International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 7, 2022, pp. 181–194
Received 20 March 2022. Published on-line 27 July 2022
web: http://www.journal-1.eu/

Inequalities Derived From Distances Between Triangle Centers

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Abstract. Using a computer, we find the formula for the distance between a number of triangle centers in terms of R, r, and s (the circumradius, inradius, and semiperimeter of a triangle). We then use the fact that this expression is always nonnegative to find upper and lower bounds for powers of the variables R, r, and s in terms of the other variables.

Keywords. triangle centers, inequalities, computer-discovered mathematics, Gerretsen's Inequality, Mathematica.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

The literature abounds with inequalities involving elements of a triangle, such as R, r, and s, the triangle's circumradius, inradius, and semiperimeter. One sometimes wonders if these inequalities have any geometrical significance.

For example, in 1765, Euler [2] found the following formula for the distance between the incenter and circumcenter of a triangle in terms of R and r:

$$d^2 = R(R - 2r).$$

Since this distance cannot be negative, this implies that

$$R \geq 2r.$$

More recently, similar relationships connecting distances between triangle centers and well-known inequalities have been found.

For example, it was noted ([7, p. 45] and [5]) that Gerretsen's Inequalities

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$$

follow from the facts that

$$|GI|^2 = \frac{1}{9}(s^2 - 16Rr + 5r^2)$$

and

$$|HI|^2 = 4R^2 + 4Rr + 3r^2 - s^2$$

where G, I, and H are the centroid, incenter, and orthocenter of a triangle and $|PQ|$ denotes the distance between points P and Q.

In 2019, it was noted [6] that Kooi's Inequality

$$s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$$

follows from

$$|OM|^2 = R^2 - \frac{2Rs^2(2R-r)}{(4R+r)^2}$$

where O and M are the circumcenter and mittenpunkt of a triangle.

In this paper, we examine the distances between various triangle centers to see if they imply similar bounds for s^2 (and other elements).

2. The Procedure

2.1. Centers used.

Let X_n denote the nth named triangle center as cataloged in the Encyclopedia of Triangle Centers [4].

We use barycentric coordinates in this study. The barycentric coordinates for 69 triangle centers in terms of the sides of the triangle, a, b, and c, are shown in Table 1. Only the first barycentric coordinate is given, because if $f(a, b, c)$ is the first barycentric coordinate for a point P, then the barycentric coordinates for P are

$$\Big(f(a, b, c) : f(b, c, a) : f(c, a, b)\Big).$$

These were derived from [4]. For this study, we included all triangle centers from the first 200 whose first barycentric coordinate is a polynomial of degree 4 or less. We also included a few additional triangle centers.

Table 1: List of Centers Used in this Study

ctr	1st barycentric coordinate
X_1	a
X_2	1
X_3	$a^2\left(a^2-b^2-c^2\right)$
X_4	$\left(a^2+b^2-c^2\right)\left(a^2-b^2+c^2\right)$
X_5	$a^2b^2+2c^2b^2-c^4+a^2c^2-b^4$
X_6	a^2
X_7	$(a+b-c)(a-b+c)$
X_8	$a-b-c$
X_9	$a(a-b-c)$
X_{10}	$b+c$
X_{11}	$(b-c)^2(b+c-a)$
X_{12}	$(a+b-c)(a-b+c)(b+c)^2$
X_{20}	$3a^4-2b^2a^2-2c^2a^2-b^4-c^4+2b^2c^2$
X_{21}	$a(a+b)(a+c)(b+c-a)$
X_{31}	a^3
X_{32}	a^4
X_{35}	$a^2\left(a^2-b^2-c^2-bc\right)$
X_{36}	$a^2\left(a^2-b^2-c^2+bc\right)$
X_{37}	$a(b+c)$
X_{38}	$a\left(b^2+c^2\right)$
X_{39}	$a^2\left(b^2+c^2\right)$
X_{40}	$a(a^3+a^2b+a^2c-ab^2-ac^2-2abc-b^3-c^3+bc^2+b^2c)$
X_{41}	$a^3(b+c-a)$
X_{42}	$a^2(b+c)$
X_{43}	$a(ab-bc+ac)$
X_{44}	$a(b+c-2a)$
X_{45}	$a(2b+2c-a)$
X_{46}	$a(a^3+a^2b+a^2c-ab^2-ac^2-b^3-c^3+bc^2+b^2c)$
X_{55}	$a^2(b+c-a)$
X_{56}	$a^2(a+b-c)(a-b+c)$
X_{57}	$a(a+b-c)(a-b+c)$
X_{58}	$a^2(a+b)(a+c)$
X_{63}	$a\left(b^2+c^2-a^2\right)$

