

Isosceles Tetrahedra Erected on the Faces of a Tetrahedron

STANLEY RABINOWITZ

545 Elm St Unit 1, Milford, New Hampshire 03055, USA

stan.rabinowitz@comcast.net

Abstract. An *isosceles tetrahedron* is a tetrahedron whose three pairs of opposite edges are equal in length. Starting with an arbitrary tetrahedron $A_1A_2A_3A_4$, we erect an isosceles tetrahedron outwardly on each face. Let B_i be the new vertex erected on the face opposite A_i , and let C_i be the centroid of that erected tetrahedron. We study three elementary questions about the lines A_iB_i and A_iC_i : where they meet the opposite faces, when the four lines meet at one point, and when the four corresponding segments have equal lengths. The projection of B_i onto the opposite face is a standard triangle center called the de Longchamps point, while the projection of C_i is the circumcenter of that face. When the original tetrahedron is isosceles, both sets of four segments have equal lengths, and both line families meet at one point exactly when the tetrahedron is regular. We also find another broad equal-length family, derive a simple criterion for equality of the four A_iC_i , and study the derived tetrahedron formed by the four points C_i .

Keywords. tetrahedron, isosceles tetrahedron, centroid, circumcenter, concurrency, de Longchamps point, dihedral angle, equal lengths, triangle centers.

Mathematics Subject Classification (2020). 51M04, 51M20.

1 Introduction

An *isosceles tetrahedron* is a tetrahedron whose three pairs of opposite edges are equal in length. Equivalently, its four faces are congruent. We use Altshiller-Court's classical term "isosceles tetrahedron" throughout [1]; see also Leech [7]. A *regular tetrahedron* is the special case in which all six edges are equal.

There is a familiar planar model for the questions in this paper. Start with a triangle ABC and erect an equilateral triangle outwardly on each of its three sides. If A', B', C' are the three new vertices, then the lines AA', BB', CC' meet at one point; this is the equilateral case of Kiepert's theorem [9]. The common point is the 1st isogonic center, X_{13} , from Kimberling's *Encyclopedia of Triangle Centers* [6]. Throughout the paper, X_n denotes Kimberling's triangle center with index n . A simple application of the cosine rule also gives

$$AA' = BB' = CC'.$$

Thus the planar construction naturally suggests asking both when the corresponding lines are concurrent and when the corresponding segments have equal lengths. Napoleon's theorem gives a further beautiful fact: the centers of the three erected equilateral triangles themselves form an equilateral triangle [4]. Figure 1 summarizes these planar features.

Thus it is natural to look both at the new vertices and at the centers of the figures that have been erected.

Our construction is a three-dimensional analog. Let $A_1A_2A_3A_4$ be a tetrahedron. On the face $A_2A_3A_4$, erect outwardly the isosceles tetrahedron $A_2A_3A_4B_1$. Thus

$$B_1A_2 = A_3A_4, \quad B_1A_3 = A_2A_4, \quad B_1A_4 = A_2A_3,$$

and B_1 is chosen on the side of the plane $A_2A_3A_4$ opposite A_1 . Define B_2, B_3, B_4 similarly.

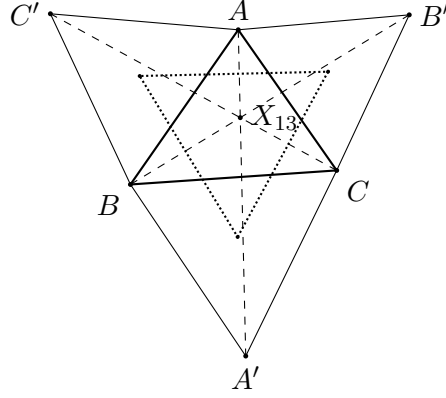


Figure 1: The planar model: AA', BB', CC' concur at X_{13} , while the centers of the three erected equilateral triangles form the dotted equilateral triangle.

For each i , let C_i be the *centroid* of the erected isosceles tetrahedron on the face opposite A_i . Thus, for example,

$$C_1 = \frac{A_2 + A_3 + A_4 + B_1}{4}.$$

As Proposition 2.4 will show, the centroid of an isosceles tetrahedron is also its circumcenter. Thus B_i and C_i are two especially natural points attached to the same erected tetrahedron. Figure 2 shows the construction on the face opposite A_1 .

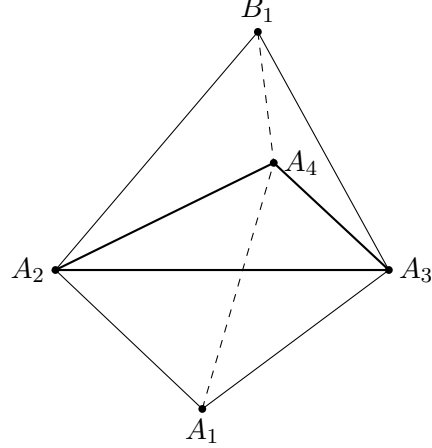


Figure 2: The basic construction.

For each of the two line families

$$A_i B_i \quad \text{and} \quad A_i C_i$$

we begin with three simple questions:

1. Where does each line meet the opposite face?
2. When are the four lines concurrent?

3. When do the four corresponding line segments have the same length?

We use the opposite-edge notation

$$\begin{aligned} A_2A_3 &= a_1, & A_3A_1 &= a_2, & A_1A_2 &= a_3, \\ A_1A_4 &= b_1, & A_2A_4 &= b_2, & A_3A_4 &= b_3. \end{aligned}$$

Thus a_i and b_i are opposite edges. This notation agrees with that used in [8].

As we shall see, an isosceles tetrahedron can be erected on a triangle as a genuine real solid exactly when the triangle is acute. A right triangle gives a flat limiting case, and an obtuse triangle gives no real Euclidean apex. Except in Section 8, we therefore assume that all four faces of the original tetrahedron are acute.

2 Erecting an isosceles tetrahedron on one face

A rectangular box gives a useful picture of the construction.

Lemma 2.1 (The box lemma). *Let XYZ be an acute triangle with side lengths*

$$x = YZ, \quad y = ZX, \quad z = XY$$

and area K . Put

$$p = \sqrt{yz \cos X}, \quad q = \sqrt{zx \cos Y}, \quad r = \sqrt{xy \cos Z}.$$

Then

$$p^2 + q^2 = z^2, \quad q^2 + r^2 = x^2, \quad r^2 + p^2 = y^2.$$

Thus X, Y, Z and the fourth vertex B of the erected isosceles tetrahedron may be taken as four alternate vertices of a rectangular box with edge lengths p, q, r . The perpendicular distance from B to the plane XYZ is

$$\frac{pqr}{K}.$$

Proof. The three displayed identities follow at once from the cosine rule. For example,

$$p^2 + q^2 = yz \cos X + zx \cos Y = \frac{y^2 + z^2 - x^2}{2} + \frac{z^2 + x^2 - y^2}{2} = z^2.$$

The other two are cyclic.

Four alternate vertices of a rectangular box with edge lengths p, q, r form an isosceles tetrahedron whose six edges are the six face diagonals of the box. Its volume is $pqr/3$. If h is the distance from B to the plane XYZ , its volume is also $Kh/3$. Hence $h = pqr/K$. $\square$

Figure 3 illustrates the rectangular-box interpretation of the preceding lemma.

The next theorem gives a direct construction of the apex.

Theorem 2.2. *Let XYZ be an acute triangle. Let O be its circumcenter (the center of its circumcircle), let H be its orthocenter (the common point of its three altitudes), and let $L = 2O - H$ be its de Longchamps point. Let R be its circumradius. If $\mathbf{n}$ is a unit normal to the plane XYZ , then the two points B satisfying*

$$BX = YZ, \quad BY = ZX, \quad BZ = XY$$

are

$$B = L \pm 4R\sqrt{\cos X \cos Y \cos Z} \mathbf{n}.$$

In particular, the orthogonal projection of B onto the base plane is the de Longchamps point L .

