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More Shapes of Central Quadrilaterals

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Abstract. Let E be a point in the plane of a convex quadrilateral $ABCD$. The lines from E to the vertices of the quadrilateral form four triangles. If we locate a triangle center in each of these triangles, the four triangle centers form another quadrilateral called a central quadrilateral. For each of various shaped quadrilaterals, and each of 1000 different triangle centers, and for various choices for E, we examine the shape of the central quadrilateral. Using a computer, we determine when the central quadrilateral has a special shape, such as being a rhombus or a cyclic quadrilateral. A typical result is the following. Let E be the centroid of equidiagonal quadrilateral $ABCD$. Let F, G, H, and I be the X_{591}-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is an orthodiagonal quadrilateral.

Keywords. triangle centers, quadrilaterals, computer-discovered mathematics, Euclidean geometry, GeometricExplorer.

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1. Introduction

In this study, $ABCD$ always represents a convex quadrilateral known as the *reference quadrilateral*. A point E in the plane of the quadrilateral (not on the boundary) is chosen and will be called the *radiator*. The radiator can be an arbitrary point or it can be a notable point associated with the quadrilateral. Lines are drawn from the radiator to the vertices of the reference quadrilateral forming four triangles with the sides of the quadrilateral as shown in Figure 1. These triangles will be called the *radial triangles*.

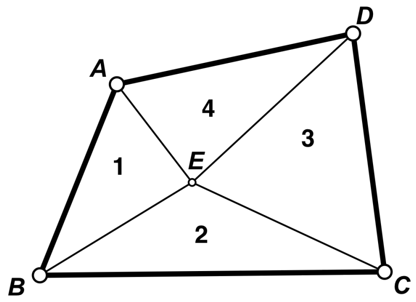

FIGURE 1. Radial Triangles

In the figure, the radial triangles have been numbered in a counterclockwise order starting with side AB: $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, $\triangle DAE$. Triangle centers (such as the incenter, centroid, or circumcenter) are selected in each triangle. The same type of triangle center is used with each radial triangle. In order, the names of these points are F, G, H, and I as shown in Figure 2. These four centers form a quadrilateral $FGHI$ that will be called the *central quadrilateral* (of quadrilateral $ABCD$ with respect to E). Quadrilateral $FGHI$ need not be convex.

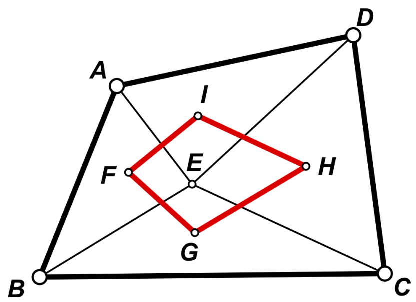

FIGURE 2. Central Quadrilateral

The purpose of this paper is to determine when a central quadrilateral has a special shape, such as being a rhombus or a cyclic quadrilateral.

2. Types of Quadrilaterals Studied

We are only interested in reference quadrilaterals that have a certain amount of symmetry. For example, we excluded bilateral quadrilaterals (those with two equal sides), bisect-diagonal quadrilaterals (where one diagonal bisects another), right kites, right trapezoids, and golden rectangles. The types of quadrilaterals we studied are shown in Table 1. The sides of the quadrilateral, in order, have lengths a, b, c, and d. The diagonals have lengths p and q. The measures of the angles of the quadrilateral, in order, are A, B, C, and D.

Table 1.

Types of Quadrilaterals Considered		
Quadrilateral Type	**Geometric Definition**	**Algebraic Condition**
general	convex	none
cyclic	has a circumcircle	$A+C=B+D$
tangential	has an incircle	$a+c=b+d$
extangential	has an excircle	$a+b=c+d$
parallelogram	opposite sides parallel	$a=c$, $b=d$
equalProdOpp	product of opposite sides equal	$ac=bd$
equalProdAdj	product of adjacent sides equal	$ab=cd$
orthodiagonal	diagonals are perpendicular	$a^2+c^2=b^2+d^2$
equidiagonal	diagonals have the same length	$p=q$
Pythagorean	equal sum of squares, adjacent sides	$a^2+b^2=c^2+d^2$
kite	two pair adjacent equal sides	$a=b$, $c=d$
trapezoid	one pair of opposite sides parallel	$A+B=C+D$
rhombus	equilateral	$a=b=c=d$
rectangle	equiangular	$A=B=C=D$
Hjelmslev	two opposite right angles	$A=C=90°$
isosceles trapezoid	trapezoid with two equal sides	$A=B$, $C=D$
APquad	sides in arithmetic progression	$d-c=c-b=b-a$

The following combinations of entries in the above list were also considered: bicentric quadrilaterals (cyclic and tangential), exbicentric quadrilaterals (cyclic and extangential), bicentric trapezoids, cyclic orthodiagonal quadrilaterals, equidiagonal kites, equidiagonal orthodiagonal quadrilaterals, equidiagonal orthodiagonal trapezoids, harmonic quadrilaterals (cyclic and equalProdOpp), orthodiagonal trapezoids, tangential trapezoids, and squares (equiangular rhombi).

So, in addition to the general convex quadrilateral, a total of 28 types of quadrilaterals were considered in this study.

A graph of the types of quadrilaterals considered is shown in Figure 3. An arrow from A to B means that any quadrilateral of type B is also of type A. For example: all squares are rectangles and all kites are orthodiagonal. If a directed path leads from a quadrilateral of type A to a quadrilateral of type B, then we will say that A is an *ancestor* of B. For example, an equidiagonal quadrilateral is an ancestor of a rectangle. In other words, all rectangles are equidiagonal.

Unless otherwise specified, when we give a theorem about a quadrilateral, we will omit an entry for a particular shape quadrilateral if the property is known to be true for an ancestor of that quadrilateral.

We do not include results where the central quadrilateral degenerates to a line segment or a point.

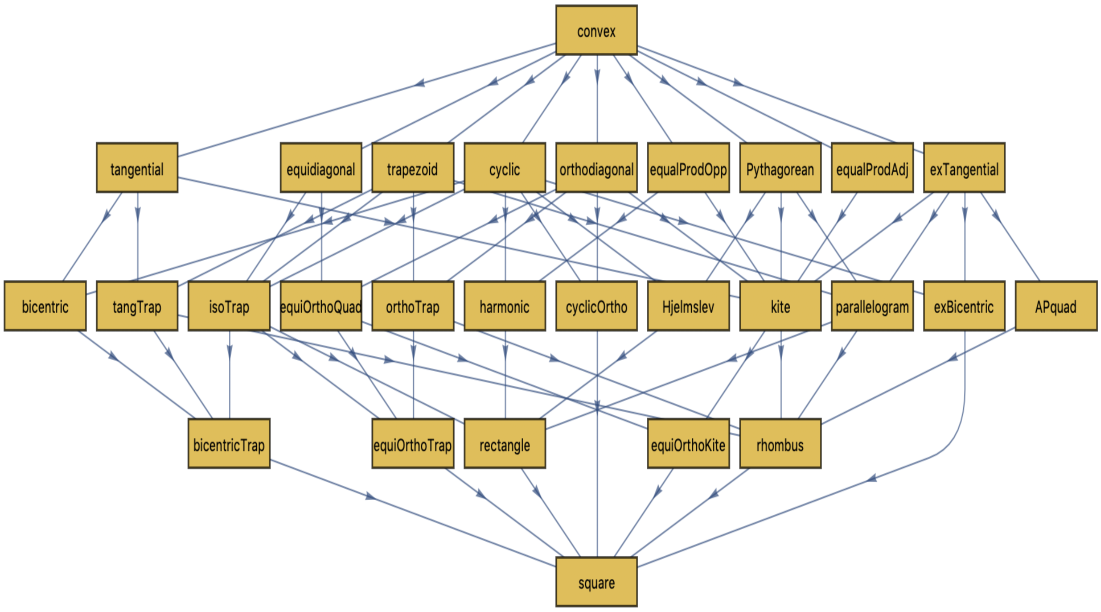

FIGURE 3. Quadrilateral Shapes

3. METHODOLOGY

In this study, we locate triangle centers in the four radial triangles. We use Clark Kimberling's definition of a triangle center [7]. More details can be found in section 3 of [10].

We used a computer program called GeometricExplorer to examine the shape of the central quadrilateral. Starting with each type of quadrilateral listed in Figure 3 for the reference quadrilateral, we picked various choices for point E, the radiator. The types of radiators studied are shown in Table 2.

TABLE 2.

Points Used as Radiators	
name	**description**
arbitrary point	any point in the plane of $ABCD$
diagonal point	intersection of the diagonals (QG–P1)
Poncelet point	(QA–P2)
Steiner point	(QA–P3)
vertex centroid	(QA-P1)
area centroid	(QG-P4)

A code in parentheses represents the name for the point as listed in the Encyclopedia of Quadri-Figures [12].

For each n from 1 to 1000, we placed center X_n in each of the radial triangles of the reference quadrilateral. The program then analyzes the central quadrilateral formed by these four centers and reports if the central quadrilateral has a special shape. Points at infinity were omitted. GeometricExplorer uses numerical coordinates (to 15 digits of precision) for locating all the points. This does not constitute a proof that the result is correct, but gives us compelling evidence for the validity of the result.

If a theorem in this paper is accompanied by a figure, this means that the figure was drawn using either Geometer's Sketchpad or GeoGebra. In either case, we used the drawing program to dynamically vary the points in the figure. Noticing that the result remains true as the points vary offers further evidence that the theorem is true.

