u^3 + 4a^4bc^3u - 4a^4bc^2u^2 - 2a^4bcu^3 + 4a^4bu^4 - 4a^4c^3u^2 + 4a^4c^2u^3 \\ &- 2a^3b^5u + 6a^3b^4u^2 + 2a^3bc^4u - 4a^3bc^3u^2 + 4a^3bc^2u^3 - 2a^3c^4u^2 + 4a^3c^3u^3 \\ &- 8a^3c^2u^4 + 2a^2b^6u + 4a^2b^5cu - 4a^2b^4cu^2 - 2a^2b^4u^3 - 4a^2b^3c^3u - 2a^2b^2c^4u \\ &+ 4a^2b^2c^3u^2 + 2a^2c^4u^3 - 4ab^6u^2 - 6ab^5cu^2 + 4ab^5u^3 + 4ab^4c^2u^2 + 6ab^4cu^3 \\ &- 4ab^4u^4 + 8ab^3c^3u^2 - 4ab^3c^2u^3 - 8ab^2c^3u^3 - 2abc^5u^2 + 2ac^5u^3 + 4ac^4u^4 \\ &+ 2b^6u^3 + 2b^5cu^3 - 4b^5u^4 - 4b^4c^2u^3 - 4b^3c^3u^3 + 8b^3c^2u^4 + 2b^2c^4u^3 + 2bc^5u^3 \\ &- 4bc^4u^4 = 0 \end{aligned} \tag{3}$$

Note that the radical form for u is too complex, so we have left u as an unevaluated variable, noting that u satisfies Equation (1).

We can eliminate u from equations (3) and (1). This can be accomplished using elimination theory and Gröbner bases. The process is straightforward, but tedious and is best left for computers. In Mathematica, the `Eliminate` function is used for this purpose. Eliminating u from equations (3) and (1) gives

$$\begin{gathered} -a^{11}b^4c^2 - a^{10}b^6c + 3a^{10}b^5c^2 - 2a^{10}b^4c^3 + a^9b^7c + 3a^9b^5c^3 - a^9b^4c^4 + 2a^8b^8c \\ -5a^8b^7c^2 + a^8b^6c^3 - 2a^7b^9c + a^7b^8c^2 + a^7b^4c^6 - a^6b^{10}c + 3a^6b^9c^2 - a^6b^6c^5 \\ -3a^6b^5c^6 + 2a^6b^4c^7 + a^5b^{11}c - 3a^5b^9c^3 - a^5b^8c^4 + 5a^5b^7c^5 - 3a^5b^5c^7 + a^5b^4c^8 \\ -a^4b^{11}c^2 + a^4b^{10}c^3 + 2a^4b^9c^4 - 2a^4b^8c^5 - a^4b^7c^6 + a^4b^6c^7 = 0 \end{gathered}$$

Polynomials of degree 17 in three variables are not easy to factor, but using the `Factor` function in Mathematica, we find that this condition is equivalent to

$$a^4b^4c(a-c)(a-b+c)(a+b+c)(a^2-b^2+c^2)(a^2c+ab^2-3abc+ac^2-b^3+b^2c) = 0.$$

The factor $a^2 - b^2 + c^2$ shows that this condition will be true for a right triangle with vertex at B. □

This proof is not satisfying. The computation involved in this proof is enormous. It seems highly unlikely that Theorem 2.1 is true merely because the determinant condition for collinearity happens to evaluate to 0. The points Y_1 and Y_2 have nice geometric definitions as shown in Figure 1. The points X_8 and X_{20} also have nice geometric definitions. (Point X_8 is the Nagel Point of a triangle and Point X_{20}, also known as the de Longchamps Point of a triangle, is the reflection of the orthocenter about the circumcenter.)

Open Question 2. *Is there a purely synthetic geometric proof of Theorem 2.1?*

Theorem 2.2. *Let ABC be a right triangle (named counterclockwise) with right angle at B and $\angle ACB = 30°$ (Figure 4). Then AY_1, BY_2, and X_8X_{21} are concurrent.*

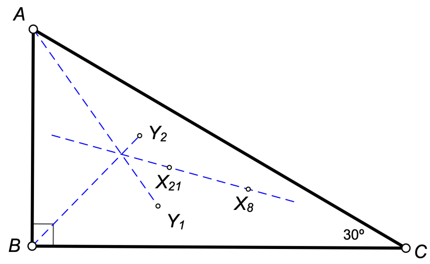


Figure 4. dashed lines are concurrent

Proof. Using the same procedure as in the proof of Theorem 2.1, we find that the condition for concurrence is $(a-b)(a^4+a^3b-2a^2c(b+c)+ab(b^2-2bc-3c^2)+(b^2-c^2)^2)(a+b-c-2u)(a-u)(b-u)u = 0$. Eliminating u from this equation and Equation (1) gives $a^4(a-b)b^4c(a+b+c)(a^2b+ab^2-3abc+ac^2+bc^2-c^3)(a^3+b^3-a^2c-abc-b^2c-ac^2-bc^2+c^3) = 0$. Setting $b=\sqrt{a^2+c^2}$ and $a=c\sqrt{3}$ shows that this condition is equivalent to $0=0$ which is always true. □

The following result comes from [5, §4.5].