ctr	1st barycentric coordinate
X_{65}	$a(a+b-c)(a-b+c)(b+c)$
X_{69}	$b^2+c^2-a^2$
X_{72}	$a(b+c)\left(b^2+c^2-a^2\right)$
X_{75}	bc
X_{76}	b^2c^2
X_{78}	$a(b+c-a)\left(b^2+c^2-a^2\right)$
X_{79}	$(a^2+ab+b^2-c^2)(a^2+ac-b^2+c^2)$
X_{80}	$(a^2-ab+b^2-c^2)(a^2-ac-b^2+c^2)$
X_{81}	$a(a+b)(a+c)$
X_{83}	$\left(a^2+b^2\right)\left(a^2+c^2\right)$
X_{85}	$bc(a+b-c)(a-b+c)$
X_{86}	$(a+b)(a+c)$
X_{88}	$a(a+b-2c)(a-2b+c)$
X_{89}	$a(2a+2b-c)(2a-b+2c)$
X_{99}	$\left(b^2-a^2\right)\left(c^2-a^2\right)$
X_{141}	b^2+c^2
X_{142}	$ab+ac-(b-c)^2$
X_{145}	$3a-b-c$
X_{190}	$(a-b)(a-c)$
X_{192}	$ab-bc+ac$
X_{194}	$a^2b^2+a^2c^2-b^2c^2$
X_{200}	$a(b+c-a)^2$
X_{210}	$a(b+c)(b+c-a)$
X_{213}	$a^3(b+c)$
X_{238}	$a(a^2-bc)$
X_{239}	a^2-bc
X_{244}	$a(b-c)^2$
X_{304}	$bc\left(b^2+c^2-a^2\right)$
X_{306}	$(b+c)\left(b^2+c^2-a^2\right)$
X_{312}	$bc(b+c-a)$
X_{319}	$b^2+c^2-a^2+bc$
X_{320}	$a^2-b^2-c^2+bc$
X_{321}	$bc(b+c)$
X_{345}	$(b+c-a)\left(b^2+c^2-a^2\right)$
X_{346}	$(a-b-c)^2$
X_{350}	$bc(a^2-bc)$

2.2. **Finding the distance.**

Using the distance formula between two points in terms of their barycentric coordinates [3], we found formulas for the distance between every pair of points in Table 1. The resulting formula (in terms of a, b, and c) was converted to an equivalent formula in terms of R, r, and s by using the Fundamental Theorem of Symmetric Polynomials [8] to express the formula in terms of the elementary symmetric polynomials, $a+b+c$, $ab+bc+ca$, and abc and then using the following well-known formulas [7, p. 7]:

$$\begin{aligned} a+b+c &= 2s \\ ab+bc+ca &= r^2+s^2+4rR \\ abc &= 4rRs. \end{aligned}$$

These distance formulas are given in the file `RrsDistances.pdf` included with the supplementary material associated with this paper. We call these the R-r-s distance formulas.

2.3. **Example 1.**

Using the barycentric distance formula, we find that the square of the distance between X_1 and X_7 is

$$|X_1X_7|^2 = \frac{E}{(a+b+c)\left(a^2-2a(b+c)+(b-c)^2\right)^2}$$

where

$$E = -(a^7-2a^6(b+c)+5a^5bc+a^4\left(b^3-2b^2c-2bc^2+c^3\right)+a^3(b-c)^2\left(b^2+c^2\right)$$
$$-2a^2bc(b-c)^2(b+c)-a(b-c)^4\left(2b^2+3bc+2c^2\right)+(b-c)^4\left(b^3+2b^2c+2bc^2+c^3\right)).$$

Writing this in terms of the elementary symmetric polynomials, shows that

$$|X_1X_7|^2 = \frac{E'}{(a+b+c)^5-8(ab+ac+bc)(a+b+c)^3+16(ab+ac+bc)^2(a+b+c)}$$

where

$$\begin{gathered} E' = -48a^2b^2c^2(a+b+c)-14abc(a+b+c)^4-(a+b+c)^7 \\ +64abc(a+b+c)^2(ab+ac+bc)+9(a+b+c)^5(ab+ac+bc) \\ -32abc(ab+ac+bc)^2-24(a+b+c)^3(ab+ac+bc)^2 \\ +16(a+b+c)(ab+ac+bc)^3. \end{gathered}$$

