Problem ideas by Stanley Rabinowitz

Problem 1.

Problem E3144 [1] shows that given a set A in a topological space, at most 34 sets can be generated using the operations {complement, interior, boundary}. Moser [2] shows that at most 13 sets can be generated from {closure, interior, union}. (See also [4].) Kuratowski [3] has shown that at most 14 sets can be generated from {closure, interior, complement}. Generate and solve other problems of this nature.

REFERENCES

[1] Jesús Ferrer, "Solution to Problem E3144 (proposed by Edwin Buchman)", *American Mathematical Monthly*. **95**(1988)353.

[2] L. E. Moser, "Closure, interior, and union in finite topological spaces", *Colloq. Math.* **38**(1977)41–51.

[3] Kuratowski, "Sur l'opération $\bar{A}$ de l'analyses situs", *Fund. Math.* **3**(1922)182–199.

[4] Arthur Smith, "Problem 5996", *American Mathematical Monthly*. **81**(1974)1034.

Problem 2.

Chang [1] showed that the sequence a_n satisfying the recurrence $n^3a_n - (34n^3 - 51n^2 + 27n - 5)a_{n-1} + (n-1)^3a_n - 2 = 0$ with initial conditions $a_0 = 1$, $a_1 = 5$, consists only of integers. Find other cubics for the coefficient of a_{n-1} that have this property.

REFERENCES

[1] Derek K. Chang, "Recurrence Relations and Integer Sequences", *Utilitas Mathematica*. **32**(1987)95–99.

Problem 3.

Let F_n denote the n-th Fibonacci number and L_n denote the n-th Lucas number. Express L_n in terms of F_n (only).

Problem 4.

Let P be a point inside $\triangle ABC$ and draw the three cevians from the vertices to the opposite sides through P. How many paths are there from A to B such that no segment is traversed more than once? such that no vertex is gone through more than once? Generalize to a simplex in E^n.

Problem 5.

Points A and B are on a horizontal line. A rocket from A is launched upwards toward B following a parabolic trajectory due to the normal laws of gravity and Newtonian physics. Assume the Earth is flat. At the same time that this rocket is launched, an interceptor missile is launched from point B which travels at a fixed rate of speed but is intelligent and can alter its trajectory. What path should the interceptor missile follow so as to hit the rocket in the minimum time? Is the path always a straight line?

Problem 6.

In an ellipse, in which direction is the average chord length largest?

Problem 7.

Let F_n denote the n-th Fibonacci number and L_n denote the n-th Lucas number. For which n do F_n and L_n have the same period (modulo n)?

Problem 8.

Find the number of divisors of $n!$.

Problem 9.

Can a triangle with sides of lengths a, b, c have three concurrent cevians of lengths a, b, c, respectively? not respectively?

Problem 10.

A triangle with sides of lengths a, b, c has three concurrent cevians of lengths x, y, z, respectively. Find a symmetrical relationship between a, b, c and x, y, z; something like $1/a + 1/b + 1/c = 1/x + 1/y + 1/z$.

Problem 11.

Let A and B be two points on diameter CD of a circle, γ, with A between B and C. With A as center and radius AC, draw a circle O internally tangent to γ. If P is the center of a circle tangent externally to O and internally to γ, and if furthermore, B is the center of γ, then it is easy to show that the locus of P is an ellipse, since $PA+PB = AC+CD$. On the other hand, suppose O is instead an ellipse tangent internally to γ with points A and B as foci. Now, what is the locus of point P, the center of a circle tangent externally to O and internally to γ?

Problem 12.

Investigate equilateral hexagons (or pentagons) inscribed in $\triangle ABC$ with one edge parallel to BC.

Problem 13.

Let P be a point inside $\triangle ABC$. Consider the proposition: "Let X, Y, Z be the 'notable point' inside triangles BPC, CPA, APB, respectively. Extend PX, PY, PZ to meet BC, CA, AB at D, E, F respectively. Then AD, BE, CF concur." Prove that this proposition is true when 'notable point' is replaced by (a) centroid, (b) incenter, (c) symmedian point, (d) isotomic conjugate of (b) or (c).

Problem 14.

Let P be a point inside non-isosceles triangle ABC. Circles inscribed in triangles PBC, PCA, PAB touch BC, CA, AB at points D, E, F respectively. Find all points P such that AD, BE, CF concur.

Problem 15.

Let P be a focus of an ellipse. It is known that for all chords passing through P, if P divides the chord into two pieces of lengths a and b, then $1/a + 1/b$ is a constant. Suppose a convex body in the plane has two points with this property. Must that body be an ellipse? Generalize to higher dimensions. What can be said about convex bodies that have one point with this property?

Problem 16.

Investigate properties of a circle that cuts each side of a triangle at the same angle.

Problem 17.

Let D be a point on side BC of $\triangle ABC$. Let C_1 be the incircle of $\triangle ABD$ and let C_2 be the incircle of $\triangle ADC$. Let C be the incircle of $\triangle ABC$. Determine when C is orthogonal to both C_1 and C_2. What if C is just orthogonal to C_1?

Problem 18.

Characterize polynomials all of whose derivatives (and itself) are unimodal. A polynomial is *unimodal* if it's coefficients first increase, then decrease.

Problem 19.

Find numbers that are 10 times their number of divisors. For example, 180 has 18 divisors. Generalize.

Problem 20.

Find n such that any subdivision of $\{1, 2, \ldots, n\}$ into two pieces contains a piece containing an n-term Fibonacci sequence (second-order linear recurrence).

Problem 21.

You have n pairs of socks. Each pair is colored with a different color. The colors are numbered from 1 to n. The colors are not completely distinguishable. To your inexperienced eye, color k looks exactly like colors $k \pm 1$. The socks are mixed up and you proceed to pull out pairs, pairing two socks if they appear to have the same color. What is the probability that all the socks will get paired up in this manner (i.e. that you are not left with a sock that does not match any other sock left over)?

Problem 22.

Are there any positive integers, $n > 1$, such that n^2 starts with n in the decimal system.

Problem 23.

Devise an alphametic to go along with the proverb "Money is the root of all evil". For example, something like

$$\text{MONEY} = \sqrt[3]{\text{ALLEVIL}}.$$

Problem 24.

