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Geometric Properties of a Hippopede

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Abstract. We survey the literature to find geometrical properties of the plane curve known as a hippopede. We also use a computer to find additional properties. We show how to construct the foci of a hippopede and determine a number of their geometrical properties as well.

Keywords. hippopede, ovals of Booth, foci, computer-discovered mathematics.

Mathematics Subject Classification (2020). 14H45, 51M04, 51-02.

1. Introduction

In this paper, we will study geometrical properties of the plane curve with equation

$$(x^2 + y^2)^2 = c^2x^2 + d^2y^2 \tag{1}$$

with $c > d > 0$. Such curves are called *hippopedes* or *ovals of Booth* [16].

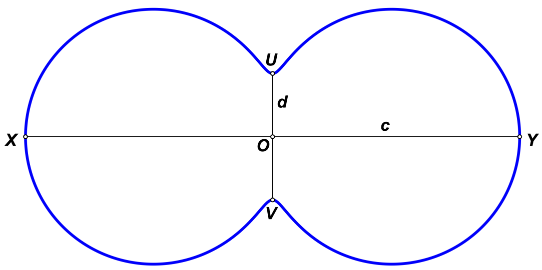


Figure 1. Hippopede axes and vertices

We start by surveying the literature for known results about the hippopede and then give additional results that were found by computer.

From Equation (1), we see that a hippopede is symmetric about the origin and symmetric about each coordinate axis. The x-intercepts, known as the *vertices* of the hippopede, are at $(\pm c, 0)$. They will be labeled $X$and Y. The y-intercepts, known as the *covertices* of the hippopede, are at $(0, \pm d)$. They will be labeled U and V as shown in Figure 1.

The values $x = 0$ and $y = 0$ satisfy Equation (1), so, technically, the origin is part of the curve. However, for our purposes, the hippopede shall consist of only the blue continuous curve shown in Figure 1. The point O is called the *center* of the hippopede. The hippopede is peanut-shaped for $\frac{d}{c} < \frac{1}{2}\sqrt{2}$ and convex otherwise.

2. Known Results

The following result comes from [3].

Theorem 1. *Let O be the center of a hippopede, H. Let C be a circle with center O as shown in Figure 2. Then the inverse of H with respect to C is an ellipse, E.*

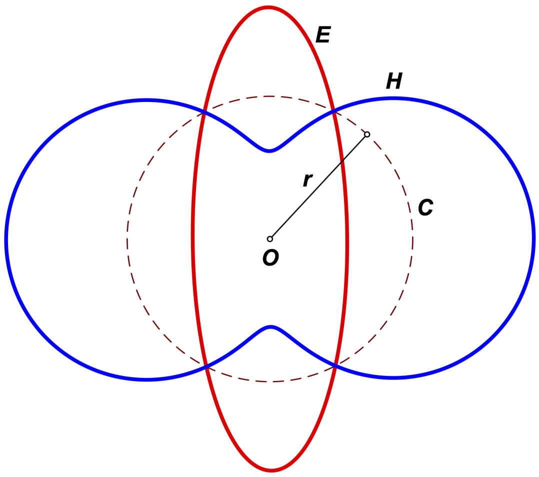


FIGURE 2. Ellipse E is inverse of hippopede H about circle C

According to [17], a hippopede is the pedal curve of an ellipse. This gives us the following result.

Theorem 2. *Let O be the center of a hippopede with axes XY and UV. Let E be the ellipse with major axis XY and minor axis UV. Let P be any point on the hippopede. Let PT be a tangent to the ellipse as shown in Figure 3. Then $OP \perp PT$.*

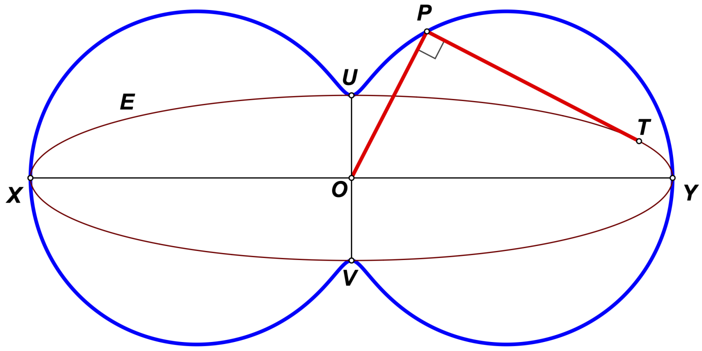


FIGURE 3. red lines are perpendicular

The equation of the hippopede can be expressed as a geometric result as follows.

Theorem 3. *A hippopede has center O and axes XY and UV. Let P be any point on the hippopede and let the feet of the perpendiculars from P to XY and UV be H_1 and H_2, respectively. Circles C_1 and C_2 are constructed on H_1Y and H_2U as diameters. Tangents OT_1 and OT_2 are drawn to circles C_1 and C_2 as shown in Figure 4. Then $(OT_1)^4 + (OT_2)^4 = (OP)^4$.*

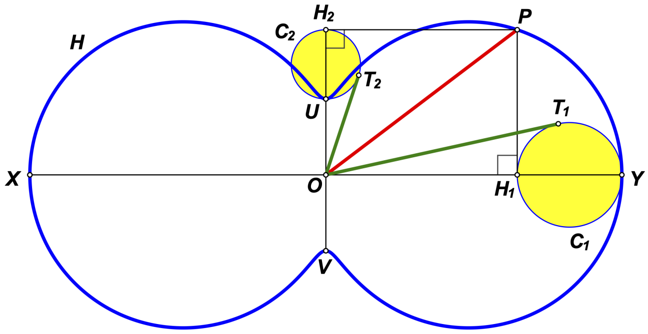


Figure 4. $(OT_1)^4 + (OT_2)^4 = (OP)^4$

Proof. Let $OH_1 = x$ and $OH_2 = y$, so that the coordinates of P are (x, y) and $(OP)^2 = x^2 + y^2$. Since $OY = c$ and $OH_1 \cdot OY = (OT_1)^2$, we have $(OT_1)^2 = cx$. Similarly, $(OT_2)^2 = dy$. From the equation of the hippopede, we have $(x^2+y^2)^2 = c^2x^2 + d^2y^2$ which implies $(OP)^4 = (OT_1)^4 + (OT_2)^4$. □

The following result comes from [17]. Related results can be found in [10].

Theorem 4. *Let O be the center of a hippopede with axes XY and UV. Let E be the ellipse with major axis XY and minor axis UV. Let P be any point on the hippopede and suppose PT is tangent to the ellipse as shown in Figure 5. Then the circle with diameter OT is tangent to the hippopede at P.*

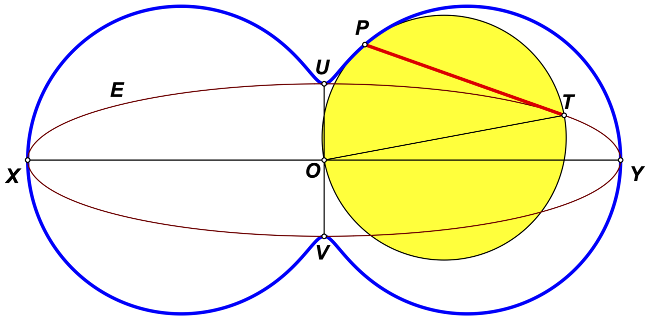


Figure 5. circle is tangent to hippopede

Note that this result gives us a way to construct the tangent to a hippopede at a given point P. Construct the ellipse E and then construct a tangent PT from P to E. The line from P to the midpoint of OT is then a normal to the hippopede and the perpendicular to the normal at P is a tangent.