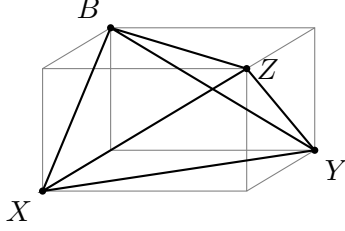


Figure 3: Four alternate vertices of a rectangular box form an isosceles tetrahedron.

Proof. Let D be the orthogonal projection of B onto the plane of XYZ . Put

$$\mathbf{d} = D - X, \quad \mathbf{u} = Y - X, \quad \mathbf{v} = Z - X.$$

Subtracting pairs of the three squared-distance equations gives

$$\mathbf{d} \cdot \mathbf{u} = |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v}, \quad \mathbf{d} \cdot \mathbf{v} = |\mathbf{v}|^2 - \mathbf{u} \cdot \mathbf{v}.$$

If $\mathbf{g} = H - X$, the altitude equations for the orthocenter give

$$\mathbf{g} \cdot \mathbf{u} = \mathbf{g} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}.$$

Hence

$$\mathbf{d} = \mathbf{u} + \mathbf{v} - 2\mathbf{g},$$

which is equivalent to

$$D = X + Y + Z - 2H.$$

Since $H = X + Y + Z - 2O$ for a triangle,

$$D = 4O - X - Y - Z = 2O - H = L.$$

By Lemma 2.1, the perpendicular height is pqr/K . Since $xyz = 4KR$,

$$\frac{pqr}{K} = \frac{xyz \sqrt{\cos X \cos Y \cos Z}}{K} = 4R \sqrt{\cos X \cos Y \cos Z}.$$

The two signs correspond to the two sides of the base plane. □

Thus the theorem is a simple recipe: reflect the orthocenter H in the circumcenter O to obtain L , and then move perpendicularly to the face by the height given above. In Kimberling's notation, the de Longchamps point is X_{20} .

Corollary 2.3. *An isosceles tetrahedron can be erected nondegenerately on a triangle if and only if the triangle is acute. For a right triangle the erection is flat, and for an obtuse triangle there is no real Euclidean erection.*

Proof. In Lemma 2.1, the three box-edge squares have the same signs as $\cos X, \cos Y, \cos Z$, respectively. They are all positive exactly when the triangle is acute, so the box construction gives a genuine three-dimensional erection in that case.

The calculation in Theorem 2.2 shows that the square of the perpendicular height is

$$16R^2 \cos X \cos Y \cos Z.$$

For a right triangle this is zero, so the fourth vertex lies in the plane of the triangle. For an obtuse triangle it is negative, so no real Euclidean apex exists. □

The calculation $D = L$ in the proof of Theorem 2.2 did not use acuteness. Thus, in the right-triangle limiting case, the flat apex itself is the de Longchamps point of the face.

The flat case is easy to visualize. For example, let XYZ be a 3–4–5 right triangle, with the right angle at X . Then the flat fourth vertex B is simply the fourth corner of the 3 by 4 rectangle. Indeed,

$$XY = BZ = 3, \quad XZ = BY = 4, \quad YZ = BX = 5,$$

so the three pairs of opposite edges are equal, but all four points lie in one plane. This flat erection is shown in Figure 4.

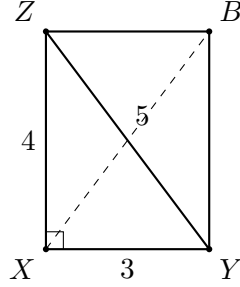


Figure 4: For a right triangle, the erected isosceles tetrahedron is flat.

Proposition 2.4 (Centroid of the erected tetrahedron). *Let C be the centroid of the isosceles tetrahedron $XYZB$ erected on an acute triangle XYZ . Then C is also the circumcenter of $XYZB$. If O is the circumcenter of XYZ , then the orthogonal projection of C onto the plane XYZ is O .*

If the perpendicular distance from B to the plane XYZ is e , then the perpendicular distance from C to the plane is $e/4$.

Proof. In the box construction of Lemma 2.1, the four vertices X, Y, Z, B are alternate vertices of a rectangular box. Their centroid is the center of the box, and that point is equally distant from all four vertices. Hence it is also the circumcenter of the erected isosceles tetrahedron.

There is also a useful direct check of the projection. By Theorem 2.2, the projection of B is $L = 2O - H$. Since

$$X + Y + Z = H + 2O,$$

the projection of

$$C = \frac{X + Y + Z + B}{4}$$

is

$$\frac{X + Y + Z + L}{4} = \frac{H + 2O + 2O - H}{4} = O.$$

Only the point B has a nonzero perpendicular component, so the distance of C from the face is one fourth the distance of B . $\square$

If G denotes the centroid of the original tetrahedron $A_1A_2A_3A_4$, then the definition gives the useful identity

$$C_i = G + \frac{B_i - A_i}{4}. \tag{1}$$

Thus C_i can be obtained from the original centroid and the already constructed point B_i . Taking lengths from G gives the particularly simple relation

$$GC_i = \frac{1}{4}A_iB_i.$$

Consequently, the four segments $A_i B_i$ have the same length exactly when the four points C_i are equidistant from G .

3 Where the lines meet the opposite faces

Let Π_i be the plane of the face opposite A_i . We shall use three familiar points in or near that face:

- P_i , the perpendicular projection of A_i onto Π_i ;
- L_i , the de Longchamps point X_{20} of the face;
- O_i , the circumcenter X_3 of the face.

By Theorem 2.2, L_i is the perpendicular projection of B_i . By Proposition 2.4, O_i is the perpendicular projection of C_i .

Since the erected tetrahedron is outward, both B_i and C_i lie on the side of Π_i opposite A_i . Hence both lines meet Π_i in unique points. Define

$$D_i = A_i B_i \cap \Pi_i, \quad E_i = A_i C_i \cap \Pi_i.$$

Proposition 3.1. *Let*

$$h_i = A_i P_i, \quad e_i = B_i L_i$$

be the perpendicular distances of A_i and B_i from the opposite face. Then

$$D_i = \frac{e_i P_i + h_i L_i}{h_i + e_i}$$

and

$$E_i = \frac{e_i P_i + 4h_i O_i}{e_i + 4h_i}.$$

Equivalently,

$$P_i D_i : D_i L_i = h_i : e_i, \quad P_i E_i : E_i O_i = 4h_i : e_i.$$

Proof. Choose a unit normal $\mathbf{n}$ to Π_i pointing toward A_i . Then

$$A_i = P_i + h_i \mathbf{n}, \quad B_i = L_i - e_i \mathbf{n}.$$

This gives the formula for D_i by taking the point of $A_i B_i$ whose normal component is zero.

Proposition 2.4 gives

$$C_i = O_i - \frac{e_i}{4} \mathbf{n}.$$

The identical calculation for the line $A_i C_i$ gives

$$E_i = \frac{(e_i/4)P_i + h_i O_i}{h_i + e_i/4} = \frac{e_i P_i + 4h_i O_i}{e_i + 4h_i}.$$

The two division ratios follow at once. □

At first sight one might hope that D_i or E_i is some fixed triangle center of the opposite face. There is a simple reason this cannot be so. Fix an equilateral face $A_2A_3A_4$, and choose A_1 so that its perpendicular projection P_1 is not the center of that face. Since the de Longchamps point and circumcenter of an equilateral triangle are both its center, Proposition 3.1 places both D_1 and E_1 strictly between P_1 and the center. Hence neither point is the center of the equilateral triangle. But every triangle center defined for an equilateral triangle must be fixed by its 120° rotational symmetry and therefore must be the center. Thus neither D_i nor E_i is a fixed triangle center for arbitrary tetrahedra.

For a concrete example, one may take

$$A_2 = (0, 0, 0), \quad A_3 = (1, 0, 0), \quad A_4 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \quad A_1 = \left(\frac{1}{4}, \frac{3}{10}, 1\right).$$

The isosceles tetrahedron case is much more attractive.

4 When the original tetrahedron is isosceles

A line through a vertex of a tetrahedron and perpendicular to the opposite face is called an *altitude*. The first positive case has a particularly simple explanation.

