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A Catalog of Properties of the Lemniscate of Bernoulli

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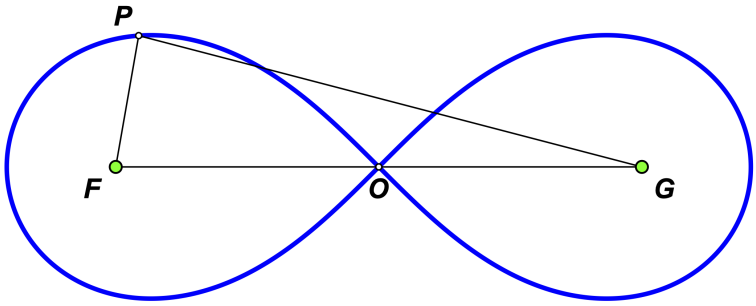
Abstract. We survey the literature to find geometrical properties of the Lemniscate of Bernoulli. We also use a computer to find additional properties.

Keywords. lemniscate, computer-discovered mathematics, Mathematica.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

The *lemniscate of Bernoulli* is the plane curve that is the locus of points such that the product of their distances from two points, F and G, is equals to a quarter of square of the distance between the two points. In the figure below, $PF \cdot PG = \frac{1}{4}(FG)^2$ for all points P on the locus.

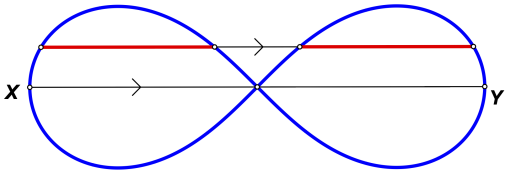


The points F and G are known as the *foci* of the lemniscate. In this paper, we survey geometrical results about the lemniscate as well as give additional results that we found by computer. Throughout this paper, the foci of the lemniscate will always be labeled F and G, and they will be colored green. The double point (center) of the lemniscate will be named O. The rays OF and OG meet the lemniscate at points X and Y, respectively, known as the *vertices* of the lemniscate.

Legend	
Symbol	**Meaning**
•	additional hypotheses not obvious from the figure
▶	conclusion
*	property was found by computer
§	article
[n]	See reference n

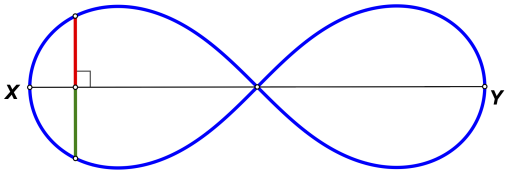
1. Basic Properties

Property 1.1 [40]



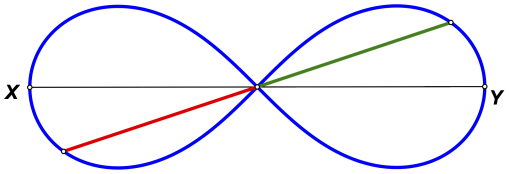
▶ red lengths are equal

Property 1.2 [40]



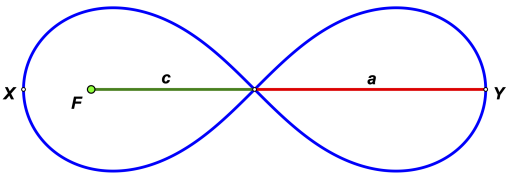
▶ red length = green length

Property 1.3 [40]



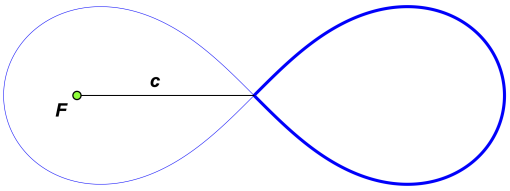
▶ red length = green length

Property 1.4 [40]



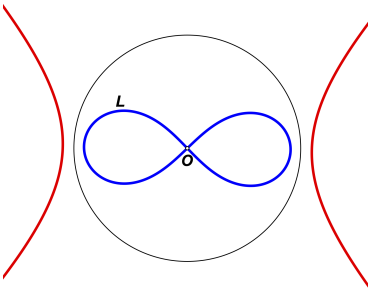
▶ $a = c\sqrt{2}$

Property 1.5 [40]



▶ area of loop = c^2

Property 1.6 [40]