Lemma 2.3. *The infinite point on the line represented by $(p:q:r)$ is $(q-r:r-p:p-q)$.*

Theorem 2.4. *Let ABC be a right triangle with right angle at B (Figure 5). Then $BX_8 \perp Y_1Y_2$.*

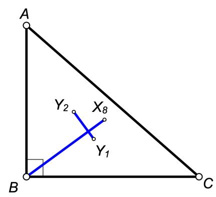


Figure 5. blue lines are perpendicular

Proof. From [1], the barycentric coordinates for X_8 are $(a-b-c : -a+b-c : -a-b+c)$. The barycentric coordinates for B are $(0:1:0)$. The line BX_8 is represented by $\texttt{Cross}[B, X_8] = (a+b-c : 0 : a-b-c)$. Using Lemma 2.3, we find that the infinite point on the line BX_8 has coordinates $(b+c-a : -2b : a+b-c)$.

Now the coordinates for X_1 are $(a:b:c)$. The circumcenter, X_3, is the midpoint of the hypotenuse in a right triangle, so its coordinates are $(1:0:1)$. The line X_1X_3 is therefore represented by $\texttt{Cross}[X_1, X_3] = (b : c-a : -b)$. So, the infinite point on the line X_1X_3 in a right triangle is $(b+c-a : -2b : a+b-c)$.

Since BX_8 and X_1X_3 have the same point at infinity, they must be parallel. But $X_1X_3 \perp Y_1Y_2$ by Theorem 1.2. Therefore, $BX_8 \perp Y_1Y_2$. □

3. 60° Triangles

Theorem 3.1. *Let ABC be a triangle (named counterclockwise) with $\angle C = 60°$ (Figure 6). Then AY_1, BY_2, and X_3X_8 are concurrent.*

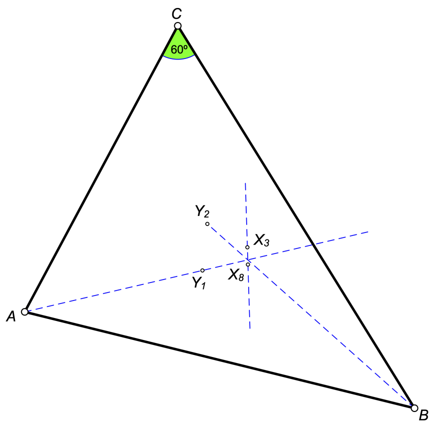


FIGURE 6. dashed lines are concurrent

Proof. As before, we find that the condition for concurrence is

$$(a-b)(a+b-c)(a^2-ab+b^2-c^2)(a+b-c-2u)u = 0.$$

From the Law of Cosines, we see that in this triangle, $c^2 = a^2+b^2-ab$, so this condition holds. □

4. Heptagonal Triangles

A *heptagonal triangle* is a triangle whose angles are $\pi/7$, $2\pi/7$, and $4\pi/7$.

Lemma 4.1. *Let ABC be a heptagonal triangle with $a > b > c$. Then*

$$a : b : c = \sin\frac{4\pi}{7} : \sin\frac{2\pi}{7} : \sin\frac{\pi}{7}.$$

Proof. This follows immediately from the Law of Sines. □

Lemma 4.2 (Reflection About BC)**.** *Let P have barycentric coordinates $(p:q:r)$ with respect to $\triangle ABC$. Then Q, the reflection of P about side BC, has coordinates*

$$\left(a^2p : (c^2-b^2)p - a^2(p+q) : (b^2-c^2)p - a^2(p+r)\right).$$

Proof. The formula for the distance between two points in barycentric coordinates can be found in [5, §7.1]. Let $Q = (p' : q' : r')$. Solving the two equations $PB = QB$ and $PC = QC$ for p', q', and r' and excluding the solution where $Q = P$ gives the required coordinates for Q. □

Another proof, not using the distance formula, is instructive.

Proof. It is immediate that infinite point of the line through the points $(p_1 : q_1 : r_1)$ and $(p_2 : q_2 : r_2)$ with the same sum $p_1 + q_1 + r_1 = p_2 + q_2 + r_2$ is given by $(p_1 - p_2 : q_1 - q_2 : r_1 - r_2)$. Thus, if (p, q, r) is a given point, we can consider the point $X = (0 : qS_B + (p + q)S_C : (p + r)S_B + rS_C)$ with sum $(p + q + r)(S_B + S_C) = (p + q + r)a^2$. Then, by subtraction

$$\begin{aligned}&(0 : qS_B + (p+q)S_C : rS_C + (p+r)S_B) - a^2\,(p : q : r)\\ =&\left(-a^2p : qS_B + (p+q)S_C - a^2q : rS_C + (p+r)S_B - a^2r\right)\\ =&\left(-a^2p : pS_C : pS_B\right),\end{aligned}$$

that is, the infinite point of PX is the infinite point of the $A-$altitude and then PX is perpendicular to BC.

