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A Solution and Generalization of Problem 2026–2

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Abstract. We give a short synthetic solution to Problem 2 of Hiroshi Okumura's *Problems 2026*. We then allow the inner circle to touch the diameter at an arbitrary point. The generalized configuration yields a simple length formula and, when the contact point divides the diameter in a rational ratio, Euclid's parametrization of Pythagorean triples.

Keywords. semicircle, tangent circles, right triangle, angle bisector, Pythagorean triples, sangaku.

Mathematics Subject Classification (2020). 51M04.

1. Introduction

Okumura [1] presented four problems taken from a Japanese source of 1812 and invited solutions together with additional results or generalizations. His Problem 2 is the following.

For semicircle α of diameter AB, let β be the maximal circle touching AB and α internally having radius b. If $C \neq A$ is a point on α such that BC touches β, show that

$$|BC| = \frac{12}{5}b.$$

Figure 1 shows the configuration with the auxiliary points used below.

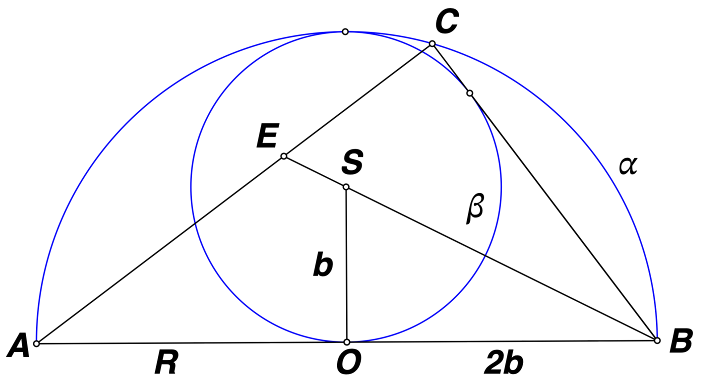

FIGURE 1. The configuration used in the solution.

2. A short solution

Let O be the center of α (equivalently, the midpoint of AB), let $R = OA$, and suppose first that a circle of radius ρ touches AB at T and α internally. If S is its center, then $ST = \rho$ and $OS = R - \rho$. Hence

$$(R-\rho)^2 = OT^2 + \rho^2,$$

so

$$\rho = \frac{R^2 - OT^2}{2R} \le \frac{R}{2}.$$

Thus the maximal circle occurs when $T = O$. For the circle β in the problem we therefore have

$$b = \frac{R}{2}, \qquad AB = 4b.$$

For the maximal circle, O is also the point where β touches AB. Let S be the center of β, and put

$$E = BS \cap AC.$$

Since BA and BC are the two tangents from B to β, the line BS bisects $\angle ABC$. Also $\angle ACB = 90^\circ$ by Thales' theorem. Hence the right triangles BOS and BCE are similar. Since $BO = 2b$ and $OS = b$,

$$\frac{EC}{BC} = \frac{OS}{BO} = \frac{1}{2}.$$

The angle-bisector theorem in triangle ABC now gives

$$\frac{AE}{EC} = \frac{AB}{BC}.$$

If $x = BC$, then $EC = x/2$. Hence the angle-bisector theorem gives

$$AE = \frac{AB}{BC}EC = \frac{4b}{x} \cdot \frac{x}{2} = 2b.$$

Thus

$$AC = 2b + \frac{x}{2}.$$

Applying the Pythagorean theorem to the right triangle ABC gives

$$(4b)^2 = x^2 + \left(2b + \frac{x}{2}\right)^2.$$

The positive root is

$$\boxed{x = \frac{12}{5}b},$$

as required.

3. A generalization

We now move the point where the inner circle touches the diameter.