Another way of looking at this result is that if T is a variable point on a fixed ellipse with center O, then the envelope of the circles with diameter OT is a hippopede.

The following result from [11] is related to the previous result.

Theorem 5. *Let P be any point on a hippopede with center O. Let T be the tangent to the hippopede at P. A circle passing through O is tangent to the hippopede at P. Let U be the tangent to the circle at O as shown in Figure 6. Then T is the reflection of U about m, the perpendicular bisector of OP.*

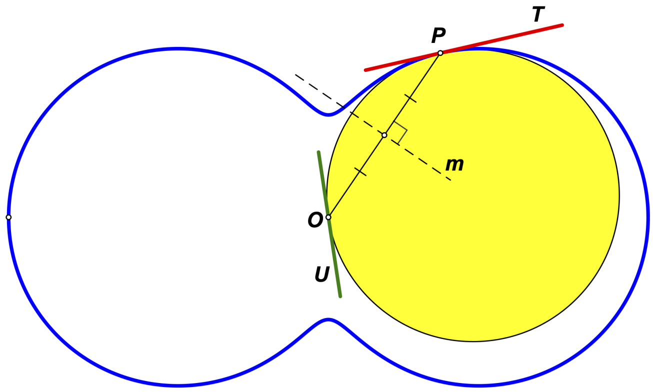


FIGURE 6. T is reflection of U about m

Proof. If T and U meet at point I, then IO and IP are both tangents to the circle from I and the result follows. □

The following theorem is a variation on a result that comes from [10].

Theorem 6. *Let O be the center of a hippopede, H, with axes XY and UV. Let E be the ellipse with major axis XY and minor axis UV. Let F and G be the foci of the ellipse. Let X' be the midpoint of XO and let F' be the midpoint of FO. Let C be the circle with center O and radius OX'. Let P be any point on C and let PF' meet C again at Q, Finally, a line through O parallel to PQ meets the hippopede at W as shown in Figure 7. Then $PQ = OW$.*

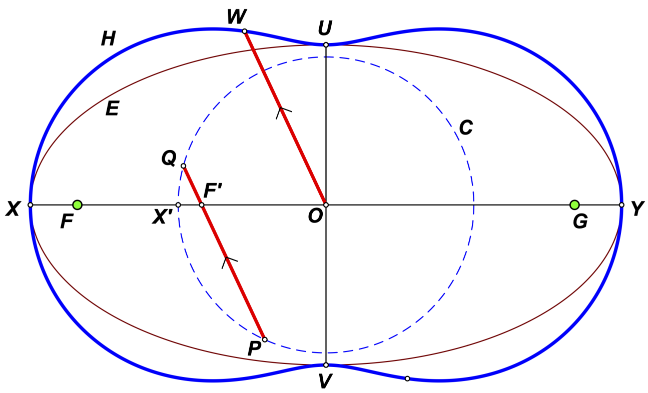


FIGURE 7. red lengths are equal

A hippopede is a bicircular quartic, so by [6, p. 307], we have the following result.

Theorem 7. *Let O be the center of a hippopede. Suppose a secant meets the hippopede at four points, A, B, C, and D as shown in Figure 8. Let M be the midpoint of AB and let N be the midpoint of CD. Then $OM = ON$.*

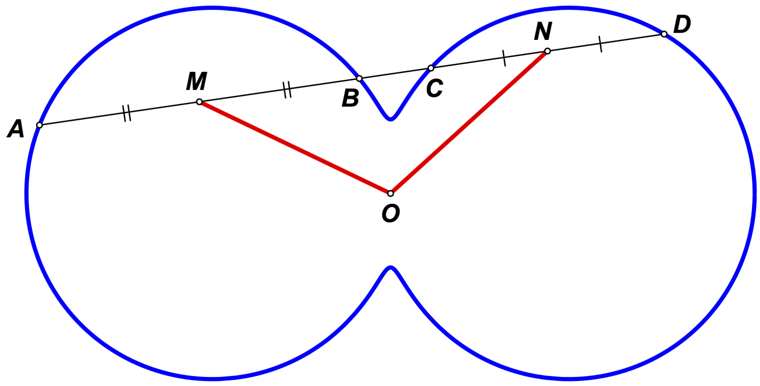


Figure 8. red lengths are equal

If we let two of the points coincide, we get the following results.

Corollary 8. *Let O be the center of a hippopede. Suppose a tangent to the hippopede touches the hippopede at P and meets it again at points A and B (with B between A and P) as shown in Figure 9. Let M be the midpoint of AB. Then $OM = OP$.*

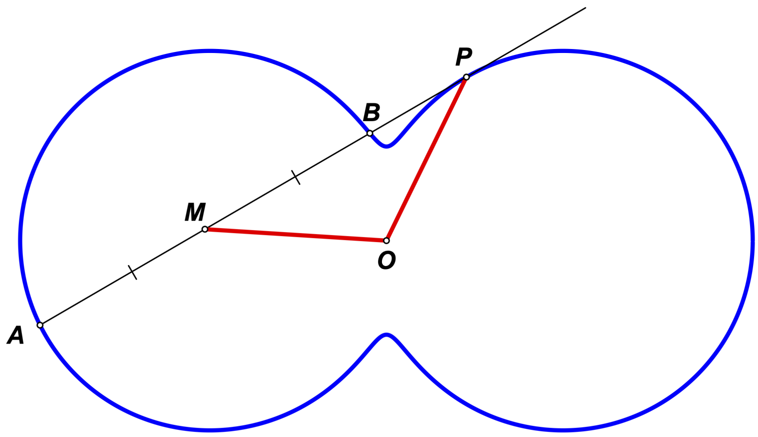


Figure 9. red lengths are equal

Corollary 9. *Let O be the center of a hippopede. Suppose a tangent to the hippopede touches the hippopede at P and meets it again at points A and B (with P between A and B) as shown in Figure 10. Let M be the midpoint of BP and let N be the midpoint of AP. Then $OM = ON$.*

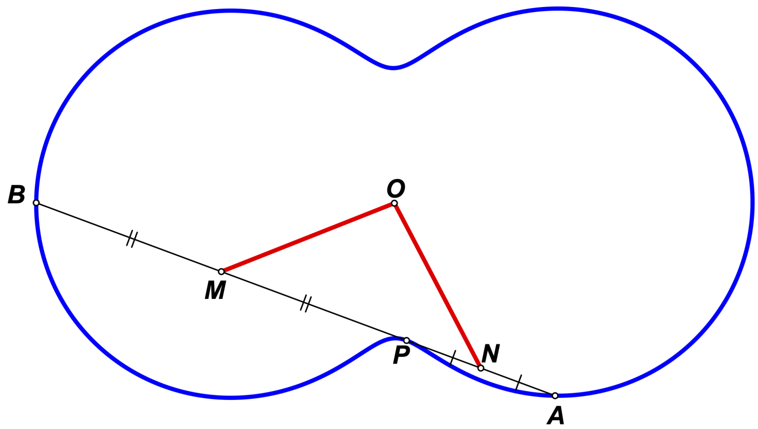


FIGURE 10. red lengths are equal

Keito Miyamoto [9] found the following result as a generalization of a similar result [15] for lemniscates.