Again, by subtraction,

$$\begin{aligned}&2\,(0 : qS_B + (p+q)S_C : rS_C + (p+r)S_B) - a^2\,(p : q : r)\\ =&\left(-a^2p : q\cdot 2S_B + (p+q)\cdot 2S_C - a^2q : r\cdot 2S_C + (p+r)\cdot 2S_B - a^2r\right)\\ =&\left(-a^2p : 2S_Cp + a^2q : 2S_Bp + a^2r\right),\end{aligned}$$

therefore the reflection of P in X is the point

$$\left(-a^2p : (b^2-c^2)p + (p+q)a^2 : (c^2-b^2)p + (p+r)a^2\right).$$ □

Theorem 4.3. *Let ABC be a heptagonal triangle (named counterclockwise) with $AB < AC < BC$ (Figure 7). Let Y_1' be the reflection of Y_1 about side BC. Then $X_1Y_2 \parallel Y_1'C$.*

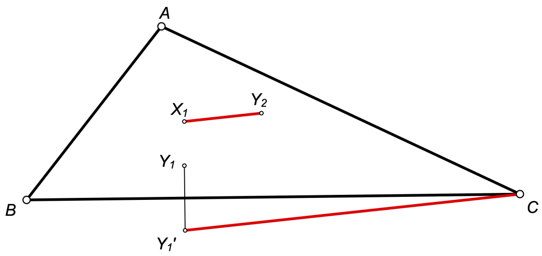


FIGURE 7. red lines are parallel

Proof. From Lemma 4.2, we find that the coordinates of Y_1' are

$$\left(a^2u^2 : a^2(b-2u)u + (-b^2+c^2)u^2 + a^3(-b+u) : u(-a^2b + (b^2-c^2)u)\right).$$

Using Lemma 2.3, we find that the infinite point on line CY_1' is

$$\begin{aligned}i_1 = \Big(&a^2u^2 : a^2(b-2u)u + (-b^2+c^2)u^2 + a^3(-b+u)\\ &: a^3(b-u) + (b^2-c^2)u^2 + a^2u(-b+u)\Big).\end{aligned}$$

Similarly, the infinite point on the line X_1Y_2 is

$$i_2 = \Big(a(b+c)(b-u) - a^2u - (b+c)(b-u)u : u(b^2+cu) + a(-b^2+u^2)$$
$$: a^2u + a(b-u)(-c+u) + u(b(c-u) - 2cu)\Big)$$

The two lines will be parallel if these infinite points are the same. The condition for i_1 and i_2 to be the same is $\texttt{Cross}[i_1, i_2] = 0$. Eliminating u from this equation and Equation (1) gives an equation that is identically true when we let

$$a = \sin\frac{4\pi}{7},\ b = \sin\frac{2\pi}{7},\ \text{and}\ c = \sin\frac{\pi}{7}.$$

Thus, the lines are parallel. □

Theorem 4.4. *Let ABC be a heptagonal triangle (named counterclockwise) with $AB < AC < BC$ (Figure 8). Then $\angle CX_1Y_2 + \angle X_1CY_1 = \frac{\pi}{7}$.*

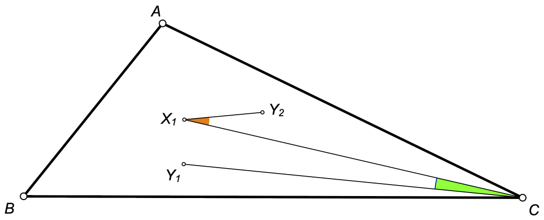


Figure 8.

Proof. Let Y_1' be the reflection of Y_1 about BC. Draw in line CX_3 as shown in Figure 9.

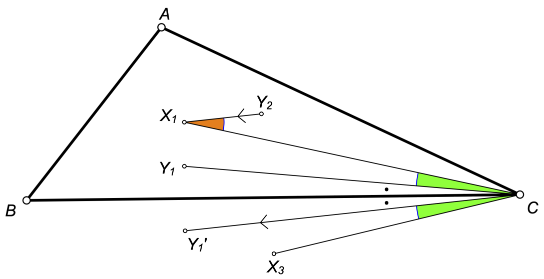


Figure 9.

Since X_3 is the center of the circle through A, B, and C, $\angle CX_3B = 6\pi/7$. This implies $\angle BCX_3 = \pi/14$. Since X_1 is the incenter, CX_1 bisects $\angle ACB$, so $\angle X_1CB = \pi/14$. Therefore $\angle BCX_3 = \angle X_1CB$.

Since Y_1' is the reflection of Y_1 about side BC, $\angle Y_1CB = \angle BCY_1'$, i.e. the dotted angles are equal (the "black angles"). Subtracting, we find that $\angle X_1CY_1 = \angle Y_1'CX_3$, i.e. the green angles are equal.

From Theorem 4.3, we know that $Y_2X_1 \parallel CY_1'$. Therefore orange = green + 2 black. Hence orange + green = 2 green + 2 black = 2(green + black) = $2\angle X_1CB = \angle ACB = \pi/7$. □

We found a number of results involving heptagonal triangles and centerX_1, but they were all geometrically equivalent to Theorem 4.4 because of the following result.

Theorem 4.5. *Let ABC be a heptagonal triangle (named counterclockwise) with $AB < AC < BC$. Let the angles be named as shown in Figure 10. If $\angle Y_2X_1A = \theta$, then*

$$\angle 1 = \frac{2\pi}{7} \qquad \angle 4 = \theta - \frac{\pi}{2} \qquad \angle 7 = \frac{\pi}{7}$$
$$\angle 2 = \frac{9\pi}{14} - \theta \qquad \angle 5 = \frac{11\pi}{14} \qquad \angle 8 = \frac{\pi}{7}$$
$$\angle 3 = \frac{\pi}{14} \qquad \angle 6 = \frac{4\pi}{7} - \theta \qquad \angle 9 = \frac{2\pi}{7}$$

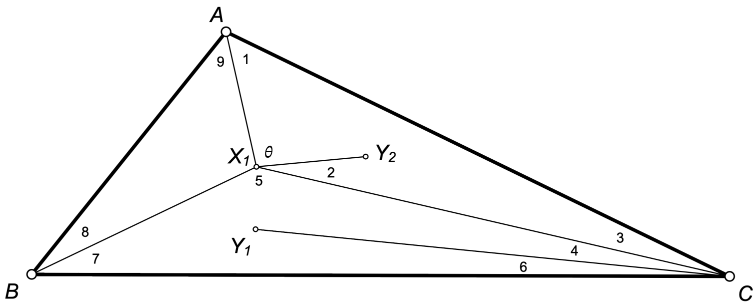


Figure 10.