Theorem 4.1. *If $A_1A_2A_3A_4$ is an isosceles tetrahedron, then B_i is the reflection of A_i in its opposite face. Consequently, A_iB_i is the altitude from A_i , and $D_i = L_i = X_{20}$.*

Proof. Consider the face $A_2A_3A_4$. Since opposite edges of the original tetrahedron are equal,

$$A_1A_2 = A_3A_4, \quad A_1A_3 = A_2A_4, \quad A_1A_4 = A_2A_3.$$

Thus A_1 itself satisfies the three distance equations that define the two possible apices erected on $A_2A_3A_4$. The other solution is its reflection in that face. Since the erection is outward, that reflected point is B_1 . The same argument applies cyclically.

The line A_iB_i is therefore perpendicular to the opposite face. By Theorem 2.2, the perpendicular projection of B_i is the de Longchamps point L_i of that face. Hence $D_i = L_i = X_{20}$. $\square$

Theorem 4.2. *If $A_1A_2A_3A_4$ is an isosceles tetrahedron, then*

$$A_1B_1 = A_2B_2 = A_3B_3 = A_4B_4.$$

Proof. By Theorem 4.1, B_i is the reflection of A_i in the opposite face. Thus A_iB_i is twice the altitude from A_i . The four faces of an isosceles tetrahedron are congruent, so they have the same area, say K . If V is the volume and h_i is the altitude from A_i , then

$$V = \frac{1}{3}Kh_i.$$

Hence all four altitudes h_i are equal, and therefore all four segments A_iB_i are equal. $\square$

For a triangle, the *Euler line* is the line through its circumcenter and orthocenter.

Proposition 4.3. *If the original tetrahedron is isosceles, then*

$$E_i = \frac{L_i + 4O_i}{5} = \frac{6O_i - H_i}{5},$$

where H_i is the orthocenter of the face opposite A_i .

Proof. In the isosceles tetrahedron case, B_i is the reflection of A_i in the opposite face. Hence $h_i = e_i$, and the projection P_i of A_i is the same point as the projection L_i of B_i . Proposition 3.1 therefore gives

$$E_i = \frac{L_i + 4O_i}{5}.$$

Since $L_i = 2O_i - H_i$, the second form follows. $\square$

Thus E_i lies on the Euler line of the opposite face, one fifth of the way from O_i to L_i .

Corollary 4.4. *In an isosceles tetrahedron, the point E_i is the triangle center X_{3522} of the opposite face.*

Proof. Proposition 4.3 gives

$$E_i = \frac{L_i + 4O_i}{5}.$$

In Kimberling's notation, $L_i = X_{20}$ and $O_i = X_3$. The corresponding ETC relation is

$$X_{3522} = \frac{X_{20} + 4X_3}{5}.$$

Hence $E_i = X_{3522}$ [6]. $\square$

The converse statement for the A_iB_i intersections is also true, and in fact only two of the four faces are needed.

Theorem 4.5. *Suppose the four faces of $A_1A_2A_3A_4$ are acute. Then the following are equivalent:*

1. *the tetrahedron is isosceles;*
2. *for every i , the line A_iB_i meets the opposite face at its de Longchamps point X_{20} .*

Moreover, in (2) it is enough that the property hold for any two distinct values of i .

Proof. The implication (1) $\Rightarrow$ (2) is Theorem 4.1. For the converse, it is enough after relabeling to assume

$$D_1 = L_1, \quad D_2 = L_2.$$

Put

$$d_i = a_i^2 - b_i^2 \quad (i = 1, 2, 3).$$

Because $D_1 = L_1$, the line A_1B_1 is the perpendicular through L_1 . Therefore differences of squared distances from A_1 to vertices of the opposite face are the same as the corresponding differences for B_1 . Comparing A_2 with A_3 gives

$$a_3^2 - a_2^2 = b_3^2 - b_2^2,$$

so $d_3 = d_2$. Comparing A_2 with A_4 gives

$$a_3^2 - b_1^2 = b_3^2 - a_1^2,$$

so $d_3 = -d_1$. Hence

$$d_2 = d_3 = -d_1.$$

Similarly, $D_2 = L_2$ gives

$$d_1 = d_3 = -d_2.$$

The two relations force

$$d_1 = d_2 = d_3 = 0.$$

Thus $a_i = b_i$ for $i = 1, 2, 3$, which is exactly the condition that the tetrahedron be isosceles. $\square$

Thus, in the isosceles case, our construction becomes a classical question about the four altitudes of a tetrahedron, with the added feature that each altitude passes through X_{20} of the opposite face. Figure 5 shows the reflection picture for one face.

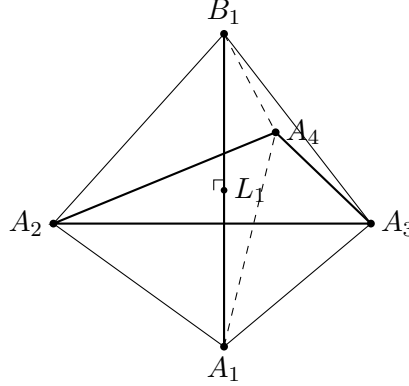


Figure 5: For an isosceles tetrahedron, B_1 is the reflection of A_1 in the opposite face.

We shall use the following elementary fact.

Lemma 4.6. *The altitudes from A_i and A_j intersect if and only if the opposite edges A_iA_j and A_kA_ℓ are perpendicular.*

This is Theorem 203 of Altshiller-Court [1, pp. 69–70].

Proof. Both altitudes are perpendicular to A_kA_ℓ . If they meet, their plane contains A_iA_j and is perpendicular to A_kA_ℓ , so $A_iA_j \perp A_kA_\ell$.

Conversely, suppose $A_iA_j \perp A_kA_\ell$. The plane through A_iA_j perpendicular to A_kA_ℓ contains both altitudes. They are not parallel, because their directions are normals to two distinct faces. Hence they meet. $\square$

An isosceles tetrahedron may be represented as

$$A_1 = (x, y, z), \quad A_2 = (x, -y, -z), \quad A_3 = (-x, y, -z), \quad A_4 = (-x, -y, z),$$

where $x, y, z > 0$. For example,

$$\overrightarrow{A_1A_2} \cdot \overrightarrow{A_3A_4} = 4(y^2 - z^2).$$

Thus one pair of opposite edges is perpendicular exactly when two of x, y, z are equal.

Theorem 4.7. *For an isosceles tetrahedron whose faces are acute, the four lines A_iB_i are concurrent if and only if the tetrahedron is regular.*

Proof. By Theorem 4.1, the four lines are the altitudes. If all four are concurrent, then every pair of altitudes intersects. Lemma 4.6 therefore makes all three pairs of opposite edges perpendicular. In the coordinates above this forces $x = y = z$, so the tetrahedron is regular. The converse is immediate by symmetry. $\square$

The centroid lines have the same simple concurrency behavior.

Theorem 4.8. *For an isosceles tetrahedron whose faces are acute, the four lines A_iC_i are concurrent if and only if the tetrahedron is regular.*

Proof. Use the coordinates above and put

$$S = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}, \quad k = \frac{2}{S}.$$

The centroid of the original tetrahedron is the origin. Reflecting each A_i in its opposite face and then using Equation (1) gives

$$\begin{aligned} C_1 &= \left(-\frac{k}{x}, -\frac{k}{y}, -\frac{k}{z} \right), & C_2 &= \left(-\frac{k}{x}, \frac{k}{y}, \frac{k}{z} \right), \\ C_3 &= \left(\frac{k}{x}, -\frac{k}{y}, \frac{k}{z} \right), & C_4 &= \left(\frac{k}{x}, \frac{k}{y}, -\frac{k}{z} \right). \end{aligned}$$

Two lines through points P, Q with direction vectors u, v are coplanar exactly when $(Q-P) \cdot (u \times v) = 0$. Applying this test to A_1C_1 and A_2C_2 gives

$$(A_2 - A_1) \cdot ((C_1 - A_1) \times (C_2 - A_2)) = -\frac{4k(k+x^2)(y^2-z^2)}{xyz}.$$

If all four lines are concurrent, every pair is coplanar. The displayed factor and its cyclic companions therefore force $x = y = z$. Conversely, in a regular tetrahedron symmetry makes all four lines pass through the common center. $\square$

Corollary 4.9. *If $A_1A_2A_3A_4$ is an isosceles tetrahedron, then*

$$A_1C_1 = A_2C_2 = A_3C_3 = A_4C_4.$$

Proof. The 180° rotational symmetries of an isosceles tetrahedron act transitively on its four vertices and carry each erected construction to the corresponding erected construction on another face. Hence they carry each segment A_iC_i to every other one. $\square$

Proposition 4.10 (A constant solid angle). *Let $A_1A_2A_3A_4$ be an isosceles tetrahedron with acute faces. Translate any three of the rays A_iB_i so that they have a common initial point. Then the solid angle determined by the three translated rays is*

$$\pi$$

steradians.

