A Geometric Proof of the Isogonal Disc Theorem

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Abstract. Let P be any point inside $\triangle ABC$ and let Q be the isogonal conjugate of P. We give a purely geometric proof that the disc with diameter PQ must always contain I, the incenter of $\triangle ABC$.

Keywords. triangle geometry, incenter, isogonal conjugate.

Mathematics Subject Classification (2020). 51M04.

1. Introduction

In a recent note, Lukarevski [2] conjectured that for any interior point P of a triangle with isogonal conjugate Q, the incenter of the triangle must lie inside the disc with diameter PQ. Rabinowitz [4] proved the conjecture using barycentric coordinates, but the proof was unsatisfying because of its extensive computation using Mathematica. In a 2025 paper [3], it was said that "this conjecture seems to be as far from a purely synthetic geometric proof as ever." We now present a purely geometric proof.

2. Three circles separating isogonal conjugates

We begin with some lemmas.

The following result comes from [1, §238].

Lemma 1. *Let P be interior to $\triangle ABC$, and let Q be its isogonal conjugate (Figure 1). Then*

$$\angle BPC + \angle BQC = 180° + A$$

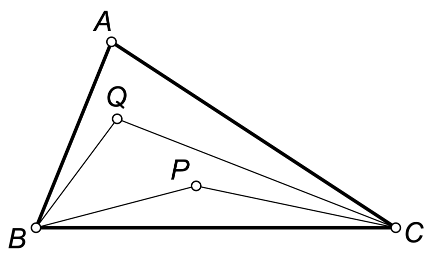

Figure 1. P and Q are isogonal conjugates

The following result is well known [1, §289].

Lemma 2. *Let I be the incenter of $\triangle ABC$ (Figure 2). Then*

$$\angle BIC = 90° + \frac{A}{2}.$$

Lemma 3. *Let P be interior to $\triangle ABC$, and let Q be its isogonal conjugate. Let I be the incenter of $\triangle ABC$. Let $\omega_A = \odot(BIC)$ (Figure 3). Then P and Q lie in opposite closed regions determined by ω_A.*

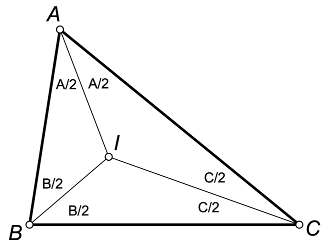

FIGURE 2. I is the incenter of $\triangle ABC$

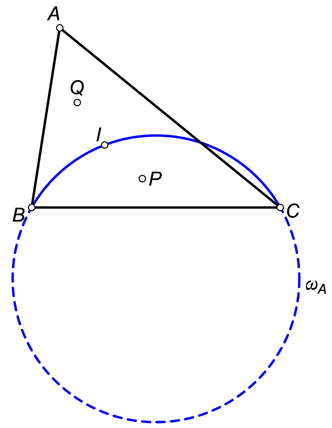

FIGURE 3. The circle ω_A separates P and Q

Proof. The isogonal conjugate of an interior point is interior, so P and Q lie on the same side of BC as A. On that side of BC, the arc BIC of ω_A is the locus of points X for which

$$\angle BXC = \angle BIC.$$

For points inside ω_A the subtended angle is larger, and for points outside it is smaller. By Lemma 1,

$$\angle BPC + \angle BQC = 180^\circ + A,$$

and by Lemma 2,

$$2\angle BIC = 180^\circ + A.$$

Thus,

$$\big(\angle BPC - \angle BIC\big) + \big(\angle BQC - \angle BIC\big) = 0.$$

Therefore, the two differences have opposite signs, unless both are zero. Hence one of P, Q is in the closed disc bounded by ω_A and the other is in the closed exterior. $\square$

The same argument applies cyclically to ω_B and ω_C.