Expressing this in terms of R, r, and s gives

$$|X_1X_7|^2 = \frac{r^2\left(r^2+8rR+16R^2-3s^2\right)}{(r+4R)^2}.$$

2.4. **Short polynomials.**

The R-r-s distance formulas where the expression is a polynomial with fewer than 4 terms are shown below.

$$
\begin{aligned}
|X_1X_2|^2 &= \frac{1}{9}(5r^2 - 16rR + s^2)\\
|X_1X_3|^2 &= R(R-2r)\\
|X_1X_5|^2 &= \frac{1}{4}(R-2r)^2\\
|X_1X_8|^2 &= 5r^2 - 16rR + s^2\\
|X_1X_{10}|^2 &= \frac{1}{4}(5r^2 - 16rR + s^2)\\
|X_1X_{11}|^2 &= r^2\\
|X_1X_{40}|^2 &= 4R(R-2r)\\
|X_1X_{80}|^2 &= 4r^2\\
|X_1X_{145}|^2 &= 5r^2 - 16rR + s^2\\
|X_2X_8|^2 &= \frac{4}{9}(5r^2 - 16rR + s^2)\\
|X_2X_{10}|^2 &= \frac{1}{36}(5r^2 - 16rR + s^2)\\
|X_2X_{145}|^2 &= \frac{16}{9}(5r^2 - 16rR + s^2)\\
|X_3X_8|^2 &= (R-2r)^2\\
|X_3X_{40}|^2 &= R(R-2r)\\
|X_3X_{99}|^2 &= R^2\\
|X_4X_8|^2 &= 4R(R-2r)\\
|X_4X_{145}|^2 &= 4(R-2r)^2\\
|X_5X_{10}|^2 &= \frac{1}{4}R(R-2r)\\
|X_5X_{11}|^2 &= \frac{1}{4}R^2\\
|X_5X_{40}|^2 &= rR + \frac{25}{4}R^2 - s^2\\
|X_5X_{80}|^2 &= \frac{1}{4}(R+2r)^2\\
|X_8X_{10}|^2 &= \frac{1}{4}(5r^2 - 16rR + s^2)\\
|X_8X_{145}|^2 &= 4(5r^2 - 16rR + s^2)\\
|X_{10}X_{145}|^2 &= \frac{9}{4}(5r^2 - 16rR + s^2)\\
|X_{11}X_{80}|^2 &= r^2\\
|X_{20}X_{145}|^2 &= 16R(R-2r)
\end{aligned}
$$

2.5. **Perfect squares.**

For some of the R-r-s distance formulas, the expression is a perfect square, meaning that the distance between the two centers can be expressed in terms of R, r, and s without using radicals. These are shown below.

$$\begin{aligned}
|X_1X_5| &= \frac{R-2r}{2}\\
|X_1X_{11}| &= r\\
|X_1X_{80}| &= 2r\\
|X_3X_8| &= R-2r\\
|X_3X_{99}| &= R\\
|X_4X_{145}| &= 2(R-2r)\\
|X_5X_{11}| &= \frac{R}{2}\\
|X_5X_{80}| &= \frac{R+2r}{2}\\
|X_{11}X_{80}| &= r
\end{aligned}$$

2.6. **Finding the inequalities.**

Each R-r-s distance formula gave us an equation of the form

$$|X_iX_j|^2 = F(R,r,s)$$

where F is a rational function of R, r, and s. Since the distance between two points cannot be negative, this give us the inequality

$$F(R,r,s) \geq 0.$$

Factoring F, if one of the factors is obviously nonnegative (using the constraints $s > 0$ and $R \geq 2r > 0$), then we discarded this factor giving an equivalent inequality of the form

$$f(R,r,s) \geq 0.$$

If one of the variables (R, r, or s) or a power thereof can be isolated from the other variables, we then transformed the inequality into the form

$$u^n \geq g(v,w) \qquad \text{or} \qquad u^n \leq g(v,w)$$

where u is the variable being isolated, and v and w are the other variables. Here n is a positive integer.

If no variable could be isolated from $f(R,r,s) \geq 0$, then we moved on to the next distance formula.

During the transformation process, if we reached an inequality of the form

$$h(v,w)u^n \geq G(v,w),$$

we examined the factor $h(v,w)$ to determine how to proceed. If $h(v,w)$ was obviously positive (using the constraints $s > 0$ and $R \geq 2r > 0$), then we formed the equivalent inequality

$$u^n \geq \frac{G(v,w)}{h(v,w)}.$$

If $h(v,w)$ was obviously negative (using the constraints $s>0$ and $R\geq 2r>0$), then we formed the equivalent inequality

$$u^n \leq \frac{G(v,w)}{h(v,w)}.$$

If $h(v,w)$ was neither always positive or always negative for all triangles, then we moved on to the next distance formula.