Proof. Use the coordinates

$$A_1 = (x, y, z), \quad A_2 = (x, -y, -z), \quad A_3 = (-x, y, -z), \quad A_4 = (-x, -y, z),$$

with $x, y, z > 0$. For the rays from A_1, A_2, A_3 toward B_1, B_2, B_3 , direction vectors may be taken as

$$\mathbf{d}_1 = (-yz, -xz, -xy), \quad \mathbf{d}_2 = (-yz, xz, xy), \quad \mathbf{d}_3 = (yz, -xz, xy).$$

All three have squared length

$$S = y^2z^2 + x^2z^2 + x^2y^2.$$

Let $\mathbf{u}_i = \mathbf{d}_i/\sqrt{S}$. Then

$$\mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_2 \cdot \mathbf{u}_3 + \mathbf{u}_3 \cdot \mathbf{u}_1 = -1,$$

while

$$|\mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3)| = \frac{4x^2y^2z^2}{S^{3/2}} > 0.$$

For three unit vectors, the solid angle Ω satisfies the standard formula

$$\tan \frac{\Omega}{2} = \frac{|\mathbf{u}_1 \cdot (\mathbf{u}_2 \times \mathbf{u}_3)|}{1 + \mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{u}_2 \cdot \mathbf{u}_3 + \mathbf{u}_3 \cdot \mathbf{u}_1}.$$

The denominator is zero and the numerator is positive, so $\Omega = \pi$. The other choices of three rays give the same calculation by symmetry. $\square$

Thus any three of the four ray directions occupy exactly one quarter of the full 4π steradians around a point.

Proposition 4.11 (Relation with the circumradius). *Let R be the circumradius of an isosceles tetrahedron. Write*

$$\ell = A_1B_1 = \cdots = A_4B_4, \quad m = A_1C_1 = \cdots = A_4C_4.$$

Then

$$m^2 = R^2 + \frac{\ell^2}{4}.$$

Moreover,

$$\ell \leq \frac{8R}{3}, \quad m \leq \frac{5R}{3},$$

and equality in either inequality occurs exactly for the regular tetrahedron.

Proof. Let h be the altitude from A_i to its opposite face. Then $\ell = 2h$. The circumcenter O of an isosceles tetrahedron is also its centroid, so its distance from any face is $h/4$. By Proposition 2.4, C_i lies on the perpendicular through the circumcenter O_i of that face. Since B_i is the reflection of A_i in the face, C_i is at distance $h/4$ on the other side. Thus C_i is the reflection of O in the opposite face.

Let P_i be the foot of the altitude from A_i , and put $u = P_iO_i$. Let F_i be the centroid of the opposite face. Since $O = G = (A_i + 3F_i)/4$ and $P_i = L_i = 4O_i - 3F_i$, the orthogonal projection of O onto the opposite face is O_i . Thus the perpendicular separations from A_i to O and to C_i are $3h/4$ and $5h/4$, respectively, while both corresponding in-plane separations have length u . Two right triangles therefore give

$$R^2 = u^2 + \left(\frac{3h}{4}\right)^2, \quad m^2 = u^2 + \left(\frac{5h}{4}\right)^2.$$

Subtracting gives

$$m^2 = R^2 + h^2 = R^2 + \frac{\ell^2}{4}.$$

The first right-triangle formula gives $h \leq 4R/3$, and hence the two stated inequalities. Equality means $u = 0$, so the altitude meets the face at its circumcenter. But in the isosceles tetrahedron case the same point is the de Longchamps point; hence the face has $O_i = H_i$ and is equilateral. Therefore the tetrahedron is regular. The converse is immediate. $\square$

Thus the circumradius gives useful bounds, but it does not by itself determine either common segment length. For a regular tetrahedron,

$$\frac{\ell}{R} = \frac{8}{3}, \quad \frac{m}{R} = \frac{5}{3}.$$

For example, the nonregular isosceles tetrahedron obtained from the coordinates above with $x = y = 1$ and $z = 2$ has acute faces and

$$\frac{\ell}{R} = \frac{16}{3\sqrt{6}}, \quad \frac{m}{R} = \sqrt{\frac{59}{27}},$$

so both ratios vary even within this equal-length family.

5 A second equal-length family

Theorem 4.2 shows that an isosceles tetrahedron has four equal segments $A_i B_i$. There is another, quite different, family with the same property. We first need the notion of a *dihedral angle*: this is the angle between two faces meeting along an edge, measured in a plane perpendicular to that edge.

We shall call a tetrahedron having right dihedral angles along a pair of opposite edges a *Hjelmslev tetrahedron*. The name is introduced here by analogy with the *Hjelmslev quadrilateral*, which has two right angles at opposite vertices [2]. We have not found this tetrahedral terminology used previously.

Theorem 5.1. *If $A_1 A_2 A_3 A_4$ is a Hjelmslev tetrahedron and all four faces are acute, then*

$$A_1 B_1 = A_2 B_2 = A_3 B_3 = A_4 B_4.$$

Proof. The proof is given in Appendix A. It uses only perpendicular cross-sections, the cosine rule, and elementary coordinate calculations. $\square$

By the observation following Equation (1), the four points C_i are therefore equidistant from the original centroid G . The more general statement is developed in Theorem 7.5.

The preceding theorem is genuinely different from the isosceles case.

Example 5.2 (A non-isosceles Hjelmslev tetrahedron). Take

$$\begin{aligned} A_1 &= (0, 0, 0), & A_2 &= (1, 0, 0), \\ A_3 &= \left(\frac{1}{3}, \frac{\sqrt{5}}{3}, 0\right), & A_4 &= \left(\frac{2}{3}, \frac{\sqrt{5}}{6}, \frac{\sqrt{15}}{6}\right). \end{aligned}$$

A dot-product check shows that the dihedral angles along $A_1 A_4$ and $A_2 A_3$ are right angles. The six edge lengths are

$$\begin{aligned} A_1 A_2 &= A_1 A_4 = A_2 A_3 = 1, \\ A_1 A_3 &= A_2 A_4 = A_3 A_4 = \sqrt{\frac{2}{3}}. \end{aligned}$$

Thus the tetrahedron is not isosceles. Its faces have side lengths either $1, 1, \sqrt{2/3}$ or $1, \sqrt{2/3}, \sqrt{2/3}$, and both of these triangles are acute. Therefore Theorem 5.1 applies and shows that its four segments $A_i B_i$ have the same length. In fact, their common length is

$$\sqrt{1 + \frac{2\sqrt{2}}{3}}.$$

A direct coordinate check shows that neither line family is concurrent in this example.

Computational searches have not produced any other nondegenerate examples with four acute faces. This suggests the following simple classification.

Conjecture 5.3. *Let $A_1A_2A_3A_4$ have four acute faces. Then*

$$A_1B_1 = A_2B_2 = A_3B_3 = A_4B_4$$

if and only if the tetrahedron is isosceles or Hjeltmslev.

The reverse implication has already been proved in Theorems 4.2 and 5.1. The open part of the conjecture is the forward implication: to prove that equality of the four segments forces the tetrahedron to be isosceles or Hjeltmslev.