Proof. Since $\triangle ABC$ is a heptagonal triangle, $\angle CBA = \pi/7$, $\angle ACB = 2\pi/7$, and $\angle BAC = 4\pi/7$. Since X_1 is the incenter of $\triangle ABC$, AX_1, BX_1, and CX_1 are angle bisectors. Thus, $\angle 1 = \angle 9 = 2\pi/7$, $\angle 3 = \angle 4 + \angle 6 = \pi/14$, and $\angle 7 = \angle 8 = \pi/7$. From $\triangle CX_1A$, we find $\angle 2 = 9\pi/14 - \theta$. From $\triangle BX_1C$, we find $\angle 5 = 11\pi/14$. From Theorem 4.4, $\angle 2 + \angle 4 = \pi/7$. Therefore $\angle 4 = \pi/7 - (9\pi/14 - \theta) = \theta - \pi/2$. Finally, $\angle 6 = \angle X_1CB - \angle 4 = \pi/14 - (\theta - \pi/2) = 4\pi/7 - \theta$. □

Theorem 4.6. *Let ABC be a heptagonal triangle (named counterclockwise) with $AB < AC < BC$ (Figure 11). Then $\angle CX_1Y_2 = \angle Y_1CX_3$.*

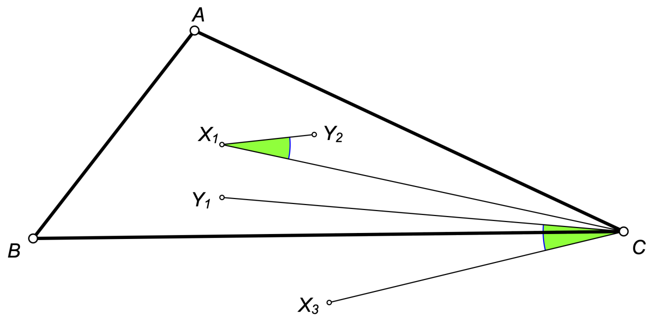


Figure 11. green angles are equal

Proof. Since X_3 is the center of the circle through A, B, and C, $\angle CX_3B = 6\pi/7$. This implies $\angle BCX_3 = \pi/14$. Then, using Theorem 4.5, we have $\angle Y_1CX_3 = \angle BCX_3 + \angle 6 = \pi/14 + 4\pi/7 - \theta = 9\pi/14 - \theta = \angle 2$. □

We also found the following three results, involving other centers. The proofs, using Mathematica, involved a lot of computation, so we have omitted the details.

Theorem 4.7. *Let ABC be a heptagonal triangle (named counterclockwise) with $AB < AC < BC$. Let P be the foot of the perpendicular from Y_1 to AC (Figure 12). Then $\angle CBX_5 = \angle Y_2Y_1P$.*

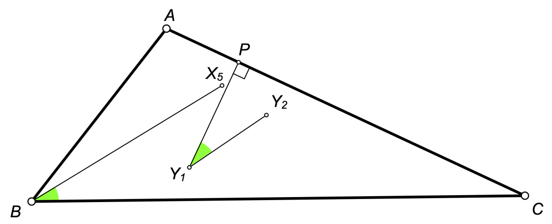


FIGURE 12. green angles are equal

Theorem 4.8. *Let ABC be a heptagonal triangle (named counterclockwise) with $AB < AC < BC$ (Figure 13). Then $\angle X_{21}X_1Y_2 = \angle CY_1Y_2$.*

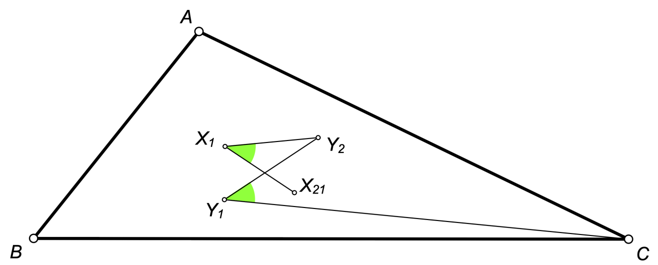


FIGURE 13. green angles are equal

Theorem 4.9. *Let ABC be a heptagonal triangle (named counterclockwise) with $AB < AC < BC$ (Figure 14). Then $\angle X_1Y_2Y_1 = \angle X_{28}CY_1$.*

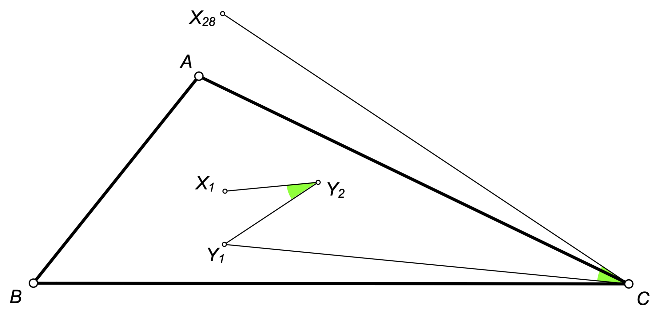


FIGURE 14. green angles are equal

Open Question 3. *Is there a purely synthetic geometric proof of any of the three preceding theorems, or a proof that does not involve massive amounts of computation?*

5. Harmonic Triangles

Theorem 5.1. *Let ABC be a triangle such that Y_1Y_2 passes through a vertex. Then Y_1Y_2 is a median of the triangle.*

Proof. Without loss of generality, assume that Y_1Y_2 passes through vertex B and $\triangle ABC$ is named counterclockwise (Figure 15).