2.7. **Example 2.**

In Example 1, we found that

$$|X_1X_7|^2 = \frac{r^2\left(r^2+8rR+16R^2-3s^2\right)}{(r+4R)^2}.$$

Since $|X_1X_7|^2 \geq 0$ and clearly r^2 and $(r+4R)^2>0$, we get the inequality

$$r^2+8rR+16R^2-3s^2 \geq 0.$$

Isolating the s^2 term gives

$$s^2 \leq \frac{(r+4R)^2}{3}.$$

2.8. **Example 3.**

Using the barycentric distance formula, we find that the square of the distance between X_2 and X_7 is

$$|X_2X_7|^2 = \frac{4}{9} \times \frac{E}{(a^2-2ab+b^2-2ac-2bc+c^2)^2}$$

where

$$E = -a^6+a^5b+5a^4b^2-10a^3b^3+5a^2b^4+ab^5-b^6+a^5c-7a^4bc+6a^3b^2c$$
$$+6a^2b^3c-7ab^4c+b^5c+5a^4c^2+6a^3bc^2-18a^2b^2c^2+6ab^3c^2+5b^4c^2-10a^3c^3$$
$$+6a^2bc^3+6ab^2c^3-10b^3c^3+5a^2c^4-7abc^4+5b^2c^4+ac^5+bc^5-c^6.$$

Converting to R–r–s form gives

$$|X_2X_7|^2 = \frac{4}{9} \times \frac{4R^2s^2-r^4-12r^3R-r^2(48R^2+5s^2)-8r(8R^3-Rs^2)}{(r+4R)^2}.$$

The factors $\frac{4}{9}$ and $(r+4R)^2$ are obviously positive, so this gives us the inequality

$$4R^2s^2-r^4-12r^3R-r^2(48R^2+5s^2)-8r(8R^3-Rs^2) \geq 0.$$

Collecting terms gives

$$(2R-r)(5r+2R)s^2-r(r+4R)^3 \geq 0.$$

Isolating the s^2 term gives

$$(2R-r)(5r+2R)s^2 \geq r(r+4R)^3.$$

The factor $5r+2R$ is obviously positive. The factor $2R-r$ is positive because of the constraint $R\geq 2r$. We can thus divide both sides of the inequality by the coefficient of s^2 without changing the sense of the inequality. We get

$$s^2 \geq \frac{r(r+4R)^3}{(2R-r)(5r+2R)}.$$

3. Inequalities

When reporting on inequalities found, we exclude duplicate inequalities.

3.1. Inequalities for r and R.

Using the procedure described in Section 2, we found the following inequalities for r and R.

Theorem 1. *The following inequalities are true for all triangles.*

$$
\begin{aligned}
|X_1X_2|^2 \geq 0 &\implies \quad R \leq \frac{5r^2+s^2}{16r}\\
|X_1X_3|^2 \geq 0 &\implies \quad r \leq \frac{R}{2}\\
|X_1X_3|^2 \geq 0 &\implies \quad R \geq 2r\\
|X_5X_{40}|^2 \geq 0 &\implies \quad r \geq \frac{4s^2-25R^2}{4R}
\end{aligned}
$$

3.2. Upper bounds for s^2.

Using the procedure described in Section 2, we found the following upper bounds for s^2 in terms of r and R.

