Coincidences of Special Points Associated with Various Shaped Quadrilaterals

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Abstract

The *Encyclopedia of Poly Geometry* (EPG) records many special points associated with configurations of four points and four lines. We investigate what happens to 93 such points when four vertices, taken in cyclic order, form a quadrilateral of a special type. No two of the 93 points are identically coincident for a general quadrilateral, but special geometric conditions can force substantial collapses. The results range from classes such as tangential, extangential, Pythagorean, and APquad quadrilaterals, which force no coincidences, to highly symmetric classes such as parallelograms, rectangles, rhombi, and squares, for which every point that remains unambiguously defined collapses to the usual center. Among the more striking intermediate phenomena are a 24-point coincidence at infinity for a trapezoid, a 39-point coincidence at infinity for several special trapezoid classes, and three additional coincidence classes for a right trapezoid. Using a characterization of Josefsson, we note that a convex tangential orthodiagonal quadrilateral is necessarily a kite; consequently a bicentric orthodiagonal quadrilateral is the same thing as a cyclic kite. All computations are exact symbolic computations in Wolfram Mathematica.

Keywords. quadrilateral, special points, point coincidences, barycentric coordinates, Encyclopedia of Poly Geometry.

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1 Introduction

There are many familiar special points associated with a triangle: the centroid, circumcenter, orthocenter, incenter, and so on. Configurations involving four points or four lines also possess a surprisingly large collection of natural special points. A systematic catalogue is maintained in the *Encyclopedia of Poly Geometry* (EPG) by Chris van Tienhoven [1, 2].

The purpose of this paper is to study a simple question about these points. Suppose that the reference figure is not a completely general convex quadrilateral, but belongs to a special class: cyclic, tangential, orthodiagonal, equidiagonal, a trapezoid, a kite, a parallelogram, and so forth. Which of the catalogued special points are then forced to coincide?

This question is well suited to exact symbolic computation. The EPG gives homogeneous coordinates for the points, so a defining condition for a special class of quadrilaterals can be imposed directly on those coordinates. Two points coincide precisely when their homogeneous coordinate triples are proportional. Thus the search can be performed exactly rather than numerically.

The main goal is not to reproduce the often lengthy coordinate formulas already available in the EPG, but to record the coincidences that occur when the reference quadrilateral acquires additional structure. Detailed coordinate verifications of every table entry would be repetitive. We therefore explain the symbolic procedure once, give representative calculations, and record the resulting coincidences in tables. Whenever a simple geometric explanation is available, it is included.

2 Terminology and the 93 points

Throughout this paper, the word *quadrilateral* is used in the ordinary elementary-geometric sense: four vertices joined successively by four sides. This is the figure ordinarily called a quadrilateral in school geometry.

The EPG uses different terminology. In the EPG [3],

- a *quadrangle* is a system of four unordered points;
- a *quadrilateral* is a system of four unordered lines; and
- a *quadrigon* is a system of four points and four connecting lines in a fixed cyclic order.

Thus the figure called a quadrigon in the EPG is exactly the kind of figure that we call a quadrilateral in this paper. We adopt the more familiar term because the emphasis here is on familiar shapes such as cyclic quadrilaterals, trapezoids, kites, rectangles, and parallelograms.

When it is necessary to refer to the EPG's four-line object, we use the standard term *complete quadrilateral*. The EPG's four-point object is called a *quadrangle*.

The EPG notation itself will be retained. Thus QA-points arise from the four-point framework, QL-points from the four-line framework, and QG-points from the cyclic four-gon framework. The present EPG lists

44 QA-points, 19 QG-points, 30 QL-points,

for a total of

$$44 + 19 + 30 = 93$$

points considered here. Their names are listed in Appendix A. Their definitions and coordinate formulas may be found in the EPG [4, 5, 6]; we do not reproduce the full coordinate catalogue.

3 Types of quadrilaterals

Let the successive side lengths of the reference quadrilateral be

$$a_1, a_2, a_3, a_4,$$

the diagonal lengths be s, t, and the successive angle measures be

$$A, B, C, D.$$

The basic classes used in this study are listed in Table 1.

Several intersections of these classes are also of interest: bicentric quadrilaterals (cyclic and tangential), exbicentric quadrilaterals (cyclic and extangential), bicentric trapezoids, cyclic orthodiagonal quadrilaterals, equidiagonal kites, equidiagonal orthodiagonal quadrilaterals, equidiagonal orthodiagonal trapezoids, harmonic quadrilaterals (cyclic and equalProdOpp), orthodiagonal trapezoids, tangential trapezoids, cyclic kites, and squares (equiangular rhombi). We shall also see that two apparently further intersections are not new classes at all: tangential orthodiagonal

Table 1: Basic types of quadrilaterals considered.

Type	**Geometric definition**	**Algebraic condition**
general	convex	none
cyclic	has a circumcircle	$A + C = B + D$
tangential	has an incircle	$a_1 + a_3 = a_2 + a_4$
extangential	has an excircle	$a_1 + a_2 = a_3 + a_4$
parallelogram	opposite sides parallel	$a_1 = a_3,\ a_2 = a_4$
equalProdOpp	product of opposite sides equal	$a_1a_3 = a_2a_4$
equalProdAdj	product of adjacent sides equal	$a_1a_2 = a_3a_4$
orthodiagonal	diagonals perpendicular	$a_1^2 + a_3^2 = a_2^2 + a_4^2$
equidiagonal	diagonals have the same length	$s = t$
Pythagorean	equal sums of squares of adjacent sides	$a_1^2 + a_2^2 = a_3^2 + a_4^2$
kite	two pairs of adjacent equal sides	$a_1 = a_2,\ a_3 = a_4$
trapezoid	one pair of opposite sides parallel	$A + B = C + D$
rhombus	equilateral	$a_1 = a_2 = a_3 = a_4$
rectangle	equiangular	$A = B = C = D$
Hjelmslev	two opposite right angles	$A = C = 90°$
isosceles trapezoid	trapezoid with two equal sides	$A = B,\ C = D$
right trapezoid	trapezoid with two adjacent right angles	$A = B = 90°$
APquad	sides in arithmetic progression	$a_4 - a_3 = a_3 - a_2 = a_2 - a_1$

quadrilaterals are precisely kites, and bicentric orthodiagonal quadrilaterals are precisely cyclic kites.

The inclusion relations among the principal classes with which we began the investigation are summarized in Figure 1. An arrow indicates specialization. Right trapezoids and the two equivalent descriptions just mentioned are discussed later in the text.