To prove the results that we have discovered, we use geometric methods, when possible. If we could not find a purely geometrical proof, we turned to analytic methods using barycentric coordinates and performing exact symbolic computation using Mathematica. All proofs can be found in the Mathematica notebooks included in the supplementary material associated with the paper.

If our only "proof" of a particular relationship is by using numerical calculations (and not using exact computation), then we have colored the center red in the table of relationships.

4. Results Using an Arbitrary Point

In this configuration, the radiator, E, is any point in the plane of the reference quadrilateral $ABCD$, not on the boundary.

Our computer analysis found only one special shape associated with all quadrilaterals when E is an arbitrary point in the plane. We examined all the types of quadrilaterals listed in Table 1 and all triangle centers from X_1 to X_{1000}. The special shape occurs only when the chosen center is X_2, the centroid. The result is shown below.

Central Quadrilateral of a General Quadrilateral	
Shape of central quadrilateral	**center**
parallelogram	2

Theorem 4.1. *Let E be an arbitrary point in the plane of convex quadrilateral $ABCD$. Let F, G, H, and I be the centroids of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 4). Then $FGHI$ is a parallelogram. The sides of the parallelogram are parallel to the diagonals of $ABCD$.*

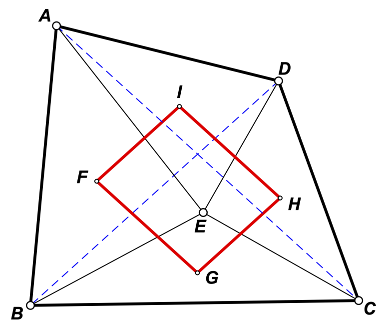

FIGURE 4. General quadrilateral: centroids $\Longrightarrow$ parallelogram

A geometric proof is straightforward. We start with a lemma.

Lemma 4.2. *Let E be an arbitrary point in the plane of $\triangle ABC$. Let F be the centroid of triangle $\triangle ABE$ and let G be the centroid of $\triangle ACE$(Figure 5). Then $FG \parallel BC$ and $FG = BC/3$.*

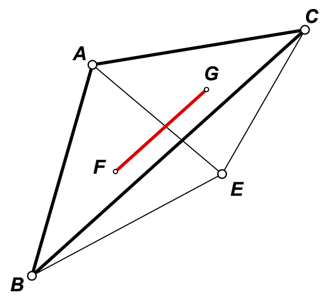

FIGURE 5.

Proof. Let AP and AQ be the medians of triangles AEB and AEC, respectively (Figure 6). Then $AF/FP = 2$ and $AG/GQ = 2$ which implies $FG \parallel PQ$ and $FG = \frac{2}{3}PQ$. Since P and Q are the midpoints of EB and EC, respectively, we have $BP/PE = 1$ and $CQ/QE = 1$ which implies that $PQ \parallel BC$ and $PQ = BC/2$. Thus, $FG \parallel BC$ and $FG = \frac{2}{3}PQ = \frac{2}{3}(\frac{1}{2}BC) = \frac{1}{3}BC$.

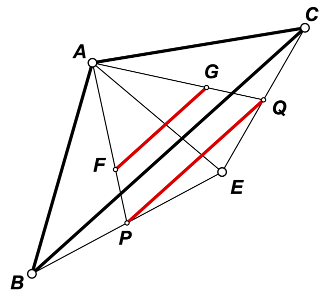

Figure 6.

□

Now on to the proof of Theorem 4.1. Refer back to Figure 4.

Proof. By Lemma 4.2, $FI \parallel BD$. Similarly, $GH \parallel BD$. Thus, $FI \parallel GH$. In the same way, $FG \parallel IH$. Hence, $FGHI$ is a parallelogram. □

Our computer study found special shapes associated with equidiagonal and orthodiagonal quadrilaterals. The results are shown in the following three tables.

Central Quadrilateral of an Equidiagonal Quadrilateral	
Shape of central quadrilateral	**center**
rhombus	2

Theorem 4.3. *Let E be an arbitrary point in the plane of equidiagonal quadrilateral $ABCD$. Let F, G, H, and I be the centroids of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 4). Then $FGHI$ is a rhombus.*

Proof. By Lemma 4.2, $FI = \frac{1}{3}BD$ and $FG = \frac{1}{3}AC = \frac{1}{3}BD$, so $FI = FG$. But a parallelogram with two equal adjacent sides is a rhombus. □

Central Quadrilateral of an Orthodiagonal Quadrilateral	
Shape of central quadrilateral	**center**
rectangle	2

Theorem 4.4. *Let E be an arbitrary point in the plane of orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the centroids of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 4). Then $FGHI$ is a rectangle.*

Proof. By Lemma 4.2, $FI \parallel BD$ and $FG \parallel AC$. Since $BD \perp AC$, we can conclude that $FI \perp FG$. But a parallelogram with two perpendicular adjacent sides is a rectangle. □

Central Quadr of an Equidiagonal Orthodiagonal Quadr	
Shape of central quadrilateral	**center**
square	2

Theorem 4.5. *Let E be an arbitrary point in the plane of equidiagonal orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the centroids of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 4). Then $FGHI$ is a square.*

Proof. By Theorem 4.3, $FGHI$ is a rhombus. By Theorem 4.4, $FGHI$ is a rectangle. But a figure that is both a rhombus and a rectangle must be a square. □

Open Question 1. *Is the centroid the only triangle center for which the central quadrilateral is a parallelogram?*

Our computer study found several interesting results for the central quadrilateral of a rectangle. These are shown in the following table.

The symbol $\mathbb{S}$ denotes the set of all triangle centers that lie on the Euler line of the reference triangle and have constant Shinagawa coefficients. Shinagawa coefficients are defined in [4]. The first few n for which X_n has constant Shinagawa coefficients are n =2, 3, 4, 5, 20, 140, 376, 381, 382, 546-550, 631, and 632.

Central Quadrilaterals of Rectangles	
Shape of central quad	**centers**
orthodiagonal	$\mathbb{S}$

Theorem 4.6. *Let E be an arbitrary point in the plane of rectangle $ABCD$. Let X be a triangle center with constant Shinagawa coefficients. Let F, G, H, and I be the X-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is orthodiagonal. The diagonals of $FGHI$ are parallel to the sides of $ABCD$. Figure 7 shows the case when X is the de Longchamps Point (X_{20}). Figure 8 shows the case when X is the X_{381} point.*

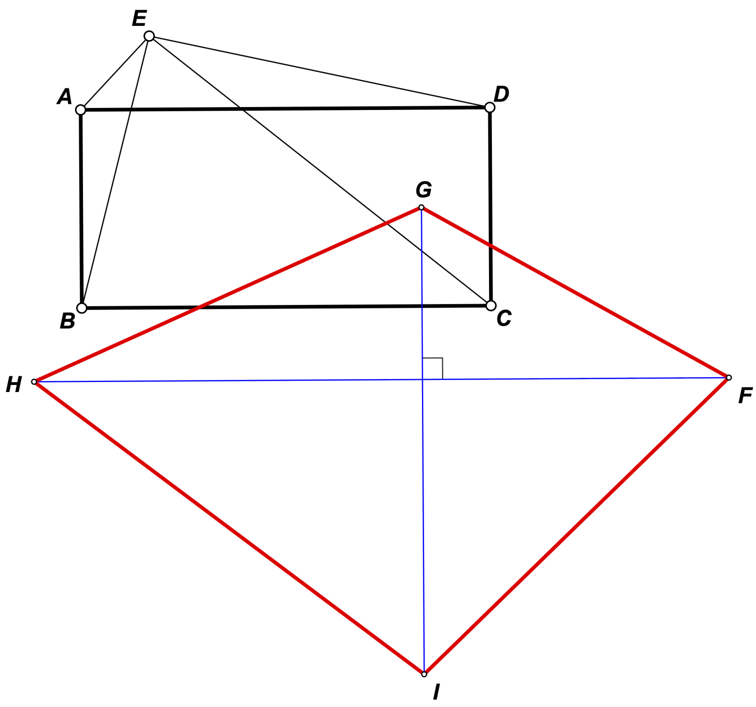

Figure 7. rectangle, X_{20}-points $\implies$ orthodiagonal

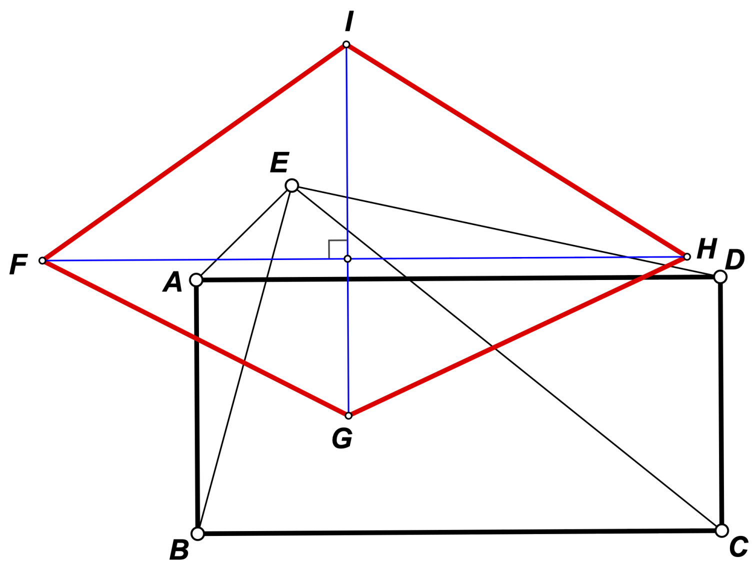

Figure 8. rectangle, X_{381}-points $\implies$ orthodiagonal

Our computer study found several interesting results for the central quadrilateral of a square. These are shown in the following table. Results that are true for rectangles or equidiagonal orthodiagonal quadrilaterals are omitted.