Theorem 2. *The following inequalities are true for all triangles.*

$$
\begin{aligned}
|X_1X_4|^2 \geq 0 &\implies \quad s^2 \leq 3r^2+4rR+4R^2\\
|X_1X_7|^2 \geq 0 &\implies \quad s^2 \leq \frac{1}{3}(r+4R)^2\\
|X_1X_{79}|^2 \geq 0 &\implies \quad s^2 \leq \frac{1}{2}(2r^2+8rR+9R^2)\\
|X_2X_{40}|^2 \geq 0 &\implies \quad s^2 \leq \frac{1}{5}(36R^2-r^2-4rR)\\
|X_3X_7|^2 \geq 0 &\implies \quad s^2 \leq \frac{R^2(r+4R)^2}{4r(r+R)}\\
|X_3X_9|^2 \geq 0 &\implies \quad s^2 \leq \frac{R(r+4R)^2}{2(2R-r)}\\
|X_3X_{12}|^2 \geq 0 &\implies \quad s^2 \leq \frac{2r^4+7r^3R+11r^2R^2+4rR^3+R^4}{r(2r+R)}\\
|X_3X_{79}|^2 \geq 0 &\implies \quad s^2 \leq \frac{4r^4+26r^3R+52r^2R^2+30rR^3+9R^4}{6r(2r+R)}\\
|X_3X_{210}|^2 \geq 0 &\implies \quad s^2 \leq \frac{R(r^2+4rR-9R^2)}{2(r-R)}\\
|X_4X_{12}|^2 \geq 0 &\implies \quad s^2 \leq \frac{2r^4+13r^3R+26r^2R^2+16rR^3+4R^4}{(r+R)(2r+R)}\\
|X_4X_{35}|^2 \geq 0 &\implies \quad s^2 \leq \frac{8r^4+42r^3R+67r^2R^2+28rR^3+4R^4}{(2r+R)(4r+R)}
\end{aligned}
$$

Upper bounds for s^2 (continued):

$$|X_4X_{63}|^2 \geq 0 \implies s^2 \leq \frac{r^4 + 6r^3R + 12r^2R^2 + 8rR^3 + 4R^4}{r(3r + 2R)}$$

$$|X_4X_{65}|^2 \geq 0 \implies s^2 \leq \frac{r^3 + 2r^2R - 4R^3}{r - R}$$

$$|X_4X_{79}|^2 \geq 0 \implies s^2 \leq \frac{2r^3 + 5r^2R - 12rR^2 - 36R^3}{3(2r - 3R)}$$

$$|X_4X_{142}|^2 \geq 0 \implies s^2 \leq \frac{(r + 4R)^2(3r^2 + 12rR + 16R^2)}{(r + 6R)(r + 10R)}$$

$$|X_4X_{210}|^2 \geq 0 \implies s^2 \leq -\frac{3r^3 + 10r^2R - 8rR^2 - 36R^3}{r + 5R}$$

$$|X_5X_{35}|^2 \geq 0 \implies s^2 \leq \frac{8r^4 + 44r^3R + 64r^2R^2 + 8rR^3 + R^4}{4r(2r + R)}$$

$$|X_5X_{40}|^2 \geq 0 \implies s^2 \leq \frac{1}{4}R(4r + 25R)$$

$$|X_5X_{55}|^2 \geq 0 \implies s^2 \leq \frac{2r^4 + 14r^3R + 25r^2R^2 + 2rR^3 + R^4}{2r(r + R)}$$

$$|X_7X_{35}|^2 \geq 0 \implies s^2 \leq \frac{R(r + 4R)^2(2r + 9R}{2r + R)(8r + 11R)}$$

$$|X_7X_{40}|^2 \geq 0 \implies s^2 \leq -\frac{((r + 4R)^2(r^2 + 4rR - 4R^2)}{r(5r + 8R))}$$

$$|X_7X_{55}|^2 \geq 0 \implies s^2 \leq \frac{R(r + 4R)^3}{(r + R)(4r + 7R)}$$

$$|X_7X_{79}|^2 \geq 0 \implies s^2 \leq -\frac{(r + 4R)^2(2r + 9R)}{5(2r - 7R)}$$

$$|X_{10}X_{20}|^2 \geq 0 \implies s^2 \leq \frac{1}{15}(21r^2 + 64rR + 64R^2)$$

$$|X_{12}X_{20}|^2 \geq 0 \implies s^2 \leq \frac{18r^4 + 81r^3R + 136r^2R^2 + 80rR^3 + 16R^4}{3(2r + R)(3r + R)}$$

$$|X_{12}X_{40}|^2 \geq 0 \implies s^2 \leq \frac{R(2r^3 + 13r^2R + 12rR^2 + 4R^3)}{2r(2r + R)}$$

$$|X_{20}X_{35}|^2 \geq 0 \implies s^2 \leq \frac{8r^4 + 38r^3R + 73r^2R^2 + 56rR^3 + 16R^4}{(2r + R)(4r + 3R)}$$

$$|X_{20}X_{55}|^2 \geq 0 \implies s^2 \leq \frac{2r^4 + 11r^3R + 28r^2R^2 + 32rR^3 + 16R^4}{(r + R)(2r + 3R)}$$

$$|X_{20}X_{65}|^2 \geq 0 \implies s^2 \leq -\frac{3r^3 - 16rR^2 - 16R^3}{r + 3R}$$

$$|X_{20}X_{79}|^2 \geq 0 \implies s^2 \leq \frac{24r^4 + 166r^3R + 409r^2R^2 + 408rR^3 + 144R^4}{(4r + 3R)(10r + 9R)}$$

$$|X_{20}X_{142}|^2 \geq 0 \implies s^2 \leq \frac{(r + 4R)^2(15r^2 + 60rR + 64R^2)}{(3r + 10R)(7r + 22R)}$$