3. INVERSION AT THE INCENTER

Lemma 4. *Let I be the incenter of $\triangle ABC$. Let $\omega_A = \odot(BIC)$, $\omega_B = \odot(CIA)$, and $\omega_C = \odot(AIB)$. Consider an inversion centered at I, with arbitrary radius. Let A', B', and C' be the inverses of A, B, and C, respectively (Figure 4). Then*

1. *The circles $\omega_A, \omega_B, \omega_C$ invert respectively to the lines $B'C'$, $C'A'$, and $A'B'$.*
2. *The point I is the orthocenter of $\triangle A'B'C'$, and this triangle is acute.*
3. *Inversion preserves every ray from I.*

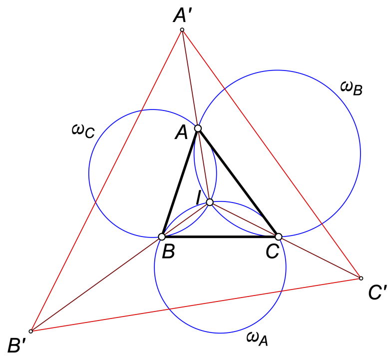

Figure 4. red lines are inverses of blue circles about incenter I

Proof. Since ω_A passes through the center of inversion, its image is a line. Since it also contains B and C, that line is $B'C'$. The other two statements in part (1) follow cyclically.

Let t be the tangent to ω_A at I (Figure 5).

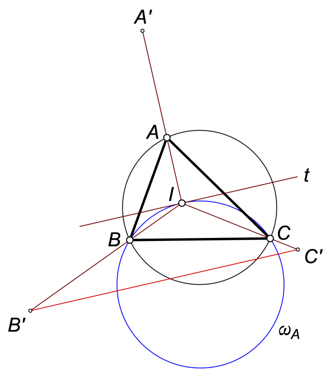

Figure 5. t is is tangent to ω_A at I

Then

$$\angle(t, IB) = \angle ICB = \frac{C}{2}.$$

But

$$\angle AIB = 90° + \frac{C}{2}$$

by Lemma 2. Hence, $t \perp AI$.

The image of a circle through the center of inversion is a line parallel to its tangent there, so $B'C' \perp AI$. Since A' lies on the ray IA, this gives $IA' \perp B'C'$. Cyclically, $IB' \perp C'A'$ and $IC' \perp A'B'$, so I is the orthocenter of $\triangle A'B'C'$.

Inversion fixes the rays from I, and therefore, by Lemma 2,

$$\angle B'IC' = \angle BIC = 90° + \frac{A}{2},$$

with the two cyclic analogues. These three angles are all less than 180° and sum to 360°. Thus, the three rays IA', IB', IC' are not contained in any closed half-plane through I, so I lies in the interior of $\triangle A'B'C'$. An interior orthocenter occurs precisely in an acute triangle. Hence $\triangle A'B'C'$ is acute, proving part (2).

Finally, inversion centered at I sends every point to another point on the same ray from I. This proves part (3). $\square$

4. A lemma for an acute triangle

Lemma 5. *Let $A_1A_2A_3$ be an acute triangle with orthocenter H, and let ℓ_i be the sideline opposite A_i. Suppose that, for each i, X and Y lie in opposite closed half-planes bounded by ℓ_i. Then*

$$\angle XHY > 90^\circ.$$

Proof. For each sideline ℓ_i, call the closed half-plane containing H the near side and the other closed half-plane the far side. Since ℓ_i weakly separates X and Y, we may classify X and Y oppositely with respect to ℓ_i: if one is on the near side, the other is on the far side. If a point lies on ℓ_i, it may be assigned to either closed half-plane. Thus the near/far patterns (N/F) of X and Y are complementary.

No point can lie on the far side of all three sidelines. Hence neither X nor Y can have the pattern (F, F, F). Since their patterns are complementary, neither can have the pattern (N, N, N). Therefore one of X, Y is on the far side of exactly one sideline and the other is on the far side of exactly two. After interchanging X and Y, we may suppose that X is on the far side of exactly one sideline. After relabelling the vertices, we may take this sideline to be ℓ_1. Thus

$$X : (F, N, N), \qquad Y : (N, F, F).$$

Consider first the ray HX. Since H and X lie on opposite sides of $\ell_1 = A_2A_3$, the segment HX meets ℓ_1. On the other hand, H and X both lie on the near sides of ℓ_2 and ℓ_3. These near half-planes are convex, so the point where HX meets ℓ_1 lies on the segment A_2A_3. Consequently the ray HX lies in the closed angular sector bounded by the rays HA_2 and HA_3.