4 Coordinates and computational method

The EPG uses homogeneous barycentric coordinates relative to a component triangle. We take

$$P_1 = (1 : 0 : 0), \qquad P_2 = (0 : 1 : 0), \qquad P_3 = (0 : 0 : 1), \qquad P_4 = (p : q : r).$$

Let

$$a = P_2P_3, \qquad b = P_3P_1, \qquad c = P_1P_2,$$

and put

$$\rho = |p + q + r|.$$

Define $L_1, L_2, L_3 \geq 0$ by

$$\begin{aligned} L_1^2 &= c^2q^2 + b^2r^2 + (b^2 + c^2 - a^2)qr, \\ L_2^2 &= c^2p^2 + a^2r^2 + (a^2 + c^2 - b^2)pr, \\ L_3^2 &= b^2p^2 + a^2q^2 + (a^2 + b^2 - c^2)pq. \end{aligned}$$

Then

$$P_4P_1 = \frac{L_1}{\rho}, \qquad P_4P_2 = \frac{L_2}{\rho}, \qquad P_4P_3 = \frac{L_3}{\rho}.$$

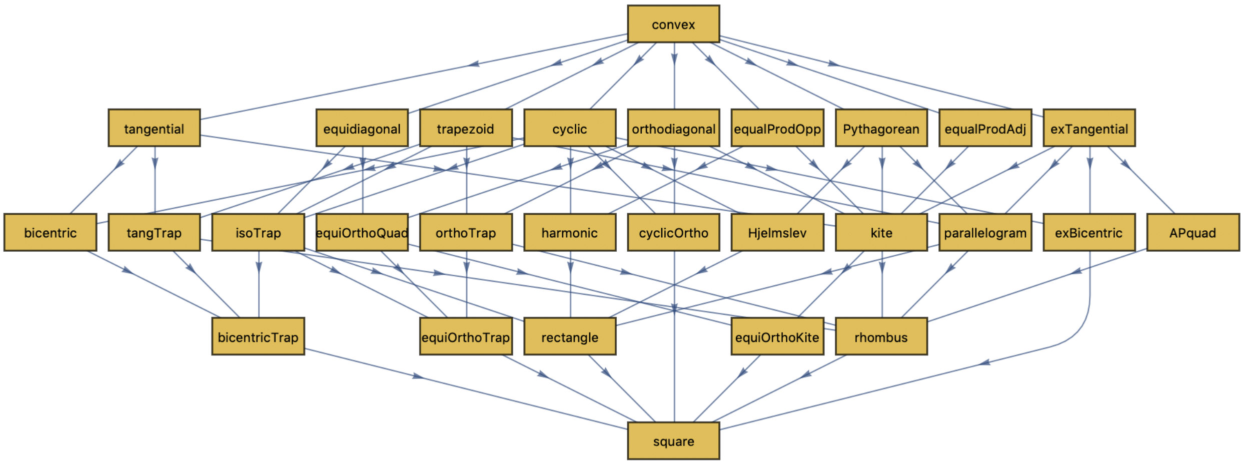

Figure 1: Relations among the principal classes considered. The diagram predates the identifications in Proposition 1 and Corollary 2; arrows are shown only for the classes represented in the diagram.

Thus the four side lengths, in order, are

$$c, \quad a, \quad \frac{L_3}{\rho}, \quad \frac{L_1}{\rho},$$

and the two diagonal lengths are

$$s = b, \qquad t = \frac{L_2}{\rho}.$$

For brevity, put

$$F_{\text{cyc}} = a^2qr + b^2rp + c^2pq$$

and

$$F_{\text{od}} = p(a^2 - b^2 - c^2) + r(a^2 + b^2 - c^2).$$

Table 2 gives conditions in the six parameters a, b, c, p, q, r. The standing assumptions are that the four vertices are finite and distinct, no three are collinear, and they occur in the stated convex order. With the scale of $(p : q : r)$ chosen accordingly, convexity gives

$$p > 0, \qquad q < 0, \qquad r > 0, \qquad p + q + r > 0,$$

so in this normalization $\rho = p + q + r$.

Table 2: Conditions in the EPG coordinate normalization.

Type	**Condition**
general	none
cyclic	$F_{\text{cyc}} = 0$
tangential	$c\rho + L_3 = a\rho + L_1$
extangential	$(a + c)\rho = L_1 + L_3$
parallelogram	$p + q = 0, \quad q + r = 0$
equalProdOpp	$cL_3 = aL_1$
equalProdAdj	$ac\rho^2 = L_1L_3$
orthodiagonal	$F_{\text{od}} = 0$

Type	Condition
equidiagonal	$L_2 = b\rho$
Pythagorean	$(a^2+c^2)\rho^2 = L_1^2 + L_3^2$
kite	$a = c, \quad p = r$
trapezoid	$q + r = 0$
right trapezoid	$q + r = 0, \quad b^2 = a^2 + c^2$
rhombus	$p + q = 0,\ q + r = 0,\ a = c$
rectangle	$p + q = 0,\ q + r = 0,\ b^2 = a^2 + c^2$
Hjelmslev	$2c^2q + (b^2 + c^2 - a^2)r = 0, \quad (a^2 + b^2 - c^2)p + 2a^2q = 0$
isosceles trapezoid	$q + r = 0, \quad a^2q + (b^2 - c^2)p = 0$
APquad	$L_1 - L_3 = L_3 - a\rho = \rho(a - c)$
bicentric	cyclic and tangential
exbicentric	cyclic and extangential
bicentric trapezoid	cyclic, tangential, and $q + r = 0$
cyclic orthodiagonal	$F_{\text{cyc}} = 0,\ F_{\text{od}} = 0$
equidiagonal kite	$a = c,\ p = r,\ L_2 = b\rho$
cyclic kite	$a = c,\ p = r,\ F_{\text{cyc}} = 0$
tangential orthodiagonal	same as kite; see Proposition 1
bicentric orthodiagonal	same as cyclic kite; see Corollary 2
equidiagonal orthodiagonal	$L_2 = b\rho,\ F_{\text{od}} = 0$
equidiagonal orthodiagonal trapezoid	$L_2 = b\rho,\ F_{\text{od}} = 0,\ q + r = 0$
harmonic	$F_{\text{cyc}} = 0,\ cL_3 = aL_1$
orthodiagonal trapezoid	$F_{\text{od}} = 0,\ q + r = 0$
tangential trapezoid	$c\rho + L_3 = a\rho + L_1,\ q + r = 0$
square	$p + q = 0,\ q + r = 0,\ a = c,\ b^2 = 2a^2$

For the isosceles-trapezoid entry, after $r = -q$ the equality of the two legs factors as

$$(p+q)\big(a^2q + (b^2 - c^2)p\big) = 0.$$

The branch $p + q = 0$ is the parallelogram specialization; the table records the other branch.

Each EPG point is represented by a homogeneous triple

$$X = (X_1 : X_2 : X_3).$$

For nonzero homogeneous triples

$$X = (X_1 : X_2 : X_3), \qquad Y = (Y_1 : Y_2 : Y_3),$$

the points X and Y coincide if and only if

$$X_1Y_2 - X_2Y_1 = X_1Y_3 - X_3Y_1 = X_2Y_3 - X_3Y_2 = 0. \tag{1}$$

For each class, its defining condition is substituted into all 93 coordinate triples, and Mathematica is used to simplify the three expressions in (1) for each of the

$$\binom{93}{2} = 4278$$

pairs. Zero triples are excluded from this test and treated as degenerate constructions, as explained in Subsection 4.1.

One point requires special care. The QL coordinates in the EPG are coordinates relative to the component triangle of the four lines, not directly the same coordinate chart as the QA and QG coordinates. Before a QL point is compared with a QA or QG point, its coordinates are therefore converted back into the common projective plane. This conversion is essential in parallel-sided cases, where a component vertex may move to infinity.

All computations for this paper were carried out with Wolfram Mathematica 15.0 [10], using exact arithmetic whenever possible and the coordinate formulas supplied by the EPG [7].

