A Sharp Form of the Isogonal Disc Theorem

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Abstract. Let P be an interior point of $\triangle ABC$, let Q be its isogonal conjugate, and let I be the incenter. We give a short purely geometric proof of the sharp inequality

$$\angle PIQ > 90^\circ + \frac{1}{2}\min(A, B, C)$$

for $P \neq I$. The Isogonal Disc Theorem follows immediately, and the bound is best possible.

Keywords. triangle geometry, incenter, isogonal conjugate.

Mathematics Subject Classification (2020). 51M04.

1. Introduction

Lukarevski and Scott [1] conjectured that if P is an interior point of a triangle and Q is its isogonal conjugate, then the incenter I lies in the disc with diameter PQ (the Isogonal Disc Conjecture). Rabinowitz [3] proved the conjecture using barycentric coordinates, but the proof involved extensive computer algebra. A later paper [2] remarked that a purely synthetic proof still seemed remote.

We prove a stronger result by a short geometric argument. Namely, for every interior $P \neq I$,

$$\angle PIQ > 90^\circ + \frac{1}{2}\min(A, B, C),$$

and the constant is sharp. Since the right-hand side is greater than 90°, the Isogonal Disc Theorem is an immediate consequence.

Throughout, ABC is a nondegenerate triangle, I is its incenter, P is an interior point, and Q is the isogonal conjugate of P. Thus the rays AP and AQ are reflections in AI, and similarly at B and C.

2. A sector lemma

We first recall the standard incenter-angle formula.

Lemma 1. *We have*

$$\angle BIC = 90^\circ + \frac{A}{2},$$

with the two cyclic analogs.

Proof. In $\triangle BIC$,

$$\angle IBC = \frac{B}{2}, \qquad \angle ICB = \frac{C}{2}.$$

Hence

$$\angle BIC = 180^\circ - \frac{B+C}{2} = 90^\circ + \frac{A}{2}. \qquad \square$$

The three angle-bisector lines AI, BI, CI divide the plane into six acute sectors. The following observation is the geometric core of the proof.

Lemma 2. *Suppose that the rays IP and IQ lie strictly inside the two vertically opposite acute sectors bounded by the lines BI and CI. Then*

$$\angle PIQ > \angle BIC = 90^\circ + \frac{A}{2}.$$

The two cyclic analogs also hold.

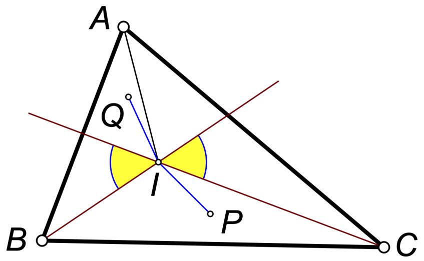

Figure 1. Opposite sectors in the proof of Lemma 2.

Proof. By Lemma 1,

$$\angle BIC = 90^\circ + \frac{A}{2}.$$

The two yellow sectors in Figure 1 are the vertically opposite acute sectors bounded by the lines BI and CI.

The nearest boundary rays of these two sectors are the rays IB and IC. The other nearest pair consists of the rays opposite to IB and IC. In either case, the angle between the two boundary rays is

$$\angle BIC = 90^\circ + \frac{A}{2}.$$

Since the rays IP and IQ lie strictly inside the two opposite sectors, the angle between them is strictly larger than the angle between either nearest pair of boundary rays. Hence,

$$\angle PIQ > \angle BIC = 90^\circ + \frac{A}{2}. \qquad \square$$

3. The sharp theorem

Theorem 3 (Sharp Isogonal Disc Theorem)**.** *Let P be interior to $\triangle ABC$ with $P \neq I$, and let Q be its isogonal conjugate. Then*

$$\angle PIQ > 90^\circ + \frac{1}{2}\min(A, B, C) = \min\big(\angle BIC, \angle CIA, \angle AIB\big),$$

and the constant cannot be increased.

