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Equivalent Figures Formed from Orthocenters

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Abstract. The diagonals of a convex quadrilateral form four "half triangles", where each half triangle is bounded by one diagonal and two sides of the original quadrilateral. We give a geometric proof that the quadrilateral formed by the orthocenters of the four half triangles of a given quadrilateral has the same area as the original quadrilateral. We also give some examples of other equivalent figures formed from orthocenters.

Keywords. orthocenters, quadrilaterals, computer-discovered mathematics, half triangles, equivalent figures.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

The following theorem comes from [9] and was discovered by computer.

Theorem 1. *Let $ABCD$ be a convex quadrilateral. Let E be the orthocenter of $\triangle BCD$. Let F be the orthocenter of $\triangle CDA$. Let G be the orthocenter of $\triangle DAB$. Let H be the orthocenter of $\triangle ABC$. Then quadrilaterals $ABCD$ and $EFGH$ have the same area (Figure 1).*

An open question in [9] asked if there was a purely geometric proof of this result. The purpose of this paper is to present such a proof.

We start with a known property of hexagons with opposite sides parallel. This result was given by Mukhopadhyaya in 1889 in the *Educational Times* [8]. The result is not very well known as it does not appear in classic papers about hexagons

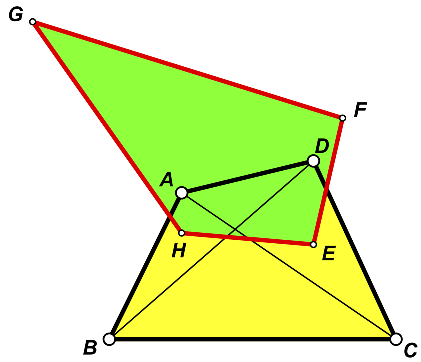

Figure 1. orthocenters $\Longrightarrow$ yellow area = green area

with opposite sides parallel such as [1] and [14]. The result appeared as a problem in 1958 in the 58th Eötvös-Kürschák Competition in Hungary [12]. A few other references are [13] and [3].

Theorem 2. *Let $ABCDEF$ be a hexagon (not necessarily simple or convex) with its opposite sides parallel. That is, $AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$ (Figure 2). Then $[ACE] = [BDF]$.*

Note. The notation $[XYZ]$ denotes the area of $\triangle XYZ$.

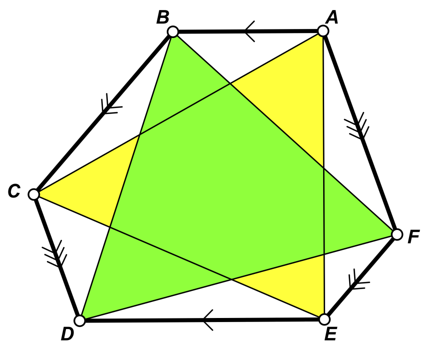

Figure 2. yellow area = green area

Proof. Place the hexagon in the complex plane and let the complex coordinates (affixes) of A, B, C, D, E, and F be a, b, c, d, e, and f, respectively. Since $AB \parallel DE$, we must have

$$a - b = p(d - e)$$

where p is a nonzero real number, [2]. Similarly,

$$b - c = q(e - f)$$

and

$$c - d = r(f - a)$$

where q and r are nonzero real numbers.

Solving these three equations for b, d, and f gives

$$\begin{pmatrix} b \\ d \\ f \end{pmatrix} = \frac{1}{q + pr} \begin{pmatrix} aq - cpq + epq + cpr - apqr + epqr \\ cq + ar - cr + epr + aqr - eqr \\ c - a + cp - ep + eq + apr \end{pmatrix}. \tag{1}$$

Using the formula for the area of a triangle in complex coordinates [4, Theorem 5], we have

$$[ACE] = \frac{i}{4}\begin{vmatrix} a & \overline{a} & 1 \\ c & \overline{c} & 1 \\ e & \overline{e} & 1 \end{vmatrix} \qquad \text{and} \qquad [BDF] = \frac{i}{4}\begin{vmatrix} b & \overline{b} & 1 \\ d & \overline{d} & 1 \\ f & \overline{f} & 1 \end{vmatrix},$$

where $\overline{z}$ denotes the complex conjugate of complex number z.

We now substitute the values of b, d, and f from equation (1) into the formula for $[BDF]$ and use the standard properties of complex numbers, $\overline{x+y} = \overline{x} + \overline{y}$ and $\overline{kz} = k\overline{z}$ when k is real. After simplifying, we find that $[BDF] = [ACE]$. □

Geometric proofs can be found in [6] and [7, pp. 42–44].

We can use Theorem 2 to prove the following result which comes from [11].

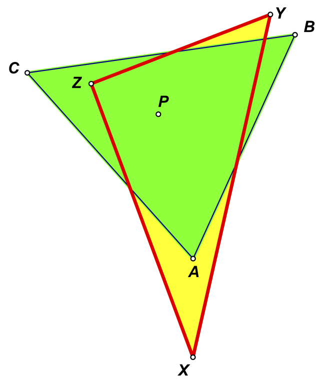

FIGURE 3. yellow area = green area

Theorem 3. *Let P be any point in the plane of $\triangle ABC$ (not on the boundary). Let X be the orthocenter of $\triangle PBC$. Let Y be the orthocenter of $\triangle PCA$. Let Z be the orthocenter of $\triangle PAB$. See Figure 3. Then $[XYZ] = [ABC]$.*

Proof. Consider (reentrant) hexagon $XBZAYC$ (Figure 4).