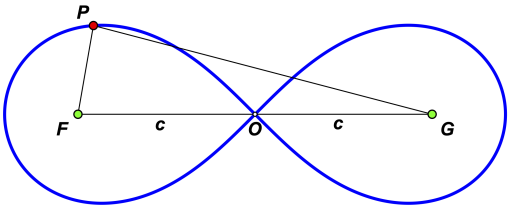


▶ The inverse of L about a circle with center O is a hyperbola

▶ If the radius of the circle is OX, then the hyperbola is rectangular

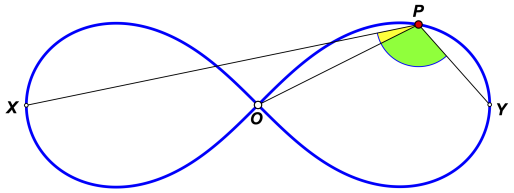
2. Point on Boundary

Property 2.1 [40]



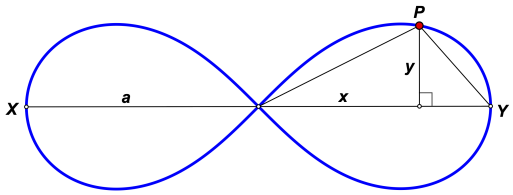
▶ $PF \cdot PG = c^2$

Property 2.2*



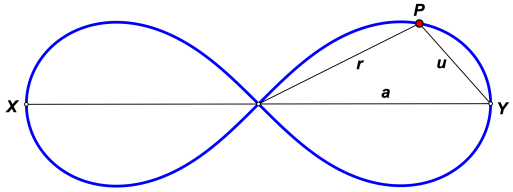
▶ green angle = yellow angle + 90°

Property 2.3 [40]



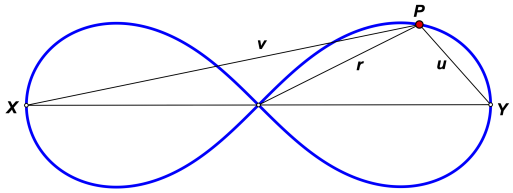
▶ $(x^2+y^2)^2 = a^2(x^2-y^2)$

Property 2.4*



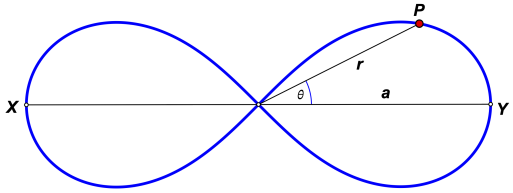
▶ $(r^2+u^2)^2 - u^4 = (a^2-u^2)^2$

Property 2.5*



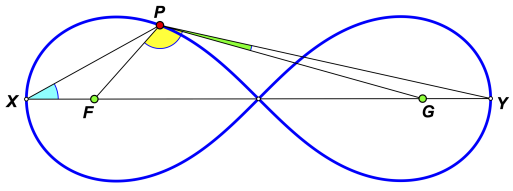
▶ $(v^2-u^2)^2 = r^2(u^2+v^2)$

Property 2.6 [40]



▶ $r^2 = a^2\cos 2\theta$

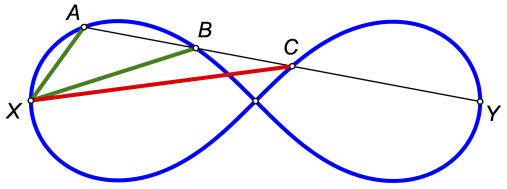
Property 2.7*



▶ $\angle FPG + 2\angle GPY + 2\angle YXP = 180°$

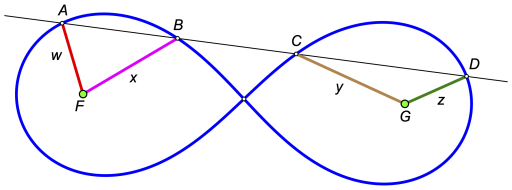
3. Secants

Property 3.1* [15].