Theorem 10. *Let O be the center of a hippopede. Suppose a variable secant meets the hippopede at four points, A, B, C, and D as shown in Figure 11. Let P be the center of $\odot OAB$ and let Q be the center of $\odot OCD$. Then $OP \cdot OQ = (c^2 - d^2)/4$.*

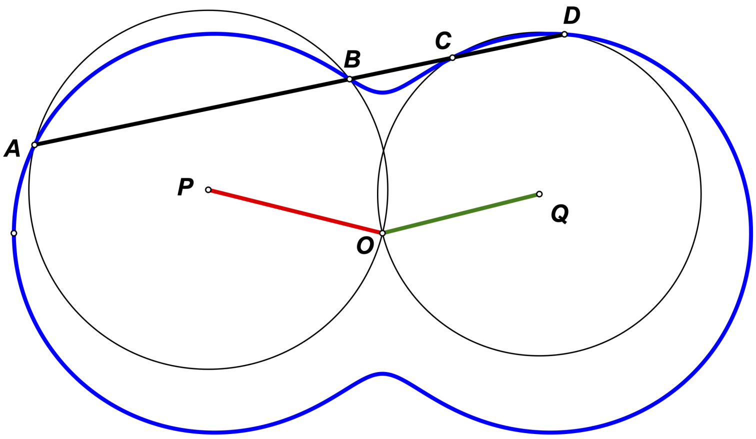


FIGURE 11. red length times green length is invariant

3. The Hippopede as a Locus

The following result comes from [10].

Theorem 11. *Let A be a fixed point inside a circle with center O. Let P be a variable point on the circle. Let line PA meet the circle again at B. Locate point Q so that OQ is equal and parallel to AB as shown in Figure 12. Then the locus of Q as P moves on the circle is a hippopede.*

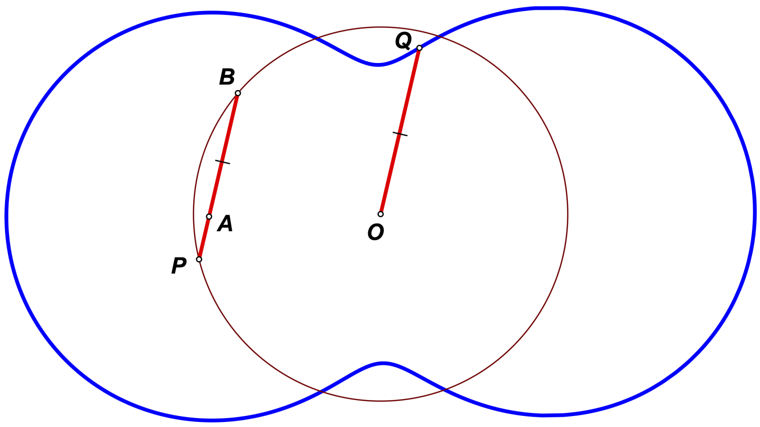


Figure 12. blue locus is a hippopede

Ferréol has noted [5] that a hippopede is the cissoid between two circles situated in a certain manner. This is explained in the following theorem.

Theorem 12. *Let O be a fixed point. Let C_1 be a fixed circle through O. Let C_2 be a fixed circle whose center, F, is the antipode of O in C_1. Let P be a variable point on C_1. Let line OP meet C_2 at Q and Q' and locate point M so that $OM = PQ$. Locate M' so that $OM' = PQ'$ as shown in Figure 13. Then the locus of M and M' as P moves on C_1 is a hippopede, H.*

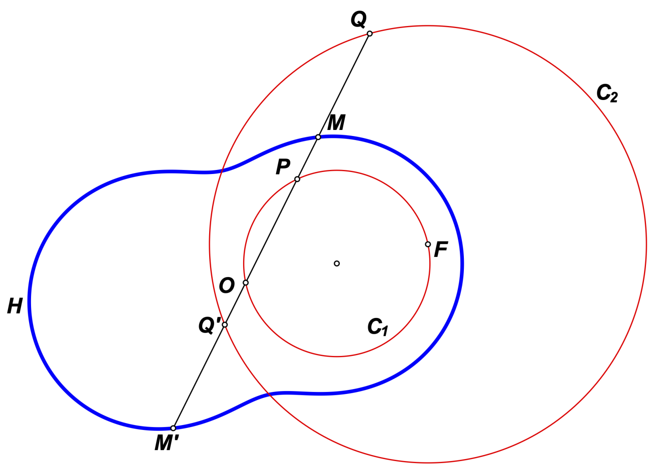


Figure 13. hippopede H is cissoid of red circles

Ferréol also notes [5] that a hippopede is the roulette formed by the center of an ellipse as it rolls along a congruent ellipse. This is explained in the following theorem.

Theorem 13. *Let E_1 be a fixed ellipse with foci F and G. Let P be a variable point on E_1. Extend GP through P to a point F' so that GF' is equal to the length of the major axis of E_1. Extend FP through P to a point G' so that FG' is equal to the length of the major axis of E_1. Note that the ellipse E_2 with foci F' and G' is congruent to E_1 and tangent to E_1 at P. Let O be the center of E_2 as shown in Figure 14. Then the locus of O as P moves on E_1 is a hippopede, H.*

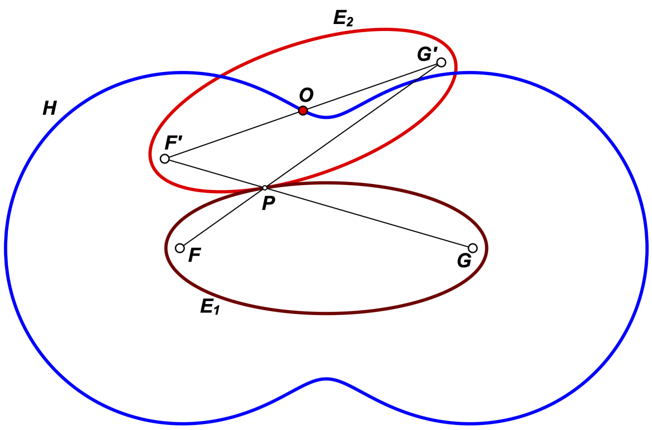


FIGURE 14. blue locus is a hippopede

Ferréol [5] also states the following two results.

Theorem 14. *Let O be a fixed point inside a fixed circle C. Let P be a variable point on C. Line PO meets the circle again at P'. Extend OP' past P' to a point M such that $PO = P'M$ as shown in Figure 15. Then the locus of M as P moves on C is a hippopede, H.*

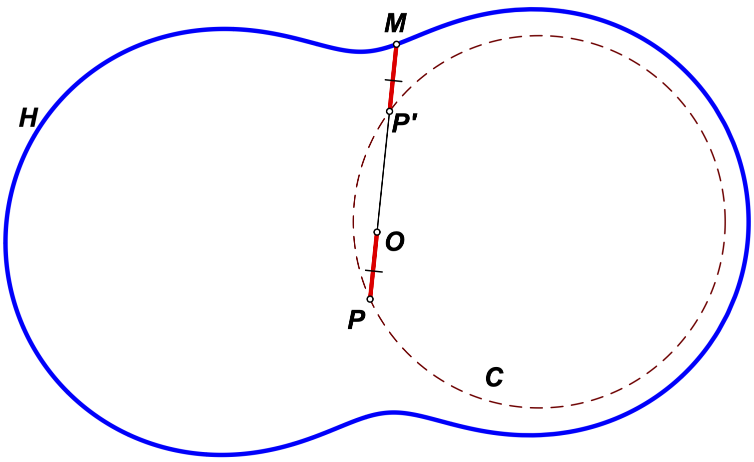


FIGURE 15. blue locus is a hippopede

Theorem 15. *Let C_1 and C_2 be fixed circles with centers O_1 and O_2, respectively. Let P be a variable point on C_2. Let Q be a point on C_1 such that $PQ = O_1O_2$. Let M be the midpoint of PQ as shown in Figure 16. Then the locus of M as P moves on C_2 is a hippopede.*