4.1 Coincidence versus degeneration

A special shape can cause a generic coordinate triple to become $(0:0:0)$, or can make a limiting point depend on the path by which the special shape is approached. Such a construction is called *degenerate* here and is not counted as a coincidence. This distinction is especially important for parallelograms and their specializations.

4.2 A note on QA-P44

QA-P44 is defined in the EPG as the point minimizing the sum of the squared distances to the six sidelines of the quadrangle. The currently listed coordinate formula appears to omit the factors a^2, b^2, c^2 from the first, second, and third coordinates, respectively. We use the corrected coordinates in this paper.

A simple check is supplied by the convex quadrangle

$$P_1 = (0,0), \qquad P_2 = (4,0), \qquad P_3 = (1,3), \qquad P_4 = (-1,1).$$

Here

$$a^2 = 18, \qquad b^2 = 10, \qquad c^2 = 16, \qquad P_4 = (3 : -1 : 1).$$

Direct minimization places QA-P44 at

$$\left(\frac{342}{1109}, \frac{1136}{1109}\right),$$

with barycentric coordinates

$$(4437 : -55 : 2272),$$

whereas the presently listed formula gives

$$(493 : -11 : 284).$$

Since

$$18(493) : 10(-11) : 16(284) = 4437 : -55 : 2272,$$

multiplying the three listed coordinates by a^2, b^2, c^2 gives the correct result. The three side-square factors are pairwise distinct in this example, so all three corrections are tested.

5 The general quadrilateral

There are no forced coincidences among the 93 points for a general quadrilateral.

Indeed, before imposing any special relation on a, b, c, p, q, r, each of the 4278 pairs is tested by (1). If two EPG points were identically the same for a general quadrilateral, all three minors would vanish identically after denominators were cleared. The symbolic calculation shows that this never occurs.

Thus every coincidence recorded below is genuinely caused by the special shape of the quadrilateral.

6 Cyclic, tangential, and related classes

6.1 Cyclic quadrilaterals

For a cyclic quadrilateral,

$$F_{\text{cyc}} = a^2qr + b^2rp + c^2pq = 0.$$

There are ten nontrivial coincidence classes.

Table 3: Coincidences for a cyclic quadrilateral.

Coincident points	**Notable names**
QA−P3 = QA−P4 = QA−P8 = QA−P12 = QA−P32 = QA−P42 = QG−P5	Gergonne–Steiner Point; Isogonal Center; 1st QG-Quasi Circumcenter.
QA−P2 = QA−P14 = QA−P15 = QA−P37 = QG−P10 = QL−P2	Euler–Poncelet Point; Orthocenter of the Morley Triangle; Morley Point.
QA−P1 = QA−P6 = QA−P36	QA-Centroid; Parabola Axes Crosspoint.
QA−P13 = QA−P28	Nine-point Center of the QA-Diagonal Triangle; Midpoint Foci of QA-Parabolas.
QA−P20 = QA−P30	
QA−P23 = QA−P41	
QA−P29 = QA−P39	
QG−P1 = QG−P16	Diagonal Crosspoint; Schmidt Point.
QG−P15 = QG−P19	Kirikami Center; Quasi Isogonal Conjugate of QG-P1.
QG−P17 = QL−P1	Projection of QG-P1 on QG-L1; Miquel Point.

6.2 Tangential quadrilaterals

Tangentiality by itself forces no coincidences among the 93 points:

$\boxed{\text{tangential: no forced coincidences.}}$

6.3 Bicentric quadrilaterals

A bicentric quadrilateral is both cyclic and tangential. The extra tangential condition creates no new relation: the coincidence classes are exactly the ten classes in Table 3.

6.4 Extangential quadrilaterals

Extangentiality by itself also forces no coincidences:

$$\boxed{\text{extangential: no forced coincidences.}}$$

For example,

$$A = (0,0), \quad B = (3,0), \quad C = (3,4), \quad D = \left(-\frac{45}{97}, \frac{336}{97}\right)$$

has side lengths

$$3, \quad 4, \quad \frac{340}{97}, \quad \frac{339}{97},$$

so the extangential relation is satisfied, and all 93 points are pairwise distinct.

6.5 Exbicentric quadrilaterals

An exbicentric quadrilateral is cyclic and extangential. Again the second condition adds nothing to the cyclic coincidence pattern. Thus the ten classes are exactly those of Table 3.

6.6 Harmonic quadrilaterals

A harmonic quadrilateral is cyclic and satisfies equalProdOpp. The number of classes remains ten, but two cyclic classes enlarge.

Table 4: Coincidences for a harmonic quadrilateral.

Coincident points	**Remark**
$\mathrm{QA-P3} = \mathrm{QA-P4} = \mathrm{QA-P8} = \mathrm{QA-P12} = \mathrm{QA-P32} = \mathrm{QA-P42} = \mathrm{QG-P5} = \mathrm{QG-P13} = \mathrm{QL-P10}$	A cyclic class enlarged by QG-P13 and QL-P10.
$\mathrm{QA-P2} = \mathrm{QA-P14} = \mathrm{QA-P15} = \mathrm{QA-P37} = \mathrm{QG-P10} = \mathrm{QL-P2}$	Inherited from the cyclic case.
$\mathrm{QA-P44} = \mathrm{QG-P1} = \mathrm{QG-P16} = \mathrm{QL-P26}$	The QA and QL least-squares points both become the diagonal crosspoint.
$\mathrm{QA-P1} = \mathrm{QA-P6} = \mathrm{QA-P36}$	
$\mathrm{QA-P13} = \mathrm{QA-P28}$	
$\mathrm{QA-P20} = \mathrm{QA-P30}$	
$\mathrm{QA-P23} = \mathrm{QA-P41}$	
$\mathrm{QA-P29} = \mathrm{QA-P39}$	
$\mathrm{QG-P15} = \mathrm{QG-P19}$	
$\mathrm{QG-P17} = \mathrm{QL-P1}$	

6.7 Hjelmslev quadrilaterals

A Hjelmslev quadrilateral has two opposite right angles and is therefore cyclic. It has twelve nontrivial coincidence classes.

Table 5: Coincidences for a Hjelmslev quadrilateral.

Coincident points	Remark
QA−P2 = QA−P14 = QA−P15 = QA−P37 = QG−P10 = QL−P2 = QL−P7 = QL−P20 = QL−P26	This common point is the midpoint of P_1P_3, the non-diameter diagonal.
QA−P3 = QA−P4 = QA−P8 = QA−P12 = QA−P32 = QA−P42 = QG−P5	This common point is the midpoint of the diameter diagonal, hence the circumcenter.
QG−P17 = QL−P1 = QL−P10 = QL−P16 = QL−P24	
QL−P3 = QL−P4 = QL−P5 = QL−P6 = QL−P27	A new Hjelmslev class.
QA−P1 = QA−P6 = QA−P36	
QA−P13 = QA−P28	
QA−P20 = QA−P30	
QA−P23 = QA−P41	
QA−P29 = QA−P39	
QA−P9 = QG−P2	A new Hjelmslev class.
QG−P1 = QG−P16	
QG−P15 = QG−P19	

7 Conditions on the diagonals

7.1 Equidiagonal quadrilaterals

There are exactly three nontrivial coincidence classes:

$$\boxed{\mathrm{QA-P15 = QG-P7 = QL-P2}}, \qquad \boxed{\mathrm{QA-P1 = QL-P7}}, \qquad \boxed{\mathrm{QA-P24 = QG-P6}}.$$

7.2 Orthodiagonal quadrilaterals

There are nine nontrivial coincidence classes.

