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Location of the Vertices of the Self-Polar Triangle of Two Ellipses

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Abstract. Two ellipses, one outside the other, divide the plane into three regions. Let T be the common self-polar triangle of the two ellipses. We show that each of the three vertices of T lies in a different region.

Keywords. ellipse, common self-polar triangle, Geometer's Sketchpad, polar conic.

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The *polar of a point* with respect to a conic and the *pole of a line* with respect to a conic are well known concepts in projective geoemtry. See, for example, [2, Chapter XX], [4], and [5].

A triangle is said to be *self-polar* with respect to a conic if each side of the triangle is the polar of the opposite vertex and each vertex is the pole of the opposite side.

A triangle that is simultaneously self-polar to two conics is called a *common self-polar triangle*. Two conics cannot have more than one self-polar triangle [3, Art. 102].

In order to study common self-polar triangles, I used Geometer's Sketchpad to draw the common self-polar triangle to two ellipses. I used the construction technique given in [3, Art. 110].

As I dragged one of the ellipses around on my computer screen, I observed the following interesting fact. If the two ellipses did not intersect, then they divided the plane into three regions. The vertices of the common self-polar triangle seemed to be such that each of the three vertices belonged to a different region. For example, in Figure 1, $\triangle PQR$ is the self-polar triangle of ellipses E_1 and E_1. Note that vertex R lies inside ellipse E_1, vertex Q lies inside ellipse E_2, and vertex P lies outside both ellipses.

The purpose of this paper is to prove this observation.

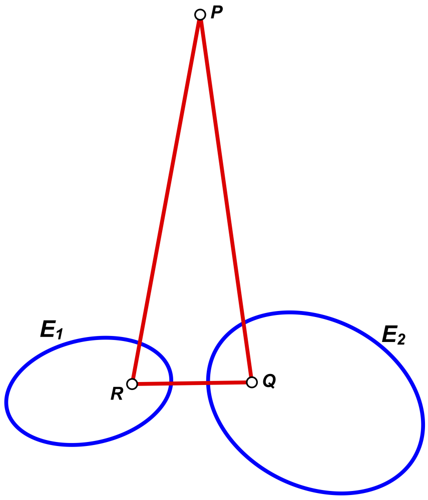

FIGURE 1. self-polar triangle of two ellipses

Theorem. *Let E_1 and E_2 be two ellipses, with one outside the other. Let T be the unique triangle that is self-polar with respect to both ellipses. Then one vertex of T lies inside each of the ellipses and the third vertex lies outside both ellipses.*

The construction process I used (from [3, Art. 110]) involved common conjugates and conjugate conics, but gave no insight as to why this result should be true. However, an alternate construction for the common self-polar triangle, given in [1] was useful for finding a simple proof.

Proof. Let V be a variable point on ellipse E_1 and let T be the tangent to the ellipse at V. The locus of the pole of T with respect to ellipse E_2 as V varies along E_1 is always a conic and is called the *polar conic* of E_1 with respect to E_2. When E_2 lies entirely outside E_1, then the polar conic of E_1 intersects E_2 in four points. Call these four points K_1, K_2, K_3, and K_4 as shown in Figure 2.

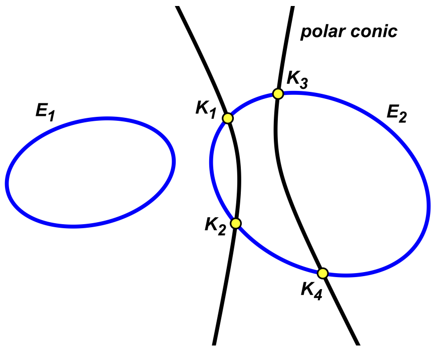

FIGURE 2. polar conic of E_1 with respect to E_2

Then [1] states that the four diagonal points of quadrilateral $K_1K_2K_3K_4$ are the vertices of the common self-polar triangle of the two ellipses. These are shown as points P, Q, and R in Figure 3.

One of the diagonal points of a quadrangle (in this case, point Q in Figure 3) lies inside the quadrangle and the other two diagonal points (P and R) lie outside the quadrangle. Since an ellipse is a convex figure, this confirms that Q must lie inside

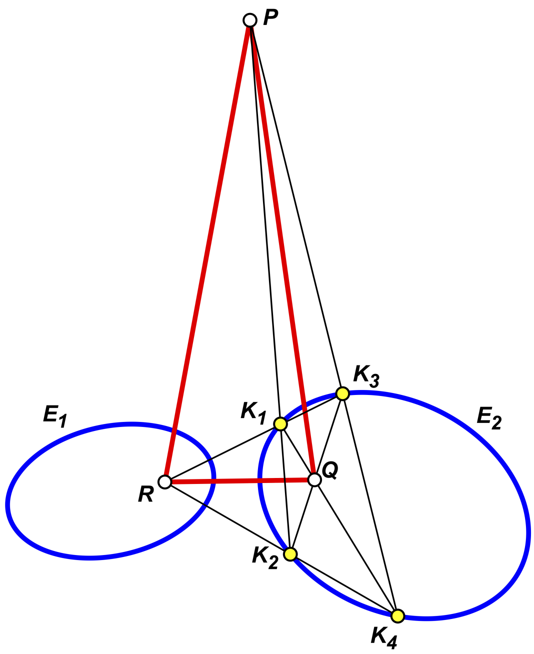

FIGURE 3. formation of common self-polar triangle from K_1, K_2, K_3, K_4

the ellipse E_2. Since a line can meet an ellipse in no more than two points, this confirms that K_1K_3 and K_2K_4 must meet outside the ellipse. Similarly, K_1K_3 and K_2K_4 must meet outside the ellipse.

Since the common self-polar triangle of two ellipses is unique, we can construct it in the same manner using the polar conic of E_2 with respect to E_1. That would show that one of the vertices of the self-polar triangle (in this case, point R) must lie inside ellipse E_1 and the other two vertices must lie outside E_1.

Thus, the remaining vertex P must lie outside both ellipses. □

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