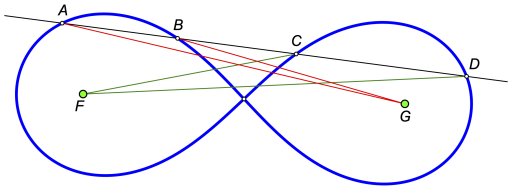
▶ sum of green lengths = red length

Property 3.2* [16]



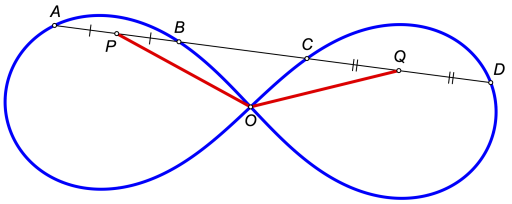
▶ $wx = yz$

Property 3.3 [33]



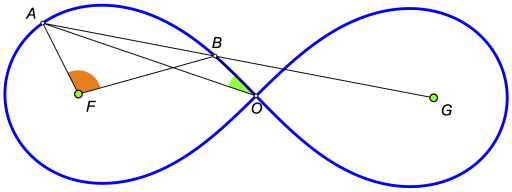
▶ $AG \cdot BG = CF \cdot DF$

Property 3.4 [7]



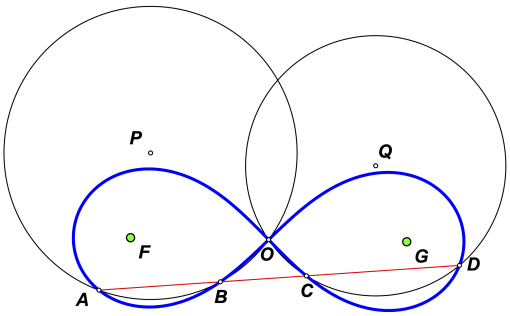
▶ red lengths are equal

Property 3.5* [17]



▶ $\angle BFA = 4 \times \angle BOA$

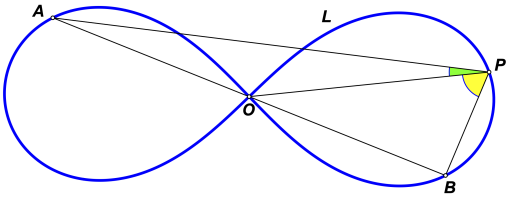
Property 3.6 [36], [38], [11].



▶ $PO \cdot QO = OF^2$

Property 3.7 [5]

- AB is a fixed chord through O
- P is a variable point on L

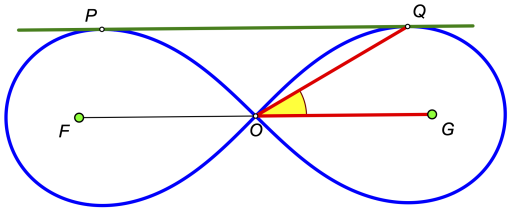


▶ $\angle OPB - \angle APO$ is invariant

4. Tangents

Property 4.1* [21]

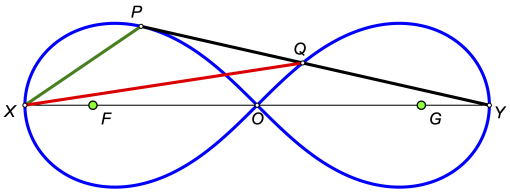
- PQ is a tangent parallel to FG



▶ red lengths are equal
▶ yellow angle is 30°
▶ $\angle OQG = 75°$

Property 4.2* [25]

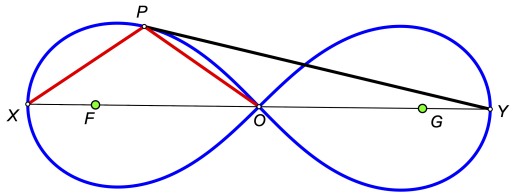
- YP is a tangent



▶ red length = 2 × green length

Property 4.3* [23]

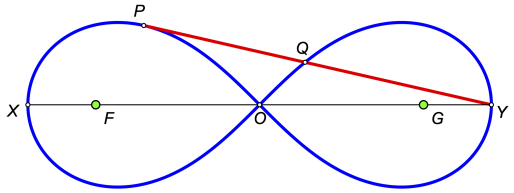
- YP is a tangent



▶ red lengths are equal

Property 4.4* [24]

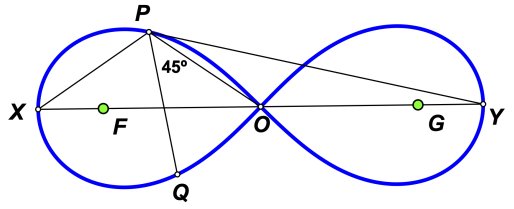
- YP is a tangent



▶ $\dfrac{YQ}{PQ} = \dfrac{2}{\sqrt{3}}$

Property 4.5 [4]