Upper bounds for s^2 (continued):

$$|X_{20}X_{210}|^2 \geq 0 \implies s^2 \leq -\frac{3r^3+44r^2R+128rR^2+144R^3}{5(r-7R)}$$

$$|X_{21}X_{40}|^2 \geq 0 \implies s^2 \leq -\frac{10r^3+17r^2R-20rR^2-36R^3}{2r+5R}$$

$$|X_{35}X_{79}|^2 \geq 0 \implies s^2 \leq \frac{4r^4+32r^3R+87r^2R^2+92rR^3+36R^4}{(2r+R)(6r+5R)}$$

$$|X_{40}X_{79}|^2 \geq 0 \implies s^2 \leq \frac{R(r^3+7r^2R+12rR^2+9R^3)}{r(4r+3R)}$$

$$|X_{40}X_{142}|^2 \geq 0 \implies s^2 \leq -\frac{(r+4R)^2(r^2+4rR-16R^2)}{(r+2R)(5r+14R)}$$

$$|X_{40}X_{210}|^2 \geq 0 \implies s^2 \leq \frac{R(5r^2+20rR+36R^2)}{2(-r+4R)}$$

$$|X_{55}X_{79}|^2 \geq 0 \implies s^2 \leq \frac{4r^4+38r^3R+126r^2R^2+170rR^3+81R^4}{2(r+R)(6r+7R)}$$

3.3. Lower bounds for s^2.

Using the procedure described in Section 2, we found the following lower bounds for s^2 in terms of r and R.

Theorem 3. *The following inequalities are true for all triangles.*

$$|X_1X_2|^2 \geq 0 \implies s^2 \geq r(16R-5r)$$

$$|X_1X_9|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2}{r+R}$$

$$|X_1X_{21}|^2 \geq 0 \implies s^2 \geq \frac{r(6r^2+19rR+16R^2)}{2r+R}$$

$$|X_1X_{142}|^2 \geq 0 \implies s^2 \geq \frac{r(3r-4R)(r+4R)^2}{(r-2R)(r+2R)}$$

$$|X_1X_{210}|^2 \geq 0 \implies s^2 \geq \frac{rR(7r+64R)}{2(r+2R)}$$

$$|X_2X_7|^2 \geq 0 \implies s^2 \geq \frac{(r(r+4R)^3}{(2R-r)(5r+2R)}$$

$$|X_2X_{12}|^2 \geq 0 \implies s^2 \geq \frac{r(2r^3+r^2R+4rR^2-16R^3)}{(r-R)(2r+R)}$$

$$|X_2X_{65}|^2 \geq 0 \implies s^2 \geq \frac{r(15r^2+28rR+16R^2)}{3r+R}$$

$$|X_2X_{72}|^2 \geq 0 \implies s^2 \geq \frac{r(12r^2+55rR+64R^2)}{2(3r+2R)}$$

$$|X_2X_{79}|^2 \geq 0 \implies s^2 \geq \frac{r(8r^3+38r^2R-3rR^2-144R^3)}{(4r-3R)(14r+3R)}$$

$$|X_3X_{63}|^2 \geq 0 \implies s^2 \geq \frac{r^3+5r^2R+6rR^2-R^3}{r}$$

Lower bounds for s^2 (continued):

$$|X_3X_{145}|^2 \geq 0 \implies s^2 \geq \frac{1}{2}(-6r^2+32rR-R^2)$$

$$|X_5X_7|^2 \geq 0 \implies s^2 \geq \frac{(r+4R)^2(2r^2+8rR-R^2)}{6r(4R-r)}$$

$$|X_5X_{63}|^2 \geq 0 \implies s^2 \geq \frac{R(2r^3+13r^2R+22rR^2-R^3)}{2(R-r)(2r+R)}$$

$$|X_5X_{65}|^2 \geq 0 \implies s^2 \geq \frac{6r^3+12r^2R+8rR^2-R^3}{2r}$$

$$|X_5X_{72}|^2 \geq 0 \implies s^2 \geq \frac{6r^3+26r^2R+28rR^2-R^3}{2(r+R)}$$

$$|X_5X_{79}|^2 \geq 0 \implies s^2 \geq \frac{R(8r^3-+6r^2R+96rR^2-9R^3)}{8r(-2r+3R)}$$