Central Quadrilaterals of Squares	
Shape of central quad	**centers**
square	2
cyclic	99, 925
equidiagonal orthodiagonal	372, 373, 640

Theorem 4.7. *Let E be an arbitrary point in the plane of square $ABCD$. Let F, G, H, and I be the centroids of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 9). Then $FGHI$ is a square.*

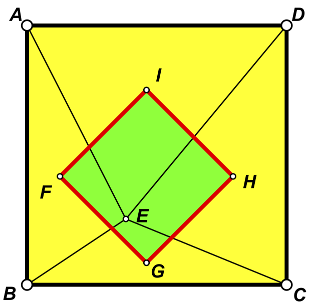

FIGURE 9. square, X_2-points $\implies$ square

Proof. This is Theorem 6.2 in [10]. □

Theorem 4.8. *Let E be an arbitrary point in the plane of square $ABCD$. Let n be 99 or 925. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is cyclic and E lies on the circle $FGHI$. (Figure 10 shows the case when $n = 99$.)*

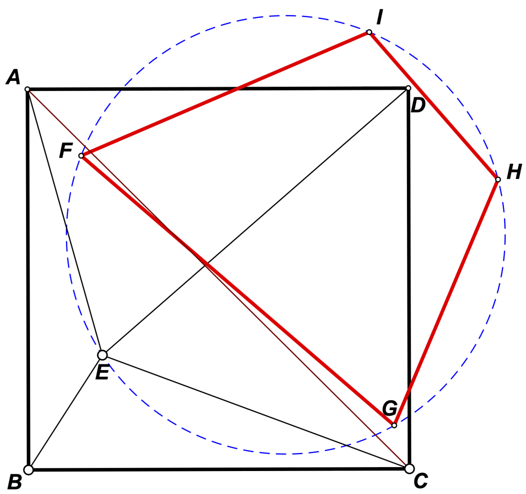

FIGURE 10. square, X_{99}-points $\implies$ cyclic

Theorem 4.9. *Let E be an arbitrary point in the plane of square $ABCD$. Let n be 372 or 640. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 11 shows the case when $n = 372$). Then $FGHI$ is an equidiagonal orthodiagonal quadrilateral. The diagonals of $FGHI$ are parallel to the sides of $ABCD$.*

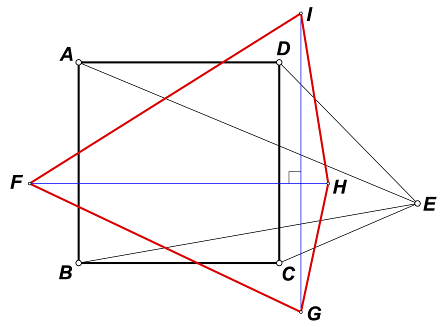

Figure 11. square, X_{372}-points $\implies$ equi-ortho

Theorem 4.10. *Let E be an arbitrary point in the plane of square $ABCD$. Let F, G, H, and I be the X_{373}-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 12). Then $FGHI$ is an equidiagonal orthodiagonal quadrilateral. (Note: The diagonals of $FGHI$ are not necessarily parallel to the sides of $ABCD$.)*

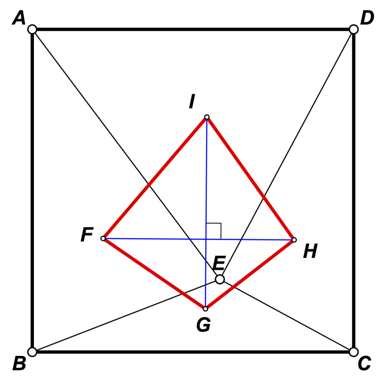

Figure 12. square, X_{373}-points $\implies$ equi-ortho

5. Point E Restricted to a Line

In the previous section, point E could be any point in the plane. If point E is restricted to be located on certain lines associated with the quadrilateral, then some interesting results are obtained. They are shown in the following two tables.

Central Quadrilaterals of Kites	
Shape of central quad	**centers**
isosceles trapezoid	all

Theorem 5.1. *Let E be any point on diagonal AC of kite $ABCD$ (with $AB = AD$). Let X be any triangle center. Let F, G, H, and I be the X-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 13). Then $FGHI$ is an isosceles trapezoid.*

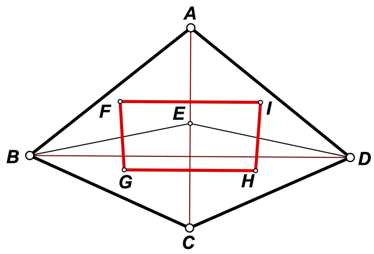

FIGURE 13. kite, X-points $\Longrightarrow$ isosceles trapezoid

Proof. Since $\triangle AED$ is the reflection of $\triangle AEB$ about AE, then I is the reflection of F about AE. In particular, $FI \perp AE$. Similarly, $GH \perp AE$. Therefore $FI \parallel GH$, so $FGHI$ is a trapezoid. Furthermore, IH is the reflection of FG about AC, so $IH = FG$. Hence, $FGHI$ is an isosceles trapezoid. □

Central Quadrilaterals of Isosceles Trapezoids	
Shape of central quad	**centers**
kite	all

Theorem 5.2. *Let E be any point on the perpendicular bisector of side BC of isosceles trapezoid $ABCD$ (with $AD \parallel BC$ and $AB = CD$). Let X be any triangle center. Let F, G, H, and I be the X-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 14). Then $FGHI$ is a kite.*

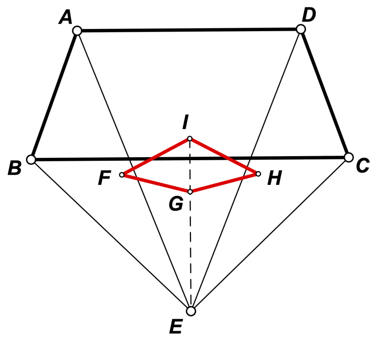

FIGURE 14. isosceles trapezoid, X-points $\Longrightarrow$ kite

Proof. Let m be the perpendicular bisector of BC. Since $\triangle DCE$ is the reflection of $\triangle ABE$ about m, then H is the reflection of F about m. Hence $FH \perp m$. Since $\triangle BEC$ is isosceles, G lies on m. Since $\triangle AED$ is isosceles, I lies on m. Therefore, IG coincides with m, and hence $IG \perp FH$. Moreover, IG bisects FH because H is the reflection of F about m. Therefore, $FGHI$ is a kite. □

6. RESULTS USING THE VERTEX CENTROID

A *bimedian* of a quadrilateral is the line segment joining the midpoints of two opposite sides.

The *centroid* (or vertex centroid) of a quadrilateral is the point of intersection of the bimedians (Figure 15). The centroid bisects each bimedian.

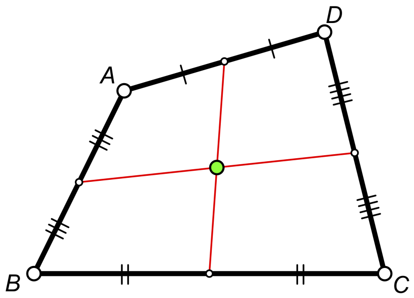

FIGURE 15. Centroid of a quadrilateral

In this section, we study the case where point E is the centroid of the quadrilateral. Results that are true when point E is arbitrary are omitted.

Our computer study found a unique shape for the central quadrilateral of an equidiagonal quadrilateral. The result is shown in the following table.

Central Quadrilaterals of Equidiagonal Quads	
Shape of central quad	**centers**
orthodiagonal	591

Theorem 6.1. *Let E be the vertex centroid of equidiagonal quadrilateral $ABCD$. Let F, G, H, and I be the X_{591}-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 16). Then $FGHI$ is an orthodiagonal quadrilateral.*

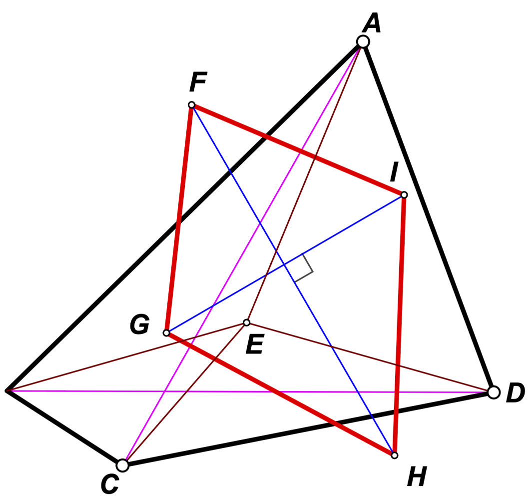

FIGURE 16. equidiagonal quadrilateral, X_{591}-points $\implies$ orthodiagonal

Our computer study found additional results if quadrilateral $ABCD$ is also orthodiagonal. The results are shown in the following table.