Proof. Suppose first that P lies on one of the angle-bisector lines through I, say AI. Since AP and AQ are isogonal rays, Q also lies on AI. Because $P \neq I$, the ray BP is not BI. The rays BP and BQ, being reflections in BI, therefore lie on opposite sides of BI. Thus P and Q, both lying on the line AI, lie on opposite sides of $I = AI \cap BI$. Hence

$$\angle PIQ = 180^\circ,$$

and the desired inequality follows.

Now suppose that P lies on none of the three angle-bisector lines. Since BP and BQ are distinct isogonal rays, P and Q lie on opposite sides of BI. Similarly they lie on opposite sides of CI and of AI. Consequently P and Q lie in a pair of vertically opposite sectors determined by the three angle-bisector lines.

The pair containing P and Q is bounded by one of

$$(BI, CI), \qquad (CI, AI), \qquad (AI, BI).$$

Lemma 2 and its cyclic analogs therefore give, respectively,

$$\angle PIQ > \angle BIC, \qquad \angle PIQ > \angle CIA, \qquad \angle PIQ > \angle AIB.$$

In every case,

$$\angle PIQ > \min\bigl(\angle BIC, \angle CIA, \angle AIB\bigr) = 90^\circ + \frac{1}{2}\min(A, B, C).$$

It remains to prove sharpness. Choose a circle tangent to AB at B, with its center on the same side of AB as C, and let P tend to B along a sufficiently short arc of the circle lying inside $\triangle ABC$ (Figure 2).

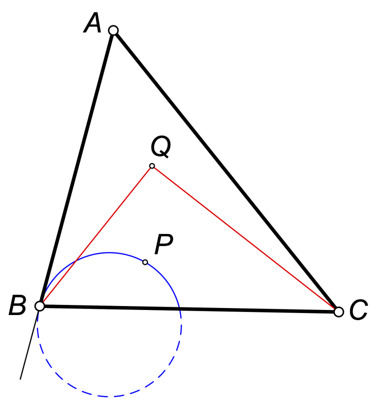

Figure 2. The geometric construction used to prove sharpness.

Since BP tends to BA, its isogonal ray BQ tends to BC. Also CP tends to CB, so its isogonal ray CQ tends to CA. Hence

$$Q = BQ \cap CQ \longrightarrow BC \cap CA = C.$$

Therefore

$$\angle PIQ \longrightarrow \angle BIC = 90^\circ + \frac{A}{2}.$$

Applying the same construction cyclically shows that $\angle PIQ$ can approach each of

$$90^\circ + \frac{A}{2}, \qquad 90^\circ + \frac{B}{2}, \qquad 90^\circ + \frac{C}{2}.$$

Consequently it can approach their minimum,

$$90^\circ + \frac{1}{2}\min(A, B, C),$$

and hence the bound cannot be increased. □

4. Consequences

The usual Isogonal Disc Theorem is now immediate.

Corollary 4 (Isogonal Disc Theorem)**.** *Let P be interior to $\triangle ABC$, and let Q be its isogonal conjugate. Then the incenter I lies in the disc with diameter PQ. If $P \neq I$, then I lies in the interior of that disc.*

Proof. If $P = I$, then $Q = I$, and the assertion is immediate. If $P \neq I$, Theorem 3 gives

$$\angle PIQ > 90^\circ + \frac{1}{2}\min(A, B, C) > 90^\circ.$$

A point sees a segment under an angle greater than a right angle exactly when it lies inside the circle having that segment as diameter. Thus I lies in the interior of the disc with diameter PQ. □

Corollary 5. *The inequality*

$$\angle PIQ > 120^\circ$$

holds for every interior $P \neq I$ if and only if $\triangle ABC$ is equilateral.

Proof. If $\triangle ABC$ is equilateral, then $\min(A, B, C) = 60^\circ$, and Theorem 3 gives the result. Conversely, if the triangle is not equilateral, then one of its angles is less than 60°. The sharpness construction in the proof of Theorem 3, applied at the corresponding vertex, produces interior points P for which $\angle PIQ < 120^\circ$. □

Acknowledgments

The author thanks OpenAI's ChatGPT for suggesting the sector-based geometric proof of the sharp inequality and for helping simplify the exposition.

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