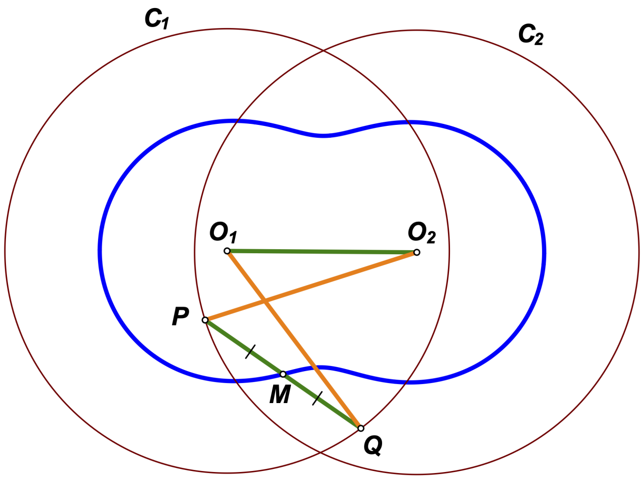


FIGURE 16. blue locus is a hippopede

4. NEW RESULTS

The following result was found by computer.

Theorem 16. *Let O be the center of a hippopede. Let A and B be two points on the hippopede such that $OA \perp OB$ as shown in Figure 17. Then $AB = \sqrt{c^2 + d^2}$.*

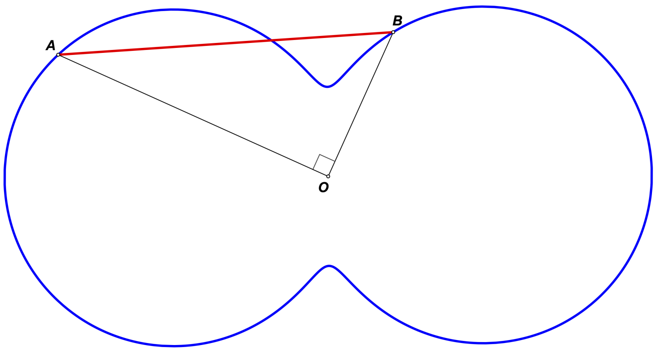


FIGURE 17. red length is invariant

Proof. Letting $x = r\cos\theta$ and $y = r\sin\theta$, we see that the polar equation for a hippopede is

$$r^2 = c^2\cos^2\theta + d^2\sin^2\theta.$$

If θ is the angle OB makes with the positive x-axis, then the angle that OA makes with the positive x-axis is $\theta + 90°$. Then $OB^2 + OA^2 = c^2\cos^2\theta + d^2\sin^2\theta + c^2\cos^2(\theta + 90°) + d^2\sin^2(\theta + 90°) = c^2\cos^2\theta + d^2\sin^2\theta + c^2(-\sin\theta)^2 + d^2\cos^2\theta = c^2(cos^2\theta + \sin^2\theta) + d^2(\sin^2\theta + \cos^2\theta) = c^2 + d^2$. □

The following result was found by computer.

Theorem 17. *Let O be the center of a hippopede with vertices X and Y. Let XD be a tangent to the hippopede as shown in Figure 18. Let $t = XD$ and $r = OD$. Then* $t^2 = \dfrac{c^2(c^2 - r^2)}{c^2 - 2r^2}$.

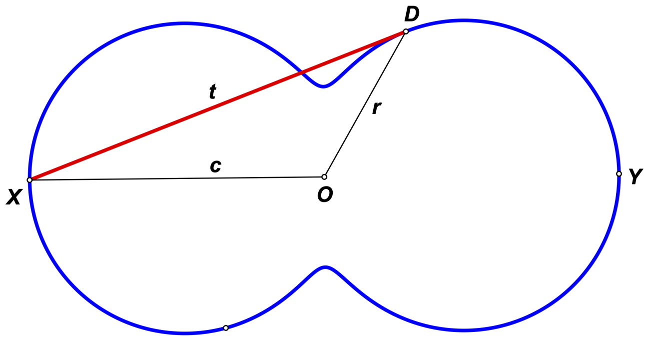


FIGURE 18. $t^2 = \frac{c^2(c^2-r^2)}{c^2-2r^2}$

Proof. Let O be at the origin, $X = (-c, 0)$, and $D = (x, y)$. Then

(2) $$x^2 + y^2 = r^2$$

and

(3) $$(x + c)^2 + y^2 = t^2.$$

Implicitly differentiating the equation of the hippopede (1) with respect to x, gives

$$2(x^2 + y^2)(2x + 2yy') = 2c^2x + 2d^2yy'$$

and solving for y' gives

(4) $$y' = \frac{c^2x - 2x^3 - 2xy^2}{y\,(2x^2 + 2y^2 - d^2)}.$$

Since y' represents the slope of line XD, we have

(5) $$\frac{x(c^2 - 2x^2 - 2y^2)}{y\,(2x^2 + 2y^2 - d^2)} = \frac{y}{x + c}.$$

Eliminating x, y, and d from Equations (1), (2), (3), and (5), using Mathematica, gives

$$t^2(c^2 - 2r^2) = c^2(c^2 - r^2)$$

which is the desired result. □

The following result was found by computer.

Theorem 18. *Let O be the center of a hippopede with vertices X and Y and suppose that $c > d\sqrt{5}$. Let XD be a tangent to the hippopede. Two possible locations for D are shown in Figure 19. Let $t = XD$. Then*

$$t^2 = c\frac{3(c^2 - d^2) \pm \sqrt{(c^2 - d^2)(c^2 - 5d^2)}}{2(c^2 - d^2)}.$$

Note that point D_2 is not on the y-axis.

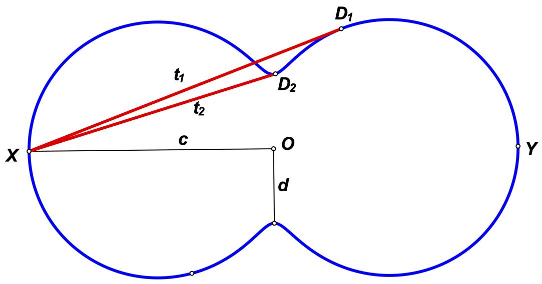


Figure 19. $t^2 = c\left(3(c^2-d^2) \pm \sqrt{(c^2-d^2)(c^2-5d^2)}\right)/2(c^2-d^2)$

The following result was found by computer.