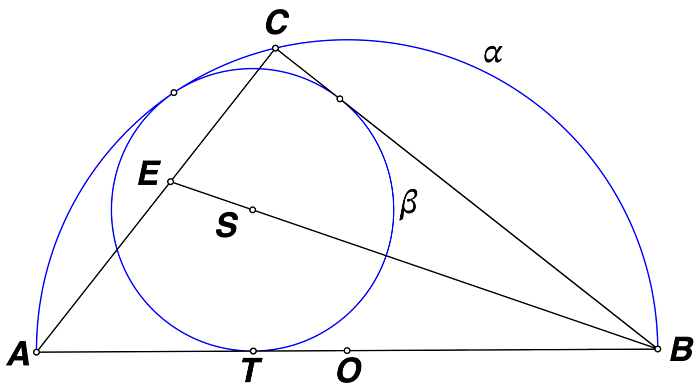

Figure 2. The generalized configuration.

Theorem 3.1. *Let T be any point in the interior of the diameter AB of a semicircle α. Let a circle β of radius r touch AB at T and touch α internally. From B, draw the other tangent to β; let it meet α again at C. Put*

$$L = AB, \qquad t = AT.$$

Then

$$r = \frac{AT \cdot BT}{AB} = \frac{t(L-t)}{L}, \tag{1}$$

$$BC = L\frac{L^2 - t^2}{L^2 + t^2}, \tag{2}$$

$$AC = \frac{2tL^2}{L^2 + t^2}. \tag{3}$$

Moreover, if S is the center of β and $E = BS \cap AC$, then

$$AE = AT. \tag{4}$$

Proof. Let O be the center of α (equivalently, the midpoint of AB), so the radius of α is $R = L/2$. Since $ST = r$ and $OS = R - r$, the right triangle OTS gives

$$(R - r)^2 = OT^2 + r^2.$$

Therefore

$$2Rr = R^2 - OT^2 = (R - OT)(R + OT) = AT \cdot BT,$$

where the two factors $R - OT$ and $R + OT$ are AT and BT in some order. Since $2R = L$, formula (1) follows.

Because BA and BC are tangents to β, BS bisects $\angle ABC$. The triangles BTS and BCE are right triangles and have equal angles at B, so they are similar. Hence

$$\frac{EC}{BC}=\frac{ST}{BT}=\frac{r}{BT}=\frac{AT\cdot BT}{2R\cdot BT}=\frac{AT}{AB}=\frac{t}{L}.$$

On the other hand, the angle-bisector theorem in triangle ABC gives

$$\frac{AE}{EC}=\frac{AB}{BC}=\frac{L}{BC}.$$

Multiplying the last two relations yields $AE=t=AT$, proving (4).

Write

$$u=\frac{t}{L},\qquad v=\frac{BC}{L}.$$

The similarity above gives $EC=u\,BC=u(vL)=uvL$, while (4) gives $AE=t=uL$. Since A,E,C are collinear with E between A and C,

$$AC=AE+EC=uL+uvL=uL(1+v),$$

and hence

$$\frac{AC}{L}=u(1+v).$$

Since $\angle ACB=90^\circ$, the Pythagorean theorem gives

$$1=\left(\frac{AB}{L}\right)^2=\left(\frac{BC}{L}\right)^2+\left(\frac{AC}{L}\right)^2=v^2+u^2(1+v)^2.$$

Hence

$$(1-v)(1+v)=u^2(1+v)^2,$$

and therefore

$$v=\frac{1-u^2}{1+u^2}.$$

This is (2). Substituting back gives (3). □

Corollary 3.1. *If*

$$\frac{AT}{AB}=\frac{m}{n},\qquad 0<m<n,$$

then

$$AC:BC:AB=2mn:(n^2-m^2):(n^2+m^2).$$

Thus the generalized configuration gives Euclid's parametrization of Pythagorean triples. The original problem is the case $m:n=1:2$, giving

$$AC:BC:AB=4:3:5.$$

Proof. Substitute $t/L=m/n$ into (2) and (3) and clear denominators. □

Remark. *The identity $AE=AT$ is a particularly simple geometric feature of the generalized configuration. It also shows directly why moving the contact point T produces the familiar rational parametrization of right triangles.*

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REFERENCES

[1] Hiroshi Okumura, *Problems 2026*, Sangaku Journal of Mathematics, 10 (2026), 31–32.