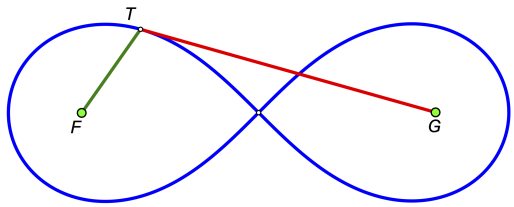
- YP is a tangent



▶ $\angle XPQ = \angle QPY$

Property 4.6* [19]

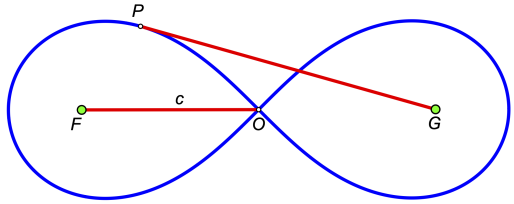
- GT is a tangent.



▶ red length = 3 × green length

Property 4.7* [19]

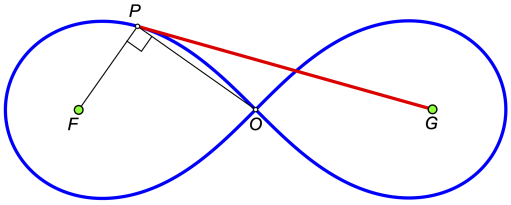
- GP is a tangent.



▶ $GP = c\sqrt{3}$

Property 4.8*

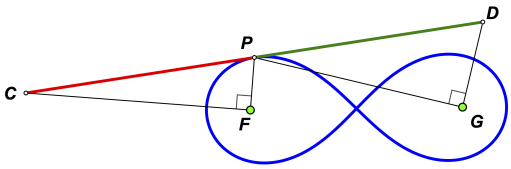
- GP is a tangent.



▶ $\angle FPO = 90°$

Property 4.9 [35], [3, §248]

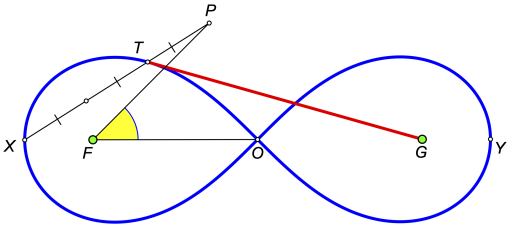
- CD is a tangent.



▶ red length = green length

Property 4.10* [19]

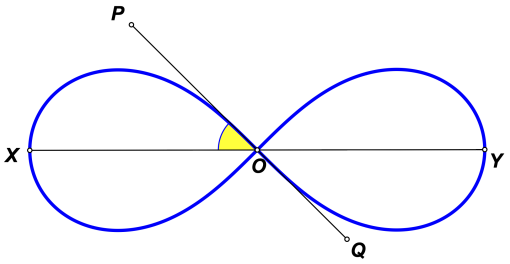
- GT is a tangent.
- $XT = 2 \cdot TP$



▶ yellow angle is 45°

Property 4.11 [40]

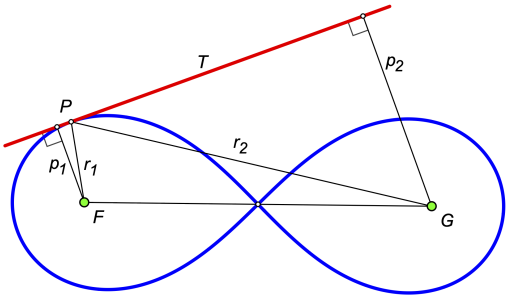
- PQ is a tangent



▶ $\angle POX = 45°$

Property 4.12 [28], [34]

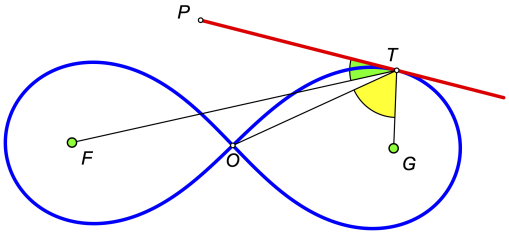
- T is a tangent



▶ $\dfrac{p_1}{r_1^2} + \dfrac{p_2}{r_2^2} = \left|\dfrac{1}{r_1} - \dfrac{1}{r_2}\right| \sqrt{2}$

Property 4.13* [22]