Theorem 19. *Let O be the center of a hippopede with vertices X and Y and suppose that PQ is a tangent parallel to XY as shown in Figure 20. Let M be the midpoint of OY and let $\triangle YMR$ be an isosceles right triangle. Then $OQ = YR$.*

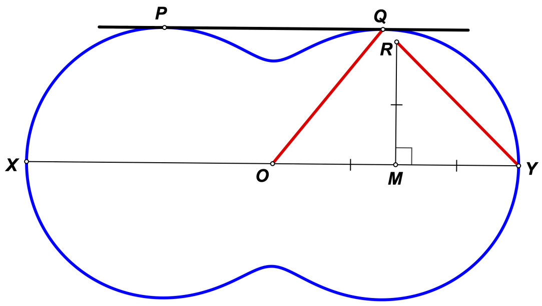


Figure 20. red lengths are equal

Proof. The tangent has slope 0, so let $y = 0$ in Equation (4). If the coordinates of Q are (x, y), then $c^2 = 2x^2 + 2y^2$, so $OQ = \sqrt{x^2+y^2} = c/\sqrt{2}$. Since $MY = MR = c/2$, RY also equals $c/2$, so $OQ = YR$. □

Incidentally, solving $c^2 = 2x^2 + 2y^2$ and $(x^2+y^2)^2 = c^2x^2 + d^2y^2$ for x and y gives

$$\begin{aligned} x &= \frac{c}{2} \cdot \frac{\sqrt{c^2-2d^2}}{\sqrt{c^2-d^2}} \\ y &= \frac{c}{2} \cdot \frac{c}{\sqrt{c^2-d^2}}. \end{aligned}$$

Thus, we see that in Figure 20, $PQ = 2x = \dfrac{c\sqrt{c^2-2d^2}}{\sqrt{c^2-d^2}}$ and the distance from P to XY is $\dfrac{c}{2} \cdot \dfrac{c}{\sqrt{c^2-d^2}}$.

The following result was found by Keita Miyamoto [9].

Theorem 20. *Let O be the center of a hippopede with vertices X and Y. A point P is located on XY, with $OP = p$, as shown in Figure 21. Then the radius of the incircle with center P is*

$$r = \sqrt{p^2 + d^2\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{p^2}{c^2 - d^2}}\right)}.$$

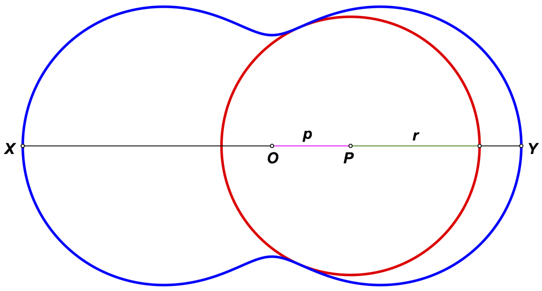


FIGURE 21. $r = \sqrt{p^2 + d^2\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{p^2}{c^2 - d^2}}\right)}$

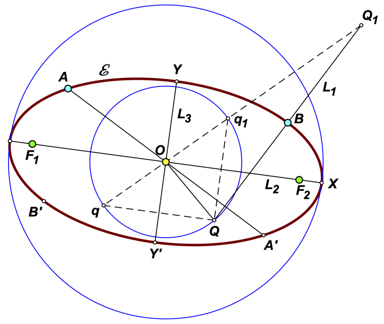
5. CONSTRUCTIONS

These constructions assume that your dynamic geometry environment allows drawing a locus and can find the intersection of a line or circle with a locus.

The following construction comes from [4, p. 110].

`Construction EllipseOAB.`

Given: Three non-collinear points, O, A, and B.
Constructs: the ellipse $\mathcal{E}$ with center O that passes through A and B.
Also constructs: the foci F_1 and F_2 of the ellipse, as well as the axes.



1. $A' = \text{reflect}(A, O)$, $\quad B' = \text{reflect}(B, O)$.
2. $L_1 = \text{perp}(B, AA')$.
3. $r = OA$.
4. $\{Q_1, Q\} = L_1 \cap \odot B(r)$.
5. $L_2 = \text{angleBisector}(OQ, OQ_1)$.
6. $L_3 = \text{perp}(O, L_2)$.
7. $\{q, q_1\} = O(Q) \cap OQ_1$.
8. $a = Q_1q/2$, $\quad b = Q_1q_1/2$.
9. $X = O(a) \cap L_2$, $\quad \{Y, Y'\} = O(b) \cap L_3$.
10. $\{F_1, F_2\} = Y(a) \cap \overleftrightarrow{OX}$.
11. $\mathcal{E} = \text{ellipse}(F_1, F_2, Y)$.

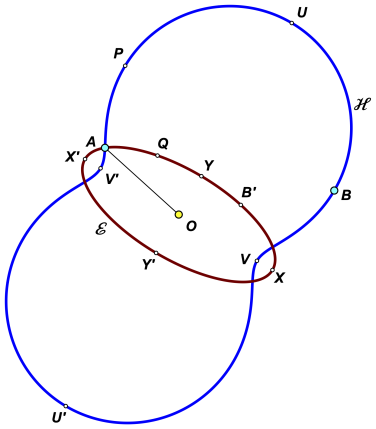
Note 1. The lengths of the semi-major and semi-minor axes of the ellipse are a and b, respectively.

Note 2. Lines AA' and BB' are conjugate diameters of the ellipse.

Since a hippopede is the inverse of an ellipse about a concentric circle, we get the following construction.

`Construction HippopedeOAB.`

Given: Three non-collinear points, O, A, and B.
Constructs: the hippopede $\mathcal{H}$ with center O that passes through A and B.
Also constructs: the vertices and covertices of the hippopede.

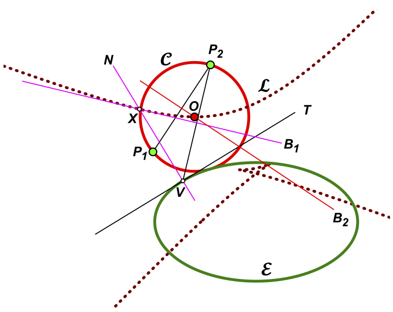


1. $B' = \text{inverse}(B, O(A))$.
2. $\{\mathcal{E}, X, X', Y, Y'\} = \text{ellipseOAB}(O, A, B')$.
3. Let $Q \in \mathcal{E}$.
4. $P = \text{inverse}(Q, O(A))$.
5. $\mathcal{H} = \text{locus}(P, Q, \mathcal{E})$.
6. $V = \text{inverse}(X, O(A))$.
7. $V' = \text{inverse}(X', O(A))$.
8. $U = \text{inverse}(Y, O(A))$.
9. $U' = \text{inverse}(Y', O(A))$.

The following construction comes from [13].

`Construction EPP.`

Given: a conic $\mathcal{E}$ and two points P_1 and P_2.
Constructs: a circle $\mathcal{C}$ with center O tangent to the conic and passing through the two points.



1. Let $V \in \mathcal{E}$.
2. $T = \text{tangentAt}(\mathcal{E}, V)$.
3. $N = \text{perp}(T, V)$.
4. $B_1 = \text{perpBisector}(VP_2)$.
5. $X = N \cap B_1$.
6. $\mathcal{L} = \text{locus}(X, V, \mathcal{E})$.
7. $B_2 = \text{perpBisector}(P_1P_2)$.
8. $O = B_2 \cap \mathcal{L}$.
9. $\mathcal{C} = O(P_1)$.

Note 1. The locus $\mathcal{L}$ represents all points that are equidistant from $\mathcal{E}$ and P_2. The perpendicular bisector B_2 represents all points equidistant from P_1 and P_2.

Note 2. The name "EPP" is a mnemonic for "Ellipse/Point/Point", however, the construction works for all conics, not just ellipses.

Note 3. There are typically two solutions. There are usually two points where $\mathcal{L}$ meets B_2. Figure 22 shows two circles tangent to an ellipse and passing through two fixed points inside the ellipse.

Note 4. This construction only constructs the circle. It does not construct the touch point with the conic. The next construction can be used to find the touch point.

