Reprinted from The American Mathematical Monthly 75.9(1968)1010–1011

A Sum of Multiples of Given Primes

E 2005 [1967, 720]. *Proposed by W. A. McWorter, Ohio State University*

Let $p_1, \cdots, p_t$ be distinct primes and n a positive integer, and let $k=p_1p_2 \cdots p_t$. Show that there exist nonnegative integers $a_1, \cdots, a_t$ such that

$$\sum_{i=1}^{t} a_i p_i = \binom{kn-1}{k-1} - 1.$$

Solution by Stanley Rabinowitz, Far Rockaway, N. Y. Claim: If M is any integer greater than or equal to k, then there exist nonnegative integers $a_1, \cdots, a_t$ such that

$$\sum_{i=1}^{t} a_i p_i = M, \qquad (t > 1).$$

Proof: The case $t=2$ is proved in problem E 1967 [1968, 675]. If it is true for t primes, then $k=p_1p_2 \cdots p_t$ is a linear combination of the p's with nonnegative coefficients. But k and p_{t+1} are relatively prime, so any number $\geqq kp_{t+1}$ is also such a linear combination of the $(t+1)$ p's. Hence by induction our claim is true for all $t \geqq 2$.

If $n>1$, $\binom{kn-1}{k-1}-1 \geqq kn-1 \geqq k$, so the theorem is true for $t>1$ by the above.

If $t=1$, we have modulo p,

$$\binom{pn-1}{p-1} \equiv (pn-1)(pn-2) \cdots (pn-p+1)/(p-1)!$$
$$\equiv (1)(2) \cdots (p-1)/(p-1)! \equiv 1.$$

Hence

$$\binom{pn-1}{p-1} - 1 = ap.$$