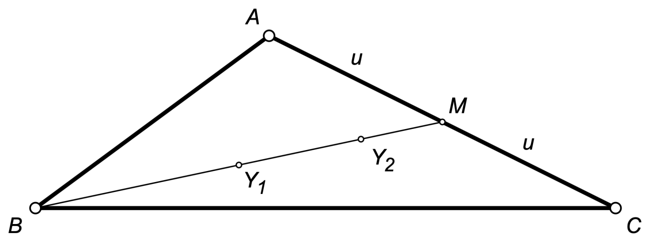
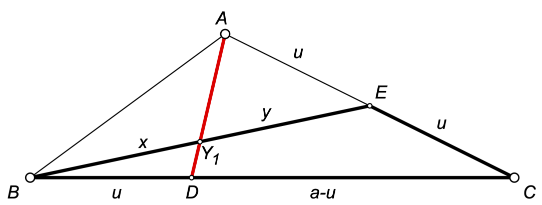


Figure 15. B, Y_1, and Y_2 are collinear

Let BY_1 meet AC at M. Then, from Figure 1 (left), we have $CM = u$. Line BY_2 is the same line, so also meets AC at M. From Figure 1 (right), we have $AM = u$. Thus, $AM = CM$ and BM is a median of the triangle. □

Theorem 5.2. *Let ABC be a triangle such that BY_1 is a median. Then Y_1 divides the median into two segments in the ratio $2b : 2a - b$.*

Proof. Let BY_1 meet AC at E and let AY_1 meet BC at D.



Then, from Figure 1 we see that $CE = BD = u$. Since BY_1 is a median, $AE = CE$, so $AE = u$. Since $BD = u$, we have $CD = a - u$. Line AD is a transversal of $\triangle EBC$, so by the Theorem of Menelaus, we have $x(a - u)u = yu(2u)$, so $x/y = 2u/(a - u)$. But $u = b/2$. Thus $x/y = b/(a - \frac{b}{2})$ which implies $x/y = 2b/(2a - b)$. □

Corollary 5.3. *If Y_1 is the midpoint of a median, then the triangle is similar to a 3–4–6 triangle.*

Theorem 5.4. *Let ABC be a triangle such that BY_2 is a median. Then Y_2 divides the median into two segments in the ratio $2(2a - b) : b$.*

Proof. The proof is similar to the proof of Theorem 5.2, so the details are omitted. □

The *harmonic mean* of x and y is $\frac{2}{1/x+1/y}$. A triangle is called a *harmonic triangle* if one side is the harmonic mean of the other two sides. We say that the triangle is *harmonic*.

Theorem 5.5. *A triangle is harmonic if and only if Y_1Y_2 passes through a vertex.*

Proof. If Y_1Y_2 passes through a vertex, assume without loss of generality, that the vertex is B and $\triangle ABC$ is named counterclockwise (Figure 15). From Theorem 5.1, BY_1 is a median and so $u = b/2$. Letting $u = b/2$ in Equation (1) gives $b(ab - 2ac + bc) = 0$ which implies $b = 2ac/(a+c)$ and the triangle is harmonic.

Conversely, if the triangle is harmonic, assume without loss of generality that $2/b = 1/a + 1/c$. The barycentric coordinates for B are $(0:1:0)$. Using the simple coordinates for Y_1 and Y_2, we find that that Y_1Y_2 passes through B if and only if

$$\begin{vmatrix} u^2 & (a-u)(b-u) & (b-u)u \\ (a-u)(b-u) & u_2 & (a-u)u \\ 0 & 1 & 0 \end{vmatrix} = 0.$$

Expanding this gives $b(b-2u)(a-u)u = 0$. From [4], it is known that $0 < u < a$, so therefore, Y_1Y_2 passes through vertex B in a harmonic triangle, if and only if $b = 2u$.

If the triangle is harmonic, then $b = 2u$, so $b - u = u$, and from Figure 1, BY_1 is a median. Similarly, BY_2 is a median. Therefore Y_1Y_2 passes through B. □

Theorem 5.6. *A triangle is harmonic if and only if Y_1Y_2 passes through X_2.*

Proof. Assume that Y_1Y_2 passes through X_2. The barycentric coordinates for X_2 are $(1:1:1)$. Using the simple coordinates for Y_1 and Y_2, the determinant condition for Y_1, Y_2 and X_2 to be collinear is

$$\begin{vmatrix} u^2 & (a-u)(b-u) & (b-u)u \\ (a-u)(b-u) & u_2 & (a-u)u \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Expanding this out gives $(a-2u)(b-2u)(bu-ab+au) = 0$. Eliminating variable u between this equation and Equation (1) gives

$$a^3b^3(ab+ac-2bc)(2ab-ac-bc)(ab-2ac+bc) = 0$$

which shows that $\triangle ABC$ is harmonic.

