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When Triangle Centers Lie Inside the Triangle

Stanley Rabinowitz
545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com/

Abstract. We use a computer to determine which Kimberling triangle centers X_1 through X_{100} must lie inside the triangle. We also determine which ones must lie outside the triangle. For those triangle centers that are sometimes inside the triangle and sometimes outside, we find conditions on the triangle so that these centers must lie inside the triangle.

Keywords. triangle geometry, triangle centers, computer-discovered mathematics.

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1. Introduction

Let X_n denote the nth named triangle center as cataloged in the Encyclopedia of Triangle Centers [1].

It is obvious that the incenter X_1 and the centroid X_2 always lie inside the triangle. The circumcenter X_3 and the orthocenter X_4 sometimes lie inside the triangle and sometimes lie outside the triangle, depending on the shape of the triangle.

In this paper, we use a computer to study triangle centers X_1 through X_{100} to determine whether they must lie inside, on the perimeter of, or outside the triangle. For those triangle centers that are sometimes inside the triangle and sometimes outside, we find conditions on the triangle so that these centers must lie inside the triangle.

2. Methodology

The barycentric coordinates for triangle centers are given in [1] in terms of the sides of the triangle, a, b, and c, and trigonometric functions of the angles of the triangle A, B, and C.

If the barycentric coordinates for a point P are $(u : v : w)$, then P is strictly outside $\triangle ABC$ if $uv < 0$, $vw < 0$, or $wu < 0$. We use the following Mathematica code to determine if P must always lie inside the triangle or on the boundary of the triangle. Here, `x[i]`, has been preloaded with the barycentric coordinates for the center X_i, expressed in terms of a, b, and c only.

```
insideOrOnTriangle = {};
For[i = 1, i <= 100, i++,
  {u, v, w} = x[i];
  triang = a > 0 && b > 0 && c > 0 &&
           a + b > c && b + c > a && c + a > b;
  inst = FindInstance[(u*v < 0 || v*w < 0 || w*u < 0) &&
         triang, {a, b, c}];
  If[inst == {}, AppendTo[insideOrOnTriangle, i]];
  ];
Print["Inside or on: ", insideOrOnTriangle];
```

This code works by looking for an instance when the point lies outside the triangle. If no such instance is found, then the point must be inside the triangle or on the boundary of the triangle.

Changing "<" to "<=" in the `FindInstance` call, gives those points that must lie strictly inside the triangle.

The point P is strictly inside $\triangle ABC$ if $u > 0$ and $v > 0$ and $w > 0$ or if $u < 0$ and $v < 0$ and $w < 0$. Using this fact and similar Mathematica code gives those points that must lie outside the triangle or on the boundary of the triangle. Changing the inequalities to strict inequalities then gives those points that must lie strictly outside the triangle

3. Results

We obtained the following results.

Theorem 1 (Points Inside the Triangle). *The Kimberling centers X_n must lie strictly inside the triangle for the following values of n.*

$$1, 2, 6, 7, 8, 9, 10, 12, 21, 31, 32, 37, 38, 39, 41, 42, 45,$$

$$55, 56, 57, 58, 60, 65, 75, 76, 81, 82, 83, 85, 86, 89$$

Theorem 2 (Points Inside or on the Triangle). *Excluding the centers noted in Theorem 1, the Kimberling centers X_n must lie inside the triangle or on the boundary of the triangle for the following values of n.*

$$11, 59$$

These points lie on the boundary of the triangle if and only if the triangle is isosceles. If $AB = AC$, then X_{11} coincides with the midpoint of BC and X_{59} coincides with A.

Theorem 3 (Points Outside the Triangle). *The Kimberling centers X_n must lie strictly outside the triangle for the following values of n.*

$$16, 23, 30, 36, 44$$

Theorem 4 (Points Outside or on the Triangle)**.** *Excluding the centers noted in Theorem 3, the Kimberling centers X_n must lie outside the triangle or on the boundary of the triangle for the following values of n.*

$$14, 67, 74, 80, 88, 98, 99, 100$$

4. Conditions for when centers must lie inside the triangle

The following two results are well-known.

Theorem 5 (location of circumcenter, X_3)**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the circumcenter of $\triangle ABC$ lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $90°$. If ABC is a right triangle, then the circumcenter coincides with the midpoint of the hypotenuse.*

Theorem 6 (location of orthocenter, X_4)**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the orthocenter of $\triangle ABC$ lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $90°$. If ABC is a right triangle, then the orthocenter coincides with the vertex of the right angle.*

We used Mathematica to find similar conditions determining when a triangle center must lie inside, on, or outside the triangle.