6 Kites and concurrence

A *kite* is a tetrahedron with an equilateral base and three equal lateral edges [5]. The symmetry behind the concurrence results below is illustrated in Figure 6.

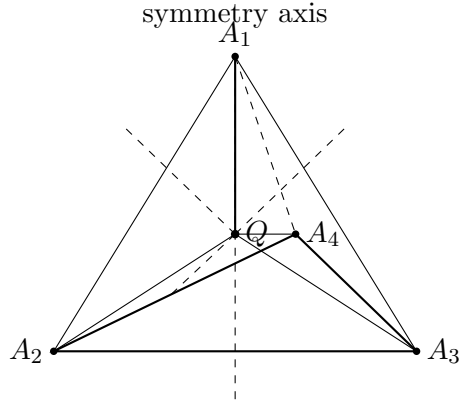


Figure 6: In a kite, symmetry forces the four lines to pass through one point Q .

Theorem 6.1. *If $A_1A_2A_3A_4$ is a kite with acute faces, then the lines $A_1B_1, A_2B_2, A_3B_3, A_4B_4$ are concurrent.*

Proof. Let $A_2A_3A_4$ be the equilateral base, and let g be the axis through A_1 and the center of that face. The construction has the full symmetry of the kite. Hence B_1 lies on g , so $A_1B_1 = g$. For $r = 2, 3, 4$, the reflection plane through g and A_r contains the line A_rB_r .

We only have to rule out the possibility that one of these three lines is parallel to g . By symmetry it is enough to consider A_2B_2 . Scale the equilateral base to side 1 and place its center at the origin, with

$$A_2 = \left(\frac{1}{\sqrt{3}}, 0, 0 \right), \quad A_1 = (0, 0, H), \quad H^2 = t^2 - \frac{1}{3},$$

where t is the common lateral edge. If $A_2B_2 \parallel g$, then

$$B_2 = \left(\frac{1}{\sqrt{3}}, 0, u \right).$$

The defining distances $B_2A_3 = t$ and $B_2A_1 = 1$ give

$$u^2 = t^2 - 1, \quad (u - H)^2 = \frac{2}{3}.$$

Since $H^2 = u^2 + 2/3$, these equations imply $u(u - H) = 0$. The second equation shows $u \neq H$, so $u = 0$ and hence $t = 1$. But then $B_2 = A_2$, the inward apex, not the outward one. This contradiction shows that A_2B_2 meets g .

The same is true for A_3B_3 and A_4B_4 . Their three intersection points with g are carried cyclically into one another by the 120° rotation about g . Every point of g is fixed by that rotation, so the three intersection points coincide. Thus all four lines are concurrent. $\square$

Theorem 6.2. *If $A_1A_2A_3A_4$ is a kite with acute faces, then the four centroid lines*

$$A_1C_1, \quad A_2C_2, \quad A_3C_3, \quad A_4C_4$$

are also concurrent.

Proof. Use the same symmetry axis g . The point C_1 lies on g , and for $r = 2, 3, 4$ the reflection plane through g and A_r contains C_r , so it contains the line A_rC_r .

We only need to see that these three lines are not parallel to g . Scale the equilateral base to side s and use coordinates with

$$A_2 = \left(\frac{s}{\sqrt{3}}, 0, 0 \right)$$

and g the z -axis. The other two base vertices have x -coordinate $-s/(2\sqrt{3})$, while A_1 has x -coordinate 0. If A_2C_2 were parallel to g , then C_2 would have x -coordinate $s/\sqrt{3}$. Since

$$C_2 = \frac{A_1 + A_3 + A_4 + B_2}{4},$$

this would force the x -coordinate of B_2 to be $5s/\sqrt{3}$. But the defining construction requires $B_2A_1 = s$, whereas the difference of their x -coordinates alone is already $5s/\sqrt{3} > s$. This is impossible.

Thus each of A_2C_2, A_3C_3, A_4C_4 meets g . The 120° rotation about g carries the three intersection points cyclically into one another, while fixing every point of g . Hence the three points coincide, and all four centroid lines are concurrent. $\square$

If the equilateral base has side s and each lateral edge has length t , the lateral faces are acute precisely when $t > s/\sqrt{2}$.

7 Further properties of the centroid construction

The preceding sections give the simplest geometric facts about the centroid lines: C_i projects to the circumcenter of the opposite face, isosceles tetrahedra have a very simple concurrence theorem, and kites give another concurrence example. We now turn to length questions and to the tetrahedron formed by the four points C_i .

7.1 Equal centroid segments

Corollary 4.9 shows that every isosceles tetrahedron has four equal centroid segments. The converse is false, but there is a simple exact criterion.

We shall use the following elementary centroid identity.

Lemma 7.1 (Centroid identity). *If V_1, V_2, V_3, V_4 have centroid G and P is any point, then*

$$\sum_{r=1}^4 PV_r^2 = 4PG^2 + \frac{1}{4} \sum_{1 \leq r < s \leq 4} V_r V_s^2.$$

Proof. Put $\mathbf{p} = P - G$ and $\mathbf{v}_r = V_r - G$. Since G is the centroid, $\sum_r \mathbf{v}_r = 0$. Hence

$$\sum_{r=1}^4 PV_r^2 = \sum_{r=1}^4 |\mathbf{p} - \mathbf{v}_r|^2 = 4|\mathbf{p}|^2 + \sum_{r=1}^4 |\mathbf{v}_r|^2.$$

Also,

$$\sum_{r < s} |\mathbf{v}_r - \mathbf{v}_s|^2 = 4 \sum_{r=1}^4 |\mathbf{v}_r|^2,$$

because $\sum_r \mathbf{v}_r = 0$. Combining the two identities gives the result. $\square$

For each i , put

$$T_i = \sum_{j \neq i} A_i A_j^2, \quad \Sigma = \sum_{1 \leq r < s \leq 4} A_r A_s^2.$$

Thus T_i is the sum of the squares of the three edges issuing from A_i , while Σ is the sum of the squares of all six edges.

Proposition 7.2. *For every i ,*

$$8A_i C_i^2 = 2A_i B_i^2 + 3T_i - \Sigma.$$

Equivalently,

$$A_i C_i^2 = \frac{1}{4} A_i B_i^2 + \frac{3}{2} A_i G^2 - \frac{\Sigma}{32}.$$

Consequently,

$$A_1 C_1 = \cdots = A_4 C_4$$

if and only if

$$A_i B_i^2 + 6A_i G^2$$

is independent of i .

Proof. Apply Lemma 7.1 to the four vertices of the erected isosceles tetrahedron opposite A_i , taking A_i as the external point. The four squared distances from A_i to those vertices sum to $T_i + A_i B_i^2$. The six edges of the erected tetrahedron consist of two copies of the three edges of the opposite face, so their squared sum is $2(\Sigma - T_i)$. Therefore

$$T_i + A_i B_i^2 = 4A_i C_i^2 + \frac{1}{2}(\Sigma - T_i),$$

which gives the first formula.

For the original tetrahedron, Lemma 7.1 with external point A_i gives

$$T_i = 4A_iG^2 + \frac{\Sigma}{4}.$$

Substitution yields the second formula. Since the last term is independent of i , equality of the four A_iC_i is equivalent to constancy of $A_iB_i^2 + 6A_iG^2$. $\square$

Theorem 7.3. *The two equalities*

$$A_1B_1 = \cdots = A_4B_4 \quad \text{and} \quad A_1C_1 = \cdots = A_4C_4$$

hold simultaneously if and only if the original tetrahedron is isosceles.

Proof. If both sets of lengths are equal, Proposition 7.2 gives

$$A_1G = A_2G = A_3G = A_4G.$$

Thus the centroid is also the circumcenter. Translating G to the origin gives $A_1 + A_2 + A_3 + A_4 = 0$ with all four $|A_i|$ equal. Squaring $A_1 + A_2 = -(A_3 + A_4)$ gives $A_1A_2 = A_3A_4$, and cyclically the other two pairs of opposite edges are equal. Hence the tetrahedron is isosceles.

