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A Sharp Form of the Isogonal Disc Theorem

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Abstract. Let P be an interior point of a triangle, let Q be its isogonal conjugate, and let I be the incenter. We prove the sharp inequality $\angle PIQ > 90^\circ + \frac{1}{2}\min(A, B, C)$ for $P \neq I$, strengthening the Isogonal Disc Theorem. The proof is elementary: the three angle-bisector lines through I divide the plane into six sectors, and isogonal conjugates lie in opposite sectors.

Keywords. isogonal conjugation, incenter, Isogonal Disc Theorem, sharp angle bound, synthetic geometry.

Mathematics Subject Classification (2020). 51M04.

1. Introduction

Let ABC be a nondegenerate triangle with incenter I, and let P be an interior point. Reflect the ray AP across the angle bisector AI; do the same at B and C. The three reflected rays are concurrent at a point Q, called the *isogonal conjugate* of P. Thus AP and AQ are reflections in AI, and similarly at the other two vertices.

Lukarevski and Scott [1] conjectured that I lies in the disc with diameter PQ. Rabinowitz [2] later proved the conjecture using barycentric coordinates and computer algebra. Lukarevski, Rabinowitz, and Scott [3] noted that a purely synthetic proof remained desirable. Subsequently, Rabinowitz [4] gave a geometric proof using circles through the incenter and inversion about the incenter. The present paper uses a different geometric argument to establish a stronger, sharp result.

Isogonal conjugation can move a point quite far across a triangle. Nevertheless, P and Q can never appear too close together when viewed from the incenter. In fact, their angular separation has a sharp lower bound determined only by the smallest angle of the triangle. We prove the stronger statement

$$\angle PIQ > 90^\circ + \frac{1}{2}\min(A, B, C), \qquad P \neq I.$$

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The bound is best possible. Since its right-hand side is greater than $90°$, the Isogonal Disc Theorem follows immediately.

2. The key picture

The proof comes from looking at the three angle-bisector lines through I. They divide the plane into six sectors. The essential point is that isogonal conjugation places P and Q in opposite sectors. Figure 1 shows one of the three possibilities.

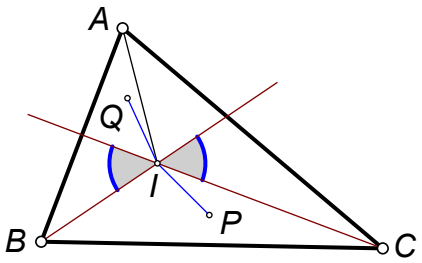


FIGURE 1. Isogonal conjugates lie in opposite sectors determined by the three angle-bisector lines.

Recall the standard incenter-angle formula

$$\angle BIC = 180° - \frac{B+C}{2} = 90° + \frac{A}{2},$$

and its cyclic analogues

$$\angle CIA = 90° + \frac{B}{2}, \qquad \angle AIB = 90° + \frac{C}{2}.$$

Thus the acute angle between the *lines* BI and CI is $90° - A/2$. Consequently, if IP and IQ lie strictly inside two vertically opposite sectors bounded by those lines, the nearest boundary rays of the two sectors are separated by

$$180° - \left(90° - \frac{A}{2}\right) = 90° + \frac{A}{2} = \angle BIC.$$

Because IP and IQ lie strictly inside the sectors, their angular separation is larger:

$$\angle PIQ > \angle BIC.$$

Exactly the same observation applies to the opposite sectors bounded by CI, AI and by AI, BI.

3. The sharp bound

We can now state the result in its sharp form.

Theorem 3.1 (Sharp Isogonal Disc Theorem)**.** *Let P be an interior point of $\triangle ABC$, let Q be its isogonal conjugate, and suppose $P \neq I$. Then*

$$\angle PIQ > 90° + \frac{1}{2}\min(A, B, C).$$

The bound is best possible.

Proof. First suppose that P lies on one of the angle-bisector lines through I, say AI. Since AP and AQ are isogonal rays, Q also lies on AI. Because $P \neq I$, the ray BP is not BI, so the distinct isogonal rays BP and BQ lie on opposite sides of BI. Thus P and Q, both lying on the line AI, are on opposite sides of $I = AI \cap BI$. Hence

$$\angle PIQ = 180^\circ,$$

and the desired inequality is immediate.

Now suppose that P lies on none of the three angle-bisector lines. At A, the distinct isogonal rays AP and AQ lie on opposite sides of AI, so P and Q lie on opposite sides of the line AI. The same reasoning applies to BI and CI. Consequently P and Q lie in a pair of vertically opposite sectors determined by the three angle-bisector lines.

If this pair is bounded by BI and CI, the observation above gives

$$\angle PIQ > \angle BIC = 90^\circ + \frac{A}{2}.$$

For the other two possible pairs we obtain, respectively,

$$\angle PIQ > 90^\circ + \frac{B}{2} \qquad \text{or} \qquad \angle PIQ > 90^\circ + \frac{C}{2}.$$

Therefore in every case

$$\angle PIQ > 90^\circ + \frac{1}{2}\min(A, B, C).$$

It remains to show that no larger universal lower bound is possible. We show how to approach $90^\circ + A/2$; cycling the vertices then gives the other two possibilities.

Choose a circle tangent to AB at B, with its center on the same side of AB as C, and let P tend to B along a sufficiently short arc of this circle contained in the triangle, as in Figure 2. Since the tangent at B is AB, the ray BP tends to BA. Its reflection BQ in BI therefore tends to BC. Also CP tends to CB, so its reflection CQ in CI tends to CA. Hence

$$Q = BQ \cap CQ \longrightarrow BC \cap CA = C.$$

It follows that

$$\angle PIQ \longrightarrow \angle BIC = 90^\circ + \frac{A}{2}.$$

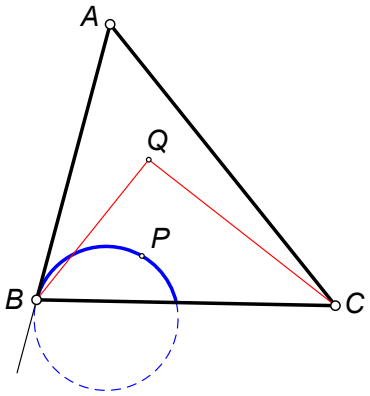


FIGURE 2. As P approaches B along the indicated circle, its isogonal conjugate Q approaches C.

The analogous constructions at the other two vertices show that $\angle PIQ$ can approach each of

$$90^\circ + \frac{A}{2}, \qquad 90^\circ + \frac{B}{2}, \qquad 90^\circ + \frac{C}{2}.$$

Thus it can approach their minimum, proving that the bound is sharp. □

4. What the sharp bound tells us

The original Isogonal Disc Theorem is now immediate. For $P \neq I$, Theorem 3.1 gives

$$\angle PIQ > 90^\circ.$$

By Thales' theorem, a point sees a segment under an angle greater than a right angle exactly when it lies inside the circle having that segment as diameter. Therefore I lies in the interior of the disc with diameter PQ. When $P = I$, its isogonal conjugate is also I, so the degenerate case causes no difficulty.

The sharp form also singles out the equilateral triangle. Namely,

$$\angle PIQ > 120^\circ$$

holds for every interior $P \neq I$ if and only if ABC is equilateral. If the triangle is equilateral, the theorem gives this immediately because $A = B = C = 60^\circ$. Conversely, a nonequilateral triangle has an angle smaller than 60°, and the sharpness construction at that vertex produces interior points P for which $\angle PIQ < 120^\circ$.

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