$$|X_5X_{145}|^2 \geq 0 \implies s^2 \geq \frac{1}{6}(-42r^2+104rR-R^2)$$

$$|X_5X_{210}|^2 \geq 0 \implies s^2 \geq \frac{6r^3+40r^2R+64rR^2-9R^3}{2(r+2R)}$$

$$|X_7X_{10}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(3r+16R)}{(4R-r)(7r+4R)}$$

$$|X_7X_{12}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(2r+9R)}{3(3R-r)(2r+R)}$$

$$|X_7X_{21}|^2 \geq 0 \implies s^2 \geq \frac{rR(r+4R)^2}{(R-r)(4r+R)}$$

$$|X_7X_{72}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(2r^2+11rR+16R^2)}{4R(r^2+8rR+4R^2)}$$

$$|X_7X_{210}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^3(3r+16R)}{(8R-r)(r^2+15rR+8R^2)}$$

$$|X_8X_{12}|^2 \geq 0 \implies s^2 \geq -\frac{r(6r^3-21r^2R-40rR^2-16R^3)}{(r+R)(2r+R)}$$

$$|X_8X_{21}|^2 \geq 0 \implies s^2 \geq -\frac{r(4r^3+5r^2R-12rR^2-16R^3)}{R(r+R)}$$

$$|X_8X_{35}|^2 \geq 0 \implies s^2 \geq -\frac{r(16r^3+2r^2R-27rR^2-16R^3)}{R(2r+R)}$$

$$|X_8X_{142}|^2 \geq 0 \implies s^2 \geq \frac{r(36R-7r)(r+4R)^2}{3(r+2R)(r+6R)}$$

$$|X_8X_{210}|^2 \geq 0 \implies s^2 \geq \frac{r(3r^2-20rR+16R^2)}{R-r}$$

$$|X_9X_{12}|^2 \geq 0 \implies s^2 \geq \frac{rR(r+4R)^2(2r^2+5rR+4R^2)}{2(r+R)(2r+R)(2R^2-r^2)}$$

$$|X_9X_{21}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(2r^2+5rR+4R^2)}{(3r+2R)(2r^2+rR+2R^2)}$$

Lower bounds for s^2 (continued):

$$|X_9X_{35}|^2 \geq 0 \implies s^2 \geq \frac{rR^2(r+4R)^2}{(2r+R)(r^2-rR+R^2)}$$

$$|X_9X_{55}|^2 \geq 0 \implies s^2 \geq \frac{rR(r+4R)^3}{2(r+R)(r^2+2R^2)}$$

$$|X_9X_{65}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(2r^2+5rR+4R^2)}{2R(r^2+6rR+2R^2)}$$

$$|X_{10}X_{55}|^2 \geq 0 \implies s^2 \geq \frac{r(3r^3+12r^2R+5rR^2-16R^3)}{(r-R)(r+R)}$$

$$|X_{10}X_{63}|^2 \geq 0 \implies s^2 \geq \frac{r(r^3+2r^2R-3rR^2-16R^3)}{(r-R)(3r+R)}$$

$$|X_{10}X_{79}|^2 \geq 0 \implies s^2 \geq \frac{r(4r^3+40r^2R+123rR^2+144R^3)}{(3R-2r)(10r+3R)}$$

$$|X_{12}X_{21}|^2 \geq 0 \implies s^2 \geq \frac{r(2r^2+5rR+4R^2)^2}{(r+R)(2r+R)^2}$$

$$|X_{12}X_{63}|^2 \geq 0 \implies s^2 \geq -\frac{rR(4r^4+28r^3R+71r^2R^2+60rR^3+16R^4)}{(2r+R)^2(r^2-rR-R^2))}$$

$$|X_{12}X_{142}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(2r^2-7rR+4R^2)}{(r-2R)(3r-2R)(2r+R)}$$

$$|X_{12}X_{145}|^2 \geq 0 \implies s^2 \geq -\frac{r(42r^3-75r^2R-76rR^2-16R^3)}{(2r+R)(3r+R)}$$

$$|X_{21}X_{65}|^2 \geq 0 \implies s^2 \geq \frac{r(8r^4+34r^3R+53r^2R^2+40rR^3+16R^4)}{R(2r^2+5rR+R^2)}$$

$$|X_{21}X_{72}|^2 \geq 0 \implies s^2 \geq \frac{r(4r^4+34r^3R+104r^2R^2+135rR^3+64R^4)}{2(r+R)(2r^2+3rR+2R^2)}$$