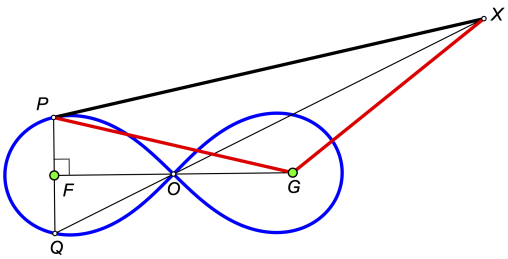
- PT is a tangent



▶ sum of colored angles is 90°

Property 4.14* [20]

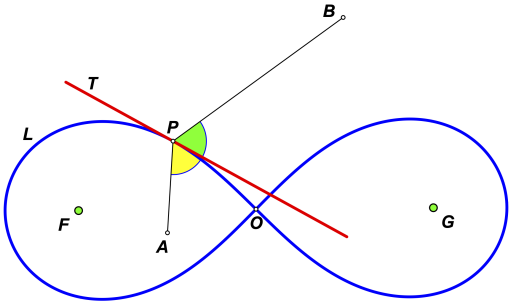
- PX is a tangent



▶ red lengths are equal

Property 4.15* [27]

- T is tangent to L at P
- A is the circumcenter of $\triangle FPO$
- B is the circumcenter of $\triangle GPO$

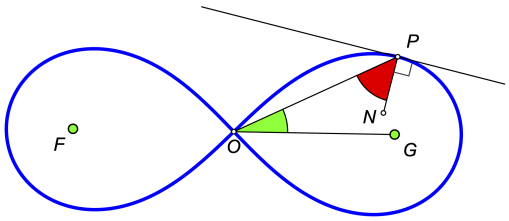


▶ T bisects $\angle APB$

5. Normals

Property 5.1 [2, §12.3]

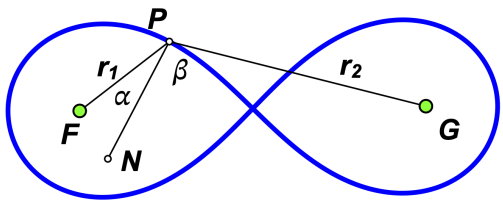
- PN is a normal



▶ red angle is twice the green angle

Property 5.2 [35]

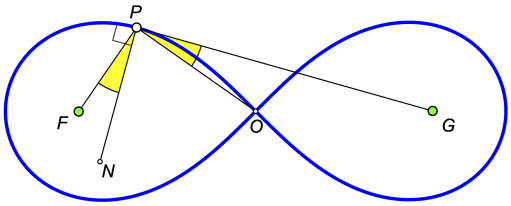
- PN is a normal
- $\angle FPN = \alpha, \quad \angle NPG = \beta$



▶ $\dfrac{r_1}{r_2} = \dfrac{\sin\alpha}{\sin\beta}$

Property 5.3 [14], [3, §249]

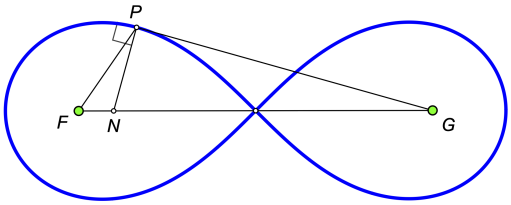
- PN is a normal



▶ yellow angles are equal

Property 5.4 [3, §249]

- PN is a normal

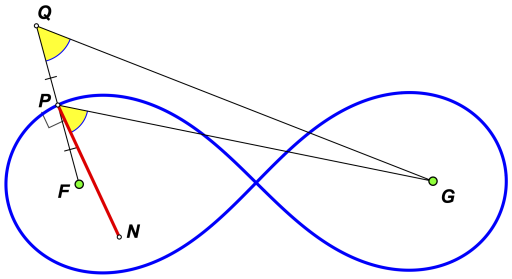


▶ $\left(\dfrac{PF}{PG}\right)^2 = \dfrac{FN}{NG}$

▶ PN is a symmedian of $\triangle FPG$

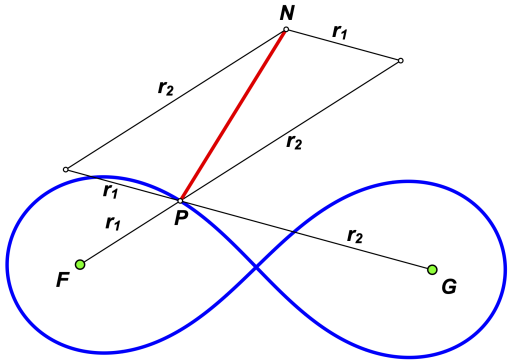
Property 5.5 [29]