Evaluate or simplify

$$\prod_{k=0}^{n} \binom{n}{k}.$$

Problem 25.

If $f(x, y, z) = x + y + z - xyz$, then it is known that $f(\tan A, \tan B, \tan C) \equiv 0$ whenever A, B, C are the angles of a triangle. Find a polynomial, $g(x, y, z)$, such that $g(\sin A, \sin B, \sin C) \equiv 0$ when A, B, C are the angles of a triangle. Do the same thing for cos.

Problem 26.

Number an $n \times n$ square with the integers from 1 to n^2 such that two consecutive integers (1 and n^2 should be considered consecutive) fall as far apart as possible (taxicab metric).

Problem 27.

Find a magic square on an $n \times n$ torus consisting of the integers from 1 to n^2 such that the numbers on *all* straight lines through n numbers sum to the magic constant. In other words, the opposite sides of the square are to be identified.

Problem 28.

Let P be inside an ellipse and let R be a variable point on the ellipse. When does the radial distance, PR, have extrema with R not being at a vertex? What is the average of the radial distances? For related information, see [1].

REFERENCE

[1] Ali R. Amir-Moéz, "The Focal Distance of a Conic Section", *Pi Mu Epsilon Journal.* **8**(1986)246–249.

Problem 29.

For a triangle with inradius r and circumradius R, we have $R \geq 2r$ or $\inf\{\frac{R}{r}\} = 2$. Find $\inf\{\frac{R}{r}\}$ where the inf is taken over all quadrilaterals. In this case, r is the radius of the largest circle contained within the quadrilateral and R is the radius of the smallest circle containing the quadrilateral.

Problem 30.

Let s be the side of an equilateral triangle inscribed in $\triangle ABC$. Find the minimum and maximum values for s. Let S be the side of an equilateral triangle circumscribed about $\triangle ABC$. Find $\inf\{\frac{S}{s}\}$.

Problem 31.

Find the area of the largest/smallest equilateral triangle circumscribing a 3-4-5 right triangle.

Problem 32.

Let P be a point inside $\triangle A_1A_2A_3$. Line segments PA_1, PA_2, PA_3 divide $\triangle A_1A_2A_3$ into three triangles. Let T_i be the triangle opposite vertex A_i. Investigate properties of the lines from A_i to the orthocenter of T_i. What happens if we replace "orthocenter" by "incenter" or "centroid"?

Problem 33.

Let P be a point inside $\triangle ABC$. Cevians AD, BE, CF all pass through P. Let C_1 be the incircle of $\triangle APF$. Let C_2 be the incircle of $\triangle BPD$. Let C_3 be the incircle of $\triangle CPE$. Investigate these three circles. Can they all have the same radius?

Problem 34.

For which n is 10^n a non-trivial binomial coefficient?

Problem 35.

When are binomial coefficients perfect squares? triangular numbers? Fibonacci numbers?

Problem 36.

An arbelos is formed from semicircles with radii a, b, and $a+b$ along a horizontal line segment of length $2a+2b$. The semicircles are drawn downward and the diameters are not drawn in. This forms a type of "cup" with two concave portions for holding water. Assume $b>a$ and fill the semicircle of radius b with water. Tip the arbelos until water spills from the big hole to the little hole. Eventually the water will drip out of the smaller hole. If

$$\frac{a}{b}=\sqrt{\frac{\pi+3}{5\pi-3}},$$

at what angle is the arbelos tilted when the water first drips out? [Answer: 15°.]

Problem 37.

From Krechmar [1], page 24, we have that if $A+B+C=\pi$, then $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$. If we let $x=\sin A$, $y=\sin B$, $z=\sin C$, then this gives

$$2x\sqrt{1-x^2}+2y\sqrt{1-y^2}+2z\sqrt{1-z^2}=4xyz.$$

Rationalize this formula.

REFERENCE

[1] V. A. Krechmar, *A Problem Book in Algebra*. Mir Publishers. Moscow: 1974.

Problem 38.

Is there an explicit formula for the finite continued fraction

$$\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots\frac{1}{n}?$$

If not, can it be expressed rationally in terms of H_n, the n-th harmonic number? ($H_n=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$.)

REFERENCES

[1] Stanley Rabinowitz, "Problem E3264", *American Mathematical Monthly*. **95**(1988)352.

[2] B. Buchberger, G. E. Collins, and R. Loos, *Computer Algebra: Symbolic and Algebraic Computation, second edition*. Springer-Verlag. New York: 1983.

[3] J. C. Lafon, Summation in Finite Terms, pages 71–77 in [2].

[4] M. Karr, "Summation in Finite Terms", *Journal of the Association for Computing Machinery*. **28**(1981)305–350.

[5] R. W. Gosper, jr., "Decision Procedure for Indefinite Hypergeometric Summation", *Proceedings of the National Academy of Sciences of the United States of America*. **75**(1978)40–42.

Problem 39.

Find the number of vertices of the smallest polytope in 4-space that has the 5 Platonic solids as facets.

Problem 40.

Given two unit squares on top of each other in parallel planes, d units apart. The center of the top square lies directly over the center of the bottom square. A side of one square is inclined by an angle of θ to the corresponding side of the lower square. The corresponding vertices are joined. Find the volume of the resulting solid in terms of d and θ. This is a generalization of problem 7B12 on page 88 of [1].

REFERENCE

[1] L. Lines, *Solid Geometry*. Dover Publications, Inc. New York: 1965.

Problem 41.

Given integers p and q (p not divisible by 3), is there an integer n with sum of digits q that is divisible by p?

Problem 42.

Let P and Q be two points inside $\triangle ABC$. The cevians through P and Q determine 6 points on the perimeter of the triangle. It is known that these 6 points lie on an ellipse (see Carnot's Theorem, page 271 in [2], combined with Ceva's Theorem, page 108 in [2]; see also [1]). What can be said about the center of this ellipse in relationship to P and Q? Where are the foci with respect to P and Q? Consider also the cases when P and Q are notable points. Are the foci of the ellipse located at any notable points?

REFERENCES

[1] Stanley Rabinowitz, "Problem 903", *Crux Mathematicorum*. **10**(1984)19.

[2] Luigi Cremona, *Elements of Projective Geometry*. Dover Publicatrions, Inc. New York: 1960.