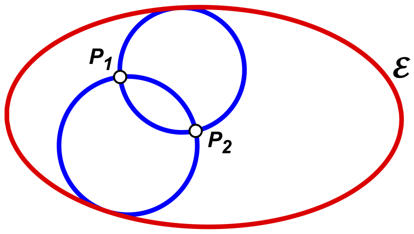


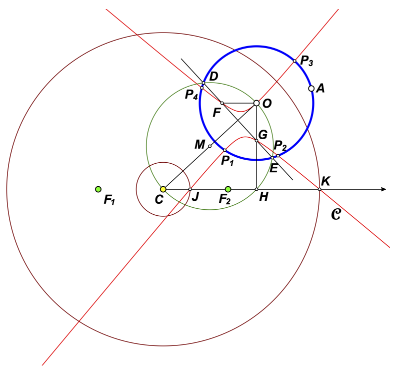
FIGURE 22. two circles tangent to $\mathcal{E}$ passing through P_1 and P_2

The following construction comes from [7].

`Construction TouchPointsOfConicWithCircle.`

Given: Two points F_1 and F_2 and a circle $O(A)$.

Constructs: the points P_1, P_2, P_3, P_4 where a central conic having foci F_1 and F_2 touches this circle



1. $k = CF_2 \quad r = OA/k$.
2. $C = \text{midpt}(F_1, F_2)$.
3. $H = \text{foot}(O, F_1F_2)$.
4. $a = CH/k, \quad b = OH/k$.
5. $M = \text{midpt}(C, O)$.
6. $\{D, E\} = \odot M(O) \cap \odot O(A)$.
7. $G = OH \cap DE, \quad F = \text{perp}(O, OH) \cap DE$.
8. $c = a^2 + b^2 - r^2$.
9. $\text{rad} = (c+1)^2 - 4a^2$.
10. $\mu = \left(c + 1 + \sqrt{\text{rad}}\right)/(2a)$.
11. $K = \odot(C, k\mu) \cap \overrightarrow{CH}, \quad J = \odot(C, k/\mu) \cap \overrightarrow{CH}$.
12. $\mathcal{C} = \text{conic}(J, K, F, G, O)$.
13. $(P_1, P_2, P_3, P_4) = \mathcal{C} \cap \odot O(A)$.

Note 1. If we set up a Cartesian coordinate system with origin at C and F_2 at $(1, 0)$, then the coordinates of O are (a, b). The value k is the unit distance.

Note 2. There are two ellipses with foci F_1 and F_2 that touch the circle; and there are two hyperbolas with foci F_1 and F_2 that touch the circle. Figure 23 shows the two ellipses.

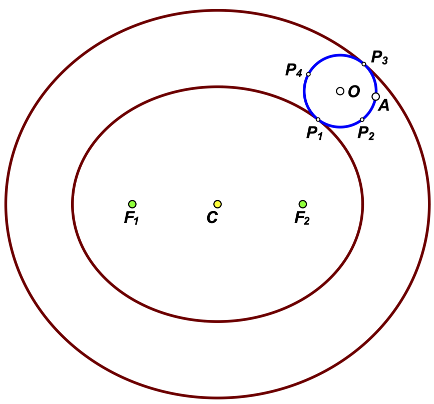
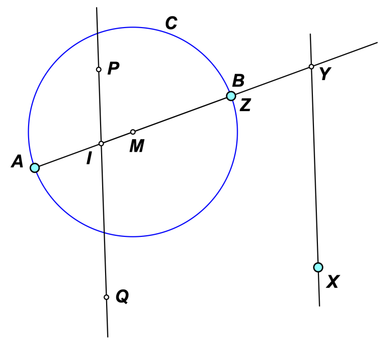


FIGURE 23. two ellipses with foci F_1 and F_2 tangent to circle $O(A)$

Note 3. The point P_1 is the one closest to C. It is the point where the ellipse with foci F_1 and F_2 touches $O(A)$ externally. The point P_3 is the one furthest from C. It is the point where the ellipse with foci F_1 and F_2 touches $O(A)$ internally.

`Construction pointOnSameSideOfLine.`

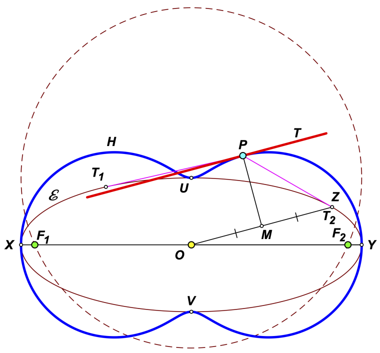
Given: Five points, P, Q, A, B, and X with A and B on opposite sides of PQ.
Constructs: the point Z (which is either A or B) that is on the same side of PQ as X.



1. $M = \text{midpt}(A, B)$.
2. $C = M(A)$.
3. $I = \overleftrightarrow{AB} \cap \overleftrightarrow{PQ}$.
4. $Y = \overleftrightarrow{AB} \cap \text{parallel}(X, PQ)$.
5. $Z = \overrightarrow{IY} \cap C$.

`Construction tangentAtHippopede.`

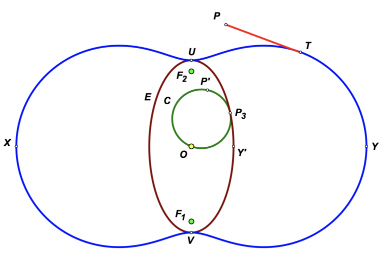
Given: Hippopede H with center O, vertex Y, and covertex U.
Also given: point P on the hippopede.
Constructs: the tangent T to the hippopede at P.



1. $\mathcal{E} = \text{ellipseOAB}(O, Y, U)$.
2. $\{F_1, F_2\} = U(OY) \cap XY$.
3. $\{T_1, T_2\} = \text{tangentFrom}(P, \mathcal{E})$.
4. $Z = \text{pointOnSameSideOfLine}(U, V, T_1, T_2, P)$.
5. $M = \text{midpt}(O, Z)$.
6. $T = \text{perp}(P, PM)$.

`Construction tangentToHippopede.`

Given: Hippopede H with center O, vertex Y, and covertex U.
Also given: point P outside the hippopede.
Constructs: the tangent PT from the point P to the hippopede.



1. $Y' = \text{inverse}(Y, O(U))$
2. $E = \text{ellipseOAB}(O, Y', U)$.
3. $\{F_1, F_2\} = Y'(OU) \cap UV$.
4. $P' = \text{inverse}(P, O(U))$.
5. $C = \text{EPP}(E, P', O)$.
6. $P_3 = \text{TouchPointsOfConicWithCircle}(F_1, F_2, C)$.
7. $T = \text{inverse}(P_3, O(U))$.

Note. There will be two solutions because there are two solutions to the EPP construction in step 5.

6. FOCI

According to [14, p. 119], a *focus* of a plane curve is a point such that the lines joining it to the two imaginary points on a circle at infinity both touch the curve. Also, [18, p. 56] and [6, p. 69] and [1, p. 47] and [14, p. 119] defines a focus as a point where tangents from the circular points at infinity meet.

We prefer the equivalent formulation given in [2]. Point F is said to be a focus of a curve C if two tangent lines having slopes i and $-i$ can be drawn from F to C.

Let us now find the foci of the curve whose equation is

$$(x^2 + y^2)^2 = c^2x^2 + d^2y^2 \tag{6}$$

with $c > d > 0$.