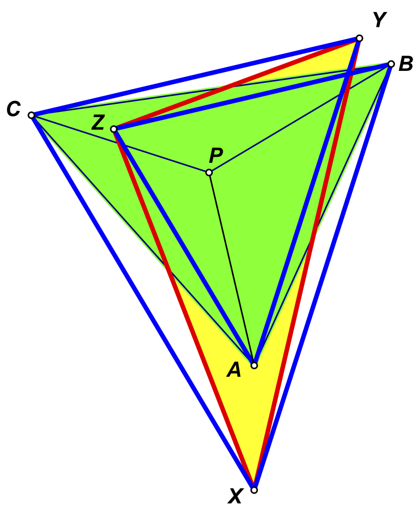

FIGURE 4.

Since Y is the orthocenter of $\triangle PCA$, the altitude CY is perpendicular to side PA. Since Z is the orthocenter of $\triangle PAB$, the altitude BZ is perpendicular to side PA. Thus, $CY \parallel ZB$.

In the same manner, $XC \parallel AZ$ and $BX \parallel YA$. So the opposite sides of hexagon $XBZAYC$ are parallel. By Theorem 2, $[XYZ] = [ABC]$. □

We can now give a proof of Theorem 1.

Proof. Note that D is a point in the plane of $\triangle ABC$. By Theorem 3, $[ABC] = [EFG]$ (Figure 5).

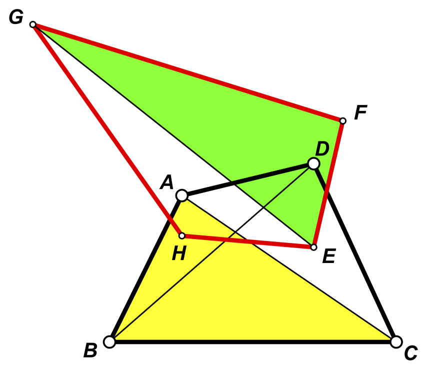

Figure 5. yellow area = green area

Similarly, note that B is a point in the plane of $\triangle ACD$. By Theorem 3, $[ACD] = [EGH]$. Therefore $[ABCD] = [ABC]+[ACD] = [EFG]+[EGH] = [EFGH]$. □

There are several other results that form equivalent figures using orthocenters that can be proven using similar techniques.

The following result comes from [10].

Theorem 4. *Let $\triangle DEF$ be inscribed in $\triangle ABC$ as shown in Figure 6. Let X be the orthocenter of $\triangle AEF$. Let Y be the orthocenter of $\triangle BFD$. Let Z be the orthocenter of $\triangle CDE$. Then $[XYZ] = [DEF]$.*

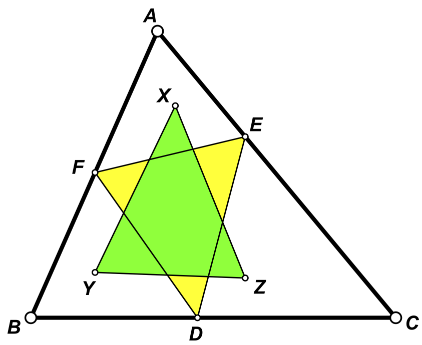

Figure 6. yellow area = green area

The following proof is by Kousik Sett [10].

Proof. Since EZ is an altitude of $\triangle EDC$, we have $EZ \perp BC$. Since FY is an altitude of $\triangle FBD$, we have $FY \perp BC$. Thus, $EZ \parallel FY$. Similarly, $EX \parallel DY$ and $XF \parallel DZ$. Therefore, the opposite sides of hexagon $DZEXFY$ are parallel and $[XYZ] = [DEF]$ by Theorem 2. □

The following result comes from [4, problem 7].

Theorem 5. *Let $ABCD$ be a cyclic quadrilateral and let E, F, G, H be the midpoints of AB, BC, CD, and DA, respectively. Let W, X, Y, Z be the orthocenters of $\triangle AHE$, $\triangle BEF$, $\triangle CFG$, and $\triangle DGH$, respectively (Figure 7). Then $[ABCD] = [WXYZ]$.*

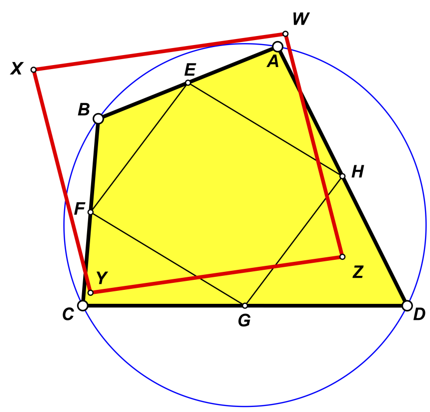

FIGURE 7.

A proof can be found in [5].

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