Table 6: Coincidences for an orthodiagonal quadrilateral.

Coincident points	Notable names
QA−P2 = QG−P1 = QG−P6 = QG−P10 = QL−P10	Euler–Poncelet Point; Diagonal Crosspoint.
QA−P1 = QG−P7 = QG−P9	QA-Centroid.
QA−P3 = QG−P5 = QG−P15	Gergonne–Steiner Point.
QA−P29 = QG−P2	
QA−P34 = QG−P4	
QA−P4 = QG−P19	Isogonal Center.
QA−P41 = QG−P18	
QA−P9 = QL−P1	Miquel point.
QG−P3 = QL−P9	

7.3 Cyclic orthodiagonal quadrilaterals

Combining cyclicity and orthodiagonality produces eleven classes. Most are obtained by merging cyclic and orthodiagonal classes, but one new relation appears:

$$\boxed{\mathrm{QA-P7} = \mathrm{QG-P8}}.$$

Table 7: Coincidences for a cyclic orthodiagonal quadrilateral.

Coincident points	**Remark**
QA−P1 = QA−P6 = QA−P36 = QG−P7 = QG−P9	
QA−P2 = QA−P14 = QA−P15 = QA−P37 = QG−P1 = QG−P6 = QG−P10 = QG−P16 = QL−P2 = QL−P10	
QA−P3 = QA−P4 = QA−P8 = QA−P12 = QA−P32 = QA−P42 = QG−P5 = QG−P15 = QG−P19	
QA−P7 = QG−P8	new
QA−P9 = QG−P17 = QL−P1	
QA−P13 = QA−P28	
QA−P20 = QA−P30	
QA−P23 = QA−P41 = QG−P18	
QA−P29 = QA−P39 = QG−P2	
QA−P34 = QG−P4	
QG−P3 = QL−P9	

7.4 Equidiagonal orthodiagonal quadrilaterals

Equidiagonal orthodiagonal quadrilaterals are sometimes called *midsquare quadrilaterals* [9]. There are eleven coincidence classes.

Table 8: Coincidences for an equidiagonal orthodiagonal quadrilateral.

Coincident points	**Remark**
QA−P1 = QA−P15 = QG−P7 = QG−P9 = QL−P2 = QL−P7	
QA−P2 = QA−P24 = QG−P1 = QG−P6 = QG−P10 = QL−P10	
QA−P3 = QG−P5 = QG−P15	
QA−P14 = QG−P8	new from combining the two conditions
QA−P29 = QG−P2	
QA−P34 = QG−P4	
QA−P4 = QG−P19	
QA−P40 = QL−P12	new from combining the two conditions
QA−P41 = QG−P18	
QA−P9 = QL−P1	
QG−P3 = QL−P9	

8 The trapezoid family

For a trapezoid we choose the orientation

$$P_1P_4 \parallel P_2P_3.$$

In our coordinate normalization this is

$$q + r = 0.$$

The common point at infinity of the parallel sides is

$$X_\infty = (0 : 1 : -1).$$

For later use define the 24-point class

$$\begin{aligned}\mathcal{T}_\infty = \{&\mathrm{QA-P5}, \mathrm{QA-P10}, \mathrm{QA-P19}, \mathrm{QA-P20}, \mathrm{QA-P22}, \mathrm{QA-P25}, \mathrm{QA-P26}, \mathrm{QA-P27},\\ &\mathrm{QA-P28}, \mathrm{QA-P29}, \mathrm{QA-P31}, \mathrm{QA-P35}, \mathrm{QA-P43}, \mathrm{QG-P2}, \mathrm{QG-P14},\\ &\mathrm{QL-P5}, \mathrm{QL-P12}, \mathrm{QL-P14}, \mathrm{QL-P15}, \mathrm{QL-P18}, \mathrm{QL-P20}, \mathrm{QL-P21},\\ &\mathrm{QL-P22}, \mathrm{QL-P23}\}.\end{aligned}$$

Every point in $\mathcal{T}_\infty$ equals X_∞.

As a representative coordinate calculation, imposing $r = -q$ on the EPG formulas gives

$$\mathrm{QA-P1} = (2p : p + q : p - q)$$

and

$$\mathrm{QG-P12} = (-2pq : -q(p + q) : -q(p - q)).$$

Thus

$$\mathrm{QG-P12} = -q\,\mathrm{QA-P1},$$

which proves the first coincidence in the trapezoid list below.

8.1 General trapezoids

There are nine nontrivial coincidence classes:

$$\mathrm{QA-P1} = \mathrm{QA-P18} = \mathrm{QG-P12},$$
$$\mathcal{T}_\infty, \qquad \mathrm{QA-P11} = \mathrm{QA-P30} = \mathrm{QA-P40} = \mathrm{QL-P6},$$
$$\mathrm{QA-P12} = \mathrm{QA-P14} = \mathrm{QA-P24} = \mathrm{QA-P36} = \mathrm{QA-P37} = \mathrm{QL-P2},$$
$$\mathrm{QA-P13} = \mathrm{QA-P39}, \qquad \mathrm{QA-P15} = \mathrm{QA-P38}, \qquad \mathrm{QA-P17} = \mathrm{QA-P21},$$
$$\mathrm{QG-P1} = \mathrm{QL-P17}, \qquad \mathrm{QG-P3} = \mathrm{QL-P1} = \mathrm{QL-P25}.$$

The largest class is the 24-point class $\mathcal{T}_\infty$. This is one of the most striking coincidences in the paper, but geometrically its common point is very simple: it is just the point at infinity of the parallel sides.

8.2 Right trapezoids

Assume

$$P_1P_4 \parallel P_2P_3, \qquad \angle P_1 = \angle P_2 = 90^\circ.$$

Then

$$q + r = 0, \qquad b^2 = a^2 + c^2.$$

A generic right trapezoid has twelve nontrivial coincidence classes.

The point at infinity of the parallel sides contains the following 23-point class:

$$\begin{aligned}\mathcal{T}_\infty^\perp = \{&\mathrm{QA-P5}, \mathrm{QA-P10}, \mathrm{QA-P19}, \mathrm{QA-P20}, \mathrm{QA-P22}, \mathrm{QA-P25}, \mathrm{QA-P26},\\ &\mathrm{QA-P27}, \mathrm{QA-P28}, \mathrm{QA-P29}, \mathrm{QA-P31}, \mathrm{QA-P35}, \mathrm{QA-P43}, \mathrm{QG-P2}, \mathrm{QG-P14},\\ &\mathrm{QL-P5}, \mathrm{QL-P12}, \mathrm{QL-P14}, \mathrm{QL-P15}, \mathrm{QL-P18}, \mathrm{QL-P22}, \mathrm{QL-P23}, \mathrm{QL-P28}\}.\end{aligned}$$

Compared with the 24-point class for a general trapezoid, QL−P20 and QL−P21 drop out because their constructions degenerate, while QL−P28 joins the class.