Let $P = (x_0, y_0)$ be a point in the plane of the curve. The equation of a line through P with slope i is

$$y - y_0 = i(x - x_0). \tag{7}$$

To find the points where this line meets the curve, we solve Equations (6) and (7) simultaneously for x and y. We find that there are two points of intersection,

$$\left(\frac{\sqrt{A} + B}{C} \quad , \quad \frac{i\sqrt{A} + B'}{C}\right)$$

and

$$\left(\frac{-\sqrt{A} + B}{C} \quad , \quad \frac{-i\sqrt{A} + B'}{C}\right)$$

where

$$\begin{aligned} A &= c^2(x_0 + iy_0)^2 \left(d^2 + (x_0 + iy_0)^2\right) - d^2(x_0 + iy_0)^4, \\ B &= -d^2(x_0 + iy_0) - 2x_0{}^3 - 6ix_0{}^2y_0 + 6x_0y_0{}^2 + 2iy_0{}^3, \\ B' &= c^2(-ix_0 + y_0) - 2ix_0{}^3 - 6x_0{}^2y_0 - 6ix_0y_0{}^2 + 2y_0{}^3, \end{aligned}$$

and

$$C = c^2 - d^2 - 4(x_0 + iy_0)^2.$$

In order for this line to be tangent to the curve, it must meet it at exactly one point (that is, a double point). The condition for that to be the case is that $A = 0$ or

$$c^2(x_0 + iy_0)^2 \left(d^2 + (x_0 + iy_0)^2\right) = d^2(x_0 + iy_0)^4. \tag{8}$$

This is the condition that x_0 and y_0 must satisfy in order for the line through P with slope i to be tangent to the curve.

Similarly, the condition that x_0 and y_0 must satisfy in order for the line through P with slope $-i$ to be tangent to the curve is

$$c^2(x_0 - iy_0)^2 \left(d^2 + (x_0 - iy_0)^2\right) = d^2(x_0 - iy_0)^4. \tag{9}$$

To find points where both conditions are satisfied, we solve Equations (8) and (9) for x_0 and y_0. Discarding imaginary solutions, we find three possible values for P, namely

$$(0, 0), \qquad \text{and} \qquad \left(0, \pm\frac{cd}{\sqrt{c^2 - d^2}}\right).$$

The point $(0, 0)$ is a singular point of the curve and is not normally considered to be a focus. In fact, $(0, 0)$ is the center of the curve.

We thus have the following theorem.

Theorem 21. *A hippopede has center at the origin and major axis XY along the x-axis. Then the coordinates of the foci of the hippopede are at $(0, \pm f)$ where $f = cd/\sqrt{c^2 - d^2}$. These are labeled J and K in Figure 24.*

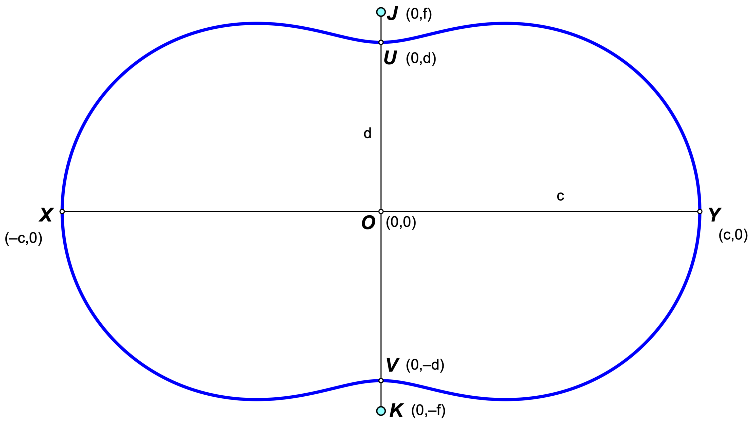


Figure 24. Foci of a hippopede

None of the usual books of special plane curves talk about the foci of a hippopede. We will now give some geometrical properties of these foci.

Theorem 22. *A hippopede has center O, foci J and K, vertices X and Y, and covertices U and V. The ellipse with axes XY and UV has foci F and G as shown in Figure 25. Then $JY \parallel UG$.*

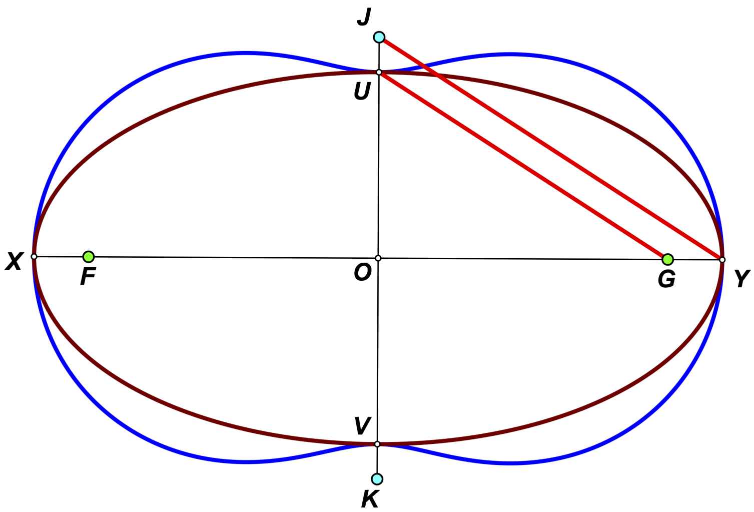
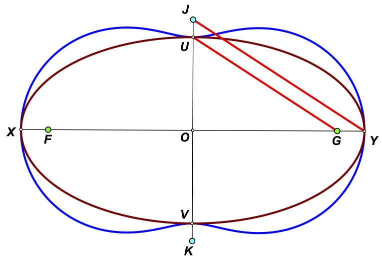


Figure 25. red lines are parallel

Proof. Note that $OY = c$, $OU = d$, and $OJ = cd/\sqrt{c^2 - d^2}$. The semi axes of the ellipse are c and d. It is a well-known property of ellipses that $UG = OY$. Therefore $OG = \sqrt{c^2 - d^2}$ and consequently, $OG \cdot OJ = cd = OU \cdot OY$. Hence, $OJ/OU = OY/OG$ which makes $JY \parallel UG$. □

`Construction fociOfHippopede.`

Given: Hippopede H with center O, vertex Y, and covertex U.
Constructs: the foci J and K of the hippopede.



1. $\mathcal{E} = \text{ellipseOAB}(O, Y, U)$.
2. $\{F, G\} = U(OY) \cap XY$.
3. $J = \text{parallel}(Y, GU) \cap UV$.
4. $K = \text{reflect}(J, O)$.

Since the inverse of a hippopede about a concentric circle is an ellipse, we can find geometrical properties of a hippopede by finding geometrical properties of this ellipse and then inverting them back to give a property of the original hippopede. Also, according to [6, p. 74], the inverses of the foci of a curve are the foci of the inverse curve.

So, for example, if we consider the optical property of an ellipse (the lines from a point on the ellipse to the foci make equal angles with the normal at that point), we get the following result by inversion. Recall that inversion preserves angles and tangency and the inverse of a line not through the center of inversion is a circle through the center of inversion.