Central Quads of Equidiagonal Orthodiagonal Quads	
Shape of central quad	**centers**
parallelogram	491, 615

Theorem 6.2. *Let E be the vertex centroid of equidiagonal orthodiagonal quadrilateral $ABCD$. Let n be 491 or 615. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a parallelogram. (Figure 17 shows the case when $n = 491$.)*

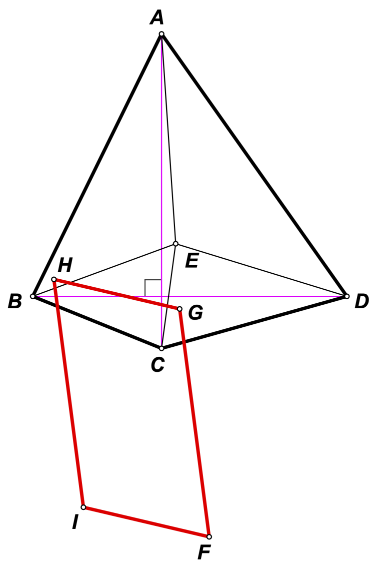

FIGURE 17. equidiagonal orthodiagonal quad, X_{491}-points $\implies$ parallelogram

7. Results Using the Steiner Point

A *midray circle* of a quadrilateral is the circle through the midpoints of the line segments joining one vertex of the quadrilateral to the other vertices.

The *Steiner point* (sometimes called the Gergonne-Steiner point) of a quadrilateral is the common point of the midray circles of the quadrilateral.

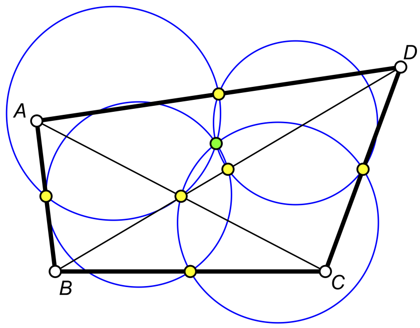

Figure 18. Steiner point of quadrilateral $ABCD$

Figure 18 shows the Steiner point of quadrilateral $ABCD$. The yellow points represent the midpoints of the sides and diagonals of the quadrilateral. The blue circles are the midray circles. The common point of the four circles is the Steiner point (shown in green).

In this section, we study the case where point E is the Steiner point of the quadrilateral. Results that are true when point E is arbitrary are omitted. Before stating the results, we give some lemmas and definitions.

Proposition 7.1. *The Steiner point of a cyclic quadrilateral is the circumcenter of the quadrilateral.*

Proof. This is Proposition 10.2 of [10]. □

The following lemma comes from [13].

Lemma 7.2. *The center X_n of $\triangle ABC$ lies on the circumcircle of $\triangle ABC$ for the following values of n:*

74, 98–112, 476, 477, 675, 681, 689, 691, 697, 699, 701, 703, 705, 707, 709, 711, 713, 715, 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 745, 747, 753, 755, 759, 761, 767, 769, 773, 777, 779, 781, 783, 785, 787, 789, 791, 793, 795, 797, 803, 805, 807, 809, 813, 815, 817, 819, 825, 827, 831, 833, 835, 839–843, 898, 901, 907, 915, 917, 919, 925, 927, 929–935, 953, 972.

The following two lemmas comes from [10].

Lemma 7.3. *Let ABC be an isosceles triangle with $AB = AC$. Then the center X_n coincides with A for the following values of n:*

59, 99, 100, 101, 107, 108, 109, 110, 112, 162, 163, 190, 249, 250, 476, 643, 644, 645, 646, 648, 651, 653, 655, 658, 660, 662, 664, 666, 668, 670, 677, 681, 685, 687, 689, 691, 692, 765, 769, 771, 773, 777, 779, 781, 783, 785, 787, 789, 791, 793, 795, 797, 799, 803, 805, 807, 809, 811, 813, 815, 817, 819, 823, 825, 827, 831, 833, 835, 839, 874, 877, 880, 883, 886, 889, 892, 898, 901, 906, 907, 919, 925, 927, 929, 930, 931, 932, 933, 934, 935.

Lemma 7.4. *Let ABC be an isosceles triangle with $AB = AC$. Let M be the midpoint of BC. Then the center X_n coincides with M for the following values of n:*

11, 115, 116, 122–125, 127, 130, 134–137, 139, 244–247, 338, 339, 865-868.

The symbol $\mathbb{M}$ denotes these points.

Lemma 7.5. *Let ABC be an isosceles triangle with $AB = AC$. Let P be the antipode of point A with respect to the circumcircle of $\triangle ABC$. Then the center X_n coincides with P for the following values of n:*

74, 98, 102–106, 111, 477, 675, 697, 699, 701, 703, 705, 707, 709, 711, 713, 715, 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 745, 747, 753, 755, 759, 761, 767, 840–843, 915, 917, 953, 972.

Proof. This follows from Lemmas 7.2 and 7.3. □

The symbol $\mathbb{T}$ denotes these points.

Our computer study found several results for the central quadrilateral of cyclic quadrilaterals. They are shown in the following table.

Central Quadrilaterals of Cyclic Quadrilaterals	
Shape of central quad	**centers**
parallelogram	$\mathbb{M}$, 148–150, 290, 402, 620, 671, 903
tangential	$\mathbb{T}$, 3, 399

Theorem 7.6. *Let E be the Steiner point of cyclic quadrilateral $ABCD$. Let n be 3 or 399. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. (Figure 19 shows the case when $n = 3$.) Then $FGHI$ is tangential with incenter E. The incircle of $FGHI$ is concentric with the circumcircle of $ABCD$ and the inradius of $FGHI$ is half the circumradius of $ABCD$.*

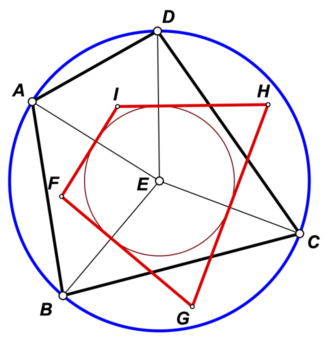

Figure 19. cyclic, X_3-points $\implies$ tangential

For the case $n = 3$, we can give a purely geometrical proof.

Proof. By Proposition 7.1, E is the circumcenter of quadrilateral $ABCD$. The points F, G, H, I are the circumcenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Line AE is the radical axis of of circles (AEB) and (AED). Let $X = AE \cap FI$. Clearly we have $AE \perp FI$ and $EX = AE/2$. Define Y, Z, and W similarly (Figure 20). Similarly we have $BE \perp FG$, $CE \perp GH$,

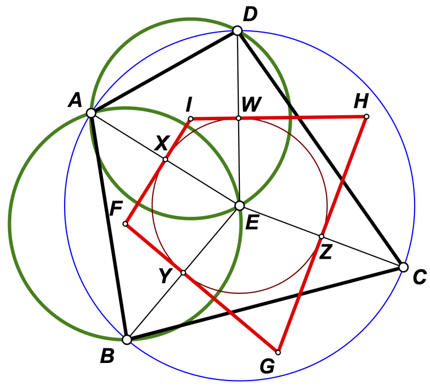

Figure 20. case when $n = 3$

and $DE \perp HI$, and $EY = EB/2$, and $EZ = EC/2$, and $EW = ED/2$. Since $EA = EB = EC = ED$, we have $EX = EY = EZ = EW$, so E is the incenter of quadrilateral $FGHI$. Therefore $FGHI$ is a tangential quadrilateral. □

Theorem 7.7. *Let E be the Steiner point of cyclic quadrilateral $ABCD$. Let $X_n \in \mathbb{T}$. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 21). Then $FGHI$ is a tangential quadrilateral with incenter E. The incircle of $FGHI$ coincides with the circumcircle of $ABCD$.*

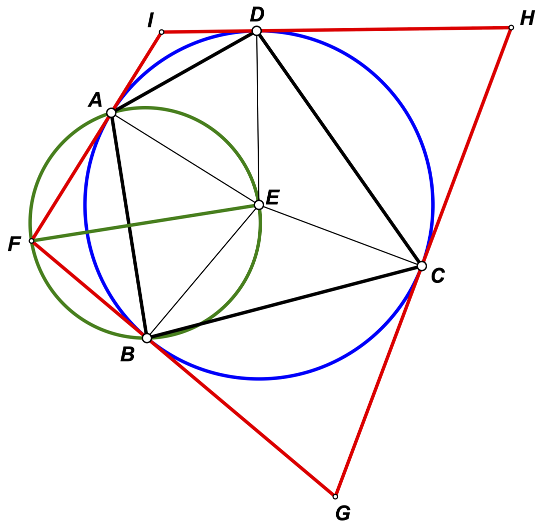

Figure 21. cyclic, $X_n \in \mathbb{T} \implies FGHI$ tangential

Proof. By Proposition 7.1, E is the circumcenter of quadrilateral $ABCD$. Since $EA = EB$, $\triangle EAB$ is isosceles with vertex E, so F is the antipode of E with respect to the circumcircle of $\triangle EAB$ by Lemma 7.5. Therefore, $\angle FAE = 90°$. Similarly, $\angle IAE = 90°$. Hence I, A, and F are collinear and $EA \perp FI$. Similarly, $EB \perp FG$, $EC \perp GH$, and $ED \perp HI$. Since $EA = EB = EC = ED$, it follows that $FGHI$ is a tangential quadrilateral and the incenter of $FGHI$ coincides with E. □

Theorem 7.8. *Let E be the Steiner point of cyclic quadrilateral $ABCD$. Let n be 11, 115, 116, 122-125, 127, 130, 134–137, 139, 148–150, 244–247, 290, 338, 339, 402, 620, 671, 865–868, or 903. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a parallelogram.*

Proof. If n is 11, 115, 116, 122–125, 127, 130, 134–137, 139, 244–247, 338, 339, 865–868, the result is true by the proof of Theorem 9.3 of [10].