- PN is a normal



▶ yellow angles are equal

Property 5.6 [31]

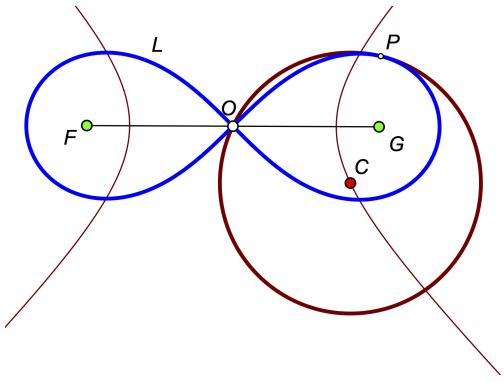


▶ red line is a normal

6. Tangent circle

Property 6.1 [1]

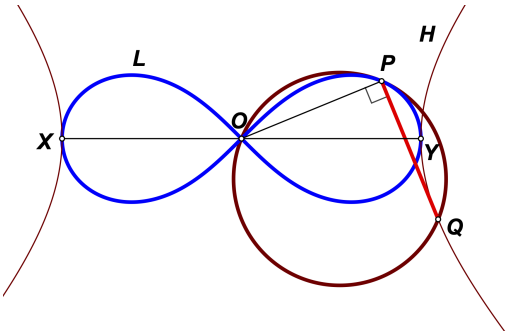
- brown circle is tangent to L at P
- C is the center of the brown circle



▶ C lies on the rectangular hyperbola with foci F and G

Property 6.2 [8]

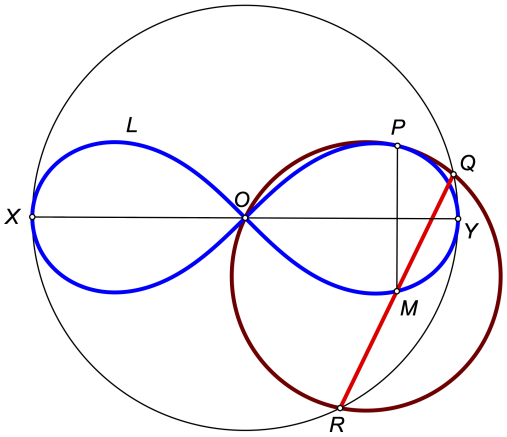
- brown circle is tangent to L at P
- H is the inverse of L about $O(X)$



▶ PQ is tangent to H at Q

Property 6.3* [13]

- brown circle is tangent to L at P
- M is the midpoint of QR

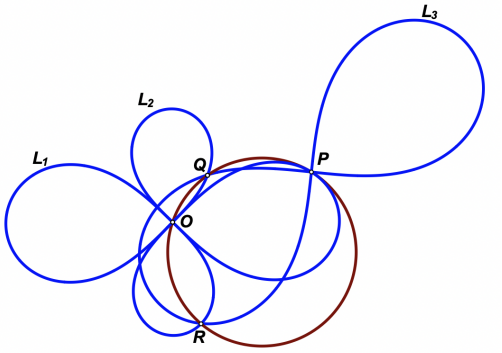


▶ M lies on L

▶ $PM \perp XY$

Property 6.4 [37]

- brown circle is tangent to L_1 at P
- Axes of L_1 and L_2 are perpendicular
- brown circle meets L_2 at Q and R
- lemnicate L_3 has center P
- L_3 passes through Q and R
- arc length of L_i is a_i, $i = 1, 2, 3$



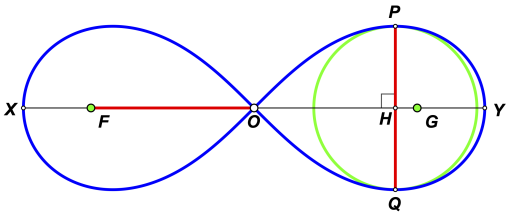
▶ $a_1^2 + a_2^2 = a_3^2$

▶ For related results, see [39]

7. Incircle

Property 7.1* [10]