Theorem 23. *Let P be a point on a hippopede with center O and foci J and K. The circle that is tangent to the hippopede at P and passes through O has center O_P. Let O_J be the center of $\odot OPJ$ and let O_K be the center of $\odot OPK$ as shown in Figure 26. Then $\angle O_J P O_P = \angle O_P P O_K$.*

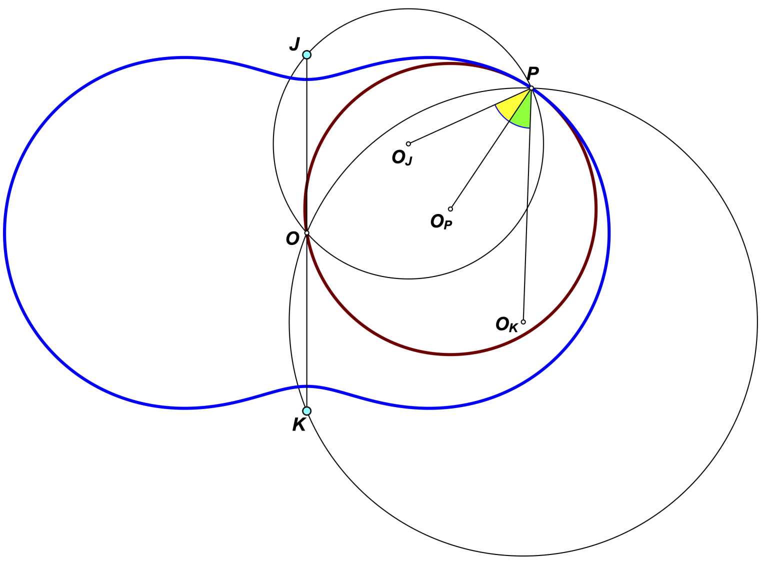


FIGURE 26. yellow angle = green angle

The following result was also found by inversion.

Theorem 24. *Let P be any point on a hippopede with center O, foci J and K, and vertices X and Y. The circle $\odot PJK$ meets XY at Q as shown in Figure 27. Then $\odot POQ$ is tangent to the hippopede at P.*

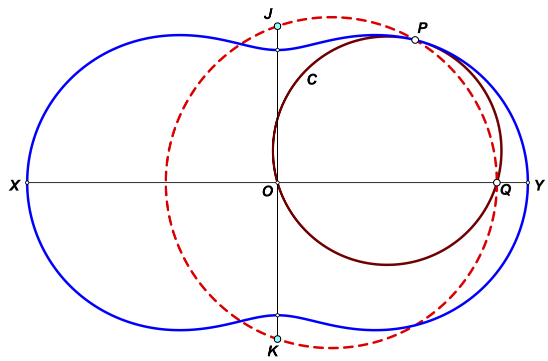


Figure 27. brown circle is tangent to hippopede

Proof. An inversion shows that the property is true by result 10.3.1 in [12]. □

The following result was found by inversion.

Theorem 25. *A hippopede has center O, foci J and K, and vertices X and Y. The circle with diameter OJ meets the hippopede at P as shown in Figure 28. Let Q be the point on segment OY such that $OQ = d$. Then $\odot POQ$ is tangent to the hippopede at P.*

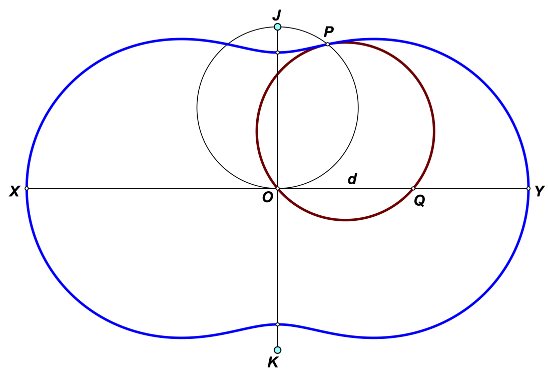


Figure 28. brown circle is tangent to hippopede

Proof. An inversion shows that the property is true by result 9.1.5 in [12]. □

Theorem 26. *A hippopede has center O, foci J and K, and vertices X and Y. The circle with diameter OJ meets the hippopede at P as shown in Figure 29. Let Q be the point on segment OY such that $OQ = d$. Let $\odot POQ$ meet OU at R. Then $\angle JQU = \angle UQR$.*

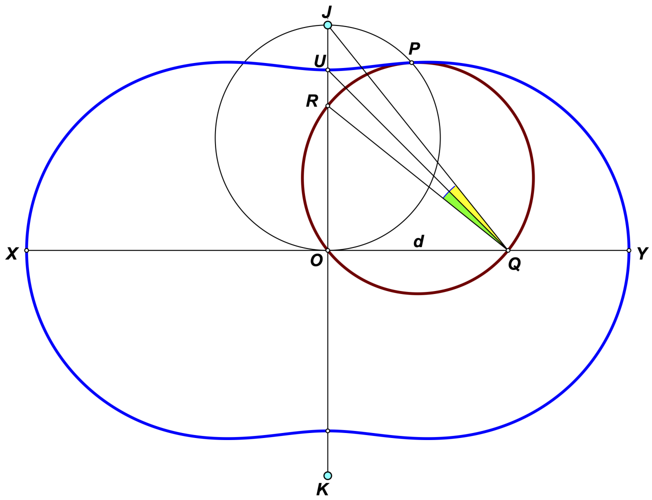


FIGURE 29. yellow angle = green angle

Proof. An inversion shows that the property is true by result 9.1.6 in [12]. □

The following result was found by computer.

Theorem 27. *A hippopede with center O has foci J and K. A tangent from J touches the hippopede at P as shown in Figure 30. Then $\angle JPK = 2\angle POJ$.*

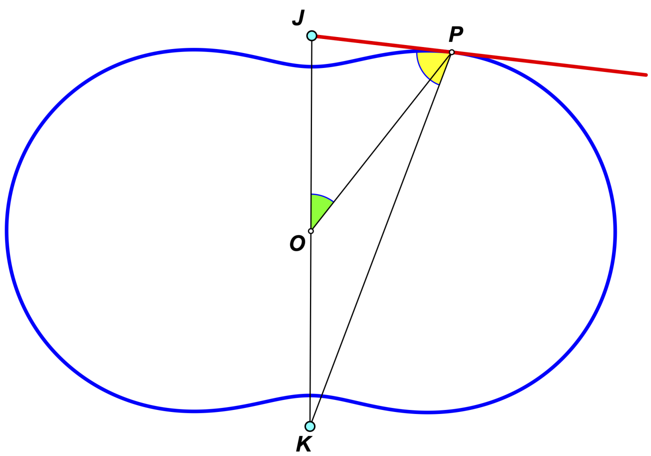


FIGURE 30. yellow angle is twice green angle

Open Question 1. *Is there a simple geometric proof for Theorem 27?*

The following result was found by computer.

Theorem 28. *A hippopede has foci J and K. A secant through J meets the hippopede at four points A, B, C, and D as shown in Figure 31. Then $KA+KB = KC + KD$.*

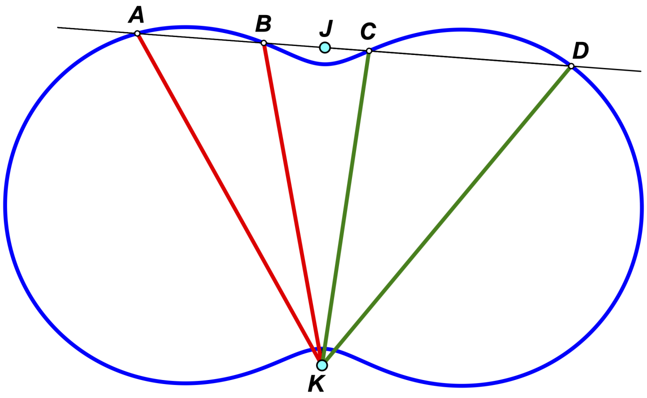


Figure 31. sum of red lengths = sum of green lengths

Open Question 2. *Is there a simple geometric proof for Theorem 28?*

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