Problem 43.

Given n points in the plane such that any three are within ϵ of some line. Must they all be within ϵ of some line? [No.] What *can* be said?

Problem 44.

Given n points in the plane. Suppose any three form a triangle with area less than ϵ. Must the area of their convex hull be less than ϵ? [No.] What *can* be said?

Problem 45.

Is there a polygonal line with all slopes different that passes through all the lattice points in the plane?

Problem 46.

Given n coins, all weighing different, except for two of them. What is the minimum number of weighings needed to find the two that weigh the same using a pan balance?

Problem 47.

Among n balls are k radioactive ones. A machine detects radioactivity. What is the maximum number of machine uses needed to locate the k radioactive ones?

REFERENCE

[1] C. D. Cox, ed., "Problem 291", *Parabola*. **12**(1976)28.

Problem 48.

Is there a polynomial $f(x,y)$ of degree $n>1$, such that $f(f(x,y),f(y,x))$ has degree less than n? has degree less than n^2? How about

$$g(g(a,b,c),g(b,c,a),g(c,a,b))?$$

Problem 49.

Can we place distinct positive integers on Z^2 such that orthogonally adjacent integers are relatively prime? [Yes, place all the primes.] Can it also be done if every positive integer must appear at some lattice point? Can we place all the positive integers on Z^2 such that if $d(a,b) \le k$, then a and b are relatively prime?

Problem 50.

Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor\sqrt{n}\rfloor}}{n}.$$

See Günter and Kusmin, volume II, problem 294 which discusses convergence. Failing this, evaluate some sum involving $(-1)^{\lfloor\sqrt{n}\rfloor}$.

Problem 51.

Given n points in the plane. Suppose any three are within ϵ of some point. Must they all be within ϵ of some point? [No.] What *can* be said?

Problem 52.

Find the centroid, orthocenter, incenter, etc. for the triangle whose vertices are the roots of $x^3+ax^2+bx+c=0$ in the complex plane.

Problem 53.

Find a condition so that the roots of a quartic equation (in the complex plane) are cyclic.

Problem 54.

Let F_i be the i-th Fibonacci number. Let $f(x)$ be a polynomial of degree n such that $f(i)=F_i$ for $i=0,1,2,\ldots,n$. What can be said about f? What can be said about $f(n+1)$?

Problem 55.

Find the number of 3×3 magic squares in Z_n.

Problem 56.

Let A be a given point inside a fixed ellipse in the plane. Vary points P and Q on the boundary of the ellipse so that $\angle PAQ$ remains constant (and equal to θ). When is PQ the largest? the smallest?

Problem 57.

Which chord of a given length of a fixed ellipse subtends the largest angle at a focus?

Problem 58.

Find the number of triangles whose sides are a, b, and $\binom{a}{b}$ with a and b integers and $a>b$.

Problem 59.

Find the square root of $9+2\sqrt{2}+2\sqrt{3}$.

Problem 60.

Do the angle bisectors of a triangle always form a triangle? If so, find an inequality for its area. Same question for other notable cevians such as the altitudes.

Problem 61.

If A, B, C are lattice points and the inradius of $\triangle ABC$ is a rational number, r, must r be integral?

Problem 62.

Does there exist an infinite set of points such that all "small" distances between them fall into k classes?

Problem 63.

Let A, B, C be convex sets (resp. circles, etc.). Let $\operatorname{dist}(P,A)$ denotes the distance from a point P to the nearest point in the set A. Consider the set of all points P such that $\{\operatorname{dist}(P,A), \operatorname{dist}(P,B), \operatorname{dist}(P,C)\}$ are not the lengths of the sides of a triangle. What can be said about the shape of this set? (e.g. is it convex?)

Problem 64.

Let F_n denote the n-th Fibonacci number and L_n denote the n-th Lucas number. Investigate F_{L_n} and L_{F_n}.

Problem 65.

Investigate the triangles whose vertices are at (F_k, L_m), (F_{k+1}, L_{m+1}), and (F_{k+2}, L_{m+2}), where F_i and L_i are Fibonacci and Lucas numbers, respectively. Find the area of this triangle.

Problem 66.

Find the volume of a tetrahedron whose edges have lengths $n+i$ as i varies from 0 to 5 (with n fixed). Does the order matter? [Yes.] Does the tetrahedron exist? Which one has largest volume? Which one has smallest volume?

Problem 67.

Find an example where two non-congruent tetrahedra with the same set of integer edge lengths has the same volume.

Problem 68.

Start with an arbitrary convex hexagon in the plane. Describe an equilateral triangle externally on each side of the hexagon. Going around in order, let the centers of these 6 equilateral triangles be A, Z, B, Y, C, X. What is the relationship, if any, between $\triangle ABC$ and $\triangle XYZ$? Are they similar?

Problem 69.

Generalize Napoleon's Theorem to 3-space. Given an arbitrary tetrahedron, suppose we describe an isosceles tetrahedron outwardly on each face. What can be said about the centers of these 4 tetrahedra?

Problem 70.

Let $f(n)$ be the smallest positive integer such that x^2-x+2 divides $x^n+x+f(n)$. Find a formula for $f(n)$.

REFERENCE

[1] A. M. Gleason, R. E. Greenwood, and L. M. Kelly, *The William Lowell Putnam Mathematical Competition Problems and Solutions: 1938–1964.* The Mathematical Association of America. USA: 1980, page 577.

Problem 71.

Find the point on the parabola $x^2+\cdots+4y^2+\cdots=0$ that is closest to the origin.

Problem 72.

Two circles are internally tangent. A chord of the outer circle is divided into three parts by the inner circle. Let the three parts (in order) be x, y, and z. Let the radii of the circles be r and R ($R>r$). Given r, R, x, and y, find z.

Problem 73.

From page 507 of [1] we know that the number of ways of picking a 3-term arithmetic progression from $\{1,2,\ldots,n\}$ is $\lfloor\frac{(n-1)^2}{4}\rfloor$. What about a 4-term arithmetic progression?

REFERENCE

[1] S. Barnard and J. M. Child, *Higher Algebra.* MacMillan and Co. limited. New York: 1955.

Problem 74.

A subset of $\{1,2,\ldots,n\}$ is chosen at random. What is the probability that it forms an arithmetic progression?