Conversely, if the triangle is harmonic, then by Theorem 5.5, Y_1Y_2 passes through a vertex. From Theorem 5.1, we can conclude that Y_1Y_2 is a median. Thus, Y_1Y_2 passes through X_2. □

Theorem 5.7. *A triangle is harmonic if and only if Y_1Y_2 bisects a side of the triangle.*

Proof. Without loss of generality. assume that Y_1Y_2 passes through E, the midpoint of AC. The barycentric coordinates for E are $(1:0:1)$. Using the simple coordinates for Y_1 and Y_2, the determinant condition for Y_1, Y_2 and E to be collinear is

$$\begin{vmatrix} u^2 & (a-u)(b-u) & (b-u)u \\ (a-u)(b-u) & u_2 & (a-u)u \\ 1 & 0 & 1 \end{vmatrix} = 0.$$

Expanding this out gives $(b-2u)(a^2(b-u)+2au(u-b)+bu^2)=0$. Eliminating variable u between this equation and Equation (1) gives

$$a^3b^4(ab-2ac+bc)(a^2b+2a^2c-2abc+2ac^2+bc^2)=0. \tag{4}$$

If $a^2b+2a^2c-2abc+2ac^2+bc^2=0$, then $b=-2ac(a+c)/(a-c)^2$. This equation is impossible since the left side is positive and the right side is negative. Thus $ab-2ac+bc=0$ which implies that the triangle is harmonic.

Conversely, if the triangle is harmonic, then by Theorem 5.5, Y_1Y_2 passes through a vertex. From Theorem 5.1, we can conclude that Y_1Y_2 is a median. Thus, Y_1Y_2 passes through E. □

6. AP Triangles

An *AP Triangle* is a triangle whose sides are in arithmetic progression.

Theorem 6.1. *Let $\triangle ABC$ (named counterclockwise), be an AP Triangle, with $2a=b+c$. Then $AX_1 \parallel Y_1Y_2$.*

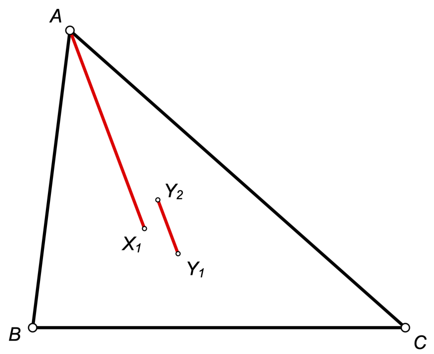


Figure 16. red lines are parallel

Proof. According to [2], the point at infinity on Y_1Y_2 is X_{513}. From [1], we find that the barycentric coordinates of X_{513} are

$$I_1=(a(b-c):b(c-a):c(a-b)).$$

Since $A=(1:0:0)$ and $X_1=(a:b:c)$, the point at infinity on AX_1 is

$$I_2=(-b-c:b:c)$$

When $b=2a-c$, we find that $I_1=(-2a:2a-c:c)$ and $I_2=(-2a:2a-c:c)$. Thus, the two lines have the same point at infinity. Therefore $AX_1 \parallel Y_1Y_2$. □

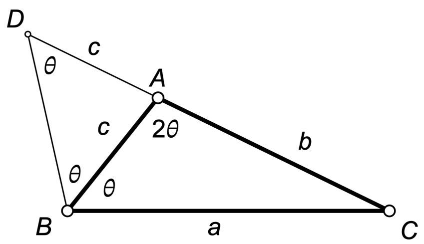
7. Double-Angle Triangles

A *double-angle triangle* is a triangle in which one angle is twice another angle. Note that all heptagonal triangles are double-angle triangles.

The following result is fairly well known.

Lemma 7.1 (Double Angle Condition)**.** *Let ABC be a triangle in which $\angle A = 2\angle B$. Then $a^2 = b(b + c)$.*

Proof. Let $\angle CBA = \theta$, so that $\angle BAC = 2\theta$. Extend CA past A to a point D such that $AD = AB$, so that ABD is an isosceles triangle and $\angle ABD = \angle BDA$.



By the Exterior Angle Theorem, $\angle BAC = \angle ABD + \angle BDA$. Thus, $\angle ABD = \angle BDA = \theta$ and $\triangle BAC \sim \triangle DBC$. Hence $AC/BC = BC/DC$ or $b/a = a/(b+c)$. □

Theorem 7.2. *Let ABC be a double-angle triangle (named counterclockwise) with $\angle A = 2\angle B$ (Figure 17). Let Y_2' be the reflection of Y_2 about side BC. Then BY_2, CX_2, and AY_2' are concurrent.*