- green circle is an incircle

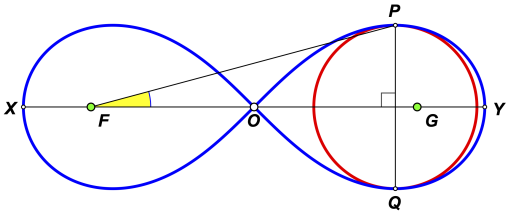


▶ red lengths are equal

▶ $\angle HOP = 30°$

Property 7.2*

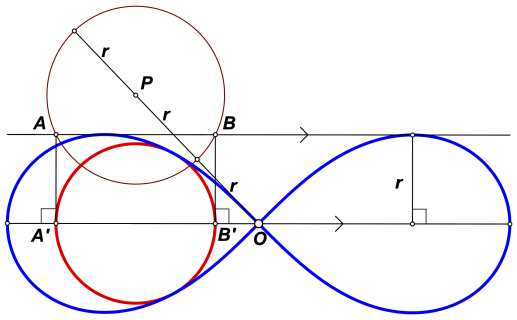
- red circle is an incircle



▶ yellow angle is 15°

Property 7.3 [26]

- $OP = 2r$

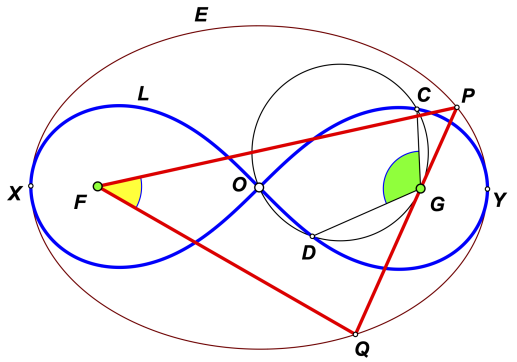


▶ circle with diameter $A'B'$ touches the lemniscate

8. Confocal Ellipse

Property 8.1 [36]

- C is any point on L
- $D = \odot OCG \cap L$
- E is an ellipse with foci F and G
- PQ is tangent to $\odot OCG$ at G

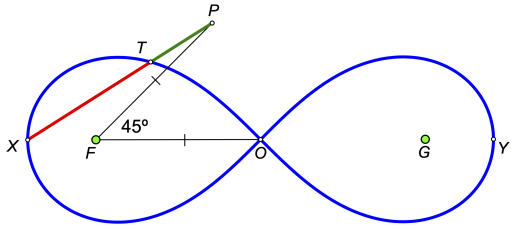


▶ $2\angle CGD - \angle QFP = 180°$

9. Miscellaneous

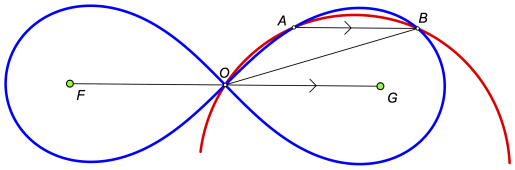
Property 9.1* [19]

- $FO = FP$



▶ red length is twice green length

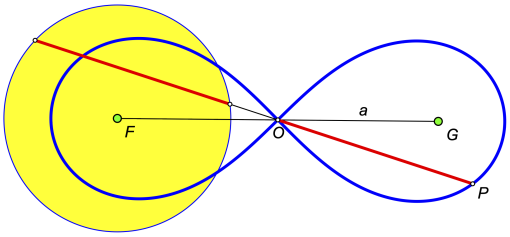
Property 9.2 [18], [38]



▶ circumradius of $\triangle OAB = OG$

Property 9.3 [1], [2, §12.2]

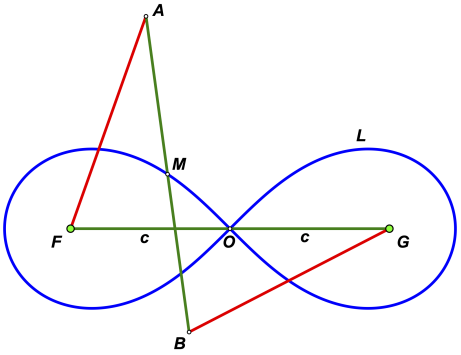
- Yellow circle has radius $a/\sqrt{2}$



▶ red lengths are equal

Property 9.4 [1]

