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A Solution and Generalization of Problem 2026–3

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Abstract. We give an elementary solution to Problem 3 of Hiroshi Okumura's *Problems 2026* after adding a tangency condition needed to determine the configuration. We then replace the four congruent circles in that problem by a chain of n congruent circles. If the additional circle touches the second circle from an end of the chain, the original ratio 4/9 becomes $4/(n-1)^2$.

Keywords. tangent circles, chord, circle chain, sangaku, Japanese temple geometry.

Mathematics Subject Classification (2020). 51M04.

1. The problem

Okumura [2] presented four problems taken from a Japanese source [1] edited by Yasuaki Aida in 1812. Problem 3 concerns a circle α, a chord AB, four congruent circles $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ on one side of AB, and a congruent circle γ on the other side. The circles γ_1 and γ_4 touch α, consecutive circles in the chain touch, and γ is the maximal circle touching AB and α internally. A circle β touches AB and α internally. If b and c are the radii of β and γ, respectively, the problem asks to prove

$$c = \frac{4}{9}b.$$

As printed, the statement does not say that β touches γ_3. Without that condition the radius b is not determined. We therefore add the condition that β touches γ_3.

The following generalization contains the problem as the case $n = 4$.

2. The generalization

Let $\gamma_1, \ldots, \gamma_n$ be congruent circles of radius c, all tangent to the chord AB on the same side and with consecutive circles tangent. Suppose that γ_1 and γ_n touch α internally. On the opposite side of AB, let γ be the maximal circle tangent to AB and α internally; assume that it too has radius c. Finally, let β be the circle, other than γ_n, that lies on the same side of AB as the chain and is tangent to AB, tangent to α internally, and tangent to γ_{n-1} externally.

Theorem 2.1. *For $n \geq 4$,*

$$\boxed{\frac{c}{b} = \frac{4}{(n-1)^2}},$$

where b is the radius of β.

Proof. Let O be the center of α, let H be the foot of the perpendicular from O to AB, and put

$$OH = h.$$

Let R be the radius of α. Since the circle γ has radius c and is maximal on the side of AB opposite the chain, its center lies on OH. Thus internal tangency of γ and α gives

$$R = h + 2c. \tag{1}$$

Let C_n be the center of γ_n, and let T_n be its point of contact with AB. The centers of γ_1 and γ_n are both at distance c from AB and at distance $R - c$ from O, so they are reflections of each other in OH. Since consecutive points of contact of the chain with AB are $2c$ apart,

$$HT_n = (n-1)c.$$

Also $C_nT_n = c$ and, because γ_n touches α internally,

$$OC_n = R - c = h + c.$$

Hence the right triangle obtained by resolving OC_n parallel and perpendicular to AB gives

$$((n-1)c)^2 + (h-c)^2 = (h+c)^2.$$

Here we have used that the chain lies on the same side of AB as O. It must: were it on the opposite side, γ would be the maximal circle on the side of O, so that $c = (R+h)/2$ and $OC_n = R - c = c - h$, and the same computation would give

$$((n-1)c)^2 + (h+c)^2 = (c-h)^2,$$

that is, $(n-1)^2c^2 = -4hc$, which is impossible. Therefore

$$(n-1)^2c^2 = 4hc,$$

and so

$$\boxed{h = \frac{(n-1)^2}{4}c}. \tag{2}$$

Let Q be the center of β and T its point of contact with AB. Write $HT = x$, taking the direction toward γ_n as positive. Since β touches α internally,

$$OQ = R - b = h + 2c - b.$$

Consequently,

$$x^2 + (h-b)^2 = (h+2c-b)^2,$$

which reduces to

$$x^2 = 4c(h+c-b). \tag{3}$$

We shall also use the familiar fact that if two externally tangent circles of radii r and s touch the same line on the same side of the line, then the distance between their points of contact with that line is

$$2\sqrt{rs}. \tag{4}$$

Indeed, the line joining their centers has length $r+s$ and perpendicular component $r-s$, so (4) follows at once from the Pythagorean theorem.

The point where γ_{n-1} touches AB is at distance $(n-3)c$ from H, on the side of γ_n. Hence (4), applied to β and γ_{n-1}, gives

$$\bigl|x-(n-3)c\bigr| = 2\sqrt{bc}. \tag{5}$$

Put

$$s = \sqrt{\frac{b}{c}}.$$

Using (2), equation (3) becomes

$$\frac{x^2}{c^2} = (n-1)^2 + 4 - 4s^2, \tag{6}$$

while (5) gives

$$\frac{x}{c} = (n-3) \pm 2s. \tag{7}$$

Substituting (7) into (6) and simplifying, the upper sign yields

$$2s^2 + (n-3)s - (n-1) = 0,$$

that is,

$$(s-1)\bigl(2s+(n-1)\bigr) = 0.$$

Its only positive root is $s=1$; this is the solution $\beta = \gamma_n$, for which $x = (n-1)c$.

The lower sign yields

$$2s^2 - (n-3)s - (n-1) = 0,$$

which factors as

$$\bigl(2s-(n-1)\bigr)(s+1) = 0.$$

Since $s > 0$,

$$s = \frac{n-1}{2}.$$

Therefore

$$\frac{b}{c} = \frac{(n-1)^2}{4},$$

which is equivalent to the stated result. For $n \geq 4$ the two solutions found above are distinct, so this second solution is different from γ_n; for $n=3$ they coincide and no new circle arises. $\square$

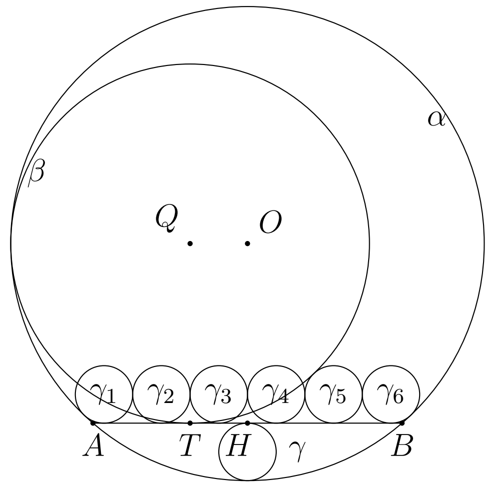

FIGURE 1. The generalized configuration for $n = 6$; the points O, Q, H, and T are those used in the proof.

For the first few values of n, the theorem gives

n	4	5	6	7	8
c/b	4/9	1/4	4/25	1/9	4/49.

Thus Aida's ratio 4/9 is the first member of the sequence

$$\frac{4}{3^2}, \frac{4}{4^2}, \frac{4}{5^2}, \frac{4}{6^2}, \ldots.$$

Remark. *The upper-sign root in the proof is precisely the circle γ_n itself. The lower-sign root, corresponding to the nontrivial circle β, also gives two simple geometric consequences. From (2) and the theorem,*

$$b = h = OH,$$

so the center Q of β is level with O and $OQ \parallel AB$. Moreover, (7) gives $x = -2c$: the point where β touches AB lies at distance $2c$ from H, on the side away from γ_n, independently of n, and $OQ = 2c$.

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