The other eight inherited trapezoid classes are

$$\begin{gathered}\mathrm{QA-P1} = \mathrm{QA-P18} = \mathrm{QG-P12},\\ \mathrm{QA-P11} = \mathrm{QA-P30} = \mathrm{QA-P40} = \mathrm{QL-P6},\\ \mathrm{QA-P12} = \mathrm{QA-P14} = \mathrm{QA-P24} = \mathrm{QA-P36} = \mathrm{QA-P37} = \mathrm{QL-P2},\\ \mathrm{QA-P13} = \mathrm{QA-P39}, \qquad \mathrm{QA-P15} = \mathrm{QA-P38}, \qquad \mathrm{QA-P17} = \mathrm{QA-P21},\\ \mathrm{QG-P1} = \mathrm{QL-P17}, \qquad \mathrm{QG-P3} = \mathrm{QL-P1} = \mathrm{QL-P25}.\end{gathered}$$

There are three further coincidence classes:

$$\boxed{\mathrm{QA-P23} = \mathrm{QL-P7}}, \qquad \boxed{\mathrm{QL-P3} = \mathrm{QL-P30}}, \qquad \boxed{\mathrm{QL-P4} = \mathrm{QL-P29}}.$$

The common point in the first of these is the midpoint of the perpendicular leg P_1P_2.

Finally,

$$\mathrm{QL-P20}, \qquad \mathrm{QL-P21}$$

degenerate. The exact example

$$P_1 = (0,0), \quad P_2 = (0,3), \quad P_3 = (5,3), \quad P_4 = (7,0)$$

gives exactly these twelve classes and no additional coincidences.

8.3 Tangential trapezoids

Tangentiality creates no new relation. A general tangential trapezoid has exactly the nine classes of the general trapezoid subsection.

8.4 Orthodiagonal trapezoids

There are fourteen nontrivial classes:

$$\begin{gathered}
\mathrm{QA-P1} = \mathrm{QA-P18} = \mathrm{QG-P7} = \mathrm{QG-P9} = \mathrm{QG-P12},\\
\mathrm{QA-P2} = \mathrm{QG-P1} = \mathrm{QG-P6} = \mathrm{QG-P10} = \mathrm{QL-P10} = \mathrm{QL-P17},\\
\mathrm{QA-P3} = \mathrm{QG-P5} = \mathrm{QG-P15}, \qquad \mathrm{QA-P4} = \mathrm{QG-P19}, \qquad \mathcal{T}_\infty,\\
\mathrm{QA-P9} = \mathrm{QG-P3} = \mathrm{QL-P1} = \mathrm{QL-P9} = \mathrm{QL-P25},\\
\mathrm{QA-P11} = \mathrm{QA-P30} = \mathrm{QA-P40} = \mathrm{QL-P6},\\
\mathrm{QA-P12} = \mathrm{QA-P14} = \mathrm{QA-P24} = \mathrm{QA-P36} = \mathrm{QA-P37} = \mathrm{QL-P2},\\
\mathrm{QA-P13} = \mathrm{QA-P39}, \qquad \mathrm{QA-P15} = \mathrm{QA-P38}, \qquad \mathrm{QA-P17} = \mathrm{QA-P21},\\
\mathrm{QA-P34} = \mathrm{QG-P4}, \qquad \mathrm{QA-P41} = \mathrm{QG-P18}, \qquad \boxed{\mathrm{QL-P16} = \mathrm{QL-P24}}.
\end{gathered}$$

The last pair is genuinely new: it is not obtained merely by merging the general trapezoid and orthodiagonal classes.

8.5 Isosceles trapezoids

An isosceles trapezoid is cyclic and equidiagonal. Define the following 39-point class:

$$\begin{aligned}
\mathcal{I}_\infty = \{&\mathrm{QA-P5}, \mathrm{QA-P10}, \mathrm{QA-P11}, \mathrm{QA-P13}, \mathrm{QA-P19}, \mathrm{QA-P20}, \mathrm{QA-P22}, \mathrm{QA-P23},\\
&\mathrm{QA-P25}, \mathrm{QA-P26}, \mathrm{QA-P27}, \mathrm{QA-P28}, \mathrm{QA-P29}, \mathrm{QA-P30}, \mathrm{QA-P31}, \mathrm{QA-P35},\\
&\mathrm{QA-P39}, \mathrm{QA-P40}, \mathrm{QA-P41}, \mathrm{QA-P43}, \mathrm{QG-P2}, \mathrm{QG-P14}, \mathrm{QG-P18},\\
&\mathrm{QL-P3}, \mathrm{QL-P4}, \mathrm{QL-P5}, \mathrm{QL-P6}, \mathrm{QL-P12}, \mathrm{QL-P14}, \mathrm{QL-P15}, \mathrm{QL-P18},\\
&\mathrm{QL-P20}, \mathrm{QL-P21}, \mathrm{QL-P22}, \mathrm{QL-P23}, \mathrm{QL-P27}, \mathrm{QL-P28}, \mathrm{QL-P29}, \mathrm{QL-P30}\}.
\end{aligned}$$

All 39 points equal the point at infinity of the parallel sides.

The nine coincidence classes are

$$\begin{gathered}
\mathrm{QA-P1} = \mathrm{QA-P6} = \mathrm{QA-P18} = \mathrm{QG-P12} = \mathrm{QL-P7},\\
\mathrm{QA-P2} = \mathrm{QG-P10},\\
\mathrm{QA-P3} = \mathrm{QA-P4} = \mathrm{QA-P8} = \mathrm{QA-P32} = \mathrm{QA-P42} = \mathrm{QG-P5},\\
\mathcal{I}_\infty, \qquad \mathrm{QA-P17} = \mathrm{QA-P21},\\
\mathrm{QG-P1} = \mathrm{QG-P16} = \mathrm{QL-P17},\\
\mathrm{QG-P3} = \mathrm{QG-P17} = \mathrm{QL-P1} = \mathrm{QL-P25},\\
\mathrm{QG-P15} = \mathrm{QG-P19}, \qquad \boxed{\mathrm{QL-P16} = \mathrm{QL-P24}}.
\end{gathered}$$

The last pair is not inherited from cyclicity, equidiagonality, or the trapezoid condition separately; it is a new coincidence for their isosceles-trapezoid intersection.

Seven constructions degenerate:

$$\mathrm{QA-P12},\ \mathrm{QA-P14},\ \mathrm{QA-P15},\ \mathrm{QA-P24},\ \mathrm{QA-P36},\ \mathrm{QA-P37},\ \mathrm{QL-P2}.$$

Although QA−P15 degenerates, QA−P38 does not. Under the isosceles-trapezoid conditions its coordinate triple reduces, up to a nonzero scalar, to

$$(-2p : p - q : p + q),$$

which is nonzero in the generic case.