If n is 402 or 620, the result is true by the proof of Theorem 9.2 of [10].

If n is 290, 671, or 903, the result is true by the proof of Theorem 9.8 of [10].

If n is 148, 149, or 150, the result is true by the proof of Theorem 9.9 of [10].

The proofs in [10] show that the central quadrilateral is homothetic to a parallelogram associated with the reference quadrilateral.

This covers all the cases. □

The following proposition is useful when giving geometric proofs of results involving the Steiner point of an orthodiagonal quadrilateral.

Proposition 7.9. *The Steiner point of an orthodiagonal quadrilateral coincides with the point of intersection of the perpendicular bisectors of the diagonals.*

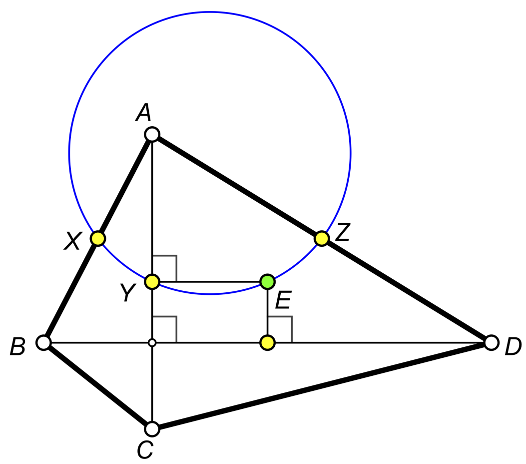

FIGURE 22. Steiner point of an orthodiagonal quadrilateral

Proof. This is Proposition 10.4 of [10]. □

We found some results for an orthodiagonal quadrilateral. They are shown in the following table.

Central Quads of an Orthodiagonal Quadrilateral	
Shape of central quad	**centers**
cyclic	3
trapezoid	4

Theorem 7.10. *Let E be the Steiner point of orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the circumcenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 23). Then $FGHI$ is a cyclic quadrilateral.*

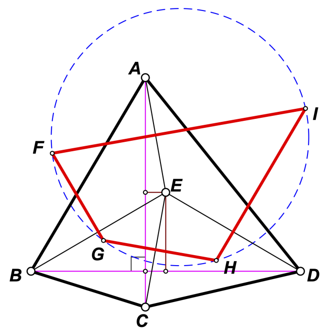

Figure 23. orthodiagonal quadrilateral, X_3-points $\Longrightarrow$ cyclic

Open Question 2. *Is there a purely geometric proof of Theorem 7.10?*

Open Question 3. *If the central quadrilateral is cyclic, must the center be the circumcenter?*

Theorem 7.11. *Let E be the Steiner point of orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the orthocenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 23). Then $FHIG$ is a trapezoid with $FH \parallel GI$.*

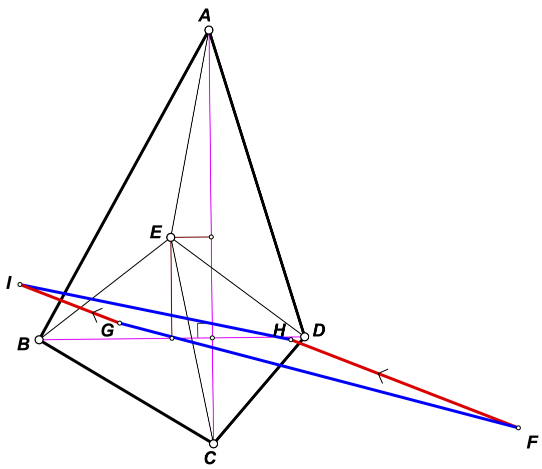

Figure 24. orthodiagonal quadrilateral, X_4-points $\Longrightarrow$ $FH \parallel GI$

Open Question 4. *Is there a purely geometric proof of Theorem 7.11?*

Open Question 5. *If the central quadrilateral is a trapezoid, must the center be the orthocenter?*

Our computer study found some additional results for an equidiagonal orthodiagonal quadrilateral. They are shown in the following table.

Central Quads of Equidiagonal Orthodiagonal Quads	
Shape of central quad	**centers**
orthodiagonal	486, 487, 642

Theorem 7.12. *Let E be the Steiner point of equidiagonal orthodiagonal quadrilateral $ABCD$. Let n be 486, 487, or 642. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Figure 25 shows an example using the inner Vecten points ($n = 486$). Figure 26 shows an example when $n = 642$. Then $FGHI$ is an orthodiagonal quadrilateral. When $n = 486$, the diagonals of $FGHI$ meet at point E.*

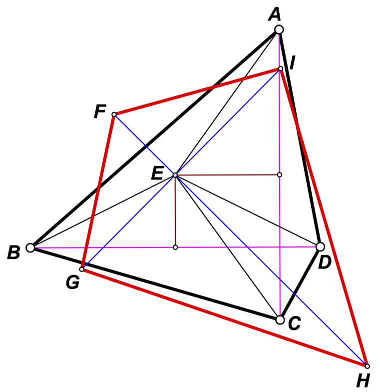

FIGURE 25. equi-ortho-quad, X_{486}-points $\Longrightarrow$ orthodiagonal

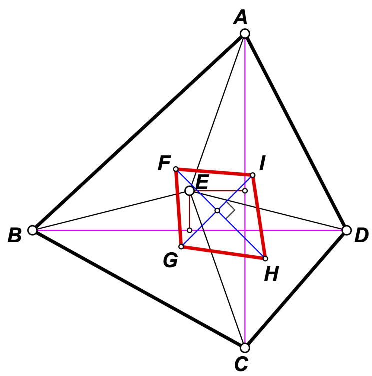

FIGURE 26. equi-ortho-quad, X_{642}-points $\Longrightarrow$ orthodiagonal

Our computer study found some additional results for bicentric quadrilaterals. They are shown in the following table.

Central Quads of Bicentric Quadrilaterals	
Shape of central quad	**centers**
cyclic	1, 165, 214

Theorem 7.13. *Let E be the Steiner point of bicentric quadrilateral $ABCD$. Let F, G, H, and I be the incenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 27). Then $FGHI$ is a cyclic quadrilateral.*

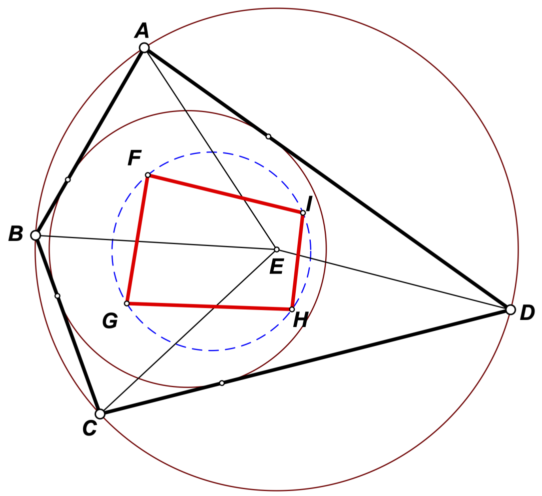

Figure 27. bicentric quadrilateral, X_1-points $\implies$ cyclic

For more information about this result, see [2], [3], and [8]. A purely geometric proof can be found in [14].

Lemma 7.14. *The X_{214}-point of a triangle is the midpoint of X_1 and X_{100}.*

Proof. See [6]. □

Theorem 7.15. *Let E be the Steiner point of bicentric quadrilateral $ABCD$. Let F, G, H, and I be the X_{214}-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a cyclic quadrilateral.*

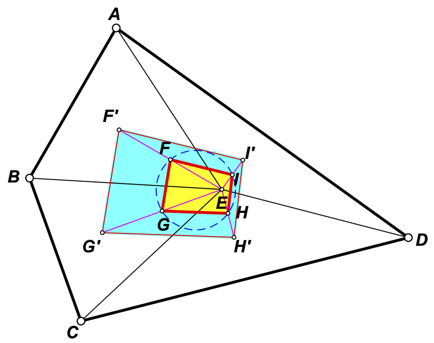

FIGURE 28. bicentric quadrilateral, X_{214}-points $\implies$ cyclic

Proof. Let F', G', H', and I' be the X_1-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively.
Let F be the midpoint of EF'.
Proposition 7.1 $\implies$ E is the circumcenter of $ABCD$.
$EA = EB$ $\implies$ EAB isosceles triangle $\implies$ F' lies on median to base, AB.
Lemma 7.3 $\implies$ E is the X_{100}-point of $\triangle EAB$.
Lemma 7.14 $\implies$ F is the X_{214}-point of triangle $\triangle EAB$.
A dilation of $\frac{1}{2}$ about E maps F' to F.
Similarly, this dilation maps $F'G'H'I'$ to $FGHI$.
Theorem 7.13 $\implies$ $F'G'H'I'$ is cyclic.
Thus, $FGHI$ is homothetic to $F'G'H'I'$ and is therefore also cyclic. ∎

Lemma 7.16. *The X_{165}-point of a triangle lies on the line X_1X_3 and $X_1X_3 = 3X_3X_{165}$.*