Problem 75.

Evaluate

$$\sum_{i=0}^{n}\frac{F_i}{i!}$$

where F_i denotes the i-th Fibonacci number. Find the limit of this value as $n\to\infty$.

Problem 76.

Find distinct primes p and q and an integer, d, ($d>2$), so that p^n+q^n is divisible by d for all positive integers n.

Problem 77.

Find integers a, b, c, d, e, f, so that $357 | 3^{an+b} + 5^{cn+d} + 7^{en+f}$ for all positive integers n.

Problem 78.

Investigate properties of expressions such as $3^{n^2} + 5^{an+b}$.

Problem 79.

An n-sided die is a homogeneous solid convex polyhedron with n faces numbered from 1 to n. Intuitively, it is said to be a *fair* die if whenever you roll it, the probability that it comes to rest on any given face is $\frac{1}{n}$.

(a) Give a precise definition of *fair*.

(b) For what n do there exist fair n-sided dice?

REFERENCE

[1] Persi Diaconis and Joseph B. Keller, "Fair Dice", *American Mathematical Monthly*. **96**(1989)337–339.

Problem 80.

Prove that

$$\sum_1^3 a^n(b-c)$$

is divisible by $\prod_1^3(c-b)$ and find the quotient. Answer:

$$\sum_{i+j+k=n-2} a^i b^j c^k.$$

This generalizes problem 188 in [1].

REFERENCE

[1] H. S. Hall and S. R. Knight, *Higher Algebra*. MacMilland & Co. Ltd. London: 1957, page 509.

Problem 81.

Find $\max\{\sin A \sin 2B \sin 4C\}$ subject to the constraint that A, B, C are the angles of a triangle. Same question but replace "4" by "3".

Problem 82.

Find $\max\{\cos A + \cos B + \cos C\}$ subject to the constraint that $A + B + C = 120°$. Same question but replace $120°$ by $60°$.

Problem 83.

Prove that you can't quadrangulate a triangle.

Problem 84.

Given n points, no 5 cyclic, what is the maximum number of cyclic quadrilaterals that can be formed?

Problem 85.

Can the first n^2 perfect squares form a magic square? How about cubes? triangular numbers?

Problem 86.

Find all functions whose graphs are congruent to the graphs of their derivatives. Same question but replace "congruent" by "similar".

Problem 87.

For a triangle of given area, which triangle has the largest Morley triangle? Napoleon triangle?

Problem 88.

Devise a problem with a finite answer with a form something like the following: Find

$$\sum \frac{1}{\text{card}^2 A}$$

as A ranges over all finite subsets of Z^+.

Problem 89.

In the game of Dox (invented by Monroe H. Berg), twenty five counters are arranged to form five rows and and five columns of five counters each. Two players alternate in removing counters. At his turn, a player must remove from one to five counters which lie in the same row or column. (They need not be adjacent.) Whoever removes the last counter loses. Find a winning strategy for this game. How does the strategy change if the center counter is removed before the start of the game? Suppose, instead, that the winner is the person removing the last counter.

Problem 90.

Let A and B be opposite vertices of an $n \times n$ square which is divided up into n^2 unit squares by lines parallel to the sides. How many non-self-intersecting paths are there from A to B travelling along the lines? (A path can go backwards, but must not cross itself.) How many such paths are of length k? Generalize to higher dimensions.

Problem 91.

Find all functions $f(x, y)$ such that

$$\left.\begin{array}{l} a = f(x,y) \\ b = f(y,x) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x = f(a,b) \\ y = f(b,a) \end{array}\right. .$$

For example,

$$f(x,y) = \frac{x}{x^2+y^2}.$$

Add any necessary conditions about continuity or differentiability.

Problem 92.

What is the envelope of all normals to an ellipse?

Problem 93.

Let $[1..n]$ denote the infinite repeating continued fraction $[1, 2, \ldots, n]$. Is there a simple formula for $[1..n]$? Here is some data.

$$[1] = \frac{1+\sqrt{5}}{2}$$
$$[1,2] = \frac{1+\sqrt{3}}{2}$$
$$[1..3] = \frac{4+\sqrt{37}}{7}$$
$$[1..4] = \frac{9+2\sqrt{39}}{15}$$
$$[1..5] = \frac{195+\sqrt{65029}}{314}$$
$$[1..6] = \frac{103+2\sqrt{4171}}{162}$$
$$[1..7] = \frac{4502+\sqrt{29964677}}{6961}$$
$$[1..8] = \frac{9280+3\sqrt{13493990}}{14165}$$
$$[1..9] = \frac{684125+\sqrt{635918528029}}{1033802}$$

Problem 94.

An ellipse moves around in the first quadrant so that it remains tangent to both axes. It is known that the locus of the center of the ellipse is a quadrant of a circle with center at the origin. What is the envelope of all these ellipses? Fix one focus in mind. Find the locus of the focus. Find the envelope of the latera recta.

Problem 95.

Find
$$\int \frac{x^n - 1}{(x-1)(x^n+1)}\,dx.$$

Problem 96.

Solve the diophantine equation $x^2 + y^2 = z^2$ subject to the constraint that xyz is of the form $2^t 3^u 5^v 7^w$. The case $w = 0$ is of special interest. The case where the diophantine equation is $x + y = z$ is given in [1]. In particular, find all Pythagorean triangles whose sides have lengths a, b, and c such that abc has at most 3 prime factors. [There must be at least 3 prime factors, since it is known that 2, 3, and 5 are all divisors of a side of a Pythagorean triangle.]

REFERENCE

[1] L. Alex, "Diophantine Equations Related to Finite Groups", *Comm. Algebra.* **4**(1976)77–100.

Problem 97.

A *multigrade of order n* is an identity of the form

$$\sum a_i^k = \sum b_i^k$$

which is true for $k = 1, 2, \ldots, n$. For example, a multigrade of order 6 is

$$7^k+18^k+23^k+29^k+42^k+44^k+48^k+54^k+67^k+73^k =$$

$$8^k+14^k+27^k+33^k+37^k+39^k+52^k+58^k+63^k+74^k$$

which is true for $k = 1, 2, \ldots 6$ (problem 100 in [1]). Typically, the number of terms in a multigrade of order n is about $2(n-1)$ on each side. However, the number of terms can be smaller, as can be seen in

$$7^k + 9^k + 25^k + 29^k + 45^k + 47^k =$$

$$5^k + 15^k + 17^k + 37^k + 39^k + 49^k$$

which is true for $k = 1, 2, \ldots, 5$ (problem 253 in [1]) but has only 6 terms on each side. For a given n, find the smallest number of terms in a multigrade of order n.