8.6 Equidiagonal orthodiagonal trapezoids

An equidiagonal trapezoid is isosceles, so this class is a specialization of the preceding one. It therefore has the same 39-point coincidence $\mathcal{I}_\infty$ and the same seven degeneracies. Its nine coincidence classes are

$$\begin{gathered}
\mathrm{QA-P1} = \mathrm{QA-P6} = \mathrm{QA-P18} = \mathrm{QG-P7} = \mathrm{QG-P9} = \mathrm{QG-P12} = \mathrm{QL-P7},\\
\mathrm{QA-P2} = \mathrm{QG-P1} = \mathrm{QG-P6} = \mathrm{QG-P10} = \mathrm{QG-P16} = \mathrm{QL-P10} = \mathrm{QL-P17},\\
\mathrm{QA-P3} = \mathrm{QA-P4} = \mathrm{QA-P8} = \mathrm{QA-P32} = \mathrm{QA-P42} = \mathrm{QG-P5} = \mathrm{QG-P15} = \mathrm{QG-P19},\\
\mathcal{I}_\infty, \qquad \mathrm{QA-P7} = \mathrm{QG-P8},\\
\mathrm{QA-P9} = \mathrm{QG-P3} = \mathrm{QG-P17} = \mathrm{QL-P1} = \mathrm{QL-P9} = \mathrm{QL-P25},\\
\mathrm{QA-P17} = \mathrm{QA-P21}, \qquad \mathrm{QA-P34} = \mathrm{QG-P4}, \qquad \mathrm{QL-P16} = \mathrm{QL-P24}.
\end{gathered}$$

8.7 Bicentric trapezoids

A bicentric trapezoid is necessarily isosceles. Again there is the same 39-point class $\mathcal{I}_\infty$ and the same seven degeneracies. The nine classes are

$$\begin{gathered}
\mathrm{QA-P1} = \mathrm{QA-P6} = \mathrm{QA-P18} = \mathrm{QG-P12} = \mathrm{QL-P7} = \mathrm{QL-P10},\\
\mathrm{QA-P2} = \mathrm{QG-P10},\\
\mathrm{QA-P3} = \mathrm{QA-P4} = \mathrm{QA-P8} = \mathrm{QA-P32} = \mathrm{QA-P42} = \mathrm{QG-P5},\\
\mathcal{I}_\infty, \qquad \mathrm{QA-P17} = \mathrm{QA-P21},\\
\mathrm{QG-P1} = \mathrm{QG-P16} = \mathrm{QL-P17},\\
\mathrm{QG-P3} = \mathrm{QG-P17} = \mathrm{QL-P1} = \mathrm{QL-P25},\\
\mathrm{QG-P15} = \mathrm{QG-P19}, \qquad \mathrm{QL-P16} = \mathrm{QL-P24}.
\end{gathered}$$

Compared with a general isosceles trapezoid, the new event is that QL−P10 joins the first class.

9 Kites

9.1 General kites

For our orientation a kite satisfies

$$a = c, \qquad p = r.$$

There are eleven nontrivial coincidence classes:

$$\begin{gathered}
\mathrm{QG-P3} = \mathrm{QL-P8} = \mathrm{QL-P9} = \mathrm{QL-P11} = \mathrm{QL-P14} = \mathrm{QL-P15} = \mathrm{QL-P18} = \mathrm{QL-P25},\\
\mathrm{QA-P3} = \mathrm{QG-P5} = \mathrm{QG-P13} = \mathrm{QG-P15} = \mathrm{QL-P17} = \mathrm{QL-P23},\\
\mathrm{QA-P2} = \mathrm{QG-P1} = \mathrm{QG-P6} = \mathrm{QG-P10} = \mathrm{QG-P14},\\
\mathrm{QA-P1} = \mathrm{QG-P7} = \mathrm{QG-P9},\\
\mathrm{QA-P29} = \mathrm{QG-P2} = \mathrm{QG-P17},\\
\mathrm{QA-P9} = \mathrm{QL-P1} = \mathrm{QL-P24},\\
\mathrm{QA-P34} = \mathrm{QG-P4}, \qquad \mathrm{QA-P4} = \mathrm{QG-P19}, \qquad \mathrm{QA-P41} = \mathrm{QG-P18},\\
\mathrm{QG-P16} = \mathrm{QL-P26}, \qquad \mathrm{QL-P2} = \mathrm{QL-P7}.
\end{gathered}$$

The first class is the point at infinity in the direction of P_1P_3. There is a simple geometric reason for this. Let

$$E = P_1P_2 \cap P_3P_4, \qquad F = P_2P_3 \cap P_4P_1.$$

The points E and F are mirror images in the symmetry axis P_2P_4. Hence $EF \perp P_2P_4$, and therefore $EF \parallel P_1P_3$. A vertex of the QL-diagonal triangle is consequently the infinite point in the direction of P_1P_3, explaining at once the occurrence of QG−P3 and the seven QL-points in the first class.

The second class is the midpoint of the symmetry diagonal, and the third is the intersection of the diagonals.

Two QL constructions degenerate:

$$\mathrm{QL-P10}, \qquad \mathrm{QL-P16}.$$

9.2 Tangential orthodiagonal quadrilaterals

The apparently broader class "tangential and orthodiagonal" is in fact exactly the kite class. This characterization is due to Josefsson [8].

Proposition 1 (Josefsson)**.** *A convex quadrilateral is tangential and orthodiagonal if and only if it is a kite.*

Proof. By Pitot's theorem and the side characterization of orthodiagonality,

$$a_1 + a_3 = a_2 + a_4, \qquad a_1^2 + a_3^2 = a_2^2 + a_4^2.$$

The second equation may be written

$$(a_1 + a_2)(a_1 - a_2) = (a_4 + a_3)(a_4 - a_3),$$

while the first gives

$$a_1 - a_2 = a_4 - a_3.$$

If $a_1 = a_2$, then the first equation gives $a_3 = a_4$. Otherwise, division gives $a_1 + a_2 = a_3 + a_4$, and together with the difference equation this yields

$$a_1 = a_4, \qquad a_2 = a_3.$$

In either case the quadrilateral has two pairs of adjacent equal sides and is a kite. The two alternatives merely correspond to the two cyclic labellings of a kite.

Conversely, the symmetry diagonal of a convex kite is perpendicular to the other diagonal, so the kite is orthodiagonal. Its side lengths satisfy Pitot's relation, and the converse of Pitot's theorem therefore makes it tangential. □

Consequently the eleven coincidence classes and two degeneracies in the preceding subsection already give the complete answer for tangential orthodiagonal quadrilaterals.

9.3 Equidiagonal kites

Adding equal diagonals gives twelve classes:

$$\begin{gathered}
\mathrm{QG-P3} = \mathrm{QL-P8} = \mathrm{QL-P9} = \mathrm{QL-P11} = \mathrm{QL-P14} = \mathrm{QL-P15} = \mathrm{QL-P18} = \mathrm{QL-P25}, \\
\mathrm{QA-P1} = \mathrm{QA-P15} = \mathrm{QG-P7} = \mathrm{QG-P9} = \mathrm{QL-P2} = \mathrm{QL-P7}, \\
\mathrm{QA-P2} = \mathrm{QA-P24} = \mathrm{QG-P1} = \mathrm{QG-P6} = \mathrm{QG-P10} = \mathrm{QG-P14}, \\
\mathrm{QA-P3} = \mathrm{QG-P5} = \mathrm{QG-P13} = \mathrm{QG-P15} = \mathrm{QL-P17} = \mathrm{QL-P23}, \\
\mathrm{QA-P29} = \mathrm{QG-P2} = \mathrm{QG-P17}, \qquad \mathrm{QA-P9} = \mathrm{QL-P1} = \mathrm{QL-P24}, \\
\mathrm{QA-P14} = \mathrm{QG-P8}, \qquad \mathrm{QA-P34} = \mathrm{QG-P4}, \qquad \mathrm{QA-P4} = \mathrm{QG-P19}, \\
\mathrm{QA-P40} = \mathrm{QL-P12}, \qquad \mathrm{QA-P41} = \mathrm{QG-P18}, \qquad \mathrm{QG-P16} = \mathrm{QL-P26}.
\end{gathered}$$

Again QL−P10 and QL−P16 degenerate.