Proof. From [5], we have that X_{165} lies on the line X_1X_3. Using the barycentric coordinates for X_1, X_3, and X_{165} along with the distance formula between two points, we find that

$$\overline{X_1X_3}^2 = -\frac{abc\,(a^3 - a^2b - a^2c - ab^2 + 3abc - ac^2 + b^3 - b^2c - bc^2 + c^3)}{(a-b-c)(a+b-c)(a-b+c)(a+b+c)}$$

and

$$\overline{X_3X_{165}}^2 = -\frac{abc\,(a^3 - a^2b - a^2c - ab^2 + 3abc - ac^2 + b^3 - b^2c - bc^2 + c^3)}{9(a-b-c)(a+b-c)(a-b+c)(a+b+c)}.$$

Thus,

$$\left(\frac{X_1X_3}{X_3X_{165}}\right)^2 = 9$$

and we are done. ∎

Theorem 7.17. *Let E be the Steiner point of bicentric quadrilateral $ABCD$. Let F, G, H, and I be the X_{165}-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a cyclic quadrilateral.*

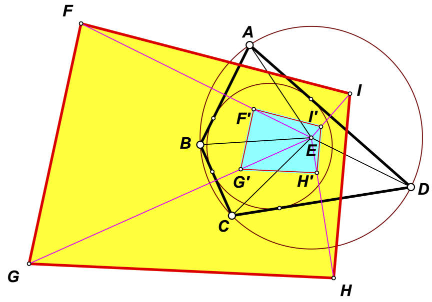

Figure 29. bicentric quadrilateral, X_{165}-points $\implies$ cyclic

Proof. Let F', G', H', and I' be the X_1-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively.
Proposition 7.1 $\implies$ E is the circumcenter of $ABCD$.
$EA = EB$ $\implies$ EAB isosceles triangle.
Lemma 7.16 $\implies$ F lies on EF' and $EF = 3EF'$.
A dilation of $3:1$ with center E maps F' to F.
Similarly, this dilation maps $F'G'H'I'$ to $FGHI$.
Theorem 7.13 $\implies$ $F'G'H'I'$ is cyclic.
Thus, $FGHI$ is homothetic to $F'G'H'I'$ and is therefore also cyclic. □

Our computer study found a number of results for a cyclic orthodiagonal quadrilateral. They are shown in the following table.

Central Quads of Cyclic Orthodiagonal Quadrilaterals	
Shape of central quad	**centers**
rectangle	$\mathbb{M}$, 2, 148–150, 402, 620, 671, 903
bicentric	$\mathbb{T}$, 3, 399
equidiagonal	13, 14, 616–619
isosceles trapezoid	154
trapezoid	26, 139, 155–157

Theorem 7.18. *Let E be the Steiner point of cyclic orthodiagonal quadrilateral $ABCD$. Let n be 2, 11, 115, 116, 122-125, 127, 130, 134–137, 139, 148–150, 244–247, 290, 338, 339, 402, 620, 671, 865–868, or 903. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a rectangle. Moreover, if $n = 148$, 149, or 150, then $FGHI$ and $ABCD$ have the same diagonal point.*

Theorem 7.19. *Let E be the Steiner point of cyclic orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the X_{154}-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is an isosceles trapezoid.*

Theorem 7.20. *Let E be the Steiner point of cyclic orthodiagonal quadrilateral $ABCD$. Let n be 26, 139, 155, 156, or 157. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a trapezoid.*

Theorem 7.21. *Let E be the Steiner point (circumcenter) of cyclic orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the circumcenters of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 30). Then $FGHI$ is a bicentric quadrilateral with incenter E.*

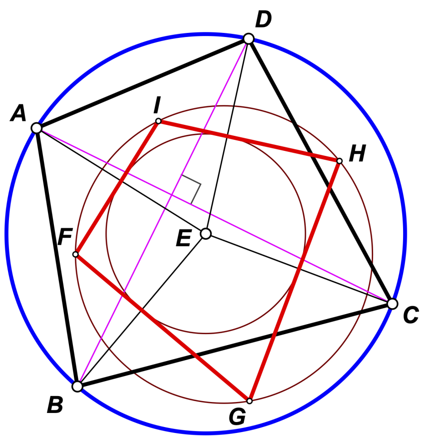

FIGURE 30. cyclic orthodiagonal quadrilateral, X_3-points $\implies$ bicentric

Proof. Since $ABCD$ is cyclic, $FGHI$ is tangential by Theorem 7.6. Since $ABCD$ is orthodiagonal, $FGHI$ is cyclic by Theorem 7.10. Thus, $FGHI$ is bicentric. □

Open Question 6. *Is there a purely geometric proof of Theorem 7.21?*

Theorem 7.22. *Let E be the Steiner point of a cyclic orthodiagonal quadrilateral $ABCD$. Let n be 13, 14, 616, 617, 618, or 619. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 31 shows the case when $n = 14$). Then $FGHI$ is equidiagonal.*

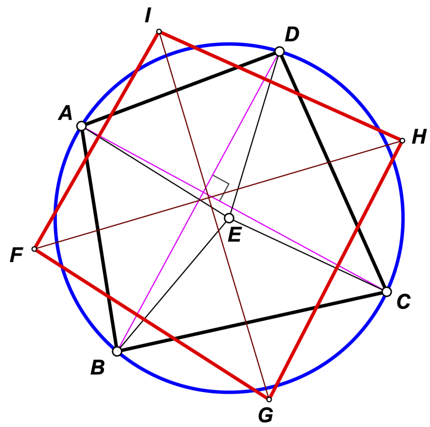

FIGURE 31. cyclic orthodiagonal quad, X_{14}-points $\implies$ $FH = GI$

Note: In Figure 31, FH is not perpendicular to GI.

Open Question 7. *Is there a purely geometric proof of Theorem 7.22 for the cases where n is 13 or 14?*

Our computer study found a number of results for Equidiagonal Orhodiagonal Trapezoids. They are shown in the following table.

Central Quads of Equidiagonal Orthodiagonal Trapezoids	
Shape of central quad	**centers**
square	$\mathbb{M}$, 2, 148–150, 290, 402, 620, 671, 903
equidiagonal kite	13, 14, 616–618
right kite	$\mathbb{T}$, 3, 399

Note: An Equidiagonal Orthodiagonal Trapezoid is an isosceles trapezoid and is cyclic. As an isosceles trapezoid, the central quadrilateral is always a kite.

Theorem 7.23. *Let E be the Steiner point of an equidiagonal orthodiagonal trapezoid $ABCD$. Let n be 13, 14, 616, 617, or 618. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively (Figure 32 shows the case when $n = 14$). Then $FGHI$ is an equidiagonal kite.*

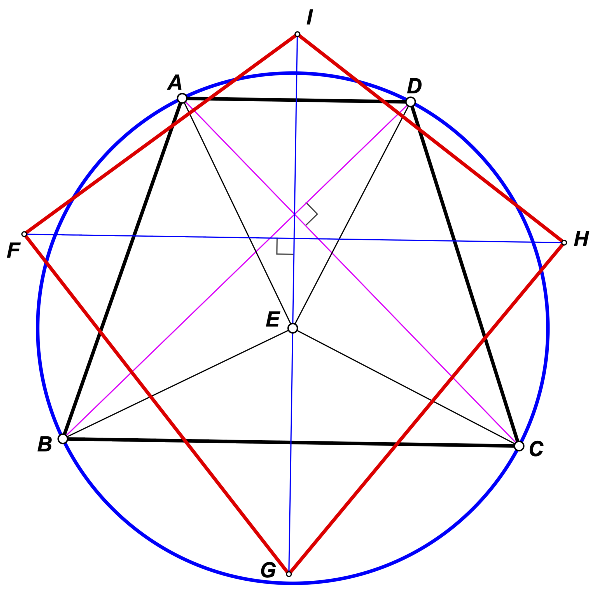

Figure 32. equi ortho trap, X_{14}-points $\implies$ equidiagonal kite

Proof. Quadrilateral $FGHI$ is a kite by Theorem 5.2. Quadrilateral $FGHI$ is equidiagonal by Theorem 7.22. Thus $FGHI$ is an equidiagonal kite. □

Theorem 7.24. *Let E be the Steiner point of an equidiagonal orthodiagonal trapezoid $ABCD$. Let n be 3 or 399 or in $\mathbb{T}$. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a right kite.*

Our computer study also found some results for Equidiagonal Kites. They are shown in the following table.

Central Quads of Equidiagonal Kites	
Shape of central quad	**centers**
square	486, 642
rectangle	586

Theorem 7.25. *Let E be the Steiner point of an equidiagonal kite $ABCD$. Let n be 486 or 642. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a square.*

Proof. The case $n = 486$ follows from Theorem 10.9 of [10].
The case $n = 642$ follows from Theorem 10.7 of [10]. □

Theorem 7.26. *Let E be the Steiner point of an equidiagonal kite $ABCD$. Let F, G, H, and I be the X_{586}-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. Then $FGHI$ is a rectangle.*

8. RESULTS USING THE PONCELET POINT

The *Poncelet point* (sometimes called the Euler-Poncelet point) of a quadrilateral is the common point of the nine-point circles of the component triangles (half-triangles) of the quadrilateral. A triangle formed from three vertices of a quadrilateral is called a *component triangle* of that quadrilateral. The *nine-point circle* of a triangle is the circle through the midpoints of the sides of that triangle.