Conversely, Theorem 4.2 and Corollary 4.9 give both equalities for every isosceles tetrahedron. $\square$

Corollary 7.4. *For an acute-faced Hjeltmslev tetrahedron,*

$$A_1C_1 = A_2C_2 = A_3C_3 = A_4C_4$$

if and only if the original tetrahedron is isosceles.

Proof. The four A_iB_i are equal by Theorem 5.1, so the result follows immediately from Theorem 7.3. $\square$

The condition $A_1C_1 = \cdots = A_4C_4$ by itself is broader. For example, in the opposite-edge notation of the Introduction, consider the family

$$(a_1, a_2, a_3) = (1, t, 1), \quad (b_1, b_2, b_3) = (1, t, t).$$

At

$$t \approx 0.7095265936$$

all four faces are acute and direct computation gives

$$A_1C_1 = A_2C_2 = A_3C_3 = A_4C_4 \approx 0.734926980.$$

This tetrahedron is not isosceles.

7.2 The derived tetrahedron

The planar Napoleon analogy suggests studying not only the four centroid lines, but also the four points C_1, C_2, C_3, C_4 themselves. Let

$$T = A_1A_2A_3A_4, \quad T' = C_1C_2C_3C_4.$$

We call T' the *derived tetrahedron* of T ; if the four points C_i are coplanar, the derived tetrahedron is degenerate. A particularly clean result occurs whenever the four A_iB_i have equal length.

We shall use two standard tetrahedral centers. If a tetrahedron has centroid G and circumcenter O , its *Monge point* is the reflection of O in G , namely

$$M = 2G - O.$$

Its *Euler point*, also called the twelve-point center, is

$$E = \frac{4G - O}{3}.$$

The line through G and O is called the *Euler line* of the tetrahedron; the points G, O, M, E all lie on it [8, Appendix E].

Theorem 7.5. *Suppose*

$$A_1B_1 = A_2B_2 = A_3B_3 = A_4B_4 = \ell.$$

Then the four vertices of the derived tetrahedron T' lie on the sphere of radius $\ell/4$ centered at the original centroid G . If T' is nondegenerate, let G', O', M', E' denote its centroid, circumcenter, Monge point, and Euler point, while O is the circumcenter of T . Then

$$O' = G, \quad G' = \frac{G + O}{2}, \quad M' = O, \quad E' = \frac{G + 2O}{3}.$$

Thus, in the non-isosceles case, T and T' have the same Euler line.

Proof. The observation following Equation (1) immediately gives the sphere assertion, with radius $\ell/4$. When T' is nondegenerate, its circumcenter is therefore $O' = G$.

It remains to find the centroid of the four C_i . Translate the origin to G , so that $A_1 + A_2 + A_3 + A_4 = 0$. Fix distinct i, j , let k, m be the other two indices, and put $\Delta = A_i - A_j$. From the defining distances of the erected tetrahedra and the common equality $A_rB_r = \ell$, subtraction of squared-distance equations gives

$$(B_i + B_j) \cdot \Delta = |A_i|^2 - |A_j|^2.$$

For the other two erected vertices, subtracting their squared distances to A_i and A_j gives

$$(B_k + B_m) \cdot \Delta = 3(|A_i|^2 - |A_j|^2).$$

Hence

$$\left(\sum_{r=1}^4 B_r \right) \cdot \Delta = 4(|A_i|^2 - |A_j|^2).$$

Since O is the circumcenter,

$$2O \cdot \Delta = |A_i|^2 - |A_j|^2.$$

The edge directions span space, so

$$\sum_{r=1}^4 B_r = 8O.$$

Now $C_i = (B_i - A_i)/4$ in these translated coordinates, and therefore

$$G' = \frac{1}{4} \sum_i C_i = \frac{1}{16} \sum_i B_i = \frac{O}{2}.$$

Restoring the original origin gives $G' = (G + O)/2$.

Finally, for every tetrahedron, the Monge point is the reflection of the circumcenter in the centroid, so

$$M' = 2G' - O' = O.$$

Finally, the Euler-point formula $E' = (4G' - O')/3 = (G + 2O)/3$ follows at once. $\square$

Corollary 7.6. *Under the hypotheses of Theorem 7.5, if T' is nondegenerate, then T' is an isosceles tetrahedron if and only if T is an isosceles tetrahedron.*

Proof. A tetrahedron is isosceles exactly when its centroid is also its circumcenter. Here $G' = O'$ is equivalent to $(G + O)/2 = G$, hence to $G = O$, which is the corresponding condition for T . $\square$

There is an additional reciprocal structure inside the isosceles class.

Proposition 7.7 (Reciprocity of the derived tetrahedron). *Let T be an isosceles tetrahedron and let T' be its derived tetrahedron. Then T' is also isosceles, and the derived tetrahedron $T'' = (T')'$ is directly similar to T .*

Moreover, T' is already similar to T after one application if and only if the three edge lengths of a rectangular box having the vertices of T as four alternate vertices are in geometric progression. Here similarity between T and T' is understood in the usual unoriented sense, so a reflection is allowed.

For a regular tetrahedron,

$$C_i - G = -\frac{2}{3}(A_i - G).$$

Proof. Use the standard coordinates

$$A_1 = (x, y, z), \quad A_2 = (x, -y, -z), \quad A_3 = (-x, y, -z), \quad A_4 = (-x, -y, z),$$

with $0 < x \leq y \leq z$. The coordinates for C_i in the proof of Theorem 4.8 are obtained from the same sign pattern as the A_i by replacing x, y, z with $k/x, k/y, k/z$ and then reflecting through the origin, where k is common to all three coordinates. Thus, up to an overall scale and a reflection, the passage $T \mapsto T'$ replaces

$$(x, y, z) \mapsto \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right).$$

In particular, T' is again isosceles, and applying the same reciprocal operation a second time returns (x, y, z) . The two reflections cancel, so T'' is directly similar to T .

Similarity after one application requires the ordered triples (x, y, z) and $(1/z, 1/y, 1/x)$ to differ by one common factor. The middle coordinate fixes that factor, and the remaining condition is

$$y^2 = xz.$$

The containing rectangular box has edge lengths $2x, 2y, 2z$, so this condition says exactly that its three edge lengths are in geometric progression.

When $x = y = z$, the value $k = 2x^2/3$ from Theorem 4.8 gives the final formula. $\square$

8 Special tetrahedra whose faces are right triangles

The preceding sections concern tetrahedra with acute faces, for which the erected isosceles tetrahedra are genuine three-dimensional solids. It is also useful to record three especially simple boundary cases involving right triangles. We shall call a face of a tetrahedron a *right face* if it is a right triangle. On a right face the erection is flat, and the geometry often becomes particularly transparent.

8.1 Trirectangular tetrahedra

A *trirectangular tetrahedron* has three pairwise perpendicular edges meeting at one vertex.

Theorem 8.1. *Let $A_1A_2A_3A_4$ be trirectangular, with the three mutually perpendicular edges meeting at A_1 . In the limiting interpretation of the three right-faced erections, the four lines A_iB_i are concurrent and*

$$A_1B_1 = A_2B_2 = A_3B_3 = A_4B_4.$$

The four centroid lines A_iC_i are also concurrent. If the perpendicular edges from A_1 have lengths x, y, z , then the A_iB_i concur at

$$\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right),$$

while the A_iC_i concur at

$$\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) = D_1.$$

Moreover, $D_1 = X_2$ on the acute face, while $D_i = X_{20}$ on its corresponding right face for $i = 2, 3, 4$.

Proof. Choose coordinates

$$A_1 = (0, 0, 0), \quad A_2 = (x, 0, 0), \quad A_3 = (0, y, 0), \quad A_4 = (0, 0, z),$$

where $x, y, z > 0$. The face $A_2A_3A_4$ is acute, so the erection giving B_1 is genuine; the other three faces are right, so the erections giving B_2, B_3, B_4 are flat. Directly from the defining distances,

$$B_1 = (x, y, z), \quad B_2 = (0, y, z), \quad B_3 = (x, 0, z), \quad B_4 = (x, y, 0).$$

Thus A_i and B_i are opposite vertices of the rectangular box $[0, x] \times [0, y] \times [0, z]$. The four lines A_iB_i are its space diagonals, so they meet at $(x/2, y/2, z/2)$, and each has length $\sqrt{x^2 + y^2 + z^2}$.