$$|X_{21}X_{79}|^2 \geq 0 \implies s^2 \geq \frac{r(2r^2+11rR+16R^2)}{6r+R}$$

$$|X_{21}X_{145}|^2 \geq 0 \implies s^2 \geq -\frac{r(6r^3-10r^2R-71rR^2-64R^3)}{2(r+R)(r+2R)}$$

$$|X_{21}X_{210}|^2 \geq 0 \implies s^2 \geq \frac{rR(10r^3+91r^2R+204rR^2+144R^3)}{(4r+3R)(2r^2+rR+3R^2)}$$

$$|X_{35}X_{63}|^2 \geq 0 \implies s^2 \geq \frac{r(4r^4+30r^3R+72r^2R^2+59rR^3+16R^4)}{(2r+R)(2r^2+2rR+R^2)}$$

$$|X_{35}X_{72}|^2 \geq 0 \implies s^2 \geq \frac{r(4r^4+24r^3R+51r^2R^2+44rR^3+16R^4)}{(2r+R)(2r^2+rR+R^2)}$$

$$|X_{35}X_{145}|^2 \geq 0 \implies s^2 \geq -\frac{r(24r^3-106r^2R-83rR^2-16R^3)}{(2r+R)(4r+R)}$$

$$|X_{35}X_{210}|^2 \geq 0 \implies s^2 \geq \frac{rR(4r^3+32r^2R+71rR^2+64R^3)}{2(2r+R)(2r^2-rR+2R^2)}$$

$$|X_{40}X_{145}|^2 \geq 0 \implies s^2 \geq \frac{1}{3}(-7r^2+52rR-4R^2)$$

Lower bounds for s^2 (continued):

$$|X_{55}X_{72}|^2 \geq 0 \implies s^2 \geq \frac{r(r^4+7r^3R+20r^2R^2+27rR^3+16R^4)}{(r+R)(r^2+rR+R^2)}$$

$$|X_{63}X_{65}|^2 \geq 0 \implies s^2 \geq \frac{r(2r^4+10r^3R+23r^2R^2+28rR^3+16R^4)}{R(r^2+3rR+R^2)}$$

$$|X_{63}X_{79}|^2 \geq 0 \implies s^2 \geq -\frac{rR(8r^4+74r^3R+240r^2R^2+315rR^3+144R^4)}{(2r^2-3R^2)(8r^2+12rR+3R^2)}$$

$$|X_{63}X_{145}|^2 \geq 0 \implies s^2 \geq -\frac{r(7r^3-28r^2R-96rR^2-64R^3)}{(r+2R)(3r+2R)}$$

$$|X_{63}X_{210}|^2 \geq 0 \implies s^2 \geq -\frac{rR(r+4R)^2}{r^2-3rR-R^2}$$

$$|X_{65}X_{142}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(6r^2+9rR+4R^2)}{(r+2R)(2r^2+7rR+2R^2)}$$

$$|X_{72}X_{79}|^2 \geq 0 \implies s^2 \geq \frac{r(8r^4+62r^3R+183r^2R^2+252rR^3+144R^4)}{3R(2r^2+9rR+3R^2)}$$

$$|X_{72}X_{142}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(6r^2+29rR+36R^2)}{(r+6R)(2r^2+11rR+6R^2)}$$

$$|X_{79}X_{142}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^2(4r^3+24r^2R+27rR^2-36R^3)}{3(2r^2+9rR-6R^2)(2r^2+11rR+2R^2)}$$

$$|X_{79}X_{145}|^2 \geq 0 \implies s^2 \geq \frac{r(56r^3+110r^2R-51rR^2-144R^3)}{(2r-3R)(4r+3R)}$$

$$|X_{79}X_{210}|^2 \geq 0 \implies s^2 \geq -\frac{r(12r^4+118r^3R+436r^2R^2+759rR^3+576R^4)}{2(r-3R)(2r^2+19rR+6R^2)}$$

$$|X_{142}X_{145}|^2 \geq 0 \implies s^2 \geq -\frac{r(39r-100R)(r+4R)^2}{(r+10R)(3r+10R)}$$

$$|X_{142}X_{210}|^2 \geq 0 \implies s^2 \geq \frac{r(r+4R)^3(6r+25R)}{(r+10R)(2r^2+15rR+10R^2)}$$

$$|X_{145}X_{210}|^2 \geq 0 \implies s^2 \geq -\frac{r(3r^2+56rR-400R^2)}{5(r+5R)}$$

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