REFERENCE

[1] A. S. Moiseenko, *1017 Problems*. Privately published. Belleville, NJ: 1987.

Problem 98.

Find the maximum number of points of intersection of a square and an ellipse. A square and a parabola. An n-gon and an ellipse.

Problem 99.

Construct an "alphametic" chess problem. Something like: each piece is replaced by a letter; the same kind of piece is replaced by the same letter. Colors are shown. White has a mate in two. Determine which piece each letter stands for.

Problem 100.

Suppose n people observe a tortoise over a period of 1 minute, such that there is total coverage. If the average speed observed is less than x mi./hr. for all observers, what is the farthest the tortoise could have travelled?

REFERENCE

[1] C. D. Cox, ed., "Solution to Problem O218", *Parabola*. **9**(Sept. 1973)36.

Problem 101.

Let P move along an ellipse. Let Q be a fixed point on the ellipse. Let R be the point such that $\triangle PQR$ is an equilateral triangle (described clockwise). What is the locus of point R?

Problem 102.

At each point P on an ellipse, draw a line segment PQ into the ellipse such that the length of PQ is twice the radius of curvature of the ellipse at point P. Find the locus of Q. Is this the envelope of the circles of curvature?

Problem 103.

A square is dissected into regular n-gons. Must they all be squares?

Problem 104.

Find a "simple" function, $f(x)$, such that the integer n is a Fibonacci number if and only if $f(n) = 0$.

Problem 105.

Find a tetrahedron with integer volume such that the lengths of the edges form an arithmetic progression.

Problem 106.

Can an $n \times n$ square be covered by $n+1$ dominoes, each an $(n-1) \times 1$ rectangle, plus one 1×1 square not located in a corner of the original square?

REFERENCE

[1] George Berzsenyi, "In Search of Colorations", *Journal of Recreational Mathematics.* **8**(1975–1976)191–194.

Problem 107.

Let k and n be given positive integers. Find the largest value for j so that k white queens and j black queens can be placed on distinct squares of an $n \times n$ chessboard such that no white queen attacks a black queen.

Problem 108.

Look for a Helly-like theorem for pairs of convex sets. For example, find a theorem like "if any 2 meet of n, then some 3 meet" or "if any k meet (of n) then all n meet".

Problem 109.

In the plane, exhibit n convex sets such that any two meet but no three meet. [Solution: just take n lines in general position.] Suppose that any k meet but not $k+1$.

Problem 110.

Let A be a convex set in the plane with width w. Prove that A is the intersection of all sets of constant width w containing A.

Problem 111.

Find all determinants which have the property that each element is equal to its

(a) minor

(b) cofactor.

Problem 112.

A set, S, of points in the plane has the property that any unit circle in the plane contains at least k of these points. What is the largest possible value that

$$M = \inf_{x,y \in S} \{d(x,y)\}$$

can have? Here, $d(x,y)$ denotes the distance from x to y.

Problem 113.

Does any row or column of $\begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}^n$ sum to a perfect square?

Problem 114.

Can we put n^2 consecutive positive integers into an $n \times n$ matrix such that the determinant of this matrix is 1?

Problem 115.

Let $ABCD$ be a square with center O. What happens to $PA + PC - PB - PD$ as P varies on a circle with center O?

Problem 116.

Fix a line L in the plane. Look at all triangles with area K and with L as their Euler line. What can be said about these triangles? Are they all congruent?

Problem 117.

Let $A = (0,0,0)$, $B = (1,1,0)$, $C = (1,1,1)$, $D = (2,3,3)$, $E = (1,2,3)$, and $F = (3,1,0)$. Find the equation of a circle in 3-space that is tangent to the lines AB, CD, and EF.

Problem 118.

Find the polygon with the smallest number of sides for which two diagonals colline. [Answer: $n = 6$.] Suppose that we want k diagonals to colline?

Problem 119.

A *crossing diagonal* of a non-convex polygon is a diagonal that meets an edge of the polygon, not at a vertex. What is the maximum number of crossing diagonals that a simple polygon of n sides can have?

Problem 120.

What is the envelope of the lines that bisect the perimeter of a given triangle? bisect the area of the triangle?

Problem 121.

Characterize all numbers such that in their base 10 representation, the number is a multiple of the number formed by removing one of its digits. For example, $(wxyz)_{10} = k(wxy)_{10}$.

Problem 122.

Find the locus of the center of all unit equilateral triangles that lie inside a unit square.

Problem 123.

Find numbers that are equal to a subset of their base k representation in another base. For example, $abcde_p = bcde_q$.

Problem 124.

Equilateral triangles $A_1A_2A_3$ and $B_1B_2B_3$ have sides of lengths a and b respectively. How should they be situated in the plane so that $\sum A_iB_i$ is minimum? How should they be situated in space to achieve a minimum?

Problem 125.

Two lines are *nearly parallel* if they are parallel or meet in an angle less than ϵ. Find the largest ϵ such that there exists a hexagon with no two sides being nearly parallel.

Problem 126.

What is the smallest possible area of a quadrilateral with sides a, b, c, and d. The quadrilateral need not be convex. Generalize to n-gons.

Problem 127.

Let $\prec$ be a relation on R^+ such that for all x, y, and z,

(i) $x \prec x$
(ii) $x \prec y,\ y \prec x \Rightarrow x = y$
(iii) $x \prec y,\ y \prec z \Rightarrow x \prec z$

and

(iv) $\sqrt{xy} \prec \frac{x+y}{2}$.

Must "$\prec$" be the usual "$\leq$"?

Problem 128.

Characterize those n for which the first n perfect squares can be partitioned into two sets with the same sum. For example, $1+4+16+49 = 9+25+36$. Same question for cubes, p-th powers, Fibonacci numbers, etc.

REFERENCES

[1] Stanley Rabinowitz, "Problem 1292", *Journal of Recreational Mathematics*. **16**(1983–1984)137.