The following code fragment determines the condition for center X_i to lie inside $\triangle ABC$ when angle A is the largest angle of the triangle.

```
{u, v, w} = x[i] ;
red = Reduce[((u >= 0 && v >= 0 && w >= 0) ||
              (u <= 0 && v <= 0 && w <= 0)) &&
              triang && a > b && a > c, {a, b, c}];
Simplify[red, triang && a > b && a > c]
```

Resulting algebraic conditions involving a^2, b^2, and c^2 can be turned into angle measurement conditions using the Law of Cosines,

$$\cos A = \frac{a^2 - b^2 - c^2}{2bc}.$$

The resulting conditions are often very complicated, so we present below results where the condition is reasonably simple.

Theorem 7 (location of first Fermat point, X_{13})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the first Fermat point of $\triangle ABC$ lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $120°$. If $\angle A = 120°$, then the first Fermat point coincides with vertex A.*

Theorem 8 (location of second Fermat point, X_{14})**.** *Let ABC be a triangle. Then the second Fermat point of $\triangle ABC$ coincides with a vertex of the triangle when the measure of the angle at that vertex is $60°$. If no angle of the triangle has measure $60°$, then the second Fermat point lies outside the triangle.*

Theorem 9 (location of first isodynamic point, X_{15})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the first isodynamic point of $\triangle ABC$ lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $120°$. If $\angle A = 120°$, then the first isodynamic point lies on side BC at the point where the angle bisector of angle A meets BC.*

Theorem 10 (location of first Napoleon point, X_{17})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the first Napoleon point of $\triangle ABC$ lies inside/on/outside the triangle depending on whether the measure of angle A is greater than/equal to/less than $150°$. If $\angle A = 150°$, then the first Napoleon point coincides with vertex A.*

Theorem 11 (location of Clawson point, X_{19})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the Clawson point of $\triangle ABC$ lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $90°$. If $\angle A = 90°$, then the Clawson point coincides with vertex A.*

Theorem 12 (location of Exeter point, X_{22})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the Exeter point of $\triangle ABC$ lies inside/on/outside the triangle depending on whether a^4 is less than/equal to/greater than $b^4 + c^4$.*

Theorem 13. *Let ABC be a triangle, named so that angle A is the largest angle. Let P be the X_n point of $\triangle ABC$ for n =25, 27, 28, 29, 33, 34, or 92. Then P lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $90°$. If $\angle A = 90°$, then P lies on side BC.*

Theorem 14. *Let ABC be a triangle, named so that angle A is the largest angle. Let P be the X_n point of $\triangle ABC$ for n =48, 63, 69, 71, 72, 73, 77, or 78. Then P lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $90°$. If $\angle A = 90°$, then P coincides with vertex A.*

Theorem 15 (location of X_{35})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the X_{35} point lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $120°$. If $\angle A = 120°$, then the X_{35} point lies on side BC.*

Theorem 16 (location of X_{50})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the X_{50} point lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $120°$. If $\angle A = 120°$, then the X_{50} point lies on side BC.*

Theorem 17 (location of X_{66})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the X_{66} point of $\triangle ABC$ lies inside/on/outside the triangle depending on whether a^4 is less than/equal to/greater than $b^4 + c^4$.*

Theorem 18 (location of X_{79})**.** *Let ABC be a triangle, named so that angle A is the largest angle. Then the X_{79} point lies inside/on/outside the triangle depending on whether the measure of angle A is less than/equal to/greater than $120°$. If $\angle A = 120°$, then the X_{79} point coincides with vertex A.*

Theorem 19 (location of X_{87})**.** *Let ABC be a triangle. Then the X_{87} point of $\triangle ABC$ lies inside/outside the triangle depending on whether $\{1/a, 1/b, 1/c\}$ forms/does not form the lengths of a triangle. If the lengths $\{1/a, 1/b, 1/c\}$ form a degenerate triangle, then the X_{87} point coincides with a vertex of $\triangle ABC$.*

Theorem 20 (location of X_{88})**.** *Let ABC be a triangle. Then the X_{88} point of $\triangle ABC$ coincides with a vertex if and only if the sides of the triangle are in arithmetic progression. In all other cases, the X_{88} point lies outside the triangle.*

Theorem 21 (location of Steiner point, X_{99})**.** *Let ABC be a triangle. Then the Steiner point of $\triangle ABC$ always lies outside $\triangle ABC$ unless $\triangle ABC$ is isosceles. If $AB = AC$, then the Steiner point coincides with vertex A. (The Steiner point is not defined if $\triangle ABC$ is equilateral.)*

Theorem 22 (location of X_{100})**.** *Let ABC be a triangle. Then the X_{100} point of $\triangle ABC$ always lies outside $\triangle ABC$ unless $\triangle ABC$ is isosceles. If $AB = AC$, then the X_{100} point coincides with vertex A. (The X_{100} point is not defined if $\triangle ABC$ is equilateral.)*

References

[1] Clark Kimberling, *Encyclopedia of Triangle Centers.* `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`