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Proof of the Isogonal Disc Conjecture

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Abstract. Let P be any point inside $\triangle ABC$ and let Q be the isogonal conjugate of P. We prove that the disc with diameter PQ must always contain I, the incenter of $\triangle ABC$. We also find a number of other discs in which the incenter must lie.

Keywords. triangle geometry, incenter, isogonal conjugate.

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1. The Isogonal Disc Conjecture

Let G, I, and K denote the centroid, incenter, and symmedian point of a triangle. It is known [4] that the incenter must lie inside the disc with diameter GK. Since G and K are isogonal conjugates, Lukarevski [4] has conjectured that if P is any point inside a triangle and Q is the isogonal conjugate of P, then I must lie inside the disc with diameter PQ.

We will give a proof of this conjecture using barycentric coordinates and Mathematica.

Theorem 1 (The Isogonal Disc Theorem)**.** *Let P be any point inside $\triangle ABC$ and let Q be the isogonal conjugate of P. Then I, the incenter of $\triangle ABC$, must lie inside the disc with diameter PQ.*

Proof.

Let the sides of $\triangle ABC$ be a, b, and c. Let the barycentric coordinates for the point P be $(u : v : w)$. The conditions $u > 0$, $v > 0$, and $w > 0$ guarantee that P lies inside $\triangle ABC$.

We use the following well-known formulas [3].

Distance Formula. Given two points $P = (u_1 : v_1 : w_1)$ and $Q = (u_2 : v_2 : w_2)$ in normalized barycentric coordinates. Denote $x = u_1 - u_2$, $y = v_1 - v_2$, and $z = w_1 - w_2$. Then the square of the distance between P and Q is

$$-a^2yz - b^2zx - c^2xy.$$

Isogonal Conjugate. The isogonal conjugate of a point with barycentric coordinates $(u : v : w)$ is

$$(a^2vw : b^2wu : c^2uv).$$

Incenter Coordinates. The coordinates of the incenter, I, are $(a : b : c)$.

We use the following Mathematica code.

```
ptI = {a, b, c};
ptP = {u, v, w};
ptQ = isogonalConjugate[ptP];
IPsq = squaredDistance[ptI, ptP] // Simplify;
IQsq = squaredDistance[ptI, ptQ] // Simplify;
PQsq = squaredDistance[ptP, ptQ] // Simplify;
triangleCondition =
  a>0 && b>0 && c>0 && a + b > c && b + c > a && c + a > b;
interiorCondition = u > 0 && v > 0 && w > 0;
Resolve[Exists[{u, v, w, a, b, c},
  IPsq+IQsq > PQsq && triangleCondition && interiorCondition]]
```

where `squaredDistance` and `isogonalConjugate` were previously defined using the formulas above.

The computer output was `False` indicating that Mathematica was unable to find a point P satisfying the condition $|IP|^2 + |IQ|^2 > |PQ|^2$, which is the condition that point I lies outside the circle with diameter PQ.

Note that the `Resolve` function eliminates ∀ and ∃ quantifiers from an expression. According to [5], `Resolve` can in principle always eliminate quantifiers if `expr` contains only polynomial equations and inequalities over the reals.

This proves the Isogonal Disc Conjecture.

2. Other Discs Containing the Incenter

Let X_i denote the i-th named center in the Encyclopedia of Triangle Centers, [2]. Using the same Mathematica code, we varied i and j from 2 to 10 and checked if the disc with diameter X_iX_j must contain the incenter. We found the following.

Theorem 2. *The incenter of a triangle must lie inside the discs whose diameters are* X_2X_4, X_2X_6, X_2X_7, X_3X_4, X_3X_6, X_3X_7, X_4X_8, X_4X_9, X_4X_{10}, X_6X_8, X_6X_9, X_6X_{10}, X_7X_8, X_7X_9, *and* X_7X_{10}.

For similar results involving triangle centers that must lie in certain discs, see [1].

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