The line A_1B_1 meets the acute face $A_2A_3A_4$ at parameter $1/3$, giving

$$D_1 = \left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right),$$

the centroid X_2 of that face. For $i = 2, 3, 4$, the opposite face is right and the erection is flat, so $D_i = B_i = X_{20}$.

The centroid points are

$$C_1 = \left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), \quad C_2 = \left(0, \frac{y}{2}, \frac{z}{2}\right), \quad C_3 = \left(\frac{x}{2}, 0, \frac{z}{2}\right), \quad C_4 = \left(\frac{x}{2}, \frac{y}{2}, 0\right).$$

A direct substitution shows that all four lines A_iC_i pass through $(x/3, y/3, z/3) = D_1$. □

8.2 Doubly special tetrahedra in the sense of Altshiller-Court

Altshiller-Court calls a vertex of a tetrahedron *special* when the foot of the altitude from that vertex lies on the circumcircle of the opposite face. A tetrahedron is *doubly special* when this happens at two vertices. An equivalent characterization is especially convenient here: the edge joining the two special vertices is a diameter of the *circumsphere*, the sphere through the four vertices of the tetrahedron [1, pp. 115–116]. Throughout this subsection, “doubly special” is used in this Altshiller-Court sense.

Theorem 8.2. *Suppose A_1A_4 is a diameter of the circumsphere of a nondegenerate tetrahedron $A_1A_2A_3A_4$. Then the erections on the faces $A_1A_3A_4$ and $A_1A_2A_4$ are flat, and*

$$A_2B_2 \parallel A_3B_3, \quad A_2B_2 = A_3B_3.$$

Consequently, the four lines A_iB_i are not concurrent. Moreover,

$$D_2 = B_2 = X_{20}, \quad D_3 = B_3 = X_{20}$$

on their respective right faces. If O_T is the circumcenter of the original tetrahedron, then

$$C_2 = C_3 = O_T.$$

Hence the two centroid lines A_2C_2 and A_3C_3 meet.

Proof. Since A_1A_4 is a diameter, Thales’ theorem gives

$$\angle A_1A_2A_4 = \angle A_1A_3A_4 = 90^\circ.$$

Thus the two indicated erections are flat.

Put the circumcenter at the origin. Then $A_4 = -A_1$. The plane of the right triangle $A_1A_3A_4$ passes through the origin, and the point diametrically opposite A_3 on its circumcircle is $-A_3$. This point has exactly the three distances required of the flat apex, so

$$B_2 = -A_3.$$

Similarly,

$$B_3 = -A_2.$$

Therefore

$$\overrightarrow{A_2B_2} = -A_3 - A_2 = \overrightarrow{A_3B_3}.$$

The two lines are distinct. Indeed, $|A_2| = |A_3|$, so $A_3 - A_2$ is perpendicular to $A_2 + A_3$, whereas the common direction of the two lines is $-A_2 - A_3$. If this direction were zero, then $A_3 = -A_2$; together with $A_4 = -A_1$, all four vertices would be coplanar, contrary to nondegeneracy. Thus the two lines are distinct and parallel, and their corresponding segments have the same length. Hence the four lines cannot be concurrent. Finally, on a right face the flat apex is its de Longchamps point, giving the statements about D_2 and D_3 . Also,

$$C_2 = \frac{A_1 + A_3 + A_4 + B_2}{4} = \frac{A_1 + A_4}{4} = 0,$$

and similarly $C_3 = 0$. Since the origin is O_T , this proves the centroid statement. $\square$

8.3 Orthoschemes

An *orthoscheme* has three successive edges that are mutually perpendicular.

Theorem 8.3. *For a nondegenerate orthoscheme, all four erections are flat. The four segments A_iB_i have the same length, but the four lines A_iB_i are not concurrent. Moreover,*

$$D_i = B_i = X_{20}$$

on every opposite face. With the standard labeling below, two centroid points coincide at the circumcenter O_T of the original tetrahedron:

$$C_2 = C_3 = O_T.$$

The four centroid lines A_iC_i are not concurrent.

Proof. Use the standard coordinates

$$A_1 = (0, 0, 0), \quad A_2 = (a, 0, 0), \quad A_3 = (a, b, 0), \quad A_4 = (a, b, c),$$

with $a, b, c > 0$. All four faces are right, and the flat apices are

$$B_1 = (a, 0, c), \quad B_2 = (0, 0, c), \quad B_3 = (0, b, c), \quad B_4 = (0, b, 0).$$

Now A_1B_1 and A_2B_2 meet at $(a/2, 0, c/2)$, while A_3B_3 and A_4B_4 meet at $(a/2, b, c/2)$. Also,

$$A_1B_1 \parallel A_4B_4, \quad A_2B_2 \parallel A_3B_3.$$

Thus the four lines are not concurrent. Each segment has length

$$A_1B_1 = A_2B_2 = A_3B_3 = A_4B_4 = \sqrt{a^2 + c^2}.$$

Since every opposite face is right, each flat apex is the de Longchamps point of that face, so $D_i = B_i = X_{20}$.

The circumcenter of the original tetrahedron is

$$O_T = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2} \right),$$

the midpoint of A_1A_4 . Direct averaging gives

$$C_2 = C_3 = O_T.$$

But

$$C_1 = \left(a, \frac{b}{2}, \frac{c}{2} \right),$$

so A_1C_1 does not pass through O_T . The two lines A_2C_2 and A_3C_3 are distinct (their direction vectors from O_T differ in the sign of their nonzero b -component), so O_T is their unique point of intersection. Any common point of all four centroid lines would therefore have to be O_T , which is impossible because A_1C_1 does not pass through it. $\square$

Thus an orthoscheme is simultaneously a doubly special tetrahedron and a right-faced limiting Hjelmslev tetrahedron, with right dihedral angles along the opposite edges A_1A_3 and A_2A_4 . Orthoschemes are classical; see Coxeter [3].

9 Concluding remarks

Erecting an isosceles tetrahedron on each face produces two natural points: the new vertex B_i and the centroid C_i . Their projections onto the opposite face have a pleasing contrast: the projection of B_i is the de Longchamps point $L_i = X_{20}$, while the projection of C_i is the circumcenter $O_i = X_3$.

For an isosceles tetrahedron, the construction becomes especially simple. Each B_i is the reflection of A_i in the opposite face, both sets of four segments are equal, and each of the two line families is concurrent exactly when the tetrahedron is regular. In addition, any three translated rays parallel to A_iB_i determine the constant solid angle π steradians. For a kite with acute faces, both line families are concurrent for a different reason: rotational symmetry.

The equal-length questions also fit together cleanly. The four A_iB_i are equal for every isosceles tetrahedron and for every acute-faced Hjelmslev tetrahedron; Conjecture 5.3 proposes that these are the only acute-faced cases. Proposition 7.2 gives a simple exact criterion for equality of the four A_iC_i , and Theorem 7.3 shows that both sets of four segments are equal simultaneously exactly for an isosceles tetrahedron.

Finally, when the four A_iB_i are equal, the vertices of the derived tetrahedron T' lie on a sphere centered at G . When T' is nondegenerate, its circumcenter is G , its centroid is $(G + O)/2$, and its Monge point is O . For an isosceles tetrahedron T , the derived tetrahedron T' is again isosceles, and its derived tetrahedron T'' is directly similar to T .

The complete concurrence loci for the two line families remain open. Even without such a classification, the construction links elementary solid geometry with familiar triangle centers and produces several natural equal-length and concurrency phenomena.

Acknowledgment

The author thanks ChatGPT, developed by OpenAI, for extensive assistance with computational exploration, symbolic checking, literature searches, figures, and preparation of this manuscript.

The author is responsible for the final mathematical statements and presentation.

A Proof of the Hjelmslev equal-length theorem

This appendix gives the calculation used in Theorem 5.1.