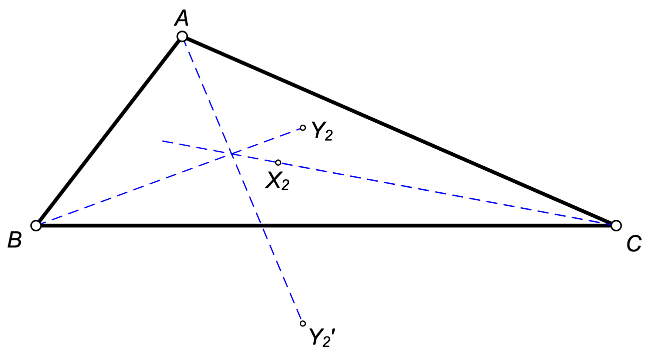


FIGURE 17. dashed lines are concurrent

Proof. From Lemma 4.2, we find that the coordinates of Y_2' are

$$\Big(a^2(a-u)(b-u) : (-b^2+c^2)(a-u)(b-u) + a^2((b-2u)u + a(-b+u))$$

$$: (-a^2b + (b^2-c^2)(b-u))(a-u)\Big).$$

We can therefore write down the equations for the three lines AY_2', BY_2, and CX_2. The determinant condition for these three lines to be concurrent simplifies to

$$(a-u)(-ab(b^2-c^2)(b-u)+a^3(b-u)^2+b(b^2-c^2)(b-u)u-a^2u(b^2-2bu+2u^2)) = 0.$$

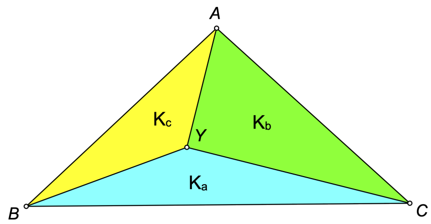
Eliminating variable u between this equation and Equation (1) gives

$$ab(b-c)(a^2-b(b+c)) = 0.$$

Thus, the three lines are concurrent if $a^2 = b(b+c)$. The theorem now follows from Lemma 7.1. □

8. ISOSCELES TRIANGLES

Theorem 8.1. *Let Y be one of the Yff points of $\triangle ABC$. Let the areas of triangles BCY, CAY, and ABY be K_a, K_b, and K_c, respectively. Then $K_bK_c = K_a^2$ if and only if $\triangle ABC$ is isosceles with $AB = AC$.*



Proof. Start with $Y = Y_1$. The areas of these triangles are proportional to the barycentric coordinates of Y_1. Then from Equation (2), without loss of generality, we have $K_a = u^2$, $K_b = (a-u)(b-u)$, and $K_c = (b-u)u$. Thus, $K_bK_c = K_a^2$ is equivalent to

$$(a-u)u(-b+u)^2 = u^4. \tag{5}$$

Eliminating u from Equations (5) and (1), this equation becomes $a^4b^4(b-c)c = 0$. This condition is true if and only if $b = c$. The proof when $Y = Y_2$ is similar. □

9. ISOSCELES RIGHT TRIANGLES

The following two results were found by computer. We omit the complicated proofs.

Theorem 9.1. *Let ABC be an isosceles right triangle (named counterclockwise) with right angle at A. Let X_1Y_2 meet BC at D (Figure 18). Then $\angle DY_2C = \angle CBX_8$.*

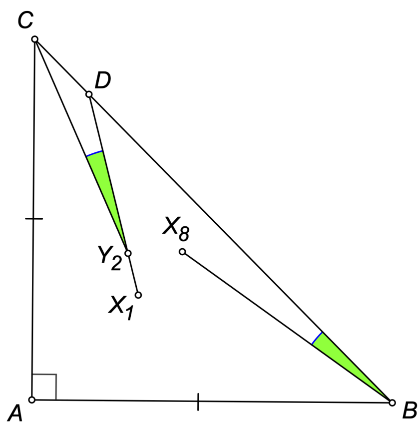


FIGURE 18. green angles are equal

Theorem 9.2. *Let ABC be an isosceles right triangle (named counterclockwise) with right angle at A. Then $\angle Y_2X_1C = \angle Y_2CX_9$ (Figure 19).*

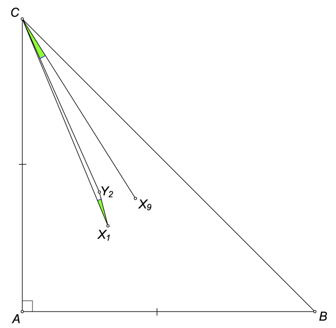


FIGURE 19. green angles are equal

10. SPECIAL TRIANGLES

Theorem 10.1. *In $\triangle ABC$ (named counterclockwise), BY_1 passes through X_8 if and only if*

$$a^2(a-b-c) = bc(b+c-3a).$$

Figure 20 shows an example.

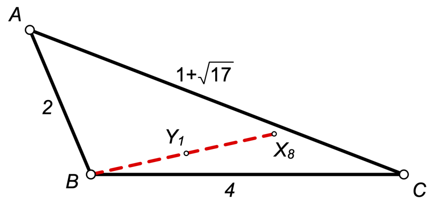
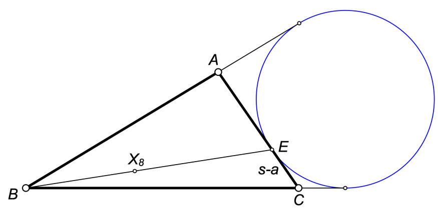


FIGURE 20. B, Y_1, and X_8 are collinear

Proof. Let BX_8 meet AC at E. Then by a well-known property of the Nagel point, $CE = s - a$ where s is the semiperimter of the triangle [5, §3.2.2].



Then Y_1 lies on BE, if and only if $s-a = u$. Eliminating u between this condition and Equation (1) gives the condition to be $a^2(a-b-c) = bc(b+c-3a)$ as required. $\square$

In the same manner, we get the following theorem.

Theorem 10.2. *In $\triangle ABC$ (named counterclockwise), CY_1 passes through X_7 if and only if*

$$a^2(a-b-c) = bc(b+c-3a).$$

11. Cevian Triangles

Let $[XYZ]$ denote the area of $\triangle XYZ$.