9.4 Cyclic kites

A generic non-square cyclic kite, sometimes called a right kite, has eleven nontrivial coincidence classes.

Table 9: Coincidences for a cyclic kite.

Coincident points	**Remark**
QA−P2 = QA−P14 = QA−P15 = QA−P37 = QA−P44 = QG−P1 = QG−P6 = QG−P10 = QG−P14 = QG−P16 = QL−P2 = QL−P7 = QL−P20 = QL−P26	diagonal crosspoint
QA−P3 = QA−P4 = QA−P8 = QA−P12 = QA−P32 = QA−P42 = QG−P5 = QG−P13 = QG−P15 = QG−P19 = QL−P17 = QL−P23	midpoint of the symmetry diagonal
QG−P3 = QL−P8 = QL−P9 = QL−P11 = QL−P14 = QL−P15 = QL−P18 = QL−P25	
QA−P9 = QA−P29 = QA−P39 = QG−P2 = QG−P17 = QL−P1 = QL−P24	
QA−P1 = QA−P6 = QA−P36 = QG−P7 = QG−P9	
QL−P3 = QL−P4 = QL−P5 = QL−P6 = QL−P27	
QA−P23 = QA−P41 = QG−P18	
QA−P7 = QG−P8	
QA−P13 = QA−P28	
QA−P20 = QA−P30	
QA−P34 = QG−P4	

The two constructions

$$\text{QL}-\text{P10}, \qquad \text{QL}-\text{P16}$$

degenerate.

The pattern is largely explained by inheritance. A cyclic kite is at once cyclic, a kite, orthodiagonal, tangential, harmonic, and Hjelmslev. Taking the transitive closure of the coincidence relations already forced by those broader classes gives exactly the eleven classes in Table 9. For example,

$$A = (-4, 0), \quad B = (0, 2), \quad C = (4, 0), \quad D = (0, -8)$$

is an exact generic cyclic-kite example and produces no additional coincidences.

Corollary 2. *A convex bicentric orthodiagonal quadrilateral is precisely a cyclic kite.*

Proof. By Proposition 1, tangentiality and orthodiagonality are equivalent here to being a kite. Adding cyclicity gives a cyclic kite. The converse is immediate. □

Thus bicentric orthodiagonal quadrilaterals have exactly the same eleven coincidence classes and the same two degeneracies as cyclic kites.

10 Parallelograms, rectangles, rhombi, and squares

Let O denote the intersection of the diagonals. The half-turn about O interchanges P_1 with P_3 and P_2 with P_4. It preserves the unordered four-point set, so it preserves every QA construction; it sends the ordered quadrigon $P_1P_2P_3P_4$ to the cyclic shift $P_3P_4P_1P_2$, so it preserves every QG construction; and it preserves the set of four sidelines, so it preserves every QL construction. Therefore any such point that is uniquely defined and finite must be fixed by the half-turn. Its only finite fixed point is O.

For a general parallelogram and for a rhombus, the following 34 points have an unambiguous finite specialization to O:

$$\begin{aligned}\mathcal{P} = \{&\mathrm{QA-P1}, \mathrm{QA-P2}, \mathrm{QA-P3}, \mathrm{QA-P4}, \mathrm{QA-P6}, \mathrm{QA-P7}, \mathrm{QA-P8}, \mathrm{QA-P16},\\ &\mathrm{QA-P17}, \mathrm{QA-P18}, \mathrm{QA-P21}, \mathrm{QA-P23}, \mathrm{QA-P32}, \mathrm{QA-P33},\\ &\mathrm{QA-P34}, \mathrm{QA-P42}, \mathrm{QA-P44},\\ &\mathrm{QG-P1}, \mathrm{QG-P4}, \mathrm{QG-P5}, \mathrm{QG-P6}, \mathrm{QG-P7}, \mathrm{QG-P8}, \mathrm{QG-P9}, \mathrm{QG-P10},\\ &\mathrm{QG-P11}, \mathrm{QG-P12}, \mathrm{QG-P13}, \mathrm{QG-P15}, \mathrm{QG-P16}, \mathrm{QG-P19},\\ &\mathrm{QL-P13}, \mathrm{QL-P17}, \mathrm{QL-P26}\}.\end{aligned}$$

Every point of $\mathcal{P}$ equals O.

For a rectangle and for a square, QA−P23 degenerates, so the corresponding unambiguous center class is

$$\mathcal{R} = \mathcal{P} \setminus \{\mathrm{QA-P23}\},$$

containing 33 points.

Thus the verified counts are

class	unambiguously defined points at O
parallelogram	34
rectangle	33
rhombus	34
square	33

The remaining generic constructions either give the zero homogeneous triple or have path-dependent or infinite limiting behavior. They are therefore not counted as coincident points.

The square is a natural endpoint for the paper. It would be misleading to say that all 93 points coincide, because many of the generic constructions degenerate. The correct statement is:

> For a square, every one of the 93 special points that remains well defined specializes to the usual center of the square.

11 One-condition classes with no coincidences

We use the label *Pythagorean* for the adjacent-side relation displayed in Tables 1 and 2, and treat the orthodiagonal relation as a separate class.

Four further broad classes turn out to force no coincidences:

$$\boxed{\text{equalProdOpp}}, \qquad \boxed{\text{equalProdAdj}}, \qquad \boxed{\text{Pythagorean}}, \qquad \boxed{\text{APquad}}.$$

Together with the tangential and extangential cases, these show that a single metric relation need not force any collapse among the 93 points.

For several of these classes one exact example is enough to rule out a universal coincidence. For equalProdOpp one may take

$$A = (0, 0), \quad B = (2, 0), \quad C = (-2, 6), \quad D = (-1, 1),$$

whose side lengths are

$$2,\ 2\sqrt{13},\ \sqrt{26},\ \sqrt{2}.$$

For equalProdAdj one may take

$$A = (0,0), \quad B = (2,0), \quad C = (-5,6), \quad D = (-4,2),$$

whose side lengths are

$$2,\ \sqrt{85},\ \sqrt{17},\ 2\sqrt{5}.$$

For a Pythagorean quadrilateral one may take

$$A = (0,0), \quad B = (2,1), \quad C = (4,4), \quad D = \left(\frac{6}{5}, \frac{7}{5}\right),$$

and for an APquad one may take

$$A = (0,0), \quad B = (3,0), \quad C = (3,4), \quad D = \left(-\frac{42}{25}, \frac{144}{25}\right),$$

whose successive side lengths are $3, 4, 5, 6$. In each example all 93 points are defined and pairwise distinct.

12 Summary

Table 10 summarizes the verified classification. A number in the second column is the number of nontrivial coincidence classes, not the number of points participating.

Table 10: Summary of the verified results.