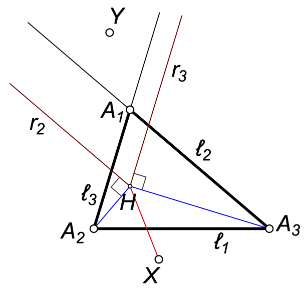

Figure 6. Two angular sectors in the proof of Lemma 5

Now consider Y. The far sides of $\ell_2 = A_1A_3$ and $\ell_3 = A_1A_2$ meet in the angle vertically opposite $\angle A_2A_1A_3$. Through H, draw rays r_2, r_3 parallel respectively to A_1A_3 and A_1A_2, directed toward the A_1-side (Figure 6). Because the triangle is acute, the altitude A_1H lies strictly inside $\angle A_2A_1A_3$. It follows that the entire vertical angle containing Y lies inside the open sector bounded by r_2 and r_3. Hence the ray HY lies strictly inside that sector.

Finally, since H is the orthocenter,

$$HA_2 \perp A_1A_3, \qquad HA_3 \perp A_1A_2.$$

Thus

$$HA_2 \perp r_2, \qquad HA_3 \perp r_3.$$

The closed sector containing HX and the open sector containing HY are therefore separated at both ends by right angles. Accordingly every ray in the first sector makes an angle at least 90° with every ray in the second, and because HY lies strictly inside the second sector, equality cannot occur. Hence $\angle XHY > 90^\circ$. $\square$

5. The Isogonal Disc Theorem

Theorem 6. *Let P and Q be distinct interior points of $\triangle ABC$, each distinct from the incenter I. Let $\omega_A = \odot(BIC)$, $\omega_B = \odot(CIA)$, and $\omega_C = \odot(AIB)$. Suppose P and Q lie in opposite closed regions determined by each of these circles. Then the incenter I lies in the disc with diameter PQ.*

Proof. Let P' and Q' be the images of P and Q through an inversion with center I and arbitrary radius. Let A', B', and C' be the images of A, B, and C, respectively, under this inversion. Under inversion, the circles ω_A, ω_B, and ω_C become the three sidelines of $\triangle A'B'C'$. Hence each sideline of $\triangle A'B'C'$ weakly separates P' from Q'.

By Lemma 4, the triangle $A'B'C'$ is acute and has orthocenter I. Lemma 5 therefore gives

$$\angle P'IQ' > 90°.$$

Since inversion centered at I preserves rays from I,

$$\angle PIQ = \angle P'IQ' > 90°.$$

A point sees a segment under an angle greater than a right angle exactly when it lies inside the circle having that segment as diameter. Hence I lies in the interior of the disc with diameter PQ. □

Theorem 7 (Isogonal Disc Theorem)**.** *Let P be a point interior to $\triangle ABC$, and let Q be its isogonal conjugate. Then the incenter I lies in the disc with diameter PQ. If $P \neq I$, then I lies in the interior of that disc.*

Proof. If $P = I$, then $Q = I$, and the assertion is immediate. Assume $P \neq I$. Since isogonal conjugation is an involution and fixes I, we also have $Q \neq I$.

Let P' and Q' be the images of P and Q through an inversion with center I and arbitrary radius. By Lemma 3, P and Q lie in opposite closed regions determined by each of the circles $\omega_A, \omega_B, \omega_C$. The result then follows by Theorem 6. □

Acknowledgments

The author gratefully acknowledges the Anthropic AI program Claude (Opus 4.8) for suggesting using an inversion about I. I also thank OpenAI's ChatGPT (version 5.5) for clarifying and simplifying some of the exposition.

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