Figure 33 shows the Poncelet point of quadrilateral $ABCD$. The yellow points represent the midpoints of the sides and diagonals of the quadrilateral. The component triangles are BCD, ACD, ABD, and ABC. The blue circles are the nine-point circles of these triangles. The common point of the four circles is the Poncelet point (shown in green).

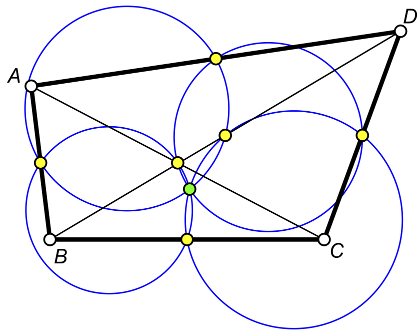

FIGURE 33. The Poncelet point of quadrilateral $ABCD$

In this section, we study the case where point E is the Poncelet point of the quadrilateral. Results that are true when point E is arbitrary are omitted.

The following two propositions come from [10].

Proposition 8.1. *The Poncelet point of a parallelogram coincides with the diagonal point.*

Proposition 8.2. *The Poncelet point of an orthodiagonal quadrilateral coincides with the diagonal point.*

Because of these propositions, we will not discuss parallelograms or orthodiagonal quadrilaterals in this section because they have been covered in Section 5 of [11].

Our computer study found a few results when the quadrilateral is not a parallelogram or orthodiagonal. They are shown in the following table.

Central Quads of Hjelmslev Quadrilaterals	
Shape of central quad	**centers**
trapezoid	3, 69
tangential trapezoid	4

Lemma 8.3. *Let E be the Poncelet point of quadrilateral $ABCD$ that has right angles at B and D. Then E is the midpoint of BD.*

Theorem 8.4. *Let E be any point on diagonal BD of quadrilateral $ABCD$. Let F, G, H, and I be the X_3 points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 34). Then $FGHI$ is a trapezoid with $FG \parallel HI$.*

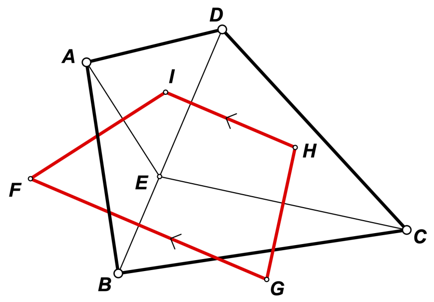

FIGURE 34. $E \in BD$, X_3-points, $\implies FG \parallel HI$

Proof. Point F is the circumcenter of $\triangle ABE$, so F lies on the perpendicular bisector of BE. Similarly G lies on the perpendicular bisector of BE. Hence $FG \perp BE$. Similarly $IH \perp DE$. Since E lies on BD, we have that FG and IH are perpendicular to BD. Therefore $FG \parallel IH$. □

Theorem 8.5. *Let E be the Poncelet point of a Hjelmslev quadrilateral $ABCD$. Let F, G, H, and I be the X_3 points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then $FGHI$ is a trapezoid with $FG \parallel HI$.*

Proof. By Lemma 8.3, E is the midpoint of BD. Then, by Theorem 8.4, $FGHI$ is a trapezoid with $FG \parallel HI$. □

Theorem 8.6. *Let E be any point on diagonal BD of quadrilateral $ABCD$. Let F, G, H, and I be the X_4 points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 35). Then $FHGI$ is a trapezoid with $FI \parallel HG$.*

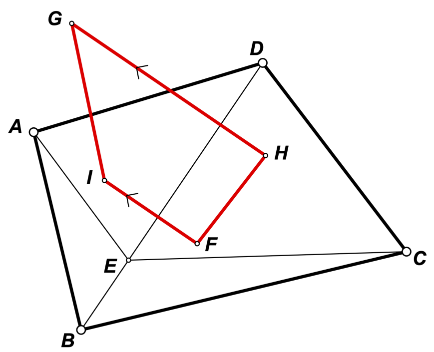

FIGURE 35. $E \in BD$, X_4-points, $\implies FI \parallel HG$

Proof. Point F is the orthocenter of $\triangle ABE$, so $AF \perp BE$. Point I is the orthocenter of $\triangle ADE$, so $AI \perp DE$. Since E lies on BD, we have that AF and AI are both perpendicular to BD. It follows that $FI \perp BD$ since F and I lie on the perpendicular from A to BD. In a similar way, it can be proved that $GH \perp BD$. Therefore $FI \parallel GH$. □

Theorem 8.7. *Let E be the Poncelet point of a Hjelmslev quadrilateral $ABCD$. Let F, G, H, and I be the X_4 points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 36). Then $FGHI$ is a tangential trapezoid with incenter E and $FI \parallel GH$. Also, diagonal BD passes through the points of contact of the incircle with sides GH and FI.*

Note that E is the midpoint of BD by Lemma 8.3.

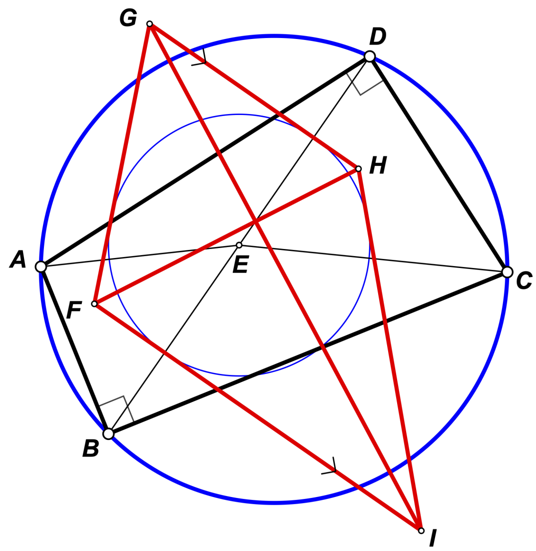

Figure 36. X_4-points $\implies$ $FGHI$ is a tangential trapezoid

Open Question 8. *Is there a purely geometric proof of Theorem 8.7?*

Open Question 9. *If E is the Poncelet point of a Hjelmslev quadrilateral $ABCD$, and the central quadrilateral is a tangential trapezoid, must the center be the orthocenter?*

9. Results Using the Area Centroid

The *Area Centroid* (also called the *quasi centroid* or 1st QG quasi centroid) of a convex quadrilateral is the the center of mass of the quadrilateral when its surface is made of some evenly distributed material.

Geometrically, it is the intersection of the diagonals of the centroid quadrilateral of the given quadrilateral.

A triangle formed from three vertices of a quadrilateral is called a *component triangle* of that quadrilateral.

The quadrilateral whose vertices are the centroids of the four component triangles of a quadrilateral is called the *centroid quadrilateral* of that quadrilateral.

Figure 37 shows the area centroid of quadrilateral $ABCD$. The yellow points represent the centroids of the component triangles of the quadrilateral. The component triangles are BCD, ACD, ABD, and ABC. The blue region is the centroid quadrilateral $G_AG_BG_CG_D$. The red point is the area centroid.

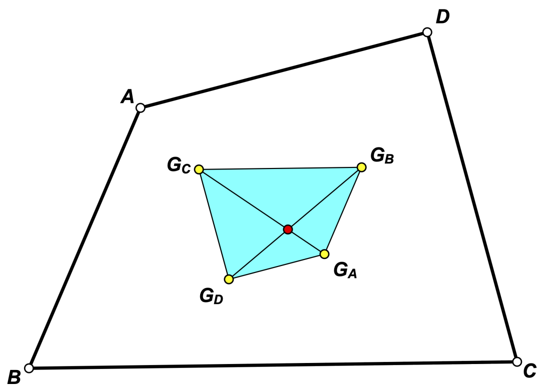

FIGURE 37. The area centroid of quadrilateral $ABCD$

In this section, we study the case where point E is the area centroid of the quadrilateral. Results that are true when point E is arbitrary are omitted.

If $ABCD$ is a kite, with $AB = AD$ and $CB = CD$, then by symmetry, the area centroid lies on diagonal AC. By Theorem 5.1, the central quadrilateral of $ABCD$ is an isosceles trapezoid.

Our computer study did not find any new results, other than the ones that are true when E is an arbitrary point or when the quadrilateral is a kite.

Central Quadrilaterals of all Quadrilaterals
No new relationships were found.

10. Areas for Future Research

There are many avenues for future investigation.

10.1. **Investigate other triangle centers.**

In our study, we only investigated triangle centers X_n for $n \leq 1000$. Extend this study to larger values of n.

10.2. **Use other shape quadrilaterals.**

In our investigation, we only studied 28 shapes of quadrilaterals as shown in Figure 3. There are many other shapes of quadrilaterals. Study these other shapes.

For example, we did find some results associated with a right kite when point E is the area centroid of quadrilateral $ABCD$. A *right kite* is a kite in which two opposite angles are right angles.

Theorem 10.1. *Let E be the area centroid of right kite $ABCD$. Let n be 68, 317, or 577. Let F, G, H, and I be the X_n-points of triangles $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, and $\triangle DAE$, respectively. (Figure 38 shows the case where $n = 68$.) Then $FGHI$ is a rectangle. The sides of the rectangle are parallel to the diagonals of the right kite.*

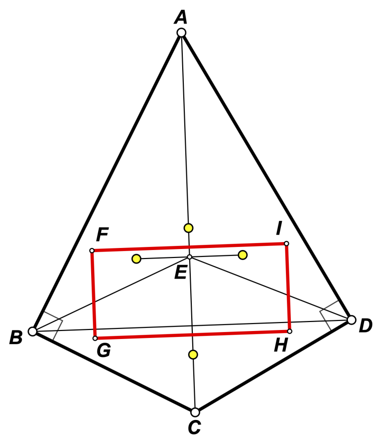

Figure 38. Right Kite, X_{68}-points $\implies$ rectangle

10.3. **Use other quadrilateral centers.**

In our study, we only used the common quadrilateral centers listed in Table 2 as radiators. Additional notable points associated with the reference quadrilateral could be used, such as the Miquel point (QL-P1), the Morley Point (QL-P2), the Newton Steiner point (QL-P7), and the various quasi points. A list of notable points associated with a quadrilateral can be found in [12].