Lemma 11.1 (The Common Angle Theorem)**.** *In $\triangle ABC$, let D be a point on BC and let E be a point on AC (Figure 21). Then*

$$[CDE] = \frac{CE \cdot CD}{CA \cdot CB}[ABC].$$

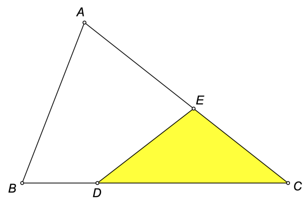


Figure 21.

Proof. This follows from the area formulas $[CDE] = \frac{1}{2}CD \cdot CE \sin C$ and $[ABC] = \frac{1}{2}CA \cdot CB \sin C$ □

Theorem 11.2 (Area of Cevian Triangle)**.** *Let P be a point inside $\triangle ABC$. Let AP meet BC at D, BP meet CA at E and CP meet AB at F. Suppose $[ABC] = K$, $AB = c$, $BC = a$, $CA = b$, $BD = a_1$, $DC = a_2$, $CE = b_1$, $EA = b_2$, $AF = c_1$, and $FB = c_2$ as shown in Figure 22. Then*

$$[DEF] = \left(1 - \frac{c_1 b_2}{bc} - \frac{a_1 c_2}{ca} - \frac{b_1 a_2}{ac}\right) K.$$

Triangle DEF is known as the *cevian triangle* of point P.

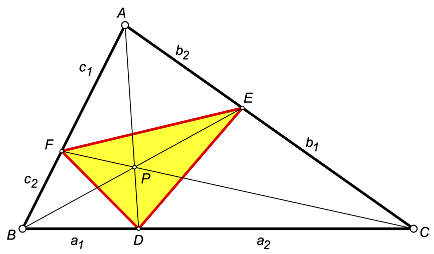


Figure 22.

Proof. From the Common Angle Theorem, we have

$$[AFE] = \frac{c_1 b_2}{bc} K,$$
$$[BDF] = \frac{a_1 c_2}{ca} K,$$
$$[CED] = \frac{b_1 a_2}{ac} K.$$

Since $[DEF] = [ABC] - [AFE] - BDF] - [CED]$, we have

$$[DEF] = \left(1 - \frac{c_1 b_2}{bc} - \frac{a_1 c_2}{ca} - \frac{b_1 a_2}{ac}\right) K$$

as required. □

Yff [4] gave a formula for the area of the cevian triangle of Y_1, but he gave no proof. We now give a purely geometric proof of his result.

Theorem 11.3 (Yff Area Formula)**.** *In $\triangle ABC$, let AY_1 meet BC at D, BY_1 meet CA at E and CY_1 meet AB at F (Figure 23). Then*

$$[DEF] = \frac{u^3}{2R}.$$

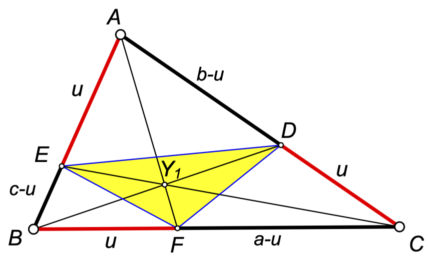


FIGURE 23.

Proof. From the definition of Y_1, the segments along the sides of $\triangle ABC$ have the lengths shown in Figure 23. From Theorem 11.2, we have

$$\begin{aligned}
[DEF] &= \left(1 - \frac{u(b-u)}{bc} - \frac{u(c-u)}{ca} - \frac{u(a-u)}{ab}\right) K \\
&= \frac{u^3 + (a-u)(b-u)(c-u)}{abc} K \\
&= \frac{u^3 + u^3}{abc} K \\
&= 2u^3 \times \frac{K}{abc} \\
&= \frac{u^3}{2R}
\end{aligned}$$

since $R = abc/(4K)$. □

In a similar manner, we find that the area of the cevian triangle of Y_2 is also $u^3/(2R)$.

Weisstein [3] calls these triangles *Yff Triangles*.

12. SPECIAL POINTS

Let Y_m be the midpoint of Y_1Y_2. The point Y_m is a triangle center, but it is not one of the ones cataloged in ETC [1] as of January 1, 2026. The first normalized trilinear coordinate of Y_m in a 6–9–13 triangle is 0.4500479513210176.

Let Y_c be the center of the unique conic that passes through the points A, B, C, Y_1, and Y_2. The point Y_c is not a triangle center listed in ETC [1] as of January 1, 2026. The first normalized trilinear coordinate of Y_c in a 6–9–13 triangle is 0.7440584232553069.

References

[1] Clark Kimberling, *Encyclopedia of Triangle Centers*, 2025.
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[2] Clark Kimberling and César Lozada, Bicentric Pairs of Points, 13th edition, *Encyclopedia of Triangle Centers*, 2024.
https://faculty.evansville.edu/ck6/encyclopedia/BicentricPairs.html

[3] Weisstein, Eric W. *Yff Points.* From MathWorld–A Wolfram Resource.
https://mathworld.wolfram.com/YffPoints.html

[4] Peter Yff, *An Analogue of the Brocard Points*, The American Mathematical Monthly, **70**(1963)495–501.
https://www.jstor.org/stable/2312058

[5] Paul Yiu, Introduction to the Geometry of the Triangle, version 13 (April 2013).
https://users.math.uoc.gr/~pamfilos/Yiu.pdf