Lemma A.1 (Pairwise comparison). *Let $\{i, j, k, m\} = \{1, 2, 3, 4\}$. Let δ be the dihedral angle of the original tetrahedron along A_kA_m . Let ϕ_i be the dihedral angle along A_kA_m in the erected isosceles tetrahedron $A_jA_kA_mB_i$, and define ϕ_j similarly. Then*

$$A_iB_i = A_jB_j$$

if and only if either $\phi_i = \phi_j$ or

$$2\delta + \phi_i + \phi_j = 360^\circ.$$

Proof. Look along the edge A_kA_m , so that it appears as a point. Let $c = A_kA_m$. For any point X , let s_X be the coordinate of its perpendicular projection onto this line, measured from A_k , and let r_X be its perpendicular distance from the line.

The defining distances for B_i are obtained from those for A_j by interchanging the distances to A_k and A_m . Subtracting the corresponding squared-distance equations gives

$$s_{B_i} = c - s_{A_j}, \quad r_{B_i} = r_{A_j},$$

and similarly

$$s_{B_j} = c - s_{A_i}, \quad r_{B_j} = r_{A_i}.$$

Thus the components of $A_i B_i$ and $A_j B_j$ parallel to $A_k A_m$ have the same length.

The perpendicular components of A_i and A_j make the angle δ . Because each erection is outward, the angle between the perpendicular components of A_i and B_i is $\delta + \phi_i$. Hence the cosine rule gives

$$A_i B_i^2 = C + r_{A_i}^2 + r_{A_j}^2 - 2r_{A_i} r_{A_j} \cos(\delta + \phi_i),$$

where C is the common squared component parallel to $A_k A_m$. The analogous formula for $A_j B_j$ has ϕ_j in place of ϕ_i . Therefore the two lengths are equal exactly when

$$\cos(\delta + \phi_i) = \cos(\delta + \phi_j).$$

The two angles $\delta + \phi_i$ and $\delta + \phi_j$ lie strictly between 0° and 360° , so equality of their cosines gives either $\phi_i = \phi_j$ or

$$2\delta + \phi_i + \phi_j = 360^\circ,$$

which are the two stated possibilities. □

Lemma A.2. *Let XYZ be an acute triangle, and let ϕ be the dihedral angle along YZ in the isosceles tetrahedron erected on XYZ . Then*

$$\tan^2 \frac{\phi}{2} = \frac{\cos Y \cos Z}{\cos X}.$$

Equivalently, if the perpendicular from X to YZ has length h and meets YZ at distance m from Y , where $YZ = c$, then

$$\tan^2 \frac{\phi}{2} = \frac{m(c-m)}{h^2 - m(c-m)}.$$

Proof. Use the rectangular box from Lemma 2.1. With its notation we may take

$$X = (0, 0, 0), \quad Y = (p, q, 0), \quad Z = (p, 0, r), \quad B = (0, q, r).$$

A direct calculation of the angle between the two faces meeting along YZ gives

$$\cos \phi = \frac{p^2 q^2 + p^2 r^2 - q^2 r^2}{p^2 q^2 + p^2 r^2 + q^2 r^2}.$$

Using $\tan^2(\phi/2) = (1 - \cos \phi)/(1 + \cos \phi)$ and the identities in Lemma 2.1, we obtain

$$\tan^2 \frac{\phi}{2} = \frac{q^2 r^2}{p^2(q^2 + r^2)} = \frac{\cos Y \cos Z}{\cos X}.$$

For the second form, write

$$\cos Y = \frac{m}{XY}, \quad \cos Z = \frac{c-m}{XZ}, \quad \cos X = \frac{h^2 - m(c-m)}{XY XZ},$$

and substitute. □

Proof of Theorem 5.1. Assume, after relabeling, that the right dihedral angles are along the opposite edges A_1A_4 and A_2A_3 . Because the dihedral angle along A_2A_3 is 90° , choose coordinates

$$\begin{aligned} A_2 &= (0, 0, 0), & A_3 &= (c, 0, 0), \\ A_1 &= (p, q, 0), & A_4 &= (r, 0, s), \end{aligned}$$

with $c, q, s > 0$. The other right dihedral angle is equivalent to

$$\frac{p(c-p)}{q^2} + \frac{r(c-r)}{s^2} = 1. \quad (2)$$

Equivalently,

$$q^2[s^2 - r(c-r)] = s^2p(c-p).$$

First consider the edge A_2A_3 . Let t_{123} and t_{234} be the values of $\tan^2(\phi/2)$ for the erected dihedral angles along A_2A_3 coming from the faces $A_1A_2A_3$ and $A_2A_3A_4$. Lemma A.2 gives

$$t_{123} = \frac{p(c-p)}{q^2 - p(c-p)}, \quad t_{234} = \frac{r(c-r)}{s^2 - r(c-r)}.$$

Equation (2) gives $t_{123}t_{234} = 1$. If the two erected dihedral angles are ϕ_1 and ϕ_4 , then both half-angles lie between 0° and 90° . Taking positive square roots therefore gives

$$\tan \frac{\phi_1}{2} \tan \frac{\phi_4}{2} = 1,$$

so $\phi_1 + \phi_4 = 180^\circ$. Since the original dihedral angle along A_2A_3 is $\delta = 90^\circ$, we have

$$2\delta + \phi_1 + \phi_4 = 360^\circ.$$

Thus the second alternative in Lemma A.1 gives

$$A_1B_1 = A_4B_4.$$

The same argument at the opposite right edge gives

$$A_2B_2 = A_3B_3.$$

It remains to connect the two equal pairs. Along A_1A_2 , put

$$\lambda = p^2 + q^2 = A_1A_2^2.$$

Let u_{123} and u_{124} be the corresponding values of $\tan^2(\phi/2)$ for the two erected dihedral angles. Lemma A.2 gives

$$u_{123} = \frac{p(\lambda - cp)}{\lambda(c-p)}, \quad u_{124} = \frac{rp(\lambda - rp)}{\lambda(r^2 + s^2 - rp)}.$$

Their difference has numerator

$$p \{ -s^2p(c-p) + q^2[s^2 - r(c-r)] \},$$

which is zero by Equation (2). Hence these two erected dihedral angles are equal, so Lemma A.1 gives

$$A_3B_3 = A_4B_4.$$

Combining the three equalities proves the theorem. $\square$

References

- [1] Nathan Altshiller-Court, *Modern Pure Solid Geometry*, 2nd ed., Chelsea Publishing Company, New York, 1964.
- [2] Martha Alvarez-Ramírez, Jaume Llibre, *Hjelmslev quadrilateral central configurations*, Physics Letters A **383** (2019), 103–109.
<https://doi.org/10.1016/j.physleta.2018.08.034>
- [3] Harold Scott MacDonald Coxeter, *Regular Polytopes*, 3rd ed., Dover Publications, New York, 1973.
- [4] Harold Scott MacDonald Coxeter, Samuel L. Greitzer, *Geometry Revisited*, New Mathematical Library, vol. 19, Mathematical Association of America, 1967.
- [5] Mowaffaq Hajja, Mostafa Hayajneh, Ismail Hammoudeh, *Pre-kites: simplices having a regular facet*, Beiträge zur Algebra und Geometrie **58** (2017), 699–721.
<https://doi.org/10.1007/s13366-017-0341-y>
- [6] Clark Kimberling, *Encyclopedia of Triangle Centers*.
<https://faculty.evansville.edu/ck6/encyclopedia/ETC.html>
- [7] John Leech, *Some properties of the isosceles tetrahedron*, The Mathematical Gazette **34** (1950), 269–271.
<https://doi.org/10.2307/3611029>
- [8] Stanley Rabinowitz, *Arrangement of central points on the faces of a tetrahedron*, International Journal of Computer Discovered Mathematics **5** (2020), 13–41.
<https://journal-1.eu/2020/TetrahedronFaces.pdf>
- [9] Eric W. Weisstein, *Kiepert’s Theorem*, MathWorld—A Wolfram Resource.
<https://mathworld.wolfram.com/KieperTsTheorem.html>