10.4. Investigate radiators lying on quadrilateral lines.

In Section 5, we studied cases where the radiator was restricted to lie on a certain line of symmetry of quadrilateral $ABCD$. We could also look at cases where the radiator lies on some notable line associated with the reference quadrilateral, such as a bimedian, the Newton line (QL-L1), the Steiner line (QL-L2), etc., or, in the case of cyclic quadrilaterals, the Euler line. A list of notable lines associated with a quadrilateral can be found in [12].

10.5. Ask about uniqueness.

Find an entry in one of our tables where there is only one center giving a particular relationship for a certain type of quadrilateral. For example, in Section 6, we found in Theorem 6.1 that for an equidiagonal quadrilateral, when the radiator is the centroid of the reference quadrilateral, the central quadrilateral is orthodiagonal only when $n = 591$. Is this because we only searched the first 1000 values of n? Expand the search and find other values of n for which the central quadrilateral is orthodiagonal or prove that X_{591} is the unique center for which the central quadrilateral is orthodiagonal when the radiator is the quadrilateral centroid.

10.6. Form quadrilaterals with the Radiator.

In the current study, we formed four triangles using the radiator and two vertices of the reference quadrilateral. Instead, form four quadrilaterals using the radiator and three vertices of the reference quadrilateral. Then consider notable quadrilateral points in each of the four quadrilaterals formed and investigate the shape of the quadrilateral determined by these four points.

The following results were found by computer.

Theorem 10.2. *Let E be an arbitrary point in the plane of quadrilateral $ABCD$ (not on the boundary). Let F, G, H, and I be the vertex centroids of quadrilaterals $EBCD$, $EACD$, $EABD$ and $EABC$, respectively (Figure 39). Then $FGHI$ is homothetic to $ABCD$. If P is the center of the homothety and M is the vertex centroid of $ABCD$, then P lies on EM and $PE/PM = 4$.*

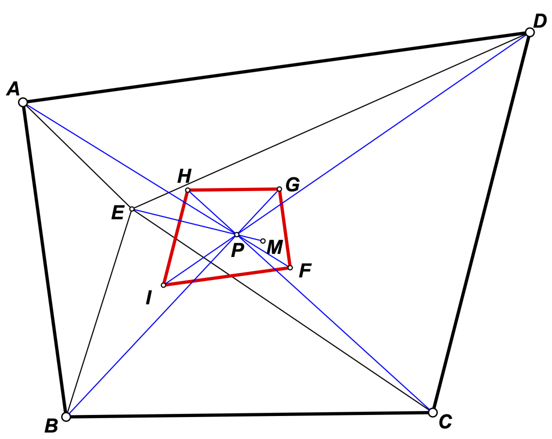

FIGURE 39. centroids $\Longrightarrow$ homothetic quadrilaterals

Theorem 10.3. *Let E be an arbitrary point in the plane of cyclic quadrilateral $ABCD$ (not on the boundary). Let F, G, H, and I be the Steiner points of quadrilaterals $EBCD$, $EACD$, $EABD$ and $EABC$, respectively (Figure 40). Then $FGHI$ is cyclic. If P is the circumcenter of $FGHI$ and O is the circumcenter of $ABCD$, then P is the midpoint of EO.*

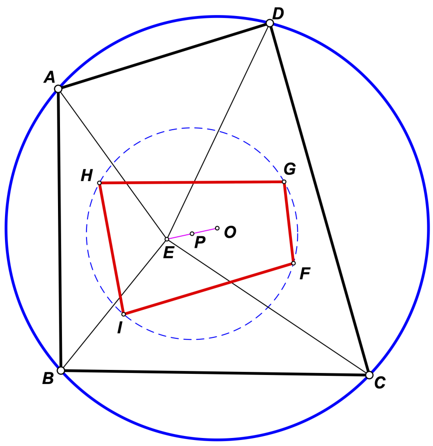

FIGURE 40. cyclic, Steiner points $\implies$ cyclic

Theorem 10.4. *Let E be an arbitrary point in the plane of cyclic quadrilateral $ABCD$ (not on the boundary). Let F, G, H, and I be the Poncelet points of quadrilaterals $EBCD$, $EACD$, $EABD$ and $EABC$, respectively (Figure 41). Then $FGHI$ is cyclic.*

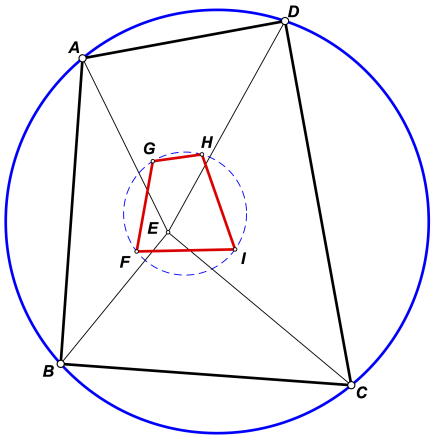

FIGURE 41. cyclic, Poncelet points $\implies$ cyclic

Open Question 10. *Are there purely geometric proofs of the previous 3 theorems?*

10.7. **Work in 3-space.**

If point D is moved off the plane of $\triangle ABC$, then the reference quadrilateral becomes a tetrahedron. Choose a point E inside this reference tetrahedron and draw lines to each of the vertices of the reference tetrahedron. This forms four tetrahedra with one vertex at E. Locate tetrahedron centers (such as the centroid, circumcenter, or Monge point) in each of these four tetrahedra. These centers form a new tetrahedron called the *central tetrahedron* of the given tetrahedron. Investigate when the central tetrahedron has a special shape (such as being isodynamic, orthocentric, or isosceles). A list of some special shape tetrahedra can be found in Section 3 of [9]. The point E can be an arbitrary point or it could be a notable point associated with the reference tetrahedron. A list of notable points can be found in Section 4 of [9].

The following result was discovered by computer.

Theorem 10.5. *Let E be any point not on the boundary of tetrahedron $ABCD$. Let F, G, H, and I be the centroids of tetrahedra $EBCD$, $EACD$, $EABD$ and $EABC$, respectively. Then tetrahedron $FGHI$ is similar to tetrahedron $ABCD$.*

The line from a vertex of a tetrahedron to the centroid of the opposite face is called a *median*. It is well known (Commandino's Theorem, [1, p. 57]) that the four medians of a tetrahedron concur at a point called the *centroid* of that tetrahedron and that the centroid divides each median in the ratio $1 : 3$.

The proof of Theorem 10.5 is similar to the proof of Theorem 4.1 and is omitted.

REFERENCES

[1] Nathan Altshiller-Court, *Modern Pure Solid Geometry*, 2nd edition. Chelsea Publishing Company. The Bronx, NY: 1964.

[2] Alexander Bogomolny, Concyclic Incenters in Bicentric Quadrilateral, Cut the Knot. https://www.cut-the-knot.org/m/Geometry/IncentersInBicentricQuad.shtml

[3] Alexander Bogomolny, Concyclic Points in Inscriptible Quadrilateral, Cut the Knot. https://www.cut-the-knot.org/m/Geometry/IncentersInBicentricQuad.shtml

[4] Clark Kimberling, *Encyclopedia of Triangle Centers*, 2025. http://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[5] Clark Kimberling, X(165), *Encyclopedia of Triangle Centers*, 2025. http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X165

[6] Clark Kimberling, X(214), *Encyclopedia of Triangle Centers*, 2025. http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X214

[7] Clark Kimberling, *Central Points and Central Lines in the Plane of a Triangle*, Mathematics Magazine, **67**(1994)163–187.

[8] Dào Thanh Oai, Four incenter lie on a circle in Bicentric quadrilateral, AoPS Online, Jan. 25, 2014. https://artofproblemsolving.com/community/c6h572848p3370150

[9] Stanley Rabinowitz, *Arrangement of Central Points on the Faces of a Tetrahedron*, International Journal of Computer Discovered Mathematics. **5**(2020)13–41. https://arxiv.org/abs/2101.02592

[10] Stanley Rabinowitz and Ercole Suppa, *Relationships between a Central Quadrilateral and its Reference Quadrilateral*. International Journal of Computer Discovered Mathematics. **7**(2022)214–287. https://arxiv.org/abs/2209.06008

[11] Stanley Rabinowitz and Ercole Suppa, *The Shape of Central Quadrilaterals*. International Journal of Computer Discovered Mathematics. **7**(2022)131–180. https://arxiv.org/pdf/2205.00870

[12] Chris van Tienhoven, Encyclopedia of Quadri-Figures. https://chrisvantienhoven.nl/mathematics/encyclopedia

[13] Eric W. Weisstein, *Circumcircle*. From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Circumcircle.html

[14] Alexey A. Zaslavsky, One Property of Bicentral Quadrilaterals. https://geometry.ru/olimp/2019/bicentr.pdf