Class	**Classes**	**Comment**
general	0	all 93 distinct
cyclic	10	
tangential	0	
bicentric	10	exactly the cyclic classes
extangential	0	
exbicentric	10	exactly the cyclic classes
harmonic	10	two cyclic classes enlarge
Hjelmslev	12	two new classes; two cyclic classes enlarge
equidiagonal	3	
orthodiagonal	9	
cyclic orthodiagonal	11	new pair QA−P7 = QG−P8
equidiagonal orthodiagonal	11	two new pairs
trapezoid	9	24-point class at infinity
right trapezoid	12	23-point infinity class; 3 new classes; 2 degeneracies
tangential trapezoid	9	exactly the trapezoid classes
orthodiagonal trapezoid	14	new pair QL−P16 = QL−P24
isosceles trapezoid	9	39-point class; 7 degeneracies

Class	Classes	Comment
equidiagonal orthodiagonal trapezoid	9	39-point class; 7 degeneracies
bicentric trapezoid	9	39-point class; 7 degeneracies
kite	11	2 degeneracies
tangential orthodiagonal	11	same class and same results as kite
equidiagonal kite	12	2 degeneracies
cyclic kite	11	2 degeneracies
bicentric orthodiagonal	11	same class and same results as cyclic kite
equalProdOpp	0	
equalProdAdj	0	
Pythagorean	0	
APquad	0	
parallelogram	–	34 unambiguous points at the center
rectangle	–	33 unambiguous points at the center
rhombus	–	34 unambiguous points at the center
square	–	33 unambiguous points at the center

The results show several distinct mechanisms. Some broad metric conditions produce no coincidences at all. Cyclicity produces a substantial but controlled collection of coincidences. Parallelism creates large classes at infinity, while adding a right angle to a trapezoid produces three additional finite coincidence classes. Reflectional and rotational symmetry can merge many otherwise different constructions, while in the most symmetric cases many generic constructions cease to be defined. Finally, the equivalences

$$\text{tangential orthodiagonal} = \text{kite}$$

and

$$\text{bicentric orthodiagonal} = \text{cyclic kite}$$

show that some apparently new intersections of familiar conditions do not produce new shape classes at all.

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A The 93 EPG points

For convenient reference, the 93 points used in this study are listed below with their current EPG names. The coordinate formulas are not reproduced here; they may be found on the corresponding EPG pages [4, 5, 6].

A.1 QA-points

Point	**Name**
QA-P1	QA-Centroid or Quadrangle Centroid
QA-P2	Euler-Poncelet Point
QA-P3	Gergonne-Steiner Point
QA-P4	Isogonal Center
QA-P5	Isotomic Center
QA-P6	Parabola Axes Crosspoint
QA-P7	QA-Nine-point Center Homothetic Center
QA-P8	Midray Homothetic Center
QA-P9	QA-Miquel Center
QA-P10	Centroid of the QA-Diagonal Triangle
QA-P11	Circumcenter of the QA-Diagonal Triangle
QA-P12	Orthocenter of the QA-Diagonal Triangle
QA-P13	Nine-point Center of the QA-Diagonal Triangle
QA-P14	Centroid of the Morley Triangle
QA-P15	Orthocenter of the Morley Triangle
QA-P16	QA-Harmonic Center
QA-P17	Involutary Conjugate of QA-P5
QA-P18	Involutary Conjugate of QA-P19
QA-P19	AntiComplement of QA-P16 wrt QA-Diagonal Triangle
QA-P20	Reflection of QA-P5 in QA-P1
QA-P21	Reflection of QA-P16 in QA-P1
QA-P22	Midpoint of QA-P1 and QA-P20
QA-P23	Inscribed Square Axes Crosspoint
QA-P24	AntiComplement of QA-P1 wrt the Morley Triangle
QA-P25	1st QA-Quasi Centroid
QA-P26	2nd QA-Quasi Centroid
QA-P27	M3D Center
QA-P28	Midpoint Foci of QA-Parabolas
QA-P29	Complement of QA-P2 wrt the QA-Diagonal Triangle
QA-P30	Reflection of QA-P2 in QA-P11
QA-P31	Complement of QA-P16 wrt the QA-Diagonal Triangle
QA-P32	Centroid of the Circumcenter Quadrangle
QA-P33	Centroid of the Orthocenter Quadrangle
QA-P34	Euler-Poncelet Point of the Centroid Quadrangle
QA-P35	Penta Point
QA-P36	Complement of QA-P30 wrt the QA-Diagonal Triangle
QA-P37	Reflection of QA-P12 in QA-P1
QA-P38	Montesdeoca–Hutson Point
QA-P39	Midpoint QA-P12 and QA-P20
QA-P40	Centroid of the QL-P6-Triple Triangle
QA-P41	Involutary Conjugate of QA-P4
QA-P42	QA-Orthopole Center
QA-P43	Least Squared Distance Point QA and QA-DT
QA-P44	Least Squared Distance Point of QA-sidelines

A.2 QG-points

Point	Name
QG-P1	Diagonal Crosspoint
QG-P2	Midpoint 3rd QA-Diagonal
QG-P3	Midpoint 3rd QL-Diagonal
QG-P4	1st QG-Quasi Centroid
QG-P5	1st QG-Quasi Circumcenter
QG-P6	1st QG-Quasi Orthocenter
QG-P7	1st QG-Quasi Nine-point Center
QG-P8	2nd QG-Quasi Centroid
QG-P9	2nd QG-Quasi Circumcenter
QG-P10	2nd QG-Quasi Orthocenter
QG-P11	2nd QG-Quasi Nine-point Center
QG-P12	Inscribed Harmonic Conic Center
QG-P13	Circumscribed Harmonic Conic Center
QG-P14	Center of the M3D Hyperbola
QG-P15	Kirikami Center
QG-P16	Schmidt Point
QG-P17	Projection QG-P1 on QG-L1
QG-P18	Quasi Isogonal Crosspoint
QG-P19	Quasi Isogonal Conjugate of QG-P1

A.3 QL-points

Point	Name
QL-P1	Miquel Point
QL-P2	Morley Point
QL-P3	Kantor-Hervey Point
QL-P4	Miquel Circumcenter
QL-P5	Clawson Center
QL-P6	Dimidium Point
QL-P7	Newton-Steiner Point
QL-P8	Centroid of the QL-Diagonal Triangle
QL-P9	Circumcenter of the QL-Diagonal Triangle
QL-P10	Orthocenter of the QL-Diagonal Triangle
QL-P11	Nine-point Center of the QL-Diagonal Triangle
QL-P12	QL-Centroid or Lateral Centroid
QL-P13	QL-Harmonic Center or Lateral Harmonic Center
QL-P14	1st QL-Quasi Centroid
QL-P15	2nd QL-Quasi Centroid
QL-P16	QL-Quasi Circumcenter
QL-P17	Adjunct QL-Quasi Circumcenter
QL-P18	Reflection of QL-P8 in QL-P12
QL-P19	Midpoint of QL-P1 and QL-P7

Point	Name
QL-P20	QL-Orthocenter Homothetic Center
QL-P21	Adjunct QL-Orthocenter Homothetic Center
QL-P22	QL-Nine-point Center Homothetic Center
QL-P23	Center of the Inscribed Midline Hyperbola
QL-P24	Intersection of QL-P1–QL-P8 and QL-P13–QL-P17
QL-P25	2nd QL-Parabola Focus
QL-P26	Least Squares Point
QL-P27	Nine-point Homothetic Center QL-X(3) Quadrangle
QL-P28	Circumcenter QL-X(186)-Quadrangle
QL-P29	Circumcenter QL-X(265)-Quadrangle
QL-P30	Morley's Second Circle Center

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