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Finding Coefficients of an Identity

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If $F_{n+2k}^2 = aF_{n+2}^2 + bF_n^2 + c(-1)^n$, where a, b, and c depend only on k but not on n, find a, b, and c.

Solution 2 by Stanley Rabinowitz, Mathpro Press, Westford, MA

All problems of this nature can be solved by the following method. We want to find when the expression $E = F_{n+2k}^2 - aF_{n+2}^2 - bF_n^2 - c(-1)^n$ is identically 0 in the variable n. First, apply the reduction formula $F_{n+x} = (F_nL_x + L_nF_x)/2$ to isolate n in subscripts. Then apply the formula $F_n^2 = (L_n^2 - 4(-1)^n)/5$ to remove any powers of F_n. The result is

$$20E = [36a + 16b - 20c - 4L_{2k}^2](-1)^n + [-30a + 10F_{2k}L_{2k}]F_nL_n + [-14a - 4b + 5F_{2k}^2 + L_{2k}^2]L_n^2.$$

This is known as the canonical form of the expression (considering n a variable and k a constant). There is a theorem that says that a polynomial expression in F_n and L_n is identically 0 if and only if its canonical form is 0. (For more details, see my paper "Algorithmic Manipulation of Fibonacci Identities" in *Proceedings of the Sixth International Conference on Fibonacci Numbers and Their Applications*.) Thus, E will be identically 0 if and only if each of the above coefficients in square brackets is 0. That is, if and only if

$$\begin{aligned} 36a + 16b - 20c - 4L_{2k}^2 &= 0, \\ -30a + 10F_{2k}L_{2k} &= 0, \\ -14a - 4b + 5F_{2k}^2 + L_{2k}^2 &= 0. \end{aligned}$$

Solving these equations simultaneously for the unknowns a, b, and c in terms of the constant k shows that E is 0 (identically in n) if and only if

$$a = \frac{F_{2k}L_{2k}}{3}, \quad b = \frac{5F_{2k}^2}{4} - \frac{7F_{2k}L_{2k}}{6} + \frac{L_{2k}^2}{4}, \quad \text{and} \quad c = F_{2k}^2 - \frac{F_{2k}L_{2k}}{3}.$$