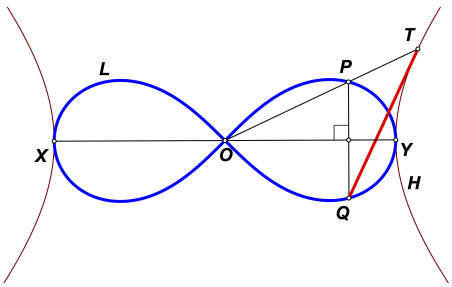
- $AF = GB = c\sqrt{2}$
- $AB = 2c$
- $AB \cap L = M$



▶ $AM = MB$

Property 9.5 [6]

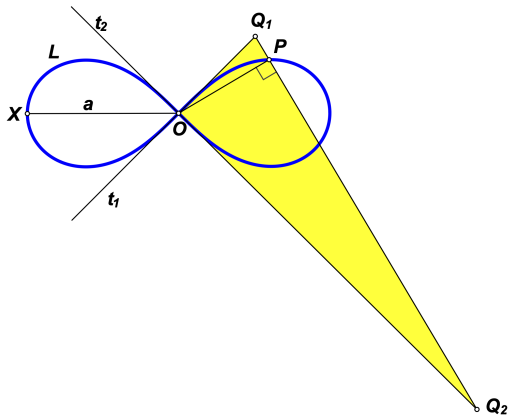
- H is the inverse of L about $O(X)$
- $P \in L$
- OP meets H at T



▶ QT is tangent to H

Property 9.6 [9]

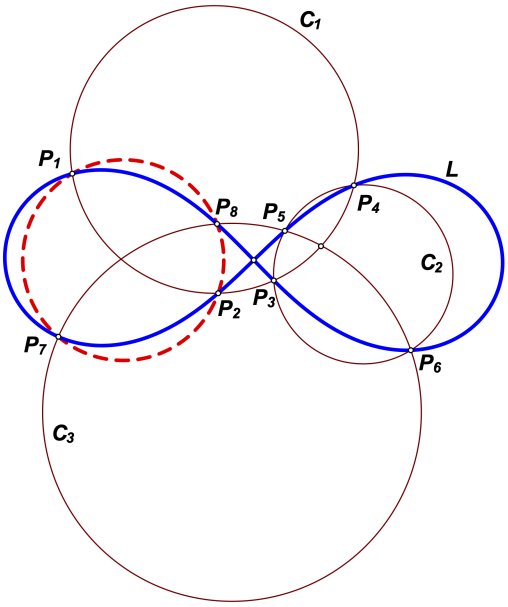
- t_1 and t_2 are tangents at O
- $P \in L$



▶ $[OQ_1Q_2] = a^2$

Property 9.7 [12]

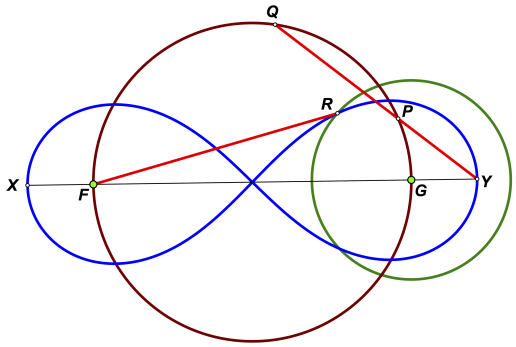
- $\odot C_1$ meets L at P_1, P_2, P_3, P_4
- $\odot C_2$ meets L at P_3, P_4, P_5, P_6
- $\odot C_3$ meets L at P_5, P_6, P_7, P_8



▶ P_1, P_2, P_7, P_8 are concyclic

Property 9.8 [32, pp. 189–190]

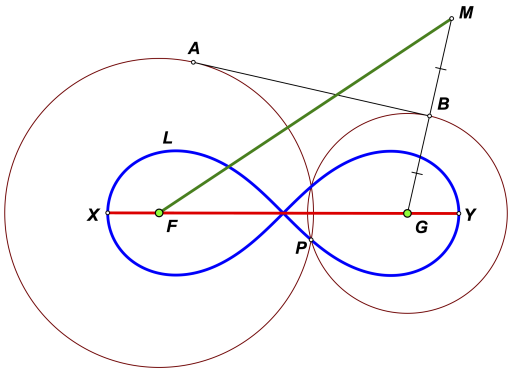
- brown circle has diameter FG
- green circle has center G, radius PY



▶ red lengths are equal

Property 9.9 [30, pp. 189–190]

- $P \in L$
- AB is common tangent to $F(P), G(P)$



▶ red length = green length

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