[2] Joseph Rosebbaum, "Problem E332", *American Mathematical Monthly*. **46**(1939).

Problem 129.

Investigate the center of a circle which bisects the circumferences of the excircles of a given triangle.

REFERENCE

[1] Howard Eves, *A Survey of Geometry, revised edition*. Allyn and Bacon, Inc. Boston, MA: 1972, page 96, problem 7.

Problem 130.

Let Z be a fixed point on side AB of $\triangle ABC$. Let X be a variable point on side AC and let Y be a variable point on side BC. Point X moves on AC toward C at k m.p.h. What speed must Y move at such that the area of $\triangle XYZ$ remains constant? You can specify where X and Y start. [probably can't be done.]

Problem 131.

Can you always pass a polygon through itself?

Problem 132.

Which non-simple n-gon whose vertices are the same as the vertices of a regular n-gon has maximum perimeter?

Problem 133.

Does the Diophantine equation $x^2+y^2 = nz^2$ have infinitely many integral solutions for each n?

Problem 134.

For each $n > 1$, does there exist an $n \times n$ matrix of distinct positive integers such that the determinant of the matrix is k for any $k \in Z$?

Problem 135.

Can you label the edges of a cube with integers from 1 to 12 so that if edges labelled i and j meet, then $|i - j| > 2$? Can you change "2" to "3"?

Problem 136.

What point of a triangle has the Euler Line of the triangle as its Simson line?

Problem 137.

What is the envelope of all Simson lines of a triangle?

Problem 138.

Is there a Heronian triangle with one angle twice another?

REFERENCE

[1] Stanley Rabinowitz, "Problem 1193", *Crux Mathematicorum.* **12**(1986)282.

Problem 139.

Is there a nice recurrence for F_{2n} in a general second-order linear recurrence?

Problem 140.

For Fibonacci and Lucas numbers, we have $F_{2n} = F_n L_n$. If U_n is the n-th term of a general second-order linear recurrence, is there an analog to the above equation?

Problem 141.

Let P be a point inside an acute triangle. Drop perpendiculars from P to the sides of the triangle. The feet of the perpendiculars divide the sides into 6 segments of lengths (in order) a, x, b, y, c, z. It follows from [1], that for an equilateral triangle, we have $a + b + c = x + y + z$. For an arbitrary triangle, if $a + b + c = x + y + z$ for all choices of P, must the triangle be equilateral? What about for one choice of P? What can be said about the triangle in that case? For how many different P must the equation be true in order to guarantee that the triangle is equilateral?

REFERENCE

[1] Stanley Rabinowitz, "Solution to Problem 1518: Perpendicular Sums Problem (proposed by Daniel Scher)", *Journal of Recreational Mathematics.* **19**(1987)313–314.

Problem 142.

Let n be a positive integer. Let A and B be real numbers such that $0 < A < B$. Is it always possible to transform an $n \times n$ matrix of real numbers by a finite number, k, of elementary row and column operations such that each element of the transformed matrix lies between A and B inclusive? Can one find a bound on k?

Problem 143.

What is the area of the largest triangle that can be inscribed in a given ellipse?

Problem 144.

Three points move around a unit circle at constant speeds a, b, and c. They all initially start at the same point. Find the area of the maximum triangle that they form.

Problem 145.

Let ABC be a fixed triangle. Find the locus of all points P such that $PA + PB + PC$ is a constant. Let the lengths of the perpendiculars dropped from P to the sides of the triangle be x, y, z. Find the locus of points P such that $x + y + z$ is a constant.

Problem 146.

Make up a problem like: "Let ABC be a triangle with centroid M and orthocenter H. If $AH + BH + CH = AM + BM + CM$, prove that the triangle is equilateral." What special points must be chosen for M and H?

Problem 147.

Which diagonals of a Platonic solid unexpectedly concur? Answer the same question in E^n for $n > 3$.

Problem 148.

Given two triangulizations by diagonals of a convex polygon. Must the fields generated by the angles (or the tangents of the angles) be the same?

Problem 149.

What is the smallest non-right Heronian triangle in the plane? (A *Heronian triangle* is a triangle with integer sides and integer area.)

Problem 150.

Let

$$S_n = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^n & b^n & c^n \end{vmatrix}.$$

Factor or simplify S_n. Find some nice relationship between the S_i.

Problem 151.

Find Heronian simplices in 4-space. (A *Heronian simplex* is a simplex with integer edges and volume.)

Problem 152.

Find all lattice polygons with a 120° angle.

Problem 153.

Find all numbers such that its decimal representation is a permutation of its representation in some other base. It would be nice if all the digits were distinct. Some examples: $13_{10} = 31_4$, $23_{10} = 32_7$, $46_{10} = 64_7$, $43_{10} = 34_{13}$, $86_{10} = 68_{13}$, $53_{10} = 35_{16}$, $21_{10} = 12_{19}$, $42_{10} = 24_{19}$, $63_{10} = 36_{19}$, $84_{10} = 48_{19}$, $73_{10} = 37_{22}$, $83_{10} = 38_{25}$, $31_{10} = 13_{28}$, $62_{10} = 26_{28}$, and $93_{10} = 39_{28}$.

Problem 154.

Two curves are said to be *equidistant* (of length a) from a point P if whenever X is on one curve, PX intersects the other curve at a unique point, Y, and $XY = a$. Prove that no two curves can be equidistant in two different ways.

Problem 155.

How many integral solutions are there to $3^n + 4^n = 5^n$? Ask the same question about similar examples. Note that $3^n + 6^n = 2^n$ has the solution $n = -1$.

Problem 156.

How many squares are there whose vertices are all lattice points with coordinates less than or equal to n in absolute value?

Problem 157.

Can polynomials $P(x)$ and $P(x)+1$ both have multiple roots?

Problem 158.

Find the 3×3 fully magic square of distinct positive integers with the smallest (in magnitude) determinant. Is it

$$\begin{pmatrix} 4 & 11 & 3 \\ 5 & 6 & 7 \\ 9 & 1 & 8 \end{pmatrix}$$

with determinant 260?

Problem 159.

If each edge of a triangle has length between x and y, what can be said about the area of the triangle?

Problem 160.

Let n be a positive integer. Is there a polynomial $f(x)$ such that $f(x+k)$ has k more terms than $f(x)$ for $k = 1, 2, \ldots, n$?

Problem 161.

For which n can one form a sequence of the first n positive integers so that no two consecutive integers are adjacent and adjacent integers are relatively prime? Example: for $n = 10$ we have 5, 2, 9, 4, 7, 10, 3, 8, 1, 6.

Problem 162.

Do the following equations always have solutions in integers?

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{abc}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{abcd}$$

See problem D28 in [1] and its errata.

REFERENCES

[1] Richard K. Guy, *Unsolved Problems in Number Theory*. Springer-Verlag. New York: 1981.

[2] L. J. Mordell, "Research Problem 6", *Canadian Mathematical Bulletin*. **17**(1974)149.

Problem 163.

In what positions can an equilateral triangle be inscribed in an ellipse?

Problem 164.

Find the condition so that the three points of inflection of $y = x^4 + ax^3 + bx^2 + cx + d$ colline. [I don't think it can happen.]

Problem 165.

Let P be a point inside $\triangle ABC$. It is known that if P is the orthocenter, then the Euler lines of triangles PAB, PBC, PCA concur. Find all P for which this is true.

Problem 166.

Find a simple formula for the n-th non-square integer. (Answer: $n + \left\lfloor \sqrt{n + \lfloor \sqrt{n} \rfloor} \right\rfloor$. See [1].)

REFERENCES

[1] H. W. Gould, "Non-Fibonacci Numbers", *Fibonacci Quarterly*. **3**(1965)177–183.

[2] J. Lambek and L. Moser, "Inverse and Complementary Sequences of Natural Numbers", *American Mathematical Monthly*. **61**(1954)454–458.

[3] H. W. Gould, "Problem H-1", *Fibonacci Quarterly*. **1**(1963)46.

Problem 167.

Find a Heronian triangle with

(a) r, a, b, c in arithmetic progression

(b) a, b, c, K in arithmetic progression

(c) a, b, c, R in arithemtic progression

(d) a, b, c in geometric progression.

Problem 168.

Find a Heronian triangle with the length of each side being a perfect square. A triangular number.

Problem 169.

A triangle with integer sides has one side of length n. What is the smallest the area can be?

Problem 170.

Find the smallest n such that some three vertices of a unit n-cube form a triangle with rational area.

Problem 171.

Given a triangle with sides of lengths a, b, c, with $a > b > c$. Prove that this triangle can be covered by an equilateral triangle of side a.

Problem 172.

Given a point P in space outside the plane of a circle, find the maximum value of $\angle APB$ where A and B lie on the circle.

Problem 173.

Find the condition so that two circles in 3-space interlock.

Problem 174.

Is every polynomial the sum of k squares in $Z[x]$?

Problem 175.

What is the sum of the entries of the Farey sequence of order n?

Problem 176.

Find a matrix, M, such that $M \uparrow 2 = M \uparrow\uparrow 2$, i.e. the square of the matrix results in a matrix whose elements are the squares of the original elements.

Problem 177.

Let $M(n)$ be the $n \times n$ matrix whose elements are the integers from 0 to n^2-1, entered consecutively, left to right, top to bottom. Let $P(n)$ be the permanent of this matrix. Is there a simple formula for $P(n)$? Some values are:

$$\begin{aligned}
P(2) &= 2\\
P(3) &= 144\\
P(4) &= 22432\\
P(5) &= 6778800\\
P(6) &= 3563641440\\
P(7) &= 3001327741440\\
P(8) &= 3805251844939776\\
P(9) &= 6921278637502113536\\
P(10) &= 17382962774600072036352
\end{aligned}$$

Problem 178.

When is $2^k > \binom{n}{k}$?

REFERENCE

[1] Stanley Rabinowitz, "Problem to appear", *American Mathematical Monthly*. **96**(1989)??.

Problem 179.

Are all members of the set $\{\binom{n}{k} - 2^k\}$ for $k = 0, 1, \ldots, n$ distinct for a given n?

Problem 180.

Given n lines in general position in the plane. Can all bounded regions have the same number of sides? Can all bounded regions have the same area?

Problem 181.

Given four concurrent lines [parallel lines], find the condition so that there exists a square with one vertex on each line.

Problem 182.

What is the smallest number of vertices of a polyhedron with k holes?

Problem 183.

If the perimeter of a quadrilateral is fixed, what is the minimum and maximum values for the sum of the diagonals?

REFERENCE

[1] Murray S. Klamkin, "Diamond Inequalities", *Mathematics Magazine*. **50**(1977)96.

Problem 184.

When can you inscribe a regular hexagon in an ellipse? Regular octagon?

Problem 185.

Are the roots of $x^2 + ax + b$ in GF(2) expressible in terms of radicals? Note that the characteristic of the field is 2.

Problem 186.

Find a primitive root in $\{a + bi | a, b \in Z_p\}$.

Problem 187.

Find the smallest matrix with integer elements whose 100th power is 0 (but not its 99th power).

Problem 188.

Find a matrix whose elements are all Egyptian fractions (of the form $\frac{1}{k}$) whose square consists only of integer entries.

Problem 189.

Let $N = pq$. Can the set $\{1, 2, \ldots, N\}$ be partitioned into p sets of q elements each so that the sum of the elements in each set are incongruent modulo p?

Problem 190.

Let ABC be an equilateral triangle. Let P be a variable point in the plane and let $PA = a$, $PB = b$, $PC = c$. Find the locus of point P so that $ab + bc + ca$ is a constant.

Problem 191.

Place the integers from 1 to $3n$ on the vertices of a regular $3n$-gon. Prove that some three numbers form an equilateral triangle and have some nice property (such as their sum is divisible by 3).

Problem 192.

How many vertex triangles of a regular octahedron (dodecahedron, etc.) are equilateral triangles? How many equivalence classes of shapes are there?

Problem 193.

Can n straight lines divide a circle up into $2^n - 1$ parts of the same area?

Problem 194.

Is there a simple formula for

$$\sum_{k=1}^{n} \binom{n}{k} \frac{1}{k}?$$

Problem 195.

Evaluate

$$\int_0^1 x^m (x+1)^n \, dx.$$

Problem 196.

Does $x^2 + y^2 + z^2$ factor over $Z_p[x, y, z]$ for any p?

REFERENCE

[1] Stanley Rabinowitz, "Problem N-3", *AMATYC Review.* **9**(1988)71.

Problem 197.

From [1], page 34 we have

$$abc + (b+c)(c+a)(a+b) \equiv (a+b+c)(bc+ca+ab).$$

Does this identity generalize?

REFERENCE

[1] J. C. Burkhill and H. M. Cundy, *Mathematical Scholarship Problems.* Cambridge University Press. Cambridge: 1962.

Problem 198.

If p and q are positive integers less than or equal to n, which polynomial, $x^p - x^q + 1$, has the largest root?

Problem 199.

Each of n people, A_1 to A_n, announce "I am A_j." Can their identities be determined if precisely k are lying? If not, what additional constraints need we put on this problem so that one can determine the identity of each person (in the presence of some number of liars)? Suppose, instead, that the n people are arranged in a circle and each man also makes the statement "The man to my right is A_k". Analyze this case too.

Problem 200.

Let

$$D_n = \begin{vmatrix} a^n & a^2 & a & 1 \\ b^n & b^2 & b & 1 \\ c^n & c^2 & c & 1 \\ d^n & d^2 & d & 1 \end{vmatrix}.$$

Prove that $D_4 = D_3(a+b+c+d)$. Does this relationship hold when we generalize to the appropriate 5×5 determinant? Also, investigate D_5, D_6, etc.

REFERENCE

[1] J. Heading, *Mathematics Problem Book 1: Algebra and Complex Numbers.* Pergamon Press. Oxford: 1964, page 15, problem 55.

Problem 201.

Suppose I pick a real number that is the root of a polynomial with "small" real coefficients and give you as many decimal places as you want for it. Devise an algorithm for finding the polynomial.

Problem 202.

Suppose I pick a real number that is a "simple" function of the integers, π, and e, using simple operators such as $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, etc. For example, $\frac{\pi}{7-\sqrt{2}}$. I will give you as many decimal places as you want for this number. Devise an algorithm that will determine the number.

Problem 203.

Let P be a variable point in the plane of a fixed ellipse. Let m and M be the minimum and maximum distance from P to the ellipse, respectively. Find a non-constant function $f(x, y)$, such that $f(m, M)$ is invariant as P varies over all points outside the ellipse.

Problem 204.

Factor

$$\begin{vmatrix} (a+x)^n & (a+y)^n & (a+z)^n \\ (b+x)^n & (b+y)^n & (b+z)^n \\ (c+x)^n & (c+y)^n & (c+z)^n \end{vmatrix}.$$

Problem 205.

Consider an expression of the form

$$\frac{1}{x + \frac{1}{x + \frac{1}{x + \frac{1}{f(x)}}}}$$

where there are n x's in the finite continued fraction. Investigate the form that $f(x)$ must take so that this expression is a constant. For example, when $n = 1$, $f(x)$ must be of the form $\frac{c}{1-cx}$. When $n = 2$, $f(x)$ must be of the form $\frac{cx-1}{x-cx^2-c}$. Ask the same question for

$$\frac{1}{1 + \frac{1}{x + \frac{1}{x^2 + \frac{1}{x^3 + \frac{1}{f(x)}}}}}.$$

Problem 206.

Evaluate

$$y_n = \cfrac{1}{1+\cfrac{1}{x+\cfrac{1}{x^2+\cfrac{1}{x^3+\cdots\frac{1}{x^n}}}}} \equiv \sum_{i=0}^{n} \frac{1}{x^i+}.$$

Also find the limit as $n \to \infty$.

Problem 207.

Evaluate the finite continued fraction

$$\sum_{i=0}^{n} \frac{1}{\binom{n}{i}+}.$$

Problem 208.

Find the average distance from a point to a circle.

Problem 209.

Let P be a fixed point in the plane of the parabola $y = x^2$. As Q varies on this parabola, find the average value for $\frac{1}{PQ}$.

Problem 210.

The altitudes of a triangle are 3, 4, and 5. Find the area of the triangle.

Problem 211.

Each vertex of a triangle is reflected about the opposite side to get three new points with coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the three vertices. [It is known that the original triangle is not constructible with Euclidean tools from the three reflection points.]

Problem 212.

Let P be a variable point inside an ellipse with equation $x^2/a^2 + y^2/b^2 = 1$. Through P draw two chords with slopes b/a and $-b/a$ respectively. The point P divides these two chords into four pieces of lengths w, x, y, z. Prove that $w^2 + x^2 + y^2 + z^2$ is independent of the location of P and in fact has the value $2(a^2 + b^2)$. Generalize to higher dimensions.

Problem 213.

Let P be an arbitrary point inside an ellipse. Draw two chords AY and BX through P. If $\angle AXB = \angle AYB$, prove that the bisector of $\angle APB$ is parallel to the major axis of the ellipse. Equivalently, the slope of AY plus the slope of BX is 0.

Problem 214.

In triangle ABC, inscribe a square with one base on side BC. Let X be the midpoint of the base of the square. Similarly, locate points Y and Z on CA and AB respectively. Prove that AX, BY, CZ concur. Do they concur at any notable point? If X', Y', Z' are the centers of the three squares, prove also that AX', BY', CZ' concur.

Problem 215.

Let $P(x)$ be a polynomial such that $P(k) = 2^k$ for $k = 1, 2, \ldots, n$. What can be said about $P(0)$ or $P(n+1)$?

Problem 216.

Let F_n denote the n-th Fibonacci number. Show that if $F_a + F_b + F_c = 3F_d$, then $\{a, b, c; d\} = \{n, n+2, n+3; n+2\}$ or $\{n, n+1, n+6; n+4\}$ for some integer n.

Problem 217.

I conjecture (via Macsyma) that if

$$x_1^n + x_2^n + \cdots + x_n^n + kx_1x_2\cdots x_n$$

factors, then either $n = 2$, $k = \pm 2$; or $n = 3$, $k = -3$.

Problem 218.

Given $\triangle ABC$, construct three circular arcs outwardly on the sides so that the resulting curve is smooth. Prove that this curve must be a circle. Ask the same question for an n-gon. How about if we